PROCEEDINGS OF THE 48TH ANNUAL MEETING OF THE NORTH AMERICAN CHAPTER OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION



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Proceedings of the Forty-Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

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NORTH AMERICAN CHAPTER OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

PME–NA History and Goals

The International Group for the Psychology of Mathematics Education (IGPME) was founded in 1976 at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany, with the first conference held in 1977 in Utrecht in the Netherlands. The North American Chapter of PME (PME-NA) was organized not long after with the first conference held in Evanston, Illinois in 1979. This initial meeting was organized and chaired by Dr. Richard Lesh following the initial National Council of Teachers of Mathematics (NCTM) Research Presession held in 1978 (Boston, Massachusetts). The plenaries at the NCTM meeting (Heinrich Bauersfeld, Efraim Fischbein, & Hans Freudenthal) were instrumental in the founding of IGPME and encouraged Lesh and others to create a North American chapter. The initial meeting in 1979 may have only featured plenary speakers. However, with the second meeting of PME-NA being a joint meeting with IGPME, individual papers published in the conference proceedings became a mainstay of the conference. Early conferences for both IGPME and PME-NA were focused on exploration of various areas of psychology in the teaching and learning of mathematics. Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically oriented foundations.

The major goals of the International Group and the North American Chapter are:

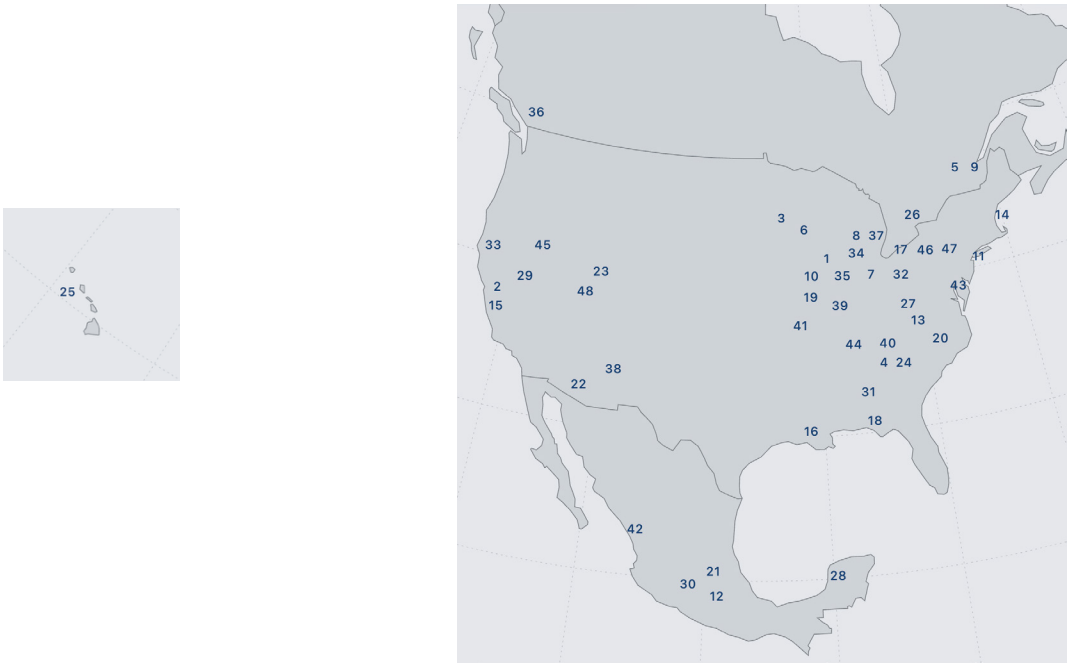
1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

PME–NA Membership

Membership is open to people who are involved in active research consistent with PME-NA's aims or who are professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at http://pmena.org. For information about membership in PME, go to http://www.igpme.org and visit the "Membership" page.

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PME–NA Conference Locations



1979	Evanston, Illinois	2003	Honolulu, Hawaii*
1980	Berkley, California*	2004	Toronto, Canada
1981	Minneapolis, Minnesota	2005	Roanoke, Virginia
1982	Athens, Georgia	2006	Mérida, Yucatán, México
1983	Montréal, Canada	2007	Lake Tahoe, Nevada
1984	Madison, Wisconsin	2008	Morelia, Michoacán, México*
1985	Columbus, Ohio	2009	Atlanta, Georgia
1986	East Lansing, Michigan	2010	Columbus, Ohio
1987	Montréal, Canada*	2011	Reno, Nevada
1988	Dekalb, Illinois	2012	Kalamazoo, Michigan
1989	New Brunswick, New Jersey	2013	Chicago, Illinois
1990	Oaxtepec Morelos, México*	2014	Vancouver, Canada*
1991	Blacksburg, Virginia	2015	East Lansing, Michigan
1992	Durham New Hampshire*	2016	Tucson, Arizona
1993	Pacific Grove, California	2017	Indianapolis, Indiana
1994	Baton Rouge, Louisiana	2018	Greenville, South Carolina
1995	Columbus, Ohio	2019	St. Louis, Missouri
1996	Panama City, Florida	2020	Mazatlán, Sinaloa, México
1997	Normal, Illinois	2021	Philadelphia, Pennsylvania
1998	Raleigh, North Carolina	2022	Nashville, Tennessee
1999	Cuernavaca, Morelos, México	2023	Reno, Nevada
2000	Tucson, Arizona	2024	Cleveland, Ohio
2001	Snowbird, Utah	2025	State College, Pennsylvania
2002	Athens, Georgia	2026	Provo, Utah

* IGPME / PME-NA Joint Conference

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PREFACE

The forty-eighth annual meeting of the International Group for the Psychology of Mathematics Education – North American Chapter was held October 11-14, 2026 on the campus of Brigham Young University in Provo, Utah, USA.

Contributions and Community

As shown in Table 1, 361 proposals were submitted for consideration. Each submitted paper was reviewed by multiple reviewers in a double-blind process. Strand Leaders examined reviewer feedback and made acceptance recommendations as well as for accepted session type. In the end, 93 posters (85% acceptance rate), 123 brief reports (77% acceptance rate), 92 full reports (60% acceptance rate), and 14 working groups / research colloquia (100% acceptance rate) were accepted.

The conference was attended by 217 faculty, 113 students, and 65 virtual registrants. These 395 participants came from the United States, México, Canada, Columbia, Ghana, Hong Kong, and Republic of Korea.

Table 1: Conference Paper Proposal Statistics

Session Type	Submitted	Scheduled	Accepted as is	Rejected	Accepted in some form
Full Research Reports	153	92	92 (60%)	12 (8%)	141 (92%)
Brief Research Reports	116	123	~89 (77%)	15 (13%)	101 (87%)
Posters	78	93	66 (85%)	12 (15%)	66 (85%)
Working Groups	13	13	13 (100%)	0	13 (100%)
Research Colloquia	1	1	1 (100%)	0	1 (100%)
Total	361	322	~261 (72%)	39 (11%)	322 (89%)

Gratitude

We are grateful to every individual who contributed to the success of the conference. The quality of the program is due the efforts of many. Reviewers (n=312) performed a critical scholarly service. Strand Leaders (n=18) assigned reviewers, examined reviews, and made recommendations for acceptance. Past conference chairs, current and past Steering Committee members, and various other colleagues provided insight, feedback, and support that contributed to the program and to the overall experience of the conference community. Many thanks to BYU undergraduates Brooke Francis, Abigail Marchant, and Katelyn Remund, who together copyedited every paper in this proceedings. And finally, thanks to Michael Holmes and his team for helping to organize the conference from beginning to end, and for creating such a wonderful, welcoming environment at the BYU Conference Center.

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PLENARY PAPERS

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QUANTITATIVE REASONING AND REDEFINING THE LEARNING-TEACHING-RESEARCH PLAYING FIELD

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Since Thompson formalized quantitative reasoning (QR), the field has extensively documented QR's role in student learning but has done comparatively less to develop analogous accounts for teaching. I argue redefining the learning-teaching-research relationship requires grounding teacher development in the "mathematics of students"—models constructed studying student thinking. Drawing on work with teachers and students, I describe how teachers' extant meanings can act as an insufficient foundation for constructing QR-based mathematics of students, and I introduce a perspective for addressing this phenomenon. I extend this discussion by connecting to decentering and arguing it is a cognitive activity teacher educators must determine how to cultivate. Collectively, I call for more attention to what it means to build teaching and teacher education "from the ground up" on the foundation of students' mathematics including their QR.

Keywords: Quantitative Reasoning, Mathematical Knowledge for Teaching, Decentering

Thompson's initial framework on *quantitative reasoning* (*QR*) defined it as, "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (Thompson, 1989, p. 13). In it, he underscored that readers should interpret the definitions as operational constructs in a system; he intended the definitions support researchers and educators in explaining mathematical ideas and student thinking. Researchers have fittingly extended his theory across the teaching and learning of mathematics at all levels (see Karagöz Akar et al., 2022 for a collection of examples). In doing so, they have identified QR's importance for students' mathematical development while simultaneously identifying the need to understand how to create educational experiences in which students' QR is honored and embraced.

The given directive for this paper was to identify future directions for QR research with respect to this need. I find addressing this directive awkward for three reasons. First, I believe future directions emerge organically. Future directions are not manufactured or created out of thin air. They reveal themselves in the process of doing research and they are realized by pursuing unexpected phenomena. So, look to those surprising or thought-provoking moments within your work for inspiration that opens future directions. Second, novel, unforeseen future directions are often introduced by individuals new to a field. The naivety and inexperience of those entering a field afford a perspective that is simultaneously blind to current research trends and not yet entrenched in a particular viewpoint. So, look to the newer entrants in the field for innovative and unexpected future directions.[1] Third, I consider much of my work to be a derivative of the work produced by those who laid the foundation for QR research. These individuals include Marilyn Carlson, Patrick Thompson, and Les Steffe, to name a few. They created and advised research directions in which I and others found footing. Researchers following their directions have also created new foci requiring additional work. This makes proposing additional future directions seem superfluous. So, look to the founders of QR for

[1] An incomplete list: Hwa Young Lee, Irma Stevens, Claudine Margolis, Amy Smith, Teo Paoletti, Halil Tasova, Kristen Frank, Joseph Antonides, Cameron Byerley, Sara Bagossi, Robert Knurek, Chris Bloom, Allie Olshefke-Clark, Anne Waswa, Yosep Dwi Kristanto, Erin Wood, Jennifer Czocher, Franklin Yu, and Hamilton Hardison.

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productive future directions. Nonetheless, what follows are my musings about areas that need to be fleshed out based on my empirical and classroom work with students and teachers.

Quantitative Reasoning and the Mathematics of Students

Although Thompson is chiefly responsible for the formal theorization of it, QR has origins in the history of human and child experience. Whether it was the Greeks' explorations of incommensurable magnitudes, Egyptians' and Hindus' construction of numeral systems, or Arabians' trigonometric studies, QR served as a tool for humans to organize and problem-solve everyday experiences (Eves, 1990; Thompson, 2011). QR is also an authentic part of children's experiential realities (Piaget, 1952; Piaget & Inhelder, 1974; Steffe, 1991). Accordingly, the specific researcher and educator activities Thompson intended his theory to support are building and using *second-order models of students' mathematics* (Thompson, 2013).

Second-order models of students' mathematics are models researchers build to explain the observable actions of students during mathematical activity (Steffe, 2004; Steffe & Thompson, 2000; Steffe et al., 1983; Thompson & Thompson, 1996; Thompson, 2008, 2013; Thompson & Saldanha, 2000). Steffe and Thompson formalized this construct to acknowledge a researcher only has access to their own mathematical thinking and thus building a viable model of student thinking requires acting in intentional and analytical ways. A researcher remains permanently skeptical as to the possibility of using their own ways of thinking to explain the thinking of a student, and they do this by designing settings in which they intentionally restrain aspects of their own mathematics (Moore et al., in press; Steffe & Thompson, 2000). For instance, a researcher iteratively seeks out unexpected or novel student behavior that breaks their model. That places the researcher in a position to construct new mathematics through the student's actions. The opportunity to construct new mathematics that they can attribute to a student is what makes second-order model building an exciting venture (Steffe & Thompson, 2000).

Because a researcher expects to construct new mathematics when modeling student thinking, second-order models are not to be bound or framed by institutionalized knowledge and learning goals found in common curricula (Steffe & Thompson, 2000). In fact, a researcher constructing second-order models should expect to generate a mathematics that has incompatibilities with long-established learning goals. Learning goals are developed by adults and thus constrained by their mathematics. Second-order models mitigate the incompatibilities existing between adults' and students' mathematics. Second-order models of students' mathematics act as our best proxy for students' mathematics, which are made more viable through iterative research.

Second-order models of students' mathematics are termed *mathematics of students* (Steffe & Thompson, 2000). For the remainder of this paper, I use mathematics of students to refer to second-order models of students' mathematics. I use *QR-based meanings* throughout to refer to mathematics of students that involve QR. The goal for this paper is to discuss pursuing the ideal of teachers teaching in ways sensitive to the mathematics of students. I underscore that when I discuss teachers constructing and leveraging QR-based meanings, the QR-meanings I refer to were co-constructed by researchers and students.

Using Quantitative Reasoning to Further Mathematical Knowledge for Teaching

At the 2005 annual meeting, Mewborn (2005) argued, "By and large, I do not yet see robust frameworks in the areas of mathematics teaching and teacher education that would parallel the frameworks in the area of learning" (p. 6). Her view remains true two decades later. The field has gained knowledge understanding the phenomena of teaching and teacher education. For instance, the introduction of *mathematical knowledge for teaching* (*MKT*) frameworks (e.g., Ball et al.,

2008; Hill et al., 2008; Silverman & Thompson, 2008; Thompson & Thompson, 1996; Thompson, 2016) has established several informative research strands. These frameworks and associated strands, however, are not as robust as their counterparts in learning. The former lack the type of developmental accounts found in the latter (Carlson, Bas-Ader, et al., 2024; Kennedy, 2016; O'Bryan, 2025; Rocha, 2023; Tallman, 2023; Thompson, 2016). Again, Mewborn's (2005) words mostly remain true: "we have predictive frameworks in the area of student learning, but we are still at the stage of explanatory frameworks in the area of teaching" (p. 5).[2]

Developmental accounts of MKT are less prevalent than those of student learning partially due to MKT being a more recent research development. If we assume that developmental accounts of MKT paralleling those of student learning are possible, their construction is imperative. Such accounts will position us to better understand teachers' realities and how to collaborate with them to improve their practice. In what follows I provide two considerations for this pursuit. I begin with a need for developmental accounts that incorporate teachers' extant knowledge in relationship with QR-based meanings. I then center the mathematics of students in MKT development with attention to QR's role in this development.

Extant Meanings and their Foundation for Learning

It is important that an educator consider an individual's extant meanings no matter with whom they are working. For instance, an educator intending to support a student's development of fractional reasoning must give significant attention to the ways in which the student can construct and coordinate units; the operations and meanings a student has available constrains how they conceive of a situation and thus the potential novel operations and meanings they can construct (Hackenberg, 2007; Hackenberg & Tillema, 2009; Norton, 2008; Steffe & Olive, 2010; Ulrich, 2014). Likewise, the operations and meanings a teacher has available constrain the novel operations and meanings they can potentially construct.

In my experience working with teachers, the relationship between their extant meanings and those QR-based meanings I intend for us to co-construct primarily occurs in two forms. In one form, their extant meanings provide a sufficient foundation for QR-based meanings. For example, reasoning about measurement as a multiplicative comparison between a magnitude and a unit provides a good foundation to understand the unit circle as representing every circle at once (Moore et al., 2016). In a second form, extant meanings provide an insufficient foundation for constructing QR-based meanings. I focus this section on this second form.

A meaning is an insufficient foundation if it does not afford a path of accommodations through which it is modified into an intended meaning.[3] Meanings that are an insufficient foundation for intended meanings can also impede the construction of them. An extant meaning can act as an impediment due to it being habitual to the point a teacher experiences difficulty reasoning in other ways. Or it can act as an impediment due to incompatibilities between it and the intended meaning. Regardless, such cases are complex because it is important to account for and honor their extant meanings while also determining how to support intended meanings (Simon, 1995). This is especially true when intended meanings reflect the mathematics of students. My conviction on this point is empirically driven.

[2] Comparing MKT perspectives is beyond the present scope. I hope that the underlying considerations I provide resonate with readers regardless of MKT commitment. I note that Silverman and Thompson (2008) defined MKT in a way that identifies potential avenues for its development.

[3] It is important that a researcher be deliberate in their characterizing a meaning as an insufficient foundation. I return to this point below to underscore they should only do so through much empirical work in which they skeptically explore the meaning's affordances and constraints.

Across two decades of work with teachers (Carlson, O'Bryan, et al., 2024; Carlson et al., 2013), my colleagues and I frequently experienced instances in which teachers' extant meanings impeded their constructing QR-based meanings (Moore et al., 2026). And when they constructed QR-based meanings, they often privileged their prior meanings both outside and inside their practice.[4] We developed the *competing meanings* perspective as a proof-of-concept response to this phenomenon (Figure 1). Competing meanings is a three-phase cognitive process: first, a teacher's extant meaning is problematized; second, the teacher constructs an alternative meaning to reconcile that perturbation; and third, the teacher reflectively compares the extant and alternative meanings, ultimately critically comparing the utility of the meanings.

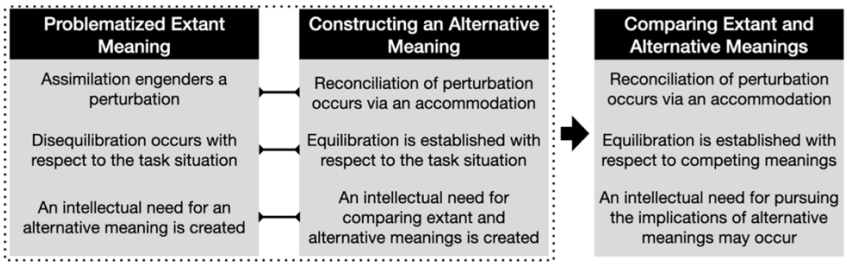


Figure 1: Three Phases of the Competing Meanings Perspective (Moore et al., 2026, p. 6)

When the alternative meaning is an intended meaning reflecting the mathematics of students, the teachers are supported in constructing a meaning pivotal to students' mathematical development. That is, teachers are supported in constructing meanings consistent with Simon's (2006) *key developmental understanding* (*KDU*), which are the meanings we should desire teachers to hold and use. In Moore et al. (2026), we illustrated the competing meanings perspective using a QR-based KDU of function and their graphs. This KDU—based on decades of QR research (e.g., Carlson et al., 2002; Ellis, 2011; Oehrtman et al., 2008; Paoletti & Moore, 2018; Paoletti et al., 2024; Thompson, 1994; Thompson & Carlson, 2017)—foregrounds quantities as simultaneously varying in an invariant relationship. An implication of this KDU is that a function and its inverse are understood as consequences of the same underlying relationship. Because a function and its inverse are derived from the same underlying relationship, representations including graphs and formulas are conceived as capturing it. Each of $y = 3x$, $x = (1/3)y$, and $y - 3x = 0$ symbolizes the same underlying relationship between covarying quantities. Each of the displayed graphs in Figure 2 is an emergent record of the same covariation. Defining the graphs as $y = 3x$ necessarily implies the graphs are defined by $x = (1/3)y$ and $y - 3x = 0$. Although their representational forms can be conceived in terms of multiple quantitative structures (Stevens, 2019, 2026), all representations capture each structure, including that each quantity's change exists in a proportional relationship with the other.

The QR-based KDU can be compared with the common extant meaning of conceiving functions and their graphs as memorized pairings of formula templates, graphical shapes, and function categories or properties (Moore, 2021). To illustrate, teachers frequently reject Figure 2b and Figure 2c as representations of $y = 3x$ because they do not adhere to their memorized pairings. With respect to Figure 2b, they infer the graph cannot be correct because of the location

[4] This was an expected outcome given research on teacher development and teacher change. Their extant meanings have been developed, used, entrenched, and become habitual over years of experience. Additionally, they teach within a system in which they perceive numerous pressures and directives to satisfy.

of x and y (Moore, Silverman, et al., 2019). With respect to Figure 2c, teachers deem it to have a "negative slope" because of its perceptual direction (Moore, Stevens, et al., 2019). In each case, they anticipate a memorized formula-shape-slope pairing and reject a representation that does not belong. Furthermore, their meanings are often incompatible with the QR-based KDU. This produces a tension when teachers construct and explore QR-based meanings. For instance, in Moore, Stevens, et al. (2019) we illustrated a teacher who recognized Figure 2c does contain the points defined by $y = 3x$. She also required that the graph be represented by $y = -3x$ because of its direction. She rejected the graph, but this discrepancy left her confused because she was unable to reconcile it on her own accord; she perceived her meanings to be incompatible with each other. She expressed emphatically that she did not understand the underlying cause of this incompatibility. Her specific words were, "this is so annoying." It is a perturbation like this that we target with the competing meanings perspective. A teacher's realization of an incompatibility they cannot reconcile affords transformative learning opportunities.

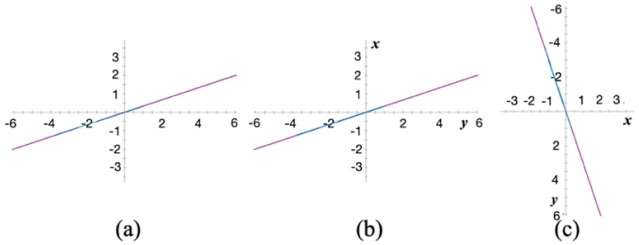


Figure 2: Three Displayed Graphs of the Same Covariational Relationship

The finer details of the competing meanings perspective are in Moore et al. (2026). I raise it to underscore that researchers and educators have an opportunity to create transformative learning experiences by critically considering whether teachers' extant meanings are sufficient for the development of intended meanings. In the context of QR-based meanings, their dominant meanings are typically not, which inhibits their ability to develop meanings researchers have derived from interacting with students.[5] This byproduct of their extant meanings requires that we, as teacher educators, find ways to perturb those meanings and create a need for the construction of QR-based meanings. Furthermore, these perturbations must occur in contexts they view as relevant to their students and practice so that the QR-based meanings have implications for their teaching (Moore et al., 2026; Wasserman et al., 2022).

I close this section by admitting that the incompatibility existing between teachers' extant meanings and QR-based meanings is natural. The incompatibility is not some deficit on the part of a teacher; it's a feature inherent to knowledge development being never-ending. The mathematics of students is not something that traditional curricula reflect, nor is it something that current teachers have likely been exposed to (Simon, 2006; Thompson, 2013). Complicating the matter, the mathematics of students is an always evolving body of mathematical knowledge. As further research occurs, novel models of students' mathematics replace or modify extant models. This is a powerful feature of the mathematics of students, yet its perpetual evolution requires supporting teachers in ongoing learning. We therefore do teachers and their students—both our

[5] I use "dominant" here to imply that, in my experience, teachers technically have the operations available to construct QR-based meanings, but those operations have not been developed, nurtured, and drawn upon repeatedly. They are thus subordinate to extant meanings and their use is somewhat (in their words) foreign and unnatural.

colleagues in co-constructing mathematics—a disservice by not problematizing the compatibility between teachers' extant meanings and the mathematics of students. We should create opportunities for incompatibilities to come to the surface in research and teacher education settings so they can be understood and addressed. The competing meanings perspective offers one proof-of-concept for creating such opportunities. Surely there are additional avenues and perspectives by which researchers and educators can accomplish this goal.

Quantitative Reasoning and Understanding the Mathematics of Students

During a pre-conference podcast, Leatham proposed a principle teacher educators should develop in teachers: "We should care more about our students' mathematics than our own mathematics" (Jones, 2026). His principle is one all educators should share. The question a teacher educator might ask is, "How do we get teachers to care about students' mathematics more than their own?" I contend there is a more productive framing question for his principle.

I anticipate most teachers would answer "Yes" if asked whether they care about their students' mathematics. If asked whether they care more about their students' mathematics than their own, I anticipate a majority would also answer "Yes." Teachers care deeply about their students' mathematics. How they operationalize that care is where we play a role as teacher educators and researchers. A productive framing question is thus, "What cognitive activity is involved in caring about students' mathematics more than one's own?" And, with respect to that question's answer, "How do we support teachers in developing that cognitive activity?"

Caring about students' mathematics. I earlier noted, "the operations and meanings a teacher has available shape how they conceive any situation or topic and they constrain their potential construction of novel operations and meanings." The statement occurred in the context of constructing intended meanings, but it can be modified to be a foundation for caring about students' mathematics. Caring about students' mathematics means attributing to them a mathematics of their own and working to understand and learn from that mathematics (Hackenberg, 2010a). The operations and meanings a teacher has available shape how they understand and learn from their students' thinking.

The act of attributing a perspective of their own to someone else and attempting to understand that perspective is generally termed *decentering*. Piaget first introduced decentering (or *decentration*) to characterize a child's transition from being egocentric to being able to consider alternative perspectives, whether those perspectives be the physical viewpoint of someone or their thoughts, feelings, and experiences (Chapman, 1988; Piaget, 1955; Piaget & Inhelder, 1967). Mathematics educators have since adapted the construct to describe the cognitive activity of constructing and acting upon a model of another's mathematics (i.e., caring about another's mathematics). Decentering is the cognitive mechanism that drives conducting a teaching experiment to build models of students' mathematics (Steffe & Thompson, 2000). Teuscher et al. (2016) identified that understanding teachers' decentering actions is necessary to explain the ways of thinking that promote quality classroom discourse against the backdrop of the mathematics of students. Hackenberg (Hackenberg, 2010a, 2010b; Hackenberg et al., 2024) extensively discussed the relationship between decentering, caring relations, and acts of equity. Ellis (2022) echoed this relationship during a recent annual meeting of this organization and in the context of learning trajectories. Still other researchers (Baş-Ader & Carlson, 2022; Carlson, Bas-Ader, et al., 2024) have detailed decentering's relevance and importance in research, teaching, and teacher education, ultimately developing an associated framework for its use. So, what of potential relationships between QR and decentering? To illustrate, I return to the competing meanings perspective and the Figure 2 graphs. I also consider the Figure 3 graphs.

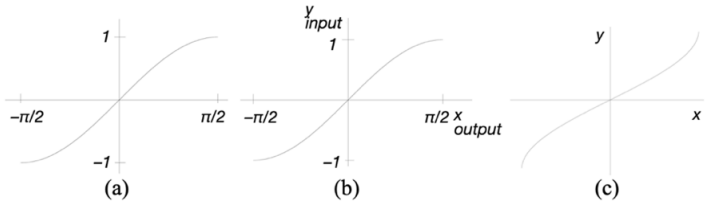


Figure 3: Three Displayed Graphs of the Sine/arcsine Relationship

Our research team has used the graphs in Figure 2 and Figure 3 across multiple studies to explore teacher meanings in relationship to practices we perceive as conventional (Moore, Silverman, et al., 2019; Moore et al., 2026). The graphs target the practices of input or x on the horizontal axis and positive y-values oriented upward. Our typical task progression involves first posing Figure 2a or Figure 3a to teachers as a student created solution to graphing $y = 3x$ or $y = \sin(x)$, respectively. We ask them to determine how a student might have been thinking to produce the solution. We ask them to determine as many ways as possible. We then provide the corresponding Figure 2b or Figure 3b and explain that a student clarified their solution by labeling the axes to indicate their thinking. We prompt the teachers to determine their thinking and comment on its validity. In some cases, we then provide the corresponding Figure 2c or Figure 3c for them to consider in the same manner. We eventually pose, "Well, what if a student claims that this graph (Figure 2c) has a negative slope so it can't be $y = 3x$?" Or, "Another student says they know this graph (Figure 3c) is *also* arcsine. Are both students correct?" Table 1 provides the two predominant response categories during these task sequences.

Table 1: Two Categories of Responses

Category	Response	Figure
I	Ohhhh… this graph is… y is three times bigger than x… x is one, y is three times bigger. Yes. Where x is two, y is three times bigger. So this graph is correct… y is three times bigger than x.	2b
	A clever kid… technically is y equals three x… not the standard way of doing it… They see the relationship between x and y.	2b
	He graphed it completely right. That's y equals three x… He just has a different perspective than… normal classroom settings. But I think it's smart of him to understand that it's not glued.	2b
	They are changing I guess what the input and output is… that is the inverse… with y as your input… this would be you know one equals sine x… you know x has to be pi over two.	3b
II	I feel like you should know your x and y, and like, know which one is which. And, yeah, you're going to get it all wrong I think.	2b
	They messed up the placement of x and y… [*rotating Figure 2b to Figure 2c*]… If you are looking at it this way, it's a negative slope and it should be a positive slope… slope is wrong.	2b-c
	I feel like he's missing the whole concept of a graph… a sine graph's… a graph like everyone knows about, you know.	2b
	This just kind of looks like the sine graph… the graph of an inverse function, like, can't be the same as the original graph.	2b

Category I responses and Category II responses reflect the prior section's KDU and corresponding extant meaning, respectively. Each Category I response involves the teacher conceiving the displayed graph as a consequence of tracking covarying quantities. Each Category II response involves the teacher conceiving the displayed graph in terms of memorized perceptual features and associations with those features. The two categories differ in the extent to which the teachers attribute a viable meaning to the student solution. Category I responses involve a teacher engaging in QR to attribute a meaning that could produce the given graph, while also valuing and considering it correct. Category II responses focus on the incorrectness of the solutions. Category II responses also foreground deficits in student meanings via the teacher imposing their image of a correct solution to identify things not upheld by the student.

Category I responses are consistent with decentering and Category II responses are not. In using tasks like those above, we consistently find teachers who foreground QR-based meanings are better supported in decentering when encountering nonnormative or unconventional student solutions and reasoning (Moore, Silverman, et al., 2019; Moore et al., 2026; Teuscher et al., 2016). Our findings are echoed by several researchers who have illustrated that QR-based meanings are critical to teachers constructing a coherent image of their instruction including interpreting or anticipating student thinking (Carlson et al., 2022; Johnson et al., 2022; Musgrave & Carlson, 2017; O'Bryan, 2025; Rocha, 2023; Tallman & Frank, 2020; Tallman et al., 2024; Thompson, 2016). This finding is sensible because QR-based meanings afford conceiving equivalence among forms of reasoning and representations that initially appear different. More generally, QR-based meanings are sufficient for accommodating to novel situations due to their operations being better suited for processes of abstraction (Moore et al., 2022; Moore et al., 2024). Returning to the examples in Table 1, QR-based meanings enabled the teachers to view the students' unconventional representations as stemming from the same QR driving anticipated conventional solutions. The other teachers merely focused on missing elements of the solutions and explaining that those absences implied the solution is incorrect.

Returning to the classroom, teachers holding QR-based meanings is important because students will create nonnormative, evolving representations in an environment that affords students' QR. diSessa et al.'s (1991) work on students' graphing inventions is an apropos example of this phenomenon. Students' QR is complex, developmental, and not always readily apparent in their solution products (Izsák, 2000, 2003; Kularajan et al., 2024; Moore & Carlson, 2012; Smith III & Thompson, 2007; Thompson & Thompson, 1996). Furthermore, authentic representational acts in a (classroom) community should involve negotiating representational practices for the purpose of establishing convention (diSessa et al., 1991; Moore, Silverman, et al., 2019; Ristroph, 2025; Thompson, 1992, 1995). The implication is that teachers must be prepared to persistently inquire into students' reasoning, identify and foreground their underlying QR, and aid them in ongoing negotiations of representational practices that convey their QR. Such teacher actions are only feasible if teachers develop the QR-based meanings and the decentering actions necessary to understand and act upon their students' thinking. And that turns us to the question, "How do we support teachers in developing that cognitive activity?"

Supporting caring about students' mathematics. The competing meanings perspective provides one answer to this question. We have found success in promoting teachers' inclination to engage in decentering alongside their construction of QR-based meanings using tasks like those discussed above. Their interacting with student products that are nonnormative or violate conventions and yet are created via QR-based meanings acts as a catalyst for the construction of those meanings. And by learning from the students' solutions, they come to value the creativity

and novelty of student reasoning in a way they can operationalize; they come to value putting aside expectations for normative or conventional solutions for the purpose of looking for creative reasoning that underlies students' solutions (see Table 1, Category I). They also express an affective response to identifying creative student thinking, including excitement.

The competing meanings perspective is only a partial response because it is limited in its scope. The cognitive processes associated with the competing meanings perspective only form a foundation for MKT construction. The development of MKT requires that those meanings and decentering orientations be transformed into meanings and orientations with pedagogical power, and a catalyst for this transformation is teachers' decentering activity through sustained work with students (O'Bryan, 2025; Rocha, 2025; Tallman, 2023; Thompson, 2013, 2016). It stands to reason that developmental accounts of MKT that account for teachers' decentering will provide us a direction for shaping experiences that help a teacher "see" their students' mathematics and support QR-based meanings. In fact, recent research provides growing evidence that teachers' decentering is the critical mechanism behind MKT development. In addition to decentering sources provided above, I point the readers to additional detailed sources on this work.

Liang (2021, 2025) drew on theory and empirical work to detail how teachers can learn through their students at a depth that specifies the cognitive mechanisms and processes by which these occur. She also contributed a methodology for investigating teachers' decentering actions. Hodkowski and Smith (Hodkowski, 2018; Smith & Hodkowski, 2025) provided compelling accounts of teachers shifting from a focus on their personal mathematics to the mathematics of students and its impact on their images of instruction. O'Bryan (2025) and Rocha (2023) have each provided a detailed case study of how the development of QR acts as a vehicle for decentering and influences a teacher's ability to create productive learning opportunities for their students that also afford MKT development. Tallman and Frank (2020) and Tallman et al. (2024) identified the importance of teachers being reflectively aware of their meanings and QR, which is supported by decentering actions. Alongside a compelling account of how decentering enables a teacher to support their students' QR, Hackenberg et al. (2025) identified patterns of decentering actions involved in responsive inquiry that give pedagogical power to the mathematics of students. In summary, whether with respect to QR-based meanings or not, the collective work on teachers' decentering actions provides future directions for aiding teachers' development of MKT that productively shapes their care for students' mathematics.

Building on Mathematics of Students to Identify and Explore KDUs

The reader might be surprised by the focus on MKT in this paper. Much of my work foregrounds investigating cognitive meanings of mathematical ideas and providing complementary conceptual analyses, and MKT is hardly mentioned in that work. My goal, however, has always been to support teachers in transforming their practice and, ultimately, their students' learning. Accomplishing that goal requires research on the mathematics of students—who are sometimes teachers—and using that research to redefine the learning goals informing our curriculum, the teaching practices that target those goals, and the research settings within which we conduct our work.

My bias is no doubt shaped by my training with Marilyn Carlson and Patrick Thompson (among others). It is also shaped by empirical experiences with students and teachers inside and outside of research projects. Those experiences continue to reveal that individuals co-constructing mathematical ideas best do so when understanding and building on each other's thinking. In the context of settings like the classroom, it is also critical that they have resources available that support their ability to do so. Unfortunately, this is not the case with most K-16

settings in North America and elsewhere (Carlson et al., 2010; Clements, 2007; McKie et al., 2025; Thompson, 2013; Visnovska & Cortina, 2022). In fact, during my formative work with Marilyn Carlson (Carlson, O'Bryan, et al., 2024), we identified that without curricular materials capturing the mathematics of students, teachers grew exasperated attempting to decenter and effectively support their students' reasoning. The mathematics targeted by their curricular materials did not invite the depth of student engagement and student-teacher interactions necessary for the teachers' intentions or student interest.

Addressing this issue is complex and requires sustained efforts. Thompson (2013) provided some guidance in this regard, and other mathematics educators have painted images of aspirational educational systems and resources (e.g., Baş-Ader & Carlson, 2022; Ellis, 2022; Hackenberg et al., 2025; Liang, 2025; Simon, 1995; Steffe & D'Ambrosio, 1995; Steffe & Wiegel, 1992). The point I underscore to end this paper is that moving toward an educational system that foregrounds the mathematics of students requires that we be more critical of our extant learning goals, topical emphases, and content structuring. The mathematics of students represents a novel and always evolving knowledge body, and thus it should be a permanent starting point and benchmark for identifying and structuring the ideas we target in education (i.e., KDUs). Moreover, we must accept that these ideas can and will run counter to institutionalized and predominant approaches to topics, thus requiring that we break free from these approaches to honor our students' mathematics as a source for our learning and instructional design. As Adiredja and Louie (Adiredja, 2019; Adiredja & Louie, 2020) emphasized, transitioning away from institutionalized narratives and knowledge as superior to students' mathematics is necessary if we wish to focus on student assets, intelligence, and creativity.

As an abridged example, in my work with students and teachers across STEM contexts, I have come to reconceive the relationship between slope and graphing linear relationships. Whereas curricular materials and research predominantly approach slope as a key understanding for linear relationships, I find an alternative framing more productive. The alternative framing is treating slope and rate-of-change as KDUs, and thus each deserving of their own framing and QR-based meanings. It is productive for students to experience the quantification of slope through modeling the inclination of physical objects and the need for quantifying differences in direction. This leads to ideas of measuring slope through percent grade, angle of inclination, and the relative measure of orthogonal change, all subject to unit choice and scale. It is also productive for students to experience the quantification of constant rate-of-change through constructing quantities' covariation and identifying proportional change. This leads to ideas of measuring relative change and representing quantities' covariation in different representations. Neither slope nor rate-of-change is more important than the other, and certainly relationships can be drawn between them. Constant rate-of-change provides a specific measure of slope restricted to coordinate system constraints. Slope applied to a produced graph of a linear function results in the same measure as rate-of-change if using relative orthogonal change under the same scale as the coordinate system. But, as KDUs, rate-of-change quantifies quantities' covariation and slope quantifies an object's physical inclination or direction. I use this suggestive example to call for critically analyzing all mathematical ideas with respect to the reasoning they afford, the goals they accomplish, and the contexts in which they are the foregrounded concept. And, returning to the paper's title, I believe it is imperative that we do this "from-the-ground-up" using the mathematics of students if we want to truly redefine in our students' image the learning-teaching-research playing field we all enjoy.

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NAVIGATING THE PREPARATION PARADOX: HOW MATHEMATICS TEACHER EDUCATORS INTEGRATE GENERATIVE AI WHILE PRESERVING PROFESSIONAL JUDGMENT DEVELOPMENT

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Generative AI can produce content that appears pedagogically sophisticated, yet mathematics teacher educators face a paradox because the professional reasoning that pre-service teachers must develop is precisely what AI threatens to bypass. We report on an educator preparation provider collaboratory in which teacher educators, the majority in mathematics education, designed, implemented, and reflected on AI-integrated coursework for pre-service teachers. Through cross-case analysis of implementation deliverables and narratives, we identify convergent design principles and organize them in the STEP-AI framework (Developmental Sequencing (S), Structural Transparency (T), and Educational Propagation (EP)). Each of its three dimensions is grounded in use cases from mathematics education courses and corroborated by cases from adjacent preparation courses. Findings inform the practical ways of AI integration in teacher preparation without sacrificing mathematics knowledge for teaching.

Keywords: Teacher Preparation, Generative AI, Mathematical Knowledge for Teaching, Technology Integration

Objectives

When a pre-service mathematics teacher prompts an AI tool to "create a lesson on fraction division for 5th graders," the resulting output may include representations, sequencing decisions, and anticipation of student misconceptions that took the field decades to articulate as expert practice (Ball et al., 2008). This use of AI creates a fundamental tension for educator preparation programs (EPPs). The tool can produce artifacts that *look* like the products of mathematical knowledge for teaching (MKT) without requiring the user to possess or develop that knowledge.

Recent large-scale research on in-service K-12 teachers' authentic AI use established that teachers primarily leverage generative AI (GenAI or AI) as an "implementation bridge," enabling practices they already value but cannot scale within available time (Liu, 2026). Most interactions operated with experienced teachers maintaining professional judgment while offloading production work. This is reassuring for practitioners with established MKT but raises an urgent question for preparation. What happens when the tool user has not yet developed the professional judgment that enables them to distinguish productive use from cognitive offloading?

This question is complicated by a cascading dynamic unique to educator preparation. Teacher educators need to simultaneously develop their own AI integration approaches, model those approaches for pre-service teachers, and prepare pre-service teachers who will carry resulting practices into K-12 classrooms. Prior research thus documents how in-service teachers with established professional knowledge use AI, but little is known about how teacher educators

design AI-integrated coursework for pre-service teachers whose professional judgment is still forming. This gap motivates two research questions:

- How do teacher educators design AI-integrated coursework that balances AI's productive capacity with the development of pre-service teachers' pedagogical reasoning?
- What design principles emerge from implementation experience that inform mathematics teacher preparation's approach to AI integration?

Perspectives

We draw on three bodies of scholarship. First, research on mathematical knowledge for teaching (Ball et al., 2008) establishes that effective mathematics instruction requires specialized knowledge domains, including specialized content knowledge (SCK) and knowledge of content and students, which develop through engagement with mathematical ideas in pedagogical contexts. When a teacher selects fraction representations, anticipates that students will invert-and-multiply without understanding why, or sequences examples to build from measurement to partitive division, they exercise MKT. These decisions resist automation because they depend on knowledge of specific students and classrooms that GenAI cannot access.

Second, practice-based teacher education (Grossman et al., 2009; Lampert et al., 2013) demonstrates that professional judgment develops through structured approximations of practice, such as rehearsals, decompositions, and coached enactments. A pre-service teacher who receives an AI-generated lesson plan has consumed a product; one who iteratively refines prompts while articulating pedagogical rationale has approximated practice.

Third, research on cognitive offloading (Risko & Gilbert, 2016) establishes that decisions to offload depend on perceived task difficulty, confidence in one's processing, and trust in external resources. For pre-service teachers, the risk of inappropriate offloading is elevated because they may trust AI outputs precisely where they lack the knowledge base to evaluate them. Without sufficient guidance and evaluation, the blind trust may occur by default rather than by choice because pre-service teachers have yet to develop the judgment that they would otherwise delegate.

Together, these perspectives converge on an insight: In EPP contexts, the critical question is not merely *how deeply* AI transforms the task or *how much* judgment is offloaded, but *when in the developmental trajectory* AI enters relative to pre-service teachers' own independent engagement with the practice they are still forming. Two design principles follow: (a) Developmental sequencing anchors each AI encounter in prior independent attempts that demonstrate mastery without AI assistance; and (b) structural transparency makes offloading decisions visible to teacher educators and therefore assessable. We use real cases of AI integration in EPP course design to illustrate both principles, tracing the boundary between AI as a shortcut and AI as a thought partner and professional development tool.

Conceptual Framework: STEP-AI

Building on the perspectives, we propose the **STEP-AI** framework to organize design principles into developmental dimensions specific to teacher preparation contexts: Developmental Sequencing (S), Structural Transparency (T), and Educational Propagation (EP) or Cascading Transfer. The framework was developed iteratively: we posited initial dimensions from the perspectives above and refined them against implementation through case analysis described in the Results section, and we present the illustrative version in the Discussion section with the interconnectivity among the dimensions.

Dimension 1: Developmental Sequencing

The first dimension captures *when* in the pre-service teacher's learning trajectory AI is introduced relative to independent engagement in the target professional practice. Unlike in-service contexts where teachers bring established expertise, pre-service contexts require teacher educators to decide about whether AI enters before, during, or after pre-service teachers have engaged in the practice on their own to establish expertise. Sequencing operates as a protective mechanism not because pre-service teachers must attain competence before AI use, but because independent attempts put their own reasoning on record: AI output then complements rather than replaces their thinking, and they have reasoning of their own against which to evaluate what AI produces instead of blindly trusting. As pre-service teachers' evaluative judgment grows across repeated cycles, the boundedness of AI use can progressively relax. The developmental question of whether *the learner (pre-service teacher) has first engaged in the reasoning that AI might bypass* is absent from frameworks designed for practicing professionals.

Dimension 2: Structural Transparency

The second dimension captures the degree to which the AI interaction process itself is made visible, documented, and assessable. Because pre-service teachers are developing professional identity alongside tool use, EPP contexts require explicit visibility into the interaction process, not merely the outputs. This goes beyond disclosure to encompass prompt documentation, revision tracking, comparative evaluation of multiple outputs, and reflective justification of human editorial decisions. To prevent inappropriate offloading, transparency must be *structural* (embedded in assessment design) rather than *aspirational* (offered as a recommendation).

Dimension 3: Educational Propagation (namely Cascading Transfer)

The third dimension captures how AI practices propagate downstream from EPP coursework into K-12 classrooms. Teacher educators design not only for immediate learning outcomes but also for dispositions that will shape how pre-service teachers eventually use AI with their own students. This dimension therefore includes both intentional transfer, such as cultivating evaluative judgment about AI tools that pre-service teachers carry into their teaching, and unintended effects, such as AI habits spreading beyond the boundaries a course was designed to hold. Because the cascade runs from teacher educators to pre-service teachers to K-12 students, from higher education institutions to K-12 classrooms, design decisions propagate across professional generations, and their consequences are harder to observe directly than outcomes within a single course. These three dimensions should operate simultaneously in well-designed AI in EPP integrations.

Methods

Context

The EPP Collaboratory on AI in Teacher Education (January–June 2026) brought together 15 teacher educators from 13 U.S. universities in a six-session design-reflection cycle spanning orientation and goal-setting, use case design, supported implementation, mid-course reflection, and synthesis of shared resources. The collaboratory was initiated by a research center focused on AI in education to produce transferable resources for EPPs nationwide. The facilitator provided free access to an education-specific GenAI platform as computational support, while teacher educators could also choose their preferred AI tools.

Participants

Eleven teacher educators designed and completed full implementations during the six-month period, which constitute the case sample, spanning mathematics education (n=6), educational

technology (n=2), assessment (n=1), linguistics/multilingual education (n=1), and clinical field experience (n=1). Mathematics teacher educators taught across the preparation spectrum: elementary methods, secondary methods, and a dedicated GenAI course (learning technology) serving mathematics pre-service teachers. Institutions ranged from R1 universities to regional comprehensives serving undergraduates, career-changers, and graduate licensure candidates. On a baseline intake survey, teacher educators reported moderate AI confidence (Mean=2.8 on a 5-point scale) and minimal prior use across two separate measures. Regarding prior integration of AI into their coursework, 46% had used AI in only one assignment or activity and 8% reported no prior integration experience. Regarding personal frequency of use, 46% used AI 1–2 times per month and 31% did not use it at all in a typical month.

Data Sources

(a) Implementation deliverables (n=11) were structured post-implementation documents capturing syllabus language, program recommendations, experience-grounded do's and don'ts, and plan-versus-reality reflections. (b) Use case narratives (n=11) were detailed accounts of actual implementations including activity designs, student-facing materials, and observed outcomes.

Analysis

We employed cross-case analysis (Miles et al., 2014), treating each implementation as a bounded case. Because teacher educators designed independently first, convergent patterns across cases represent independent discovery rather than imitation. Within-case analysis identified design logic and observed outcomes; cross-case pattern-matching identified principles that recurred across three or more independent implementations. Insights observed in a single case are reported as illustrative and labeled as such rather than treated as cross-case principles. Consistent with the abductive approach described above, we iteratively refined the STEP-AI dimensions against observed practices through constant comparison (Corbin & Strauss, 2014).

Results

We present findings organized by the three intertwined framework dimensions, grounded in concrete use cases. Regarding the first research question, teacher educators' designs shared a common logic despite disciplinary variety; we identified in each case for where AI was permitted, what independent work preceded it, and how the interaction would be documented and assessed. Mathematics education cases serve as primary evidence, and cases from adjacent preparation courses serve as corroboration interpreted through the lens of MKT. Identifying information has been removed; we reference teacher educators by content area and course type.

Dimension 1: Developmental Sequencing (When AI Enters)

This group of teacher educators' designs converged on requiring pre-service teachers to engage in the target practice independently before AI entered, so that pre-service teachers' initial reasoning was documented and gaps in it became visible to both the pre-service teachers themselves and the teacher educators. The sequencing was cyclical rather than one-shot: independent work and AI-supported work interleaved across repeated cycles, so AI operated *during* development while each encounter remained anchored in and built upon prior independent reasoning process. This pattern was manifested in mathematics education, where the distinction between correct-looking artifacts and underlying pedagogical and mathematical understanding is long-established.

Use Case A: Student work analysis in mathematics methods (Grades 3–9). A mathematics methods instructor designed a four-stage sequence for developing pre-service teachers' capacity to diagnose student mathematical thinking:

Stage 1: Pre-service teachers independently analyze written student work on a fraction task, identifying evidence of understanding, misconceptions, and productive starting points for instruction, with no AI access.

Stage 2: Pre-service teachers engage in an AI simulation where they role-play as the teacher asking diagnostic questions to an AI-simulated K-12 student who exhibits specific mathematical thinking patterns (e.g., treating fractions as whole numbers).

Stage 3: Pre-service teachers receive teacher educator's feedback on their Stage 1 analysis and revise, articulating what they noticed after the AI simulation that they missed initially.

Stage 4: Extended AI simulation with more complex student reasoning (e.g., partially correct conceptual understanding applied inconsistently across problem types).

The critical design insight lies in Stage 1 where, by requiring independent analysis first, gaps in pre-service teachers' diagnostic reasoning became detectable to the pre-service teachers themselves and to the teacher educator. The teacher educator reported that AI integration "revealed gaps in [pre-service teachers]' ability to evaluate the quality of instructional materials and explain why a teaching decision was appropriate." Without sequencing from independent work to AI integration, these gaps would have been masked by the AI-generated products and therefore could not have been purposefully closed. The teacher educator added that "even though students know how to use AI, they should be re-taught how to prompt, evaluate, revise, and reflect," suggesting that general AI familiarity does not guarantee pedagogically purposeful use automatically.

Use Case B: Lesson plan enhancement in secondary mathematics methods. In a secondary mathematics methods course, pre-service teachers used AI to enhance an existing high-quality instructional materials probability lesson plan aligned to 7th-grade Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010). The implicit yet critical sequencing choice was that pre-service teachers began with a *complete, published lesson plan* and used AI to *modify and extend* it, instead of generating one from scratch. This meant pre-service teachers needed to understand the lesson's mathematical goals, anticipate student reasoning about probability, and evaluate whether AI-suggested modifications maintained and enhanced mathematical rigor. Starting from a known-quality artifact, the iterative prompts that were used for modifications required pedagogical reasoning about what to accept, reject, or revise, thus making reasoning processes tractable and assessable.

Use Case C: Bounded integration in linguistics for educators. Though not a mathematics course, this corroborating case illustrates sequencing. In a course on linguistics and language acquisition, AI was introduced only *after* pre-service teachers independently analyzed transcriptions, identified linguistic features in student speech, and formulated initial instructional recommendations. Only then could pre-service teachers use AI to "brainstorm instructional strategies, generate alternative scaffolds, or identify possible blind spots in their analysis [and initial recommendations]."

The teacher educator observed a clear behavioral split. Pre-service teachers who engaged productively with AI described it as a "second opinion," "thinking partner," or "starting point [to revise and build upon]." Those who struggled "initially approached AI as an answer-generator" but shifted with scaffolding. The design principle was explicit: "Define *why* AI is being used, *when* it can be used, when it cannot be used, and *what* human learning you are trying to protect."

Dimension 2: Structural Transparency (Making Reasoning Visible)

Teacher educators who reported pre-service teachers accepting AI outputs uncritically had initially offered transparency as a *recommendation* rather than an assessment *requirement*. Across the cases that addressed critique, structural requirements and intentional design were necessary to produce critical engagement; expecting pre-service teachers to spontaneously evaluate AI outputs was not sufficient. This is not because pre-service teachers deliberately take shortcuts or use AI irresponsibly, but because they are still forming the professional judgment that evaluation requires. Provoking and building that judgment therefore depends on conditions the teacher educators designed to elicit it. The use cases below illustrate this point.

Use Case D: Contrasting multiple outputs in assessment education. An assessment course instructor's experience, which concerned the architecture of critical evaluation, reported as a single-case insight with immediate design utility. When pre-service teachers only reviewed a single AI-generated assessment item or rubric, they "tended to evaluate it as acceptable for classroom use without modification, regardless of its quality." When asked to contrast multiple AI outputs for the same instructional purpose, they "were better able to distinguish certain outputs as better or worse for a given intended use."

Expanding this pattern to the mathematics methods courses, a single AI-generated lesson plan on, for example, fraction division may appear reasonable in isolation. In contrast, comparing three AI-generated plans for the *same* objectives would instead force pre-service teachers to articulate criteria and pedagogical reasoning, such as *which representation sequence better builds conceptual understanding? which plan better anticipates the invert-and-multiply misconception? which provides more productive struggle rather than premature proceduralization?* These are MKT questions that a single AI-generated, seemingly high-quality output can hardly provoke.

The instructor also found that "with little instruction about prompting strategies, the AI tool output that pre-service teachers produced tended to be clearly suboptimal—too general, or too broad." Pre-service teachers lacked the specialized content knowledge to constrain prompts effectively. A prompt like "make an assessment for 4th grade fractions" produces qualitatively different output than "create three items assessing whether students can locate unit fractions on a number line, distinguishing area-model from linear measurement understanding." The latter requires SCK the student has not yet developed.

Use Case E: Process assessment through mini-lesson presentation. In a technology-in-education course, pre-service teachers delivered five-minute mini-lessons to peers. Peers and the teacher educator assessed the live mini-lesson on standards alignment, clarity of objectives, instructional strategies, and student engagement. In addition, the teacher educator designed an extra assessment layer focused on an artifact each pre-service teacher submitted to *document their reasoning process*, which comprised the following elements:

- Original prompts submitted to the AI
- Explicit identification of where the AI's initial output was "flawed or biased"
- Documentation of iterative refinement decisions
- Justification of manual edits made to the final product

This dual structure ensured that "students are developing true critical agency and professional autonomy rather than just learning how to copy and paste from a machine." Some pre-service teachers "simply accepted the first response that AI produced" until required to present their reasoning publicly.

Use Case F: Polished products without understanding. In an educational technology course, the instructor observed a pattern familiar to mathematics educators, namely surface correctness masking conceptual absence: "Some students turned in polished products but struggled to explain why instructional choices were appropriate.... Some students initially viewed AI as a shortcut for completing tasks rather than a tool that still requires professional judgment."

Requiring reflection on the AI process made the gap between production and understanding evaluable; pre-service teachers who could not justify their choices revealed that AI had substituted for reasoning rather than supported it. In this case, the pre-service teachers had drifted into offloading by default; the reflection requirement illustrated by the previous cases could make the offloading decision visible and assessable rather than hidden behind a polished product. The instructor concluded, "Teach [pre-service teachers] how to think *with* AI, not just how to *use* AI."

Dimension 3: Cascading Transfer (Designing for Future Classrooms)

Teacher educators, and mathematics teacher educators in particular, explicitly grappled with how pre-service teachers' current AI experiences would shape their future K-12 practice. In-service teachers shape their students' AI use directly within their current classrooms, where the effects of their decisions are immediately observable and adjustable. In teacher preparation, by contrast, the influence travels a longer path, from teacher educators to pre-service teachers and only later to K-12 students, so design decisions propagate across a professional generation before their consequences appear.

Use Case G: Developing evaluative judgment for professional practice. In a secondary mathematics methods course, pre-service teachers engaged in two parallel AI activities throughout the semester. First, they practiced with AI-simulated students, role-playing as the teacher while the AI exhibited specific mathematical thinking patterns (procedural fluency without conceptual understanding, productive but incomplete reasoning, common misconceptions). Second, they evaluated AI-powered educational technology tools with the NCSM Educational Technology Guidance Rubric for Mathematics Instruction (NCSM, 2025), assessing mathematical accuracy, cognitive demand, and student engagement.

This dual design positioned pre-service teachers as both *users* of AI for their own instructional development and *evaluators* of AI tools for their future students. The instructor noted a dispositional challenge: "Of all the technology tools I taught about in my course, [pre-service teachers] had the most preconceived notions and associated dispositions about AI." One pre-service teacher "was dead set against any use of AI from the beginning and didn't change." This dispositional range, from uncritical adoption to principled refusal, required designs that created space for developing *judgment* rather than mandating a particular stance.

Use Case H: Sustained cultivation of critical dispositions. In a full-semester GenAI course serving pre-service teachers (predominantly in mathematics education), the instructor required hallucination audits, bias audits, iteration evidence, and ethical disclosure for every assignment.

Despite semester-long scaffolding, first-assignment submissions were "full of ethical issues." The instructor observed, "[Pre-service teachers] won't do hallucination audits or iterate with AI. Unless explicitly required, they won't think critically about hallucinations." By mid-semester, with sustained structural requirements, pre-service teachers demonstrated more sophisticated critical engagement. The finding is sobering for single-module approaches. Critical dispositions toward AI required *sustained, repeated* practice across multiple assignments rather than a single workshop or introductory activity.

The teacher educator also raised a cascading concern with K-12 implications: "Don't encourage chatbots for teachers of children below 8th grade. We don't yet know enough about the impact of chatbots on children." Teacher preparation thus carries a distinct responsibility, preparing future practitioners not only to use tools effectively but to judge whether and when those tools should reach their students at K-12 developmental stages.

Use Case I: Unintended habit transfer. An instructor who designed AI-supported simulations for rehearsing instructional decision-making during field placement observed unintended transmitting: "Because I allowed the use of AI for instructional support, [pre-service teachers] also started to rely on it for other assignments that did not allow for AI use."

This unintended transfer created productive classroom conversations about authorship, voice, and human experience, but it also revealed that once established, AI habits can transfer beyond their intended boundaries and harden into overreliance. For EPPs, AI integration policies must therefore consider not only what is explicitly taught but what dispositions are inadvertently cultivated. The dual position of pre-service teachers makes this consideration actionable. Because they are simultaneously students who use AI and future teachers whose students will use AI, their current experience previews what may unfold in their own classrooms. Teacher educators can make this mirror explicit, guiding pre-service teachers to examine their own AI habits, for instance the drift toward overreliance they may themselves exhibit, as evidence of what their future K-12 students will likely experience without deliberate boundaries.

Discussion

These findings address a gap at the intersection of mathematics teacher education and educational technology, namely, how to integrate GenAI without undermining the development of professional reasoning and mathematical knowledge for teaching. The STEP-AI framework organizes design principles that recurred across independently designed implementations.

Developmental Sequencing establishes that each AI encounter should be anchored in and build upon pre-service teachers' prior independent reasoning with the practice. In mathematics methods courses, pre-service teachers first analyzed student work and formulated instructional responses independently before AI complemented that practice. This design made gaps in diagnostic reasoning visible (Use Case A and B) and positioned AI as a second opinion rather than an answer generator (Use Case C). The parallel in mathematics learning is instructive precisely because it is not a fixed order in which one kind of knowledge precedes another. Just as conceptual understanding and procedural fluency develop as intertwined strands rather than in sequence (National Research Council [NRC], 2001), independent reasoning capability and AI-supported work interleaved across iterative improvement cycles in these designs. The principle is not to withhold AI until mastery is attained but to ensure that pre-service teachers always have reasoning of their own against which AI output can be weighed.

Whereas developmental sequencing ensures that pre-service teachers have reasoning of their own to bring to each AI encounter, structural transparency ensures that this reasoning, and how it changes through the encounter, remains visible and assessable for pre-service teachers themselves and the teacher educators. **Structural Transparency** establishes that assessment structures, not instructions, produce critical engagement with AI. The most effective designs made the documented *reasoning process* of prompts, critiques, revisions, and justifications an assessed artifact alongside, or instead of, the final product. The single-case observation that comparing multiple AI outputs provokes deeper evaluation than examining one output has immediate practical utility for methods course design and warrants systematic testing (Use Case D). When transparency was embedded, the public and explicit presentation of reasoning process

ended the acceptance of first responses (Use Case E), and required reflection made the gap between polished products and actual understanding evaluable (Use Case F).

Cascading Transfer establishes that teacher educators design not only for immediate learning but for downstream K-12 classroom practice. This includes intentional design (positioning pre-service teachers as evaluators of AI tools; Use Case G) and attention to unintended effects (AI habits transferring beyond intended contexts; Use Case I). The cascading responsibility, as teacher educators shaping pre-service teachers who will eventually shape K-12 students, requires a longer design horizon than in-service professional development. More importantly, judgments about whether AI tools should reach young learners at all (Use Case H) are ones pre-service teachers must be prepared to make for their future K-12 students, and preparation means learning to ground such judgments in comprehensive information, institutional policy, and criteria oriented toward students' benefit, rather than in personal opinion or belief alone.

Developmental sequencing builds pre-service teachers' own reasoning, and structural transparency makes that reasoning visible; together they supply the grounded confidence pre-service teachers need to model AI integration in their future classrooms through the cascaded transformation, a practice their K-12 students will carry into a workforce where working with AI becomes a baseline expectation (World Economic Forum, 2025). Although teacher educators designed for a variety of courses, they arrived at the same core principle, one that stands at the center of the STEP-AI framework illustrated in Figure 1 below: AI must be bounded, sequenced, and made structurally accountable if it is to produce professional learning rather than professional bypass.

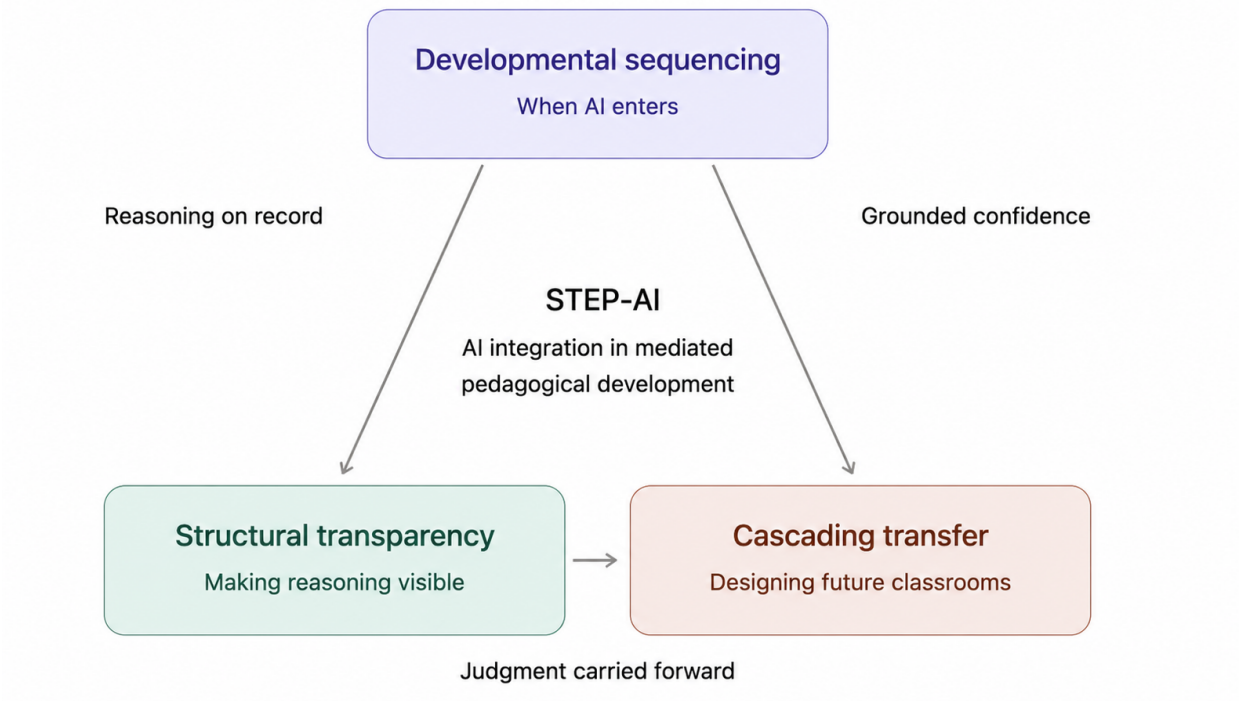


Figure 1: The STEP-AI Framework and Its Three Dimensions for AI Integration in Educator Preparation

Limitations. The collaboratory comprised self-selected teacher educators. Findings reflect their observations rather than direct measurement of pre-service teacher outcomes. Implementations occurred during a six-month period, and the STEP-AI framework was refined against the same cases used to illustrate it, so it requires validation in independent samples. Future work should analyze pre-service teachers' actual AI interactions and track them longitudinally into classroom practice. Because the framework's developmental claim implies that pre-service teachers' professional judgment should grow across iterative cycles of independent and AI-supported work, measuring that professional growth is crucial.

Conclusion. Our results indicate that AI integration in mathematics teacher preparation cannot be treated as an add-on technology module. Across eleven designed implementations, teacher educators converged on the same core logic, organized in the STEP-AI framework: anchor each AI encounter in pre-service teachers' prior independent reasoning (Use Cases A through C), make the interaction process visible and assessable through structural requirements rather than instruction alone (Use Cases D through F), and design for the K-12 classrooms pre-service teachers will eventually lead (Use Cases G through I). AI integration must therefore be embedded within the pedagogical structures through which mathematical knowledge for teaching has always developed in existing study, including approximations of practice, coached rehearsals, and structured reflection. The tool is new; the developmental principles are not.

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DESIGNING FOR ARTIFICIAL INTELLIGENCE: RETHINKING CURRICULUM TASKS, TOOLS, AND STUDENT THINKING IN MATHEMATICS

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Generative artificial intelligence (AI) is rapidly entering mathematics classrooms, yet much of its current use focuses on generating instructional tasks and evaluating answers rather than supporting students' mathematical thinking. I argue that designing for AI requires rethinking curriculum tasks, digital tools, and the mathematical thinking they are intended to elicit. Using proportional reasoning as a case study, I draw on design-based research involving the Connected Mathematics®4 curriculum, the CLUE digital collaborative platform, and a human-developed interpretive framework for students' reasoning. Together, this work illustrates how curriculum and technology can make students' mathematical thinking visible, shareable, and interpretable over time. I conclude by proposing design principles for developing AI that augments teachers' interpretation of student thinking and supports meaningful mathematics learning.

Keywords: Curriculum, Technology, Rational Numbers and Proportional Reasoning, Problem-Based Learning

Introduction

Curriculum has long been recognized as more than written instructional materials. It is realized through the design of mathematical tasks and their enactment in classrooms as teachers, students, and instructional resources interact around mathematical ideas. Remillard and Heck (2014) described curriculum as a dynamic process in which written curriculum is interpreted, enacted, and transformed through instruction, while Choppin et al. (2022) argued that curriculum materials should function as thinking devices that support teachers' and students' mathematical reasoning rather than simply deliver content. Together, these perspectives suggest that curriculum design extends beyond creating instructional materials to designing learning experiences, tools, and supports that shape mathematical thinking over time.

Building on this perspective, my work has examined curriculum as an iterative process in which design, classroom enactment, field testing, teacher feedback, and revision continually shape one another (Edson, 2025; Edson et al., 2025). This work has also informed the design of coherent learning experiences that support students' developing mathematical understanding (Edson et al., 2019) and digital curriculum resources that help teachers access and build on students' mathematical thinking during instruction (Edson & Phillips, 2021). In this paper, I extend this line of work by asking a different question: What aspects of curriculum must be intentionally redesigned if AI is to support meaningful mathematics learning?

The emergence of generative artificial intelligence presents a new challenge for this curriculum enactment process. Much of the current discussion has centered on AI's ability to generate instructional tasks, solve mathematics problems, or evaluate student work. Although these applications may improve efficiency, they do little to deepen our understanding of how students develop mathematical ideas or how teachers interpret that thinking during instruction. If AI is to become a meaningful part of mathematics education, it must be considered as part of the curriculum design process rather than simply another instructional technology.

I argue that designing for AI requires rethinking three interconnected aspects of curriculum: the mathematical tasks students engage in, the digital tools that mediate those experiences, and the ways student thinking is made visible and interpreted. These three dimensions are inseparable. Rich curriculum tasks are needed to develop important mathematical ideas while eliciting meaningful mathematical reasoning. Well-designed tools can preserve, connect, and make visible that reasoning. Human-developed interpretive frameworks provide principled ways of characterizing students' mathematical reasoning and establish the educational foundation for AI systems that support, rather than replace, teacher judgment. In other words, designing for AI begins with designing curriculum that both develops mathematical understanding and reveals students' mathematical thinking. The challenge is no longer simply how to incorporate AI into existing curriculum materials, but how to design curriculum that intentionally creates opportunities for students to develop important mathematical ideas and produce rich mathematical thinking that AI can help teachers interpret and build upon.

Using proportional reasoning as a case study, I draw on a decade of design research involving the Connected Mathematics Project's curriculum, a digital collaborative platform and its tools, and a human-developed framework for interpreting students' reasoning. Collectively, this work illustrates how the focus of curriculum design has expanded, from designing rich mathematical tasks, to designing digital collaborative learning environments, to developing tools and interpretive frameworks that make students' proportional reasoning visible and understandable. I argue that these same principles should guide the design of AI so that it augments teachers' understanding of students' developing mathematical thinking rather than simply automating existing instructional practices.

Rethinking Curriculum Tasks

The mathematical tasks students engage in have long been recognized as the heart of curriculum. They shape not only what students learn, but also how they reason, communicate, and develop mathematical understanding. Tasks can help students develop important mathematical ideas by providing opportunities to grapple with mathematical relationships, justify their thinking, compare strategies, and build conceptual understanding (Stein et al., 1996; Henningsen & Stein, 1997). In problem-based curriculum, a worthwhile task is therefore designed not only to elicit student thinking, but also to advance particular mathematical conceptions and connect them to ideas developed in subsequent tasks. Consequently, the design of worthwhile mathematical tasks has been a central concern of mathematics curriculum research and development for decades.

The emergence of AI does not diminish the importance of curriculum tasks; rather, it heightens it. Current AI systems can generate an endless supply of mathematical exercises, many of which emphasize procedural fluency and the production of correct answers. Yet generating more tasks is not the same as designing better mathematical experiences. If AI is to contribute meaningfully to mathematics education, curriculum developers must first ask what mathematical ideas students should develop and what kinds of mathematical thinking would provide evidence of that development. Only then can AI be designed to recognize, interpret, and respond to that thinking in ways that support instruction.

Over four decades of curriculum development, one lesson has become increasingly clear: worthwhile mathematical tasks do more than develop mathematical understanding—they reveal how students think mathematically. A particular task can help students develop an important mathematical conception, while a carefully sequenced set of tasks can support students in connecting, refining, and eventually consolidating that idea. Successive cycles of classroom field

testing and teacher feedback have continually reinforced this lesson, demonstrating that the quality of curriculum tasks is reflected not only in student learning but also in the mathematical thinking those tasks elicit (Edson et al., 2025). These experiences also reinforced that students' mathematical understanding develops across carefully sequenced investigations rather than through isolated tasks, making the organization of learning opportunities as important as the design of individual problems (Edson et al., 2019). This idea has guided the design of Connected Mathematics®4 (CMP4; Phillips et al., 2025), where individual tasks are intentionally organized into coherent sequences that support the development of important mathematical ideas over time (Edson et al., 2019). Rather than viewing curriculum tasks as isolated exercises, CMP4 organizes learning around coherent sequences of problem-based investigations that invite students to explore meaningful mathematical situations, represent relationships in multiple ways, compare solution strategies, and reflect on what they have learned. The CMP4 STEM Problem Format extends this vision through an "Initial Challenge," followed by "What If…?" questions that encourage students to explore the embedded mathematics within contextual situations, and "Now What Do You Know?" reflections that consolidate important mathematical ideas. Field-test teachers identified the STEM Problem Format as one of CMP4's most effective design features, reporting that it promoted student engagement, extended students' problem solving and mathematical discourse, and created opportunities for students to compare multiple solution strategies (Edson et al., 2025). Collectively, these experiences are designed not simply to produce correct answers, but to develop and connect important mathematical ideas while also making students' mathematical thinking visible through multiple strategies, representations, and explanations.

To better understand how current generative AI supports curriculum design, I compared a proportional reasoning problem from CMP4 with tasks generated by ChatGPT and Microsoft Copilot from a common description of the mathematical goals. All three tasks encouraged students to use multiple strategies, representations, and reflection. The comparison, however, revealed important differences. The AI-generated tasks focused exclusively on proportional situations, whereas the CMP4 problem intentionally juxtaposed proportional and nonproportional relationships to help students develop criteria for determining when proportional reasoning applies. The CMP4 problem also embedded examples of students' mathematical reasoning that students were asked to analyze and critique, while the AI-generated tasks primarily described or referenced possible strategies. Finally, although the AI tasks included broad discussion prompts that encouraged reflection and participation, the CMP4 questions were intentionally designed to advance specific mathematical ideas by directing students' attention to key relationships across strategies, representations, and contexts. Together, these differences illustrate that curriculum design involves more than generating engaging classroom tasks. It requires intentionally designing mathematical experiences that develop conceptual understanding, leverage student thinking as a resource for learning, and guide students toward important mathematical generalizations.

Proportional reasoning provides a particularly compelling illustration (Lamon, 1994; Ben-Chaim et al., 1998). Rich proportional reasoning tasks rarely produce a single pathway to a solution. Students may reason additively or multiplicatively, build up through repeated relationships, use unit rates, scale quantities, construct tables, graphs, equations, or diagrams, and justify their reasoning in different ways. From a curriculum perspective, these variations are not incidental; they are a strength rather than a complication. They are precisely the evidence teachers need to understand students' developing mathematical thinking. Different approaches

reveal different understandings of multiplicative relationships, creating opportunities for teachers to build on students' existing reasoning while also defining the kinds of evidence AI must eventually learn to interpret. They also provide the kinds of rich mathematical activity that AI must eventually learn to interpret if it is to support classroom instruction in meaningful ways.

Designing rich curriculum tasks, however, is only the beginning. Once students begin generating diverse strategies, representations, and explanations, a new challenge emerges: how can this thinking be preserved, shared, and interpreted as it develops? The lesson for curriculum design is that worthwhile tasks should be judged not only by the mathematics they teach, but also by the mathematical thinking they elicit. That thinking becomes the foundation upon which teachers, and eventually AI, can interpret, respond to, and extend students' learning. Addressing this challenge requires rethinking the tools that support, preserve, and make visible students' mathematical thinking.

Rethinking Classroom Tools

Rich mathematical tasks alone are not enough. Although carefully designed tasks can elicit rich mathematical thinking, much of that thinking disappears unless it can be preserved, shared, and interpreted. As students engage in meaningful problems, they produce a wide range of strategies, representations, explanations, and questions. Yet much of this thinking is fleeting. Ideas shared during small-group discussions may never reach the whole class. Intermediate representations are often erased or discarded. Teachers frequently make instructional decisions based on only a fraction of what students know and understand. If curriculum is to support students' developing mathematical thinking, the tools that mediate classroom activity must be intentionally designed to preserve, connect, and make visible that thinking.

Conventional classroom tools have served mathematics education well. Paper and pencil, whiteboards, graph paper, manipulatives, and calculators each provide important ways for students to explore mathematical ideas. However, these tools were not designed to capture the evolution of students' thinking over time or to support teachers in interpreting patterns of reasoning across individuals, groups, and classroom discussions. The lesson for curriculum design is not that traditional tools are inadequate, but that new forms of mathematical activity require new kinds of tools.

Building on the problem-based design of CMP4, we sought to design digital collaborative tools that extend, rather than replace, students' mathematical activity while also supporting teachers' enactment of problem-based mathematics curriculum (Edson & Phillips, 2021). Figure 1 illustrates this shift. Students engage with the CMP4 curriculum in a shared digital workspace where they solve problems using text, tables, graphs, drawings, equations, and images while documenting, revising, and communicating their mathematical thinking. Unlike traditional classroom tools, their work persists over time, allowing ideas to be revisited, refined, connected across problems and investigations, and shared with peers. Rather than producing isolated solutions, students create a growing collection of mathematical artifacts that support collaboration and make their thinking visible throughout the learning process.

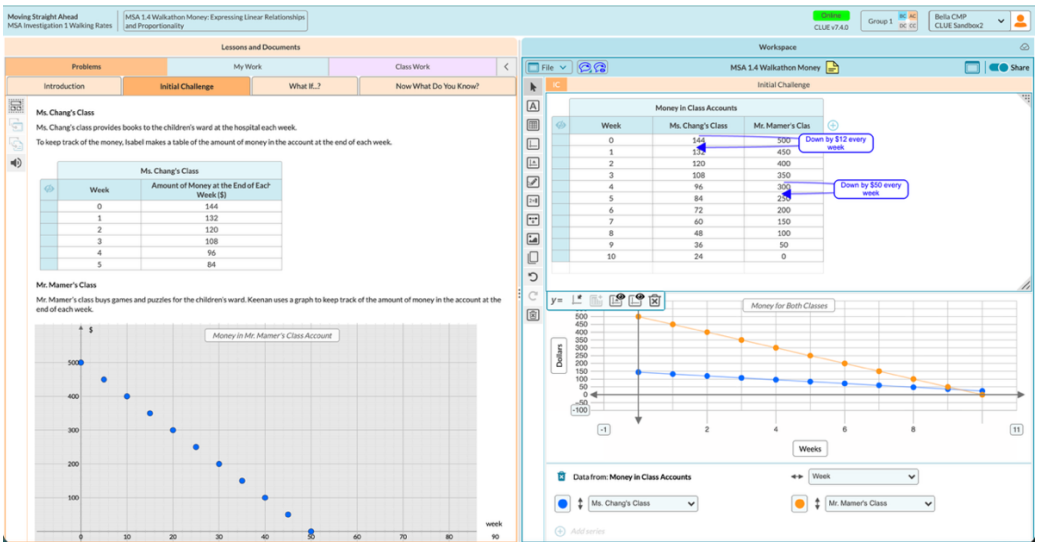


Figure 1: A Digital Collaborative Environment that Enables Students to Represent, Revise, and Share Mathematical Thinking Using Multiple Representations

Figure 2 illustrates the complementary role of the teacher. Teachers can access curriculum materials alongside students' developing work, monitor individuals, groups, or the entire class in real time, identify emerging strategies, compare approaches, select student work for discussion, and incorporate that work into subsequent instruction. In this way, the digital tools become part of the operational curriculum, supporting teachers' ongoing interpretation of students' mathematical thinking rather than simply delivering digital content (Edson & Phillips, 2021; Edson, 2024).

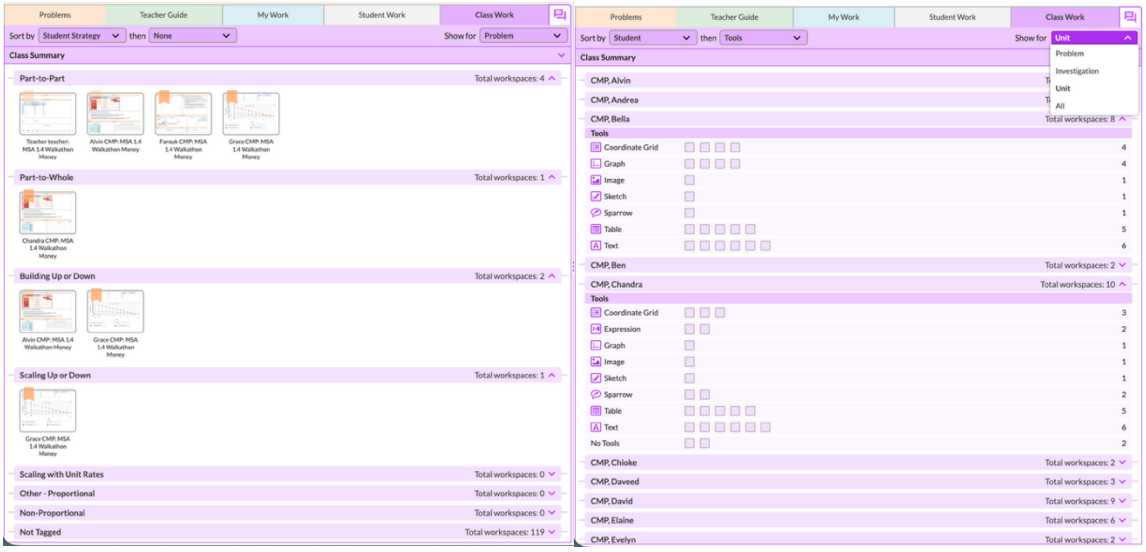


Figure 2: Teacher Dashboard that Supports the Interpretation of Students' Mathematical Thinking During Classroom Instruction

Subsequent classroom studies demonstrated that these environments not only preserved students' mathematical work but also transformed that work into an instructional resource for

both students and teachers (Park et al., 2024). Students could revisit, refine, and build upon their own and their peers' mathematical ideas across problems and investigations, while teachers used these persistent records to identify emerging strategies, orchestrate classroom discussions, and inform subsequent instructional planning. Rather than disappearing when group work ended, students' inscriptions became shared classroom resources that teachers could revisit during instruction, planning, and reflection (Park et al., 2024). Teachers, in turn, can monitor students' work in real time, identify emerging strategies, select work for whole-class discussion, and build on students' mathematical ideas as instruction unfolds through dashboard tools specifically designed to support teacher decision making during instruction (Edson & Phillips, 2021). Research also suggests that access to persistent evidence of student thinking influences teachers' collaborative planning, prompting them to reconsider when and how digital and non-digital resources best support students' mathematical understanding (Park & Edson, 2024). Rather than simply delivering content, these tools become part of the operational curriculum, supporting the interactions among curriculum materials, teachers, and students envisioned by Remillard and Heck (2014), while making students' mathematical thinking persistently visible throughout the learning process (Edson, 2024). Recent work has begun extending these digital collaborative environments with curriculum-aligned AI that uses interpretive rubrics to organize and summarize students' mathematical thinking rather than simply generate answers or evaluate correctness (Bondaryk et al., 2026).

Designing these tools also revealed an important lesson. Making students' work visible did not necessarily make their mathematical reasoning interpretable. During our analyses of students' proportional reasoning, we found that critical mathematical relationships often remained implicit, even when multiple representations were available. This realization led to the development of Student Proportional Reasoning Arrows (SPArrows), a representational tool that enables students to explicitly communicate proportional relationships within and across representations (Edson et al., 2024). Rather than functioning as an AI technology, SPArrows was designed to strengthen students' mathematical communication while making their reasoning more visible to teachers, researchers, and future AI systems (Edson et al., 2024). The broader lesson was that visibility alone is insufficient; curriculum tools must also help make students' mathematical reasoning interpretable.

These experiences suggest that the role of classroom tools must also be reconsidered in the age of AI. Their purpose extends beyond delivering content or collecting answers. Well-designed tools mediate mathematical activity, support collaboration, preserve students' evolving thinking, and provide richer evidence for instructional decision making. Empirical studies suggest that when students' mathematical thinking becomes persistently visible and accessible, teachers are better able to notice emerging strategies, orchestrate discussions around student-generated ideas, and use evidence of student thinking in subsequent instructional decisions (Park et al., 2024). The lesson for curriculum design is that tools should be judged not only by the activities they support, but also by how well they make students' mathematical thinking visible and interpretable. Only when rich curriculum tasks are coupled with tools that make students' mathematical thinking visible and increasingly interpretable can AI begin to support teachers' understanding of students' developing reasoning.

Rethinking Student Thinking

Making students' mathematical thinking visible is an important step, but visibility alone is not enough. Teachers must still interpret what they see. A student's table, graph, diagram, equation, or written explanation does not automatically reveal the mathematical ideas that

underlie the work. Instead, teachers draw on their mathematical knowledge and professional judgment to recognize emerging understandings, misconceptions, and opportunities for extending learning (Jacobs et al., 2010; Sherin et al., 2011). If AI is to support this process, it must be designed around principled interpretations of student thinking rather than simply identifying correct or incorrect answers.

Proportional reasoning illustrates the interpretive challenge particularly well. The same student work can provide evidence of multiple dimensions of mathematical thinking, and those dimensions may not be immediately apparent from an answer alone. For example, a correct solution may reflect different understandings of the relationships among quantities, just as an incorrect solution may reveal productive reasoning that can be extended. Interpreting such work therefore requires attention not only to what students produce, but also to the mathematical relationships their representations and strategies reveal. This complexity makes proportional reasoning a useful context for developing principled ways to characterize student thinking, ways that can support teachers' interpretation of student work, and ultimately, provide a foundation for AI systems designed to assist rather than replace that interpretation.

This interpretive challenge led us to develop a human-centered framework for interpreting students' proportional reasoning. Rather than beginning with AI models or automated classification systems, we began with mathematics. Through iterative analyses of authentic student work, we identified three complementary dimensions of students' reasoning: the mathematical representations students used to communicate their thinking, the approaches they took to proportional situations, and the strategies they employed to solve problems. To operationalize this framework, we developed a rubric (without AI) that distinguishes these dimensions of students' reasoning and provides a principled basis for analyzing authentic student work before any AI analysis is undertaken (Edson et al., in press). Table 1 provides an at-a-glance summary of the rubric. Rather than reducing student work to correct or incorrect answers, the rubric captures meaningful differences in students' mathematical reasoning while remaining grounded in mathematics education research and classroom practice.

Table 1: Human-Developed Rubric for Students' Proportional Reasoning

Criterion and Associated Elements		
A. Student Representation – Student proportional reasoning representations based on their inscriptional resource used in the digital collaborative CLUE platform:		
1. Coordinate Grid	**4.** Image (uploaded photo)	**7.** SPArrow
2. Expression (or equation)	**5.** Number Line	**8.** Table
3. Graph	**6.** Sketch (drawing)	**9.** Text
B. Student Approach – Student proportional reasoning approaches to the contextual problem using reasoning about:		
1. Part-to-Part	**2.** Part-to-Whole (or Whole-to-Part)	
C. Student Strategy – Student proportional reasoning strategies include:		
1. Building Additively	**3.** Using a Multiplicative Relationship	
2. Scaling Up or Down Multiplicatively	**4.** Other Proportional Reasoning	

The rubric serves a second purpose beyond supporting teachers' interpretation of student thinking. It is also being used to construct a large human-coded dataset of students' proportional

reasoning drawn from authentic classroom work across the seventh-grade CMP4 curriculum. Researchers code each instance of student work according to its mathematical representation, proportional reasoning approach, and solution strategy, producing structured data that preserve meaningful distinctions in students' reasoning. Rather than training future AI systems directly from raw student responses or correctness, this work establishes an educationally grounded dataset through which AI can learn from mathematics educators' interpretations of student thinking (Edson et al., in press).

This progression, from curriculum tasks to student work, to human interpretation, to a human-coded dataset, is intentional. Before AI can interpret students' thinking, educators must first establish what counts as meaningful mathematical thinking. Human-developed interpretive frameworks provide this foundation. They reflect decades of research on proportional reasoning, classroom teaching, and curriculum design while making explicit the features of student work that matter instructionally. The resulting human-coded dataset preserves these educational judgments so that future AI systems learn from principled interpretations of students' mathematical thinking rather than from correctness alone. This design philosophy is now being operationalized within CLUE through AI that uses curriculum-specific interpretive rubrics to identify students' approaches, representations, and strategies while supporting teacher noticing and student reflection (Bondaryk et al., 2026).

This perspective reframes the role of AI in mathematics education. Rather than generating tasks or evaluating answers, AI has the potential to assist teachers by recognizing patterns in students' strategies, representations, and approaches as they develop over time. Such systems would not replace teachers' professional judgment; instead, they would extend teachers' capacity to notice, interpret, and respond to students' mathematical thinking across individuals, groups, and classrooms. The lesson for curriculum design is that designing for AI requires designing principled ways to understand student thinking before designing AI systems to interpret it.

Design Principles for AI-Enhanced Curriculum

The preceding discussion suggests that the emergence of generative artificial intelligence requires more than new technologies; it requires a reconsideration of how mathematics curriculum is designed. AI should not be viewed as an instructional add-on or as a replacement for teachers' professional judgment. Rather, it should be understood as one component of a broader curriculum system in which mathematical tasks, classroom tools, and human interpretations of student thinking work together to support learning. Drawing on the lessons from this work, I propose four design principles for AI-enhanced mathematics curriculum.

Design Principle 1

Curriculum tasks should be judged not simply by whether they produce correct answers, but how well they support the development of important mathematical ideas and elicit evidence of students' developing thinking. As the comparison between the CMP4 investigation and AI-generated tasks illustrated, generating mathematically appropriate problems is not the same as designing curriculum. Curriculum tasks should intentionally develop and connect important mathematical ideas through carefully sequenced questions, meaningful contexts, opportunities to compare strategies, and structured reflection. Individual tasks can introduce or advance particular mathematical ideas, while sequences of tasks can support students in connecting, refining, and consolidating those ideas over time. When designed in this way, tasks both support meaningful mathematical learning and produce the rich evidence of student thinking that both teachers and future AI systems require to support learning.

Design Principle 2

Rich mathematical tasks alone are insufficient if students' thinking disappears once the lesson ends. Curriculum tools should therefore be designed to preserve, connect, and communicate students' mathematical reasoning across individuals, groups, and classroom discussions. Digital collaborative environments, representational tools, and teacher dashboards should not simply digitize existing instruction; they should transform students' work into persistent instructional resources that teachers can revisit, interpret, and build upon during and after instruction. AI becomes valuable only when it has access to rich, persistent records of students' developing mathematical thinking.

Design Principle 3

Before AI can recognize meaningful mathematical thinking, mathematics educators must first determine what meaningful understanding and evidence of that understanding matter instructionally. Human-developed interpretive frameworks and rubrics make explicit the mathematical distinctions that matter instructionally. These frameworks also provide the foundation for constructing educationally grounded datasets that preserve expert interpretations of student reasoning. Rather than learning directly from correctness or large collections of student responses, future AI systems should learn from these principled interpretations of students' approaches, representations, and strategies. In this way, AI becomes grounded in mathematics education research rather than statistical patterns alone.

Design Principle 4

Designing for AI is fundamentally a curriculum challenge rather than a technological one. AI should augment, not replace, the expertise of teachers and curriculum developers. Rather than generating lessons or evaluating answers in isolation, AI should help teachers interpret evidence of students' developing mathematical understanding, notice patterns in students' mathematical thinking, organize evidence across time, identify emerging understandings, and support instructional decision making. This vision requires sustained collaboration among curriculum developers, mathematics educators, teachers, learning scientists, and AI researchers so that advances in AI remain grounded in how students actually learn mathematics.

Taken together, these principles represent an evolution in curriculum design. The focus shifts from designing instructional materials alone to designing integrated curriculum systems in which tasks elicit meaningful thinking, tools preserve that thinking, interpretive frameworks make it understandable, and AI extends teachers' capacity to build on students' developing ideas. The future of AI in mathematics education therefore depends less on increasingly sophisticated algorithms than on increasingly sophisticated curriculum design. By intentionally designing these elements together, curriculum developers can create learning environments in which AI meaningfully supports students' mathematical learning while strengthening, rather than replacing, teachers' professional judgment.

Conclusion

Generative artificial intelligence is often presented as a technological innovation that will transform mathematics education. I have argued for a different perspective. The more significant challenge is not designing increasingly sophisticated AI systems, but designing curriculum that elicits, preserves, and supports the interpretation of rich mathematical thinking.

Using proportional reasoning as a case study, I have illustrated how decades of curriculum design research have progressively expanded what can be learned about students' mathematical thinking. Carefully designed tasks develop important mathematical ideas while eliciting diverse reasoning. Thoughtfully designed classroom tools preserve and extend that reasoning. Human-

developed interpretive frameworks provide principled ways of understanding students' approaches, representations, and strategies. Together, these developments establish the educational foundation upon which meaningful AI systems can be built.

This progression also represents an evolution in curriculum design. The focus is no longer solely on designing instructional materials, but on designing integrated curriculum systems in which tasks elicit rich mathematical thinking, tools preserve that thinking, interpretive frameworks make it understandable, and AI extends teachers' capacity to interpret and respond to students' developing ideas.

The future of AI in mathematics education will depend less on advances in algorithms than on advances in curriculum design. As curriculum developers, mathematics educators, teachers, and AI researchers work together, the goal should not be to automate instruction, but to strengthen teachers' capacity to notice, interpret, and respond to students' developing mathematical thinking. Doing so requires intentionally designing mathematical tasks, classroom tools, interpretive frameworks, and AI systems as complementary components of a coherent curriculum. When these elements are designed together, AI becomes not the driver of curriculum, but a curriculum partner that helps teachers understand, interpret, and build on students' mathematical thinking while preserving the central role of human judgment in mathematics teaching and learning.

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FROM "NO" TO "KNOW" AI: IN-SERVICE TEACHERS' AGENCY ACROSS AN AI-INTEGRATED CURRICULUM

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This paper introduces the conceptual framework "AI Integration in K-12 Mathematics: A Framework Linking Curriculum, Classroom Enactment, and Student Learning (AIM-CEL)." It outlines the relationships among the official, teacher-intended, and enacted mathematics curriculum specific to AI integration, along with related student outcomes and affecting factors. The framework was used to analyze survey data from in-service secondary mathematics teachers enrolled in a graduate program about their perspectives on AI integration in mathematics. The results reveal that teachers' perspectives toward AI integration vary from reluctance and resistance to support for strategic, intentional use of AI. Both groups demonstrated agency, framing their positions in terms of supporting students' learning outcomes. The discussion covers the relationships among factors that influence teachers' decisions about AI integration.

Keywords: teacher agency, artificial intelligence (AI), mathematics curriculum

Artificial intelligence (AI) and its subset, generative artificial intelligence (GenAI), are catalyzing transformative changes to the mathematics curriculum by providing new possibilities and opportunities to meet student needs (Arslan et al., 2024; Batta, 2024; Opesemowo et al., 2024). AI systems can perform cognitively demanding tasks, engage in complex reasoning that mirrors intelligent behavior, while GenAI uses complex models to create new content, facilitate more natural human-like interactions, and support a variety of tasks (Peñalvo & Ingelmo, 2023; Van Vaerenbergh & Pérez-Suay, 2022). In the United States (U.S.), national mathematics organizations and federal policies promote the use of AI in K-12 education, with the assumption that AI can support instruction and efforts to promote student learning without diminishing opportunities for students to engage in problem solving or exhibit critical thinking skills (Department of Labor, 2026; White House, 2026; NCTM, 2024).

As AI becomes more prevalent in educational settings and policy mandates continue to encourage its integration into mathematics curricula, the role of teachers becomes particularly critical since they are tasked with implementing these curricula and assessment policies (Long & Lampen, 2014). When teachers take stances and make decisions that can affect their professional lives within their sociocultural contexts, they exercise their agency (Sang, 2022). Teachers' decision-making and actions affect how mathematical tasks are set up and differentiated, how the physical and cognitive classroom spaces are organized, and how student engagement with mathematical content is coordinated, all of which ultimately impact students' learning (Long & Lampen, 2014; Smith & Stein, 2023). However, little is known about how advancements in AI educational technology interact with and affect teacher agency. Therefore, there is a need to examine the agency of secondary mathematics teachers regarding AI integration to understand the uptake or lack thereof of AI and GenAI in mathematics classrooms, and to tailor professional support for teachers to facilitate that uptake in meaningful and ethical ways (Alsina & Vasquez, 2024; Biesta et al., 2015). To understand the interplay of teacher agency in integrating AI into the mathematics curriculum, we developed a conceptual framework (described below) and used it in a study with in-service secondary teachers. The following research questions guided our inquiry: *How do secondary mathematics in-service teachers (who are graduate students)*

perceive AI integration into the official, teacher-intended, and enacted mathematics curriculum, and how do they see AI integration impacting student learning outcomes? What factors do they perceive as contributing to the integration of AI? To what extent do teachers perceive having agency in an AI-integrated mathematics curriculum?

Theoretical and Empirical Backgrounds

Teacher Agency in the Era of AI

Sang (2022) defined teacher agency as "the capacity of teachers to act purposefully and constructively to direct their professional growth and contribute to the growth of education quality" (p. 2). Teachers may exhibit resistance to change or hesitation to conform to top-down policies due to distrust in the extent to which the policies align with their classroom realities and skepticism about how these changes affect their efforts to support students' learning of mathematics (Jita et al., 2025; Wright, 2021). To advance sustainable educational change, teachers should be empowered to facilitate and lead it, reflecting a bottom-up approach and aligning with top-down policy shifts (Wright, 2021).

With respect to AI integration, teachers' agency may be influenced by their AI literacy, ethical considerations, and knowledge about the suitability of AI tools (Jita et al, 2025). Teachers' epistemic agency is a subset of teacher agency and encompasses teachers' ability to construct and evaluate knowledge, enabling them to critique AI and its output rather than passively accept AI-generated content (Chu & Wang, 2025; Jose et al., 2025; Nieminen & Ketonen, 2024Urbani, 2024;). When exhibiting epistemic agency, teachers engage in "trust audits," questioning and critically examining AI outputs (Jose et al., 2025). Thus, as teachers exercise teacher agency, there may be evidence that they also demonstrate epistemic agency.

Mathematics Teachers Perspectives towards AI

Studies have shown that teachers perceive that AI can promote personalized learning, enhance classroom instruction and discourse, provide immediate feedback on assessment, improve students' problem-solving skills, increase efficiency by automating repetitive tasks, and promote collaboration (Albalawi, 2025; Ballenas & Lasco, 2025; Clarke, 2025; Korkmaz et al., 2025; Meylani, 2024; Sears, Castro-Minnehan et al., 2026; Walkington, 2025). On the other hand, teachers also acknowledge the need to address ethical considerations of AI use and to be cautious of students becoming over-reliant on AI tools, thus reducing the cognitive rigor of their mathematical engagement (Busuttil & Calleja, 2025; Chounta et al., 2022; Clarke, 2025; Egara & Mosimege, 2024; Meylani, 2024; Sears et al., 2025). Additionally, teachers expressed concerns about their own limited personal experience and lack of professional training on AI have been identified as barriers to AI integration in mathematics classrooms (Korkmaz et al., 2025; Meylani, 2024).

While prior research catalogs teachers' perceptions about AI-related benefits or challenges, in their current form, these findings offer a limited view of teacher reasoning. To make these insights usable for researchers and mathematics educators, it is necessary to understand how these perceptions can be classified and characterized. Toward this aim, we propose a conceptual framework to map teachers' AI-related perceptions across the domains of official, teacher-intended, and enacted curricula, as well as student learning outcomes. Because curricular systems structure and constrain what teachers do, or perceive they cannot do, in practice, situating perceptions within these domains provides a more coherent and actionable account of how teachers make sense of AI in their work.

Conceptual Framework

To capture various aspects of AI integration in school mathematics, we (the authors) propose

a conceptual framework, which we term *AI Integration in K-12 Mathematics: A Framework Linking Curriculum, Classroom Enactment, and Student Learning,* or the *AIM-CEL Framework* for short. Drawing on two existing frameworks: the *Curriculum Policy, Design and Enactment System* (Remillard & Heck, 2014) and the *Instructional Technology Tetrahedron* (ITT) model (Buchbinder, Liu, & Towle, 2025; Liu & Buchbinder, 2024), we envision how AI is integrated in and affects various aspects of curriculum, classroom enactment, and student learning. By curriculum, we mean both the planned and the actual learning experiences designed to help learners reach particular objectives (Remillard & Heck, 2014). Additionally, the *AIM-CEL* Framework considers various factors affecting AI integration within official, teacher-intended, and enacted curriculum as well as students' learning outcomes (see Figure 1).

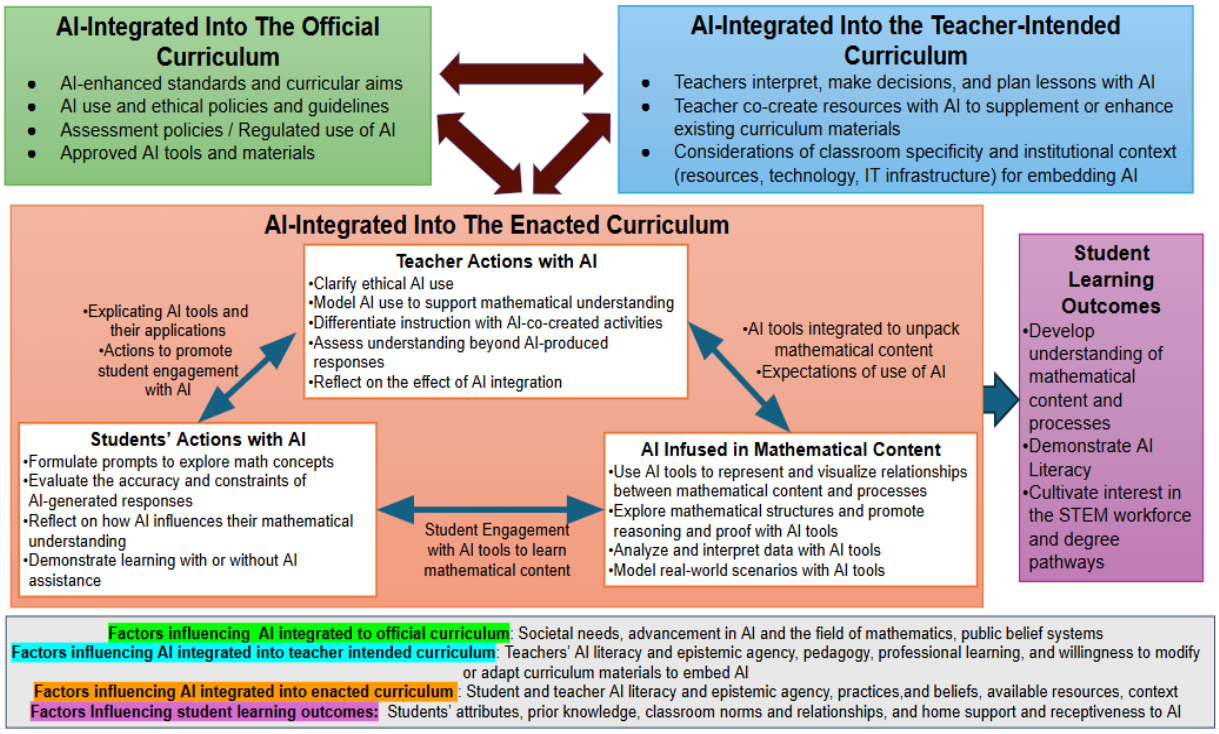


Figure 1: AI Integration in K-12 Mathematics: A Framework Linking Curriculum, Classroom Enactment, and Student Learning

The first component of the *AIM-CEL* Framework involves the *official* curriculum, which, in the US, involves curricular aims and objectives, standards, and assessment structures authorized at the national, state, or district levels. The official curriculum provides overarching policies and guidelines for integrating AI into mathematical standards, curriculum materials, assessment structures, and approved tools. The second *AIM-CEL* Framework component concerns a teacher-intended curriculum, examining how individual teachers interpret official AI policies and integrate them into their planned curricula. In this process, teachers make decisions about which AI tools to use, if at all, for what purposes, and in what ways by considering the specificity of their classroom and institutional context. Based on these considerations, teachers devise lesson plans and instructional resources, often with the aid of AI (e.g., co-creation), to enhance or replace existing curricular materials.

The *enacted* curriculum corresponds to the interactions among students, teachers, and

instructional materials as they unfold in mathematics classrooms. To conceptualize AI integration within this component of the *AIM-CEL* Framework, we draw on the Instructional Technology Tetrahedron (ITT) model (Buchbinder et al., 2025; Liu & Buchbinder, 2024). The ITT model emphasizes the role of technology as both a tool and a medium for classroom interactions. In the *AIM-CEL* Framework, AI technology is diffused throughout the teacher, students, and content vertices, and the arrows represent connections among them. Teacher actions with AI involve promoting ethical use of AI to support students' mathematical understanding, differentiating instruction, and modeling efficient AI use. The student actions involve directly interacting with AI by formulating prompts, evaluating AI responses in terms of mathematical accuracy, and possible constraints or biases. The framework also considers how students' learning with or without AI is demonstrated during the lesson. The content vertex of the framework represents AI-infused instructional materials. It considers the use of AI to represent mathematical concepts, processes, relationships, data, and real-world phenomena, as well as how AI is used to explore and reason about these mathematical objects and relationships.

The AIM-CEL Framework further outlines the bi-directional relationships among the vertices within the enacted space. The teacher facilitates the integration of AI content and sets expectations for its use. The teacher also explicates AI tools and promotes their use by students, who, in turn, engage with AI-infused content in their learning environment. Since "digital technology is saturated with humanity through its design and conception to the extent that it can be seen as having its own agency" (Liu & Buchbinder, 2024, p. 437), we conceptualize AI-infused mathematical content as "interacting" with both teachers and students. The official, teacher-intended, and enacted curricula mutually affect each other, as captured by the brown double-sided arrows (Figure 1). The ultimate goal of the curriculum network is to support students' learning outcomes, which may include developing mathematical understanding, skills, practices, communicative skills, strategic competence, and productive dispositions (NASEM, 2018; NRC, 2001). In the era of AI, student learning outcomes should also include developing AI literacy and cultivating curiosity about and interest in STEM disciplines.

In the following section, we describe a study that used the *AIM-CEL* Framework to analyze data from secondary mathematics teachers. We highlight the affordances of the framework for illuminating complex relationships in teachers' perspectives on AI integration across the curriculum, as well as the factors affecting this process.

Methods

Study Setting and Participants

The study was conducted with two cohorts of in-service secondary mathematics teachers pursuing a master's degree in education (M.Ed) at a large state university in the Southeastern US. Data were collected from seven in-service teachers participating in a secondary methods course in Spring 2025 and eight in-service teachers enrolled in a middle school methods course in Fall 2025. Of these latter eight teachers, three were international teachers on professional leave, funded by their governments to pursue advanced training. Thus, all participants had teaching experience and were positioned to demonstrate a degree of agency in their practices.

Data Collection

The first group of teachers completed the *Perspectives about AI in Secondary Mathematics Teaching and Learning* (PAISMTL) survey, which included 12 items, nine of which were open-response (Sears, Castro-Minnehan et al., 2026). This survey was administered in class before the GenAI workshop. The teachers were asked to describe the extent to which they use AI to plan and teach lessons, which AI tools they use and how, and what ethical considerations they

employ. The participants were also asked to share their perspectives on using AI to support mathematics teaching and learning, on the extent to which AI can promote equitable learning outcomes, on the professional development they had received, and on the support teacher preparation programs should offer relative to AI.

The second group of teachers completed the *Ethics in AI in Mathematics Education* (EAIME) survey immediately following a workshop on ethics in AI in mathematics education (Sears, Castro-Minnehan et al., 2026). Of the 16 survey items, 13 were open-ended, asking teachers to share their perspectives on how ethical considerations related to AI integration affect teaching, learning, and assessment, to describe opportunities for teaching mathematics with GenAI, and to identify potential challenges and risks. The survey also asked teachers to share their views on how capable they feel of ensuring students' learning, discuss what they consider as evidence of learning when AI is used, outline the ethical rules related to GenAI use they want to establish for themselves and their students, contemplate the cognitive, physical, informational, and governance ethical dimensions of Ashok et al. (2022)'s framework, which was provided to them, and comment on what kind of professional support they might need to make ethical decisions regarding AI in mathematics education.

Data Analysis

The initial analysis of data from the PAISMTL and EAIME surveys provided valuable insights, indicating that teachers have polarized opinions on whether AI should be integrated into the mathematics curriculum (Sears, Castro-Minnehan et al., 2026; Sears, Franco et al., 2026). To better understand this phenomenon, we conducted a retrospective analysis of the open responses from both surveys, using the AIM-CEL Framework as an analytic lens. A retrospective analysis (Simon, 2019) occurs after data collection is complete and aims to inform models of individual thinking, social interactions, or instructional interventions, thereby extending initial interpretations of the data.

The first level of the retrospective analysis (Simon, 2019) involved a line-by-line analysis to identify instances of the phenomenon in question, here, the teachers' perspectives on integrating AI into the official, teacher-led, enacted curriculum, and the resulting impact on student learning outcomes. These perspectives, and how teachers envision they may evolve, shed light on teacher agency while also capturing factors that may contribute to AI integration. Thus, we imported the dataset into NVivo 15 (Lumivero, 2025) and coded it at the utterance level, capturing how various domains of the AIM-CEL Framework were appraised, using the following categories: *no AI use/AI-reluctant/AI-resistant,* and *AI-supportive/conscientious/ strategic intentional use*.

The *no AI use/AI-reluctant/AI-resistant* code reflected instances where the teachers acknowledged intentionally not using AI tools, actively avoided them, expressed distrust, skepticism, or concerns for negative implications on student learning (such as reducing cognitive rigor), expressed a preference for solely human activities or for limits or bans in the workplace and in policies, or expressed explicit reluctance toward AI integration into their curriculum. The code *AI supportive/conscientious/strategic intentional use* was applied when teachers acknowledged the intentional and purposeful use of AI tools, indicated being knowledgeable and responsive to the AI capabilities, described instances of integrating AI or facilitating discussions about it, and promoted ethical and appropriate use of AI. This code also captured teachers noting potential benefits of using AI and expressing a desire to enhance their own knowledge and instructional practices involving AI, while being conscious of the need for the human in the loop to verify the accuracy of AI-generated responses. After coding, matrices were generated to identify dominant codes and explore relationships among them.

For the second level of analysis, we considered the nature of teachers' understanding of AI, the extent to which they exhibited agency, and, potentially, its subset of epistemic agency. Thus, we scrutinized the codes, assessing the extent to which they suggested that teachers had agency in decision-making regarding AI integration across the curriculum. The third level of analysis aimed to develop explanatory constructs to account for how teacher perceptions may influence the nature of curriculum enactment in support of students' mathematical learning. We also identified factors that teachers perceived as supporting or hindering their use of AI.

Results

There were different views on the extent to which AI should be integrated into the mathematics curriculum. In the official curriculum, 5 codes expressed reluctance toward AI, while 15 codes supported the strategic and intentional use of AI. Regarding the teacher-led curriculum, 14 codes opposed using AI, whereas 34 supported it. In the enacted curriculum, 32 codes reflected a reluctant stance toward AI integration, while 69 supported it. Overall, 51 coded utterances reflected a negative view of AI integration (no AI, resistance, or reluctance), whereas 118 coded utterances supported incorporating AI across the official, teacher-led, and enacted curricula.

Nesting the utterances within the participants suggested that overall, teachers were more likely to support AI integration in the official, operationalized, and enacted curricula, although approximately one-third of the responses took a strong negative stance toward AI integration. We also identified factors that influenced the extent to which they use AI. The teachers demonstrated teacher agency and epistemic agency in some instances.

Teachers' Perspectives on AI Integration According to the AIM-CEL Framework

Official curriculum. Teachers who objected to AI integration in the official curriculum expressed concerns about data privacy and uncertainty about the details of AI operation. For instance, Teacher 1 explained,

> I am constantly concerned about how my data and the data of my students are being used. I also strongly believe in the concept of productive struggle. The over-dependence on these tools feels like the social contract between student and teacher has been broken; students are no longer obligated to work to learn. This is a problem. (EAIME)

Similarly, Teacher 13 noted, "I wouldn't encourage it currently, because I think there is a lot that is not fleshed out with how it is used in a math classroom. I think it is still too messy to use" (PAISML). The teachers' reasoning suggests that if AI integration in classrooms were to become a reality, more clarity is needed on data usage and AI tools, and on ensuring that all teachers are adequately prepared to use them.

Teachers who supported AI integration into the official curriculum reported using school guidelines and facilitating productive conversations with students. For instance, Teacher 3 indicated, "I am already having these conversations with students using the [district] guidelines" (EAIME), while Teacher 11 indicated that

> AI is here, and as teachers, we must stress the ethics and trust that students are doing what we ask as educators. We must also know how AI is being used and what we should be expecting for AI answers. (PAISML)

These teachers' remarks highlight students' role in the educational setting as critical partners rather than consumers, while also acknowledging the importance of clear expectations and guidelines for responsible AI use.

Teacher-intended curriculum. Teachers who were resistant to integrating AI into the teacher-intended curriculum were generally concerned about the repetitiveness and stagnation in AI-generated responses, as well as mathematical inaccuracies. However, some teachers simply preferred doing things the traditional way. For instance, Teacher 1 noted, "AI is notoriously bad at math. Until it improves, I am highly skeptical" (EAIME). Teacher 1 also noted, "I do not use it at all. I prefer to do my assignments the old-fashioned way" (EAIME). Other reasons for a reluctant stance included limited opportunities for teachers to learn about and experience using AI for instructional planning. For instance, Teacher 10 indicated, "I am not sure as I have not had time to sit and read/learn about using AI in planning and learning" (PAISML).

Teachers who supported using AI in the teacher-intended curriculum emphasized that AI tools can help them brainstorm, develop lesson plans, and communicate with stakeholders. For instance, Teacher 2 indicated, "Gen AI can help with lesson plans, communication with parents, [and] newsletters" (PAISML), while Teacher 3 noted, "I use it to brainstorm when I have not taught a subject before and to find resources for me to vet and consider" (EAIME).

Enacted curriculum. Resistance to AI integration into the enacted curriculum was grounded in the desire that classroom instruction should prepare students for assessments that do not currently include the use of AI, students may rely on the AI tool more than on the teachers, and students may not exhibit any effort to engage in mathematical tasks by treating AI as a "surrogate knower" (Jose et al., 2025). For example, Teacher 12 expressed this concern, saying, "Students think they don't need to learn ANYTHING [emphasis in the original] because AI can do it for them" (PAISML), while another teacher suggested that there is a need to "make exams offline, ask questions alive, ask them to present on their work" (Teacher 2, EAIME).

Teachers who support integrating AI into the enacted curriculum suggested that AI can be used strategically to verify the accuracy of students' responses, reflect on multiple problem-solving approaches, pose high-level questions, and engage in error analysis. Given the possibilities and opportunities that can be explored with AI, Teacher 8 indicated, "AI has so much power, but it's imperative that students are exposed to ethical use and taught how learning can be enhanced/expanded with it" (EAIME).

Student learning outcomes. All the teachers sought to promote student learning outcomes. Interestingly, both reluctance and support toward AI were motivated by teachers' desire to support students' mathematical learning. Teachers objecting to AI integration wanted to ensure that students did not become lazy, unable to demonstrate problem-solving skills, and overly reliant on AI tools. For instance, Teacher 2 noted, "This raises concerns about overreliance on AI, potential misinformation, and the loss of critical thinking and problem-solving skills if students depend too heavily [on AI]" (EAIME). Teachers who favor AI integration asserted that, if used responsibly, AI can support learning in and out of the classroom. For instance, Teacher 3 envisioned using AI as a personal tutor for students with limited financial means or limited access to technology outside the school, saying, "Students having a personal [AI] tutor seems like an amazing opportunity for students who cannot afford one. After [school] hours, they would still need the technology, which is not always possible" (EAIME).

Factors Teachers Perceive as Affecting AI Integration across the Curriculum

Various factors contributed to teachers' perspectives on AI across the mathematics curriculum. The factors included teachers' own knowledge and beliefs (23 coded instances), concerns for academic integrity and assessment (22 coded instances), access and equity issues (19 codes), understanding of AI tools (18 codes), teachers' own confidence and comfort level

with AI (15 codes), pedagogical considerations (8 codes), concerns for mathematical accuracy or AI hallucinations (7 codes), and students' attitudes and dispositions (6 codes).

Teachers' Perceived Agency in Integrating AI in the Curriculum

The results showed that teachers perceived having agency in determining whether and how to use AI. It was noted that teachers who were more willing to use AI reported receiving some professional development or training and felt more knowledgeable about AI tools. Additionally, several teachers exhibited *epistemic* agency in co-constructing knowledge and co-creating resources with AI. For instance, Teacher 15 noted,

> I think AI provides an excellent support role. I have used it to create a short reading quiz on an assignment, then refined the result with additional input. It was just as good as I would have made on my own, but it took minutes. As a teacher, my time is my most precious resource. I feel no guilt in using AI to support, but not to prevent myself from completing required learning. If I didn't understand a concept in my statistics class this fall, Copilot gave me a brief synopsis, and could elaborate and give me examples to try. I now understand the material, so I was not using AI to prevent learning, more to facilitate. (PAISML)

This quote illustrates how Teacher 15 exercised epistemic agency, using AI judiciously and responsibly. Such use of AI tools in the mathematics curriculum can increase productivity and promote efficiency in supporting learning.

Discussion and Implications

The results of the study align with prior research that suggests that teachers' perspectives of perceived benefits and constraints of AI may impact their willingness to integrate AI into their teaching practices (Albalawi, 2025; Ballenas & Lasco, 2025; Busuttil & Calleja, 2025; Chounta et al., 2022; Clarke, 2025; Egara & Mosimege, 2024; Korkmaz et al., 2025; Meylani, 2024; Sears et al., 2025; Sears, Castro-Minnehan et al., 2026; Walkington, 2025). Our contribution to this literature is the specification, through the AIM-CEL Framework, that AI integration spans all components of the mathematics curriculum: official, teacher-intended, and enacted, with implications for students' learning outcomes.

The study findings further show that teachers' reluctance to adopt AI or their support for it is expressed across all components of the AIM-CEL Framework. This indicates that teachers engage thoughtfully and deeply with the meaning and impacts of AI integration on society, the teaching profession, classroom interactions, and students' mathematical learning. Interestingly, both positive and reluctant attitudes toward AI integration were grounded in perceived benefits or negative impacts on student learning outcomes. This reveals complex connections between teachers' views of AI's advantages or risks and their ideas of what constitutes learning mathematics, whether it's demonstrating competencies on traditional assessments or developing flexible knowledge and reasoning skills. Although these relationships need further investigation, our findings underscore the importance of clarifying what it means to learn mathematics in an AI-enhanced curriculum and re-envisioning assessment strategies that effectively support AI integration, particularly to address concerns about how AI tools might influence cognitive rigor and students' demonstration of mathematical processes and practices.

Our findings also indicate that teachers can demonstrate both teacher agency and epistemic agency (a subset) regarding AI integration into the mathematics curriculum (Chu & Wang, 2025; Jose et al., 2025; Urbani, 2024). As AI continues to play an increasingly central role in education, it is essential for teachers to develop epistemic agency when engaging with AI and when involving students in these interactions. Therefore, our findings emphasize the importance

of supporting teachers in implementing AI-related policies and practices. Ongoing professional development is needed to increase the likelihood that teachers can implement AI strategically and responsibly, as well as to reflect on and address potential resistance to AI tools. Future research should examine changes in teacher agency and epistemic agency related to AI use across the mathematics curriculum following continued professional development.

In closing, we highlight that the AIM-CEL Framework can be used to examine how AI is integrated into the mathematics curriculum by helping researchers unpack the official, teacher-intended, and enacted curricula to promote student learning. The generalizability of our findings is limited by the relatively small sample of teachers from a single locale. Nevertheless, the findings provide insights into teachers' stances on "no AI" and on when it is essential to "know AI" to support mathematics teaching and learning.

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CURRICULUM, ASSESSMENT, AND RELATED TOPICS

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DEVELOPMENT OF A MULTIDIMENSIONAL SURVEY OF UNDERGRADUATE MATHEMATICS STUDENT AND TUTOR REASSESSMENT EXPERIENCES

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Despite growing use of reassessment in undergraduate mathematics, few validated instruments exist that measure the multidimensional experiences of students and tutors involved in reassessment. In this study, we developed a survey to explore motivational, affective, and cognitive dimensions of undergraduate student participation in reassessment from both students' perspective and that of the tutors responsible for preparing students for reassessments. Preliminary analyses of student and tutor responses at one southeast university provide evidence of survey item's reliability and content validity, although further analyses are needed to validate the survey.

Keywords: Assessment; Affect, Emotion, Beliefs, and Attitudes; Instructional Activities and Practices; Undergraduate Education

Reassessments, or exam retakes, in undergraduate mathematics courses, are typically discussed in the context of grading systems. While reassessment is often part of mastery-based, techniques-based, and standards-based grading, these grading systems also often include pre-defined topics that need to be mastered, retake opportunities on homework and other course assignments, and a grading scale based on number of topics mastered rather than percentage points (Cooper, 2020; Halperin, 2020; Harsy, 2020; Linhart, 2020). While the existing literature often reports positive outcomes for students participating in alternative assessments, much of this work relies on unvalidated instruments or researcher-generated surveys, and many studies fail to provide the survey tools, validation efforts, or instructor feedback (Hackerson et al., 2024).

The literature highlights a clear gap: a need for validated instruments to comprehensively assess student and tutor perspectives on reassessment. Accordingly, the purpose of this study is to develop two surveys measuring these perspectives, providing data that can inform refinement of the reassessment process, support potential scalability, and address gaps in the literature.

Context of the Study

This study is situated in the context of College Algebra and Intermediate Algebra courses at a university in the Southeastern US, where reassessment was implemented beginning in 2022. The primary goal of this initiative was to improve student success and persistence in high DFW gateway math courses. The reassessment structure was designed to emphasize more than grade recovery; students were required to complete remediation activities (such as worksheets and tutoring sessions) in the university mathematics tutoring center prior to retaking unit exams for the purpose of deepening their mathematics understanding. Courses were divided into 14 to 20 topics, allowing students to reassess individual topics following unsatisfactory performance on initial exams although the final exam could not be reassessed. Retakes were then coordinated with the center and returned to the instructor for grading. This context provides a unique opportunity to explore how both students and tutors experience reassessment, offering insights that could enhance the effectiveness of alternative assessment strategies in gateway mathematics courses.

In this study, reassessment is distinctive due to its wide-scale implementation across lower-level mathematics courses, its traditional exam structure, and its close integration with a mathematics tutoring center. This design helps reduce the remediation burden for instructors while supporting students who are working or commuting by providing flexible tutoring and reassessment timeslots. Figure 1 illustrates the reassessment process.

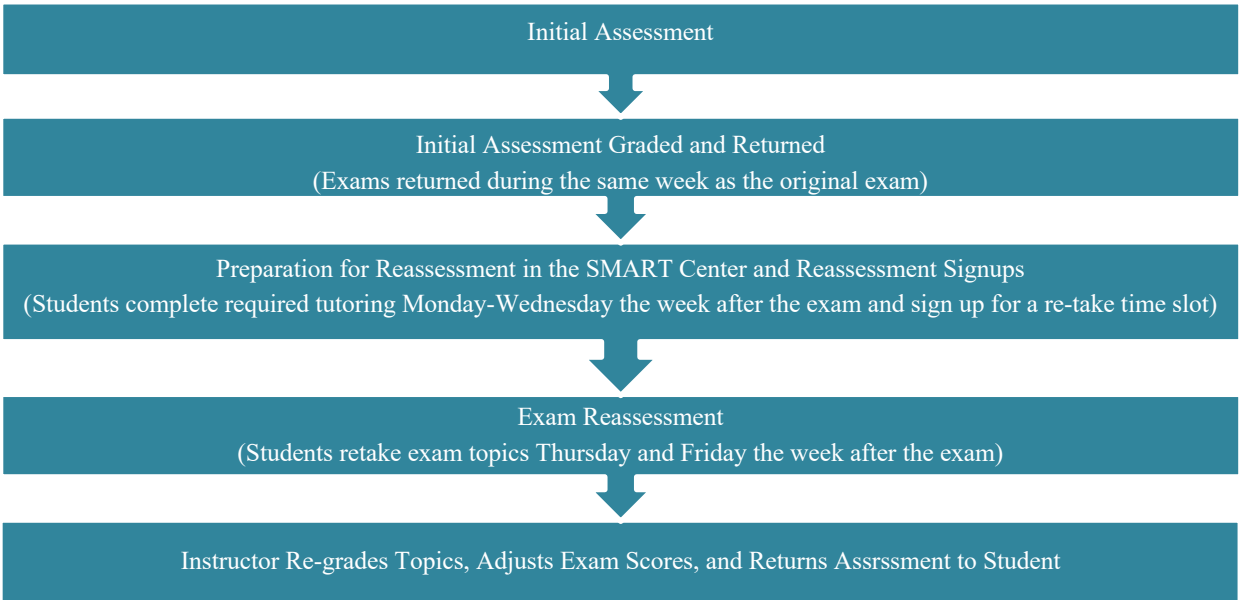


Figure 1: Student Reassessment Process

Conceptual Framework

The developed surveys measure constructs related to motivation and participation (i.e., how often students engage in reassessment and why), self-efficacy, self-regulation, anxiety, and program evaluation (i.e., how reassessments serve students and opportunities for improvement). Affective constructs, including self-efficacy, motivation, and anxiety, as well as attitudes and student behaviors, were selected based on their prominence and frequency in the existing literature (Hackerson et al., 2024). Prior research has also documented positive effects of reassessment on motivation, effort, confidence, and growth mindset (Barhite, 2024; Elsinger & Lewis, 2020; Harsy, 2020; Jones & Lanaghan, 2021; Mangum, 2020). These constructs are central to understanding the impacts of reassessment on both students and tutors. They inform potential modifications to increase participation, enhance buy-in, and improve the overall reassessment experience, ultimately supporting student success in lower-level mathematics courses. Figure 2 presents the conceptual framework, illustrating how reassessment influences motivation, participation, self-efficacy, self-regulation, and anxiety, which together shape students' perceptions of the initiative.

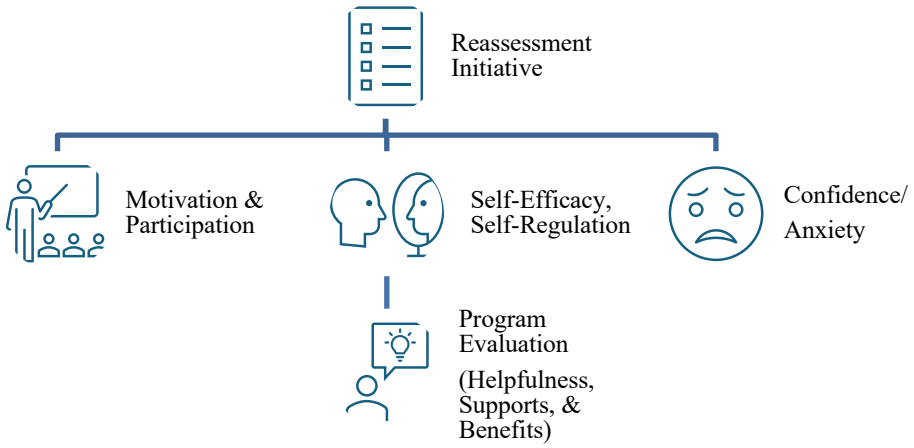


Figure 2: Conceptual Framework for Constructs

Methods

This study employed a quantitative approach to develop surveys aimed at capturing both qualitative and quantitative data on students' and tutors' perspectives of reassessments in lower-level college mathematics courses. Survey development followed the procedures outlined by Gehlbach and Brinkworth (2011) to support content and construct validity.

Instrument Selection

A systematic literature review of studies surrounding alternative assessment in college-level mathematics courses following the year 2014 was conducted. Forty-four studies were identified for inclusion in the literature review. Each study's outcomes and utilized instruments were noted. Furthermore, Hackerson et al.'s (2024) scoping review of alternative assessment practices in college-level STEM courses served as a helpful synthesis of existing survey tools.

Instrument Adaptation

Following the literature review, the Situational Motivation Sale (SIMS) and Mathematics Motivation Questionnaire (MMQ) were selected as the basis for questions measuring student motivation, self-efficacy, self-regulation, and anxiety due to their alignment with the desired constructs and availability. The Community College Survey of Student Engagement (CCSE) was selected as the foundation for the program evaluation component of the reassessment initiative due to its alignment with postsecondary education and the desired constructs. Questions from these existing surveys were compiled and adapted to fit the reassessment initiative.

While constructs were shared across the tutor and student surveys, questions on the tutors' survey relating to motivation, anxiety, and self-regulation focused on tutors' perceptions of how reassessments impact students. Program evaluation questions, in contrast, focused on tutors' own experiences, perceived benefits, and perspectives regarding the reassessment process.

Situational Motivation Sale (SIMS). The SIMS, which contained 16 items measured intrinsic motivation, identified regulation, external regulation, and motivation. The original Likert-scale design was maintained, while "not sure" or "not applicable" were added as options.

Community College Survey of Student Engagement (CCSE). Eight questions from the CCSE, a large community college evaluation survey, were adapted to fit the reassessment context and to avoid numerical labels in accordance with best practices (Gehlbach & Brinkworth, 2011). The remaining questions, which did not pertain to a classroom context, were not adopted.

Mathematics Motivation Questionnaire (MMQ). Fifteen of the MMQ items were included in the reassessment survey. The MMQ was shortened, and some similar ideas were removed to reduce the length of the reassessment survey. In alignment with survey best practices, the MMQ questions were altered from an agree/disagree Likert-scale format to an amount/frequency Likert-scale format (Gehlbach & Brinkworth, 2011).

Once compiled, a Qualtrics survey was developed and adjusted for flow. The student survey was adapted with minor adjustments to measure tutors' perspectives. Following the revision of selected items, both surveys were examined by content experts to establish content validity.

Content Expert Review

Three content experts were consulted to ensure content validity for both tutor and student surveys. Experts were selected based on their experience with reassessment and their familiarity with the reassessment process. They were asked to review each survey item for appropriateness and clarity, provide feedback on question wording, and suggest additional elements to include. Expert A is an instructor with extensive experience teaching College Algebra courses using reassessment. Expert B serves as the director of the mathematics tutoring center responsible for preparing students and administering reassessments. Expert C is a doctoral student and College Algebra instructor who also teaches courses utilizing reassessment. Each expert received links to both surveys and met in person with the researcher to provide feedback.

The feedback from content experts resulted in revisions to question wording and response scales to enhance clarity and reduce complexity. Experts also recommended removing items related to enjoyment of mathematics, adding "please explain" prompts to free-response questions, and including items assessing how reassessment supports students in managing responsibilities across other courses. All recommendations were incorporated into both the student and tutor surveys. Following the content expert review, cognitive interviews were conducted with both students and tutors to further refine the instruments.

Cognitive Interviews

Student cognitive interviews. Cognitive interviews were conducted with six students who had completed a reassessment in a College Algebra course within the past year. Students met with the researcher and were asked to answer each question aloud, describing their thought process behind answering the question. Additionally, students were asked if there is anything else they feel should be included in the survey. Iterative improvements were made and tested with subsequent participants.

Insights from the cognitive interviews led to several changes including adding definitions or rewording for clarity and the inclusion of a free-response question asking what motivates students to participate in reassessment. Following cognitive interviews, all suggested revisions were incorporated. Additionally, questions which were misleading were re-worded to ensure answers were in line with those desired.

Tutor cognitive interviews. Cognitive interviews were conducted with three mathematics tutoring center tutors. These tutors, a mix of undergraduate and graduate students, were selected based on their role, length of employment with the mathematics tutoring center, and familiarity with the reassessment preparation and proctoring process. Tutors were also asked to answer the survey out loud while including their thought process behind their answer and to provide feedback on the clarity of questions and any additional questions they felt should be added.

Several common themes emerged from the interviews, including recommendations to split lengthy questions and revise wording for clarity, adjust numerical intervals on two items, and add questions addressing how reassessment promotes better study habits, clarifies tutor roles, and

reflects support from administration. Iterative revisions were made following each interview, and when appropriate, parallel changes were applied to the student version of the survey.

Instrument Pilot and Refinement

Following cognitive interviews, tutor survey was distributed to all mathematics tutoring center tutors via the center's GroupMe page. Within the first day, four responses were received. One tutor participant reported confusion regarding the wording of a particular question. Based on this feedback, the question was revised and split into two separate items. Although this modification resulted in the loss of a small portion of previously collected data for that question, all other data remained intact, and the survey was improved for clarity for future respondents.

The revised survey was subsequently redistributed to tutors via Canvas at the end of the Fall 2025 semester and the beginning of the Spring 2026 semester, yielding additional responses. Students were invited by proctors to participate after completing their third exam retake. Upon collection of survey responses, data was analyzed to assess internal consistency and reliability.

Results

Forty-five tutors were invited to participate in the survey via a Canvas announcement in the Fall 2025 and Spring 2026 semesters and 40 consenting responses were collected. To track repeat responses, tutors were asked to provide the last four digits of their phone number. In cases where a tutor submitted multiple responses, only the most recent submission was retained for analysis.

Approximately 100 students who participated in the third exam reassessment were asked to take the student perception of reassessment survey. 57 students engaged with the survey by scanning the QR code. Following the removal of fully incomplete surveys, 20 complete responses were recorded, and 25 partial responses were gathered.

Following cleaning of data, results were exported into R (R Core Team, 2024) for analysis and mean and standard deviation were collected for each item.

Tutor Perspectives of Reassessment: Internal Consistency and Descriptive Data

Table 1 displays each Likert-scale question and the corresponding participation rate and standard deviation. Based on information gathered from the survey, two questions were altered to add a sixth Likert-scale option.

Table 1: Likert-Scale Items - Tutor Perspectives

Construct	Number of Responses	Question Type	Mean Range	Standard Deviation Range
Motivation	40	Likert (6-point scale ranging from Not At All–Very Much) 5 Questions	3.34–4.9	0.99–1.71
Perceived Course-Related Benefits	38	Likert (5-point scale ranging from Not At All–Very Much) 5 Questions	3.21–4.51	0.96–1.56
Self-Regulation	38	Likert (6-point scale ranging from Not At All–Very Much) 4 Questions	3.27–4.08	1.21–1.63
Confidence	38	Likert (6-point scale ranging from Not At All–Very Much) 4 Questions	3.49–4.41	1.36–1.69

Anxiety	38	Likert (6-point scale ranging from Not At All–Very Much) 4 Questions	3.26–4.20	0.98–1.50
Perceived Impacts on Learning Outcomes	38	Likert (6-point scale ranging from Did not Emphasize–Emphasized Very Much) 6 Questions	2.25–3.14	1.20–1.56
Perceived Level of Challenge	38	Likert (6-point scale ranging from Did Not Challenge–Challenged Very Much) 1 Question	3.10	1.64
Perceived Student Support	38	Likert (6-point scale ranging from Did Not Emphasize–Emphasized Very Much) 6 Questions	2.27–3.86	1.33–1.65
Perceived Long-Term Benefits and Growth	38	Likert (5-point scale ranging from Contributed Very Little–Contributed Very Much) 5 Questions	1.37–1.43	2.19–2.86

Although many statements exhibited high variability, this was expected due to the opinion-based nature of the items. Free-response question responses at the end of the survey further contextualized this variability, with some tutors expressing positive views of reassessment and others noting challenges. Therefore, the observed variability in opinion-based items is reasonable and does not indicate a need for item revision.

Table 2: Cronbach's Alpha and McDonald's Omega Values - Tutor Perspectives

Construct	McDonald's Omega	Cronbach's Alpha
Motivation	0.89	0.81
Perceived Course-Related Benefits	0.96	0.92
Self-Regulation	0.93	0.92
Confidence	0.88	0.83
Anxiety	0.81	0.53
Perceived Impacts on Learning Outcomes	0.96	0.93
Perceived Student Support	0.91	0.86
Perceived Long-Term Benefits and Growth	0.96	0.94

Table 2 presents alpha and omega values for each construct. All values were high enough to suggest good internal consistency except for the alpha value relating to the anxiety construct. In this case, the low alpha value was likely due to violations of tau-equivalence, and the omega indicated acceptable internal consistency.

While 40 responses is a relatively large response rate given a lack of incentive for completion and the small population size, the small sample size impacted McDonald's Omega values. Factor-based reliability estimation produced diagnostic warnings for four constructs (motivation, anxiety, confidence, and self-regulation), likely due to this small sample. As a result, McDonald's omega values were interpreted alongside Cronbach's alpha. As new tutors are added and the sample size grows, further analysis will be conducted.

Student Perspectives of Reassessment: Internal Consistency and Descriptive Data

Table 3 displays each construct along with its number of responses and standard deviation in response for each item.

Table 3: Likert-Scale Questions - Student Perspectives

Construct	Number of Responses	Question Type	Mean Range	Standard Deviation Range
Motivation	39	Likert (6-point scale ranging from Not At All–Very Much) 5 Questions	2.62–4.69	0.51–1.88
Perceived Course-Related Benefits	35	Likert (5-point scale ranging from Not At All–Very Much) 5 Questions	3.91–4.47	0.79–1.11
Self-Regulation	33	Likert (6-point scale ranging from Not At All–Very Much) 4 Questions	3.85–4.30	0.65–1.05
Confidence	24	Likert (7-point scale ranging from Not At All–Very Much) 4 Questions	4.00–4.20	0.79–1.1
Anxiety	24	Likert (7-point scale ranging from Not At All–Very Much) 4 Questions	3.68–4.05	0.69–1.22
Perceived Impacts on Learning Outcomes	26	Likert (6-point scale from Did not Emphasize–Emphasized Very Much) 7 Questions	1.96–2.79	0.81–1.42
Perceived Level of Challenge	28	Likert (6-point scale ranging from Did Not Challenge–Challenged Very Much) 1 Question	2.64	1.29
Perceived Student Support	27	Likert (6-point scale from Did not Emphasize- Emphasized Very Much) 6 Questions	2.44–2.89	1.45–1.77
Perceived Long-Term Benefits and Growth	26	Likert (6-point scale from Did Not Contribute–Contributed Very Much) 6 Questions	2.31–3.31	1.29–1.56
Intended Future Use	25	Multiple choice with 3 options 1 Question	1.96	0.34

The variability of survey responses was likely due to the "not sure" option as well as differing opinions of reassessment. Students were unanimous in their opinion that reassessment was helpful because it improves grades. But some students felt that reassessments were great as-is, while others mentioned it was too easy, possibly explaining why such great variation existed for the level of challenge construct. Table 4 displays alpha and omega values for each construct.

Table 4: McDonald's Omega Values - Student Perspectives

Construct	McDonald's Omega	Cronbach's Alpha
Motivation	0.64	0.36
Perceived Course-Related Benefits	0.94	0.86
Self-Regulation	0.89	0.87
Confidence	0.77	0.61
Anxiety	0.90	0.75
Perceived Impacts on Learning Outcomes	0.92	0.87
Perceived Student Support	0.97	0.93
Perceived Long-Term Benefits and Growth	0.97	0.92

The alpha values for motivation and confidence were small, as shown in table 4. While this suggests poor internal consistency under the assumptions of Cronbach's alpha, alpha can underestimate reliability when the tau-equivalence assumption is violated. Because the included items may measure different aspects of the construct, McDonald's Omega was also examined. The McDonald's Omega value for both constructs is much higher, suggesting that both constructs have an acceptable internal consistency.

All other constructs included in the instrument that utilized a Likert-scale had good internal consistency according to both the alpha and omega values. While this suggests internal consistency for the instrument, the sample from which data was collected was small. Each item ranged from 25 to 39 responses. While the data preliminarily suggests reliability and internal consistency, a larger sample is necessary to draw concrete conclusions.

Conclusion

In this study, we developed surveys to measure student and tutor perspectives of reassessment. Information collected on the survey included mostly Likert-scale questions, with additional free-response and demographic questions to further understand perspectives and gather feedback. Both instruments were examined for content validity via content experts. Cognitive interviews were conducted with tutors and students, resulting in iterative changes to both instruments. Following cognitive interviews, both surveys were disseminated to mathematics tutoring center tutors and reassessing College Algebra students.

Using classical test theory, analyses indicate the survey items show promise for further refinement. Future studies include gathering additional responses for factor analysis. Future item refinement will focus on separating program-evaluation items from items evaluating affective latent constructs relating to the concept of reassessment in undergraduate mathematics.

Given increasing gaps in prerequisite skills necessary for success in undergraduate mathematics courses, reassessment structures that promote student success and persistence are essential. However, these structures cannot be developed by faculty and administration in isolation. The voices of students and tutors are essential in meeting the aims of these reassessment initiatives and increasing the likelihood of student participation and deeper mathematics learning.

This study addresses a critical gap in reassessment research by providing empirically grounded instruments to measure student and tutor perspectives. These tools can inform both local program improvement and broader research with the goal of designing equitable, learning-centered reassessment systems.

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FROM THEORY TO PRACTICE: THE DEVELOPMENT OF A SURVEY TO MEASURE TEACHERS' CURRICULAR REASONING

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As teachers interact with textbooks and other curricular materials to plan instruction, they rely heavily on their curricular reasoning (CR) to make decisions regarding what tasks and problems to use in their mathematics lesson. Teachers' CR may include a number of aspects focusing on differing components of the classroom environment. To assist teachers in considering these various CR aspects, we developed a self-assessment survey to help teachers in instructional planning and enacting lessons. In this paper we describe the development of this survey, the process by which the survey was validated, and consider how the survey can be used to impact teacher practice.

Keywords: Curriculum, Instructional Activities and Practices, Teacher Beliefs, Research Methods

Teachers and students learn from a variety of mathematics curricula, and teachers use these curricula in numerous ways as they plan and enact instruction. Different types of curricula provide various opportunities for students to learn mathematics (e.g., Tran et al., 2016; Tarr et al., 2008). Teachers also use curricula differently depending on the context (e.g., Remillard & Kim, 2017). However, little is known *about teachers' cognitive work as they use mathematics curriculum to plan and enact lessons*. This cognitive work includes making myriad mathematical decisions and reasoning as they plan and enact lessons with curricula. Roth McDuffie and Mather (2009) defined teachers' curricular reasoning (CR) as "a specific form of pedagogical reasoning that teachers employ while working with curriculum materials to plan, implement and reflect on instruction" (p. 303). As teachers utilize a variety of curricula, they act on many decisions including the degree to which the curriculum provides students the opportunity to learn targeted mathematical content, and whether it is at the level needed to meet learning goals. Investigating teachers' CR can elucidate why teachers make specific mathematical decisions such as skipping or reordering lessons, revising tasks, or modifying definitions. While some decisions may appear to an observer as inappropriate for a given situation, these decisions may be much more complex than simply correct or incorrect. In understanding teachers' reasoning behind decisions, teachers can be better equipped to deliberately make productive decisions and strengthen or avoid decisions that lead to less productive outcomes for teaching and learning.

In this paper we share results regarding the development process for a survey to assist teachers in considering various CR aspects in their instructional planning. We describe how we partnered with teachers in the development process, provide survey results from teachers who have used the survey across the four states, and amplify teachers' voices in how the survey assisted them reason through mathematical decisions as they plan, enact, and reflect on their teaching. The development of this survey was part of a larger NSF-funded project *Quantifying*

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Curricular Reasoning as a Critical Practice in Teaching Mathematics (Grant Nos. 2201164, 2201165, 2201167, 2525831, and 2201169). The goal of the project is to design and validate tools for middle school mathematics teachers to use to identify and quantify their CR. The survey is one of these tools and is aimed at formatively assessing and providing feedback to teachers on their use of CR as they plan and enact mathematics lessons.

Cohen et al. (2003) introduced the Instructional Triangle to depict the interactions among teachers, students and the mathematical content in the classroom. However, given the different representations of, and approaches to, mathematical content seen in mathematics curricula, we argue that the mathematical content should be two distinct classroom elements to reflect two qualitatively different components (i.e., the Mathematics and the Curriculum) that impact the classroom learning environment. Building on the Instructional Triangle (Cohen et al., 2003) our team developed the Instructional Pyramid (Dingman et al., 2021; Teuscher et al., 2025), as shown in Figure 1, to depict the interaction among four major classroom elements (i.e., Teacher, Student, Mathematics, Curriculum). Each edge of the pyramid represents one of the six key CR aspects that teachers reason with as they plan, enact, and reflect on their mathematics lessons. Table 1 lists operational definitions for each CR aspect. The self-assessment survey developed in our project unpacks the six CR aspects for teachers and instructional coaches to identify which of these CR aspects teachers use and which ones they may want to develop further. Initial findings from our pilot work with the survey indicate that when teachers reason with all CR aspects on the Instructional Pyramid, they provide their students with learning opportunities that enhance student learning.

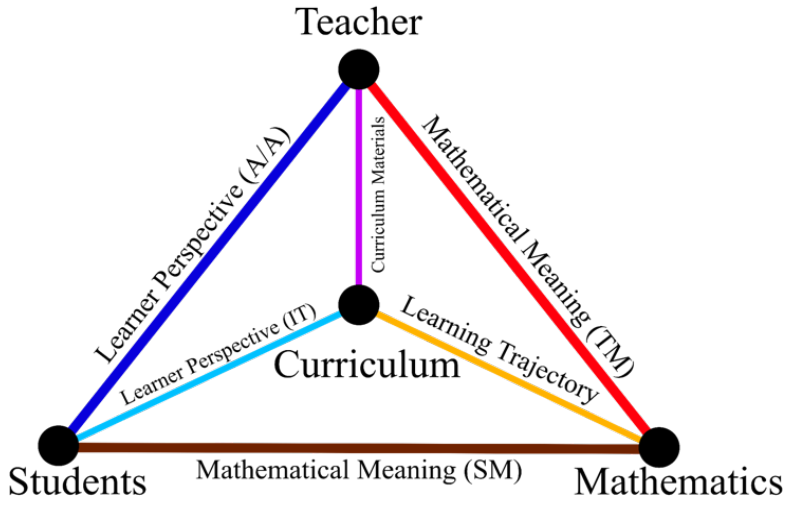


Figure 1: Instructional Pyramid with Curricular Reasoning Aspects

Table 1: Aspects of Mathematics Teacher Curricular Reasoning Operationally Defined

CR Aspect	Operational Definition
Viewing Mathematics from the Learner Perspective – Anticipating and Assessing (LP-AA)	Teachers "reason about the specific details of the mathematics or structure of the lesson to consider student interpretations of the mathematics, anticipate how students will perform or perceive the mathematics in the lesson, or identify student misconceptions that will need to be addressed as the lesson progresses" (p. 276).
Viewing Mathematics from the Learner Perspective – Intentionality of Tasks (LP-IT)	Teachers "reason about the intentionality of the task (IT), suggesting that teachers have a well-defined purpose for giving students specific questions or tasks that will guide them in learning the mathematics concepts that are intended" (pp. 276-277).
Mapping Learning Trajectory (LT)	Teacher "reasoning about how the mathematical concepts will unfold in a lesson or how the mathematics of the current lesson connects to past or future topics" (p. 274).
Considering Mathematical Meaning – Students (SM)	"…centers on what the teacher thinks students should understand (SM) based on the teacher's own mathematical meanings" (p. 278)
Considering Mathematical Meaning – Teachers (TM)	"…teachers' reasoning with their schemes for how they understand the mathematics under study" (p. 278).
Analyzing Curriculum Materials (CM)	Teachers "analyze or compare curriculum materials to other materials, providing analyses of the strengths and limitations of the competing materials and offering details of the various approaches" (p. 273).

Note. Operational definitions directly quoted from Dingman et al., 2021.

While much of the research around curriculum use has focused on in-depth interviews with individual teachers using a particular curriculum, as well as a labor-intensive qualitative analysis to determine how teachers use these specific curricula, our project pursued developing a tool that is valid and reliable, and usable for many teachers, regardless of their mathematics curricula. Therefore, this survey allows teachers to work with other colleagues to improve their decision making.

Development and Piloting of the CR Survey

Building on our prior work delineating the six CR aspects shown in Table 1, the survey development began with forming teams within our project personnel and outlining key items for each of the CR aspects. Three teams were assigned two CR aspects each and used existing definitions for the CR aspects as well as data from previous teacher interviews (Dingman et al., 2021) to draft the items of each aspect to guide survey item development. After several rounds of internal revisions, the list of survey items was sent out to external reviewers for additional input to inform *content validity evidence*. In total, 97% of items were identified as aligned with their CR aspect by the Expert Panel. Before moving into field testing with typical participants, survey language was modified based on expert panel review information.

We further refined the survey by conducting four rounds of cognitive interviews with 12 middle grades teachers, followed by administering the survey through three pilot studies with larger teacher samples across seven states. Each round informed revisions to survey items, language for items and instructions, and response scale, resulting in a final 38-item survey. At the start of the cognitive interview, teachers were asked to consider a mathematics lesson they taught within the past two weeks as they answered the items in the survey. Each teacher was asked to read the survey items aloud and to answer according to their experience planning the lesson under consideration. Teachers were also asked to comment on the introductory narrative, the survey response scale, and the demographic information collected on the survey. Information collected from cognitive interviews informed *response processes validity evidence* and *consequences of testing validity evidence*. Across aspects, 11 of 12 (92%) teacher participants indicated feeling no negative consequences related to completing the survey and all participants (100%) noted being able to use the survey scale well.

After the four rounds of teacher cognitive interviews, we administered the survey through three pilot studies with groups of middle grades mathematics teachers to see how the items were capturing teachers' CR. The pilot studies focused on establishing survey validity and removing items that were redundant. Teachers were invited to complete the survey in a manner consistent with how we anticipated teachers may interact with the survey, namely as a reflection upon a past lesson they had taught and the instructional planning methods they used to create and enact that lesson. The first two pilot studies included 103 and 184 middle grades mathematics teachers, respectively, from the four states represented in the project. The third and final pilot study used email contacts and popular listserv connections to invite teachers nationally to complete the survey. A total of 219 mathematics teachers across seven states responded to the survey. The results of these pilot studies are summarized in the next section, which details the evolution of the survey across the four rounds of pilot studies.

Evolution of CR Survey Based on Quantitative Field-Testing

At the conclusion of the teacher cognitive interviews, the CR survey consisted of 61 items across the six CR aspects. One of the goals of the pilot studies was to identify items that were duplicative in terms of consistently reporting the same results across teachers. Based on the Rasch rating scale model (Andrich, 1978) results of the first pilot study, we removed 15 items from the survey due to redundancy in content and difficulty measures. In addition, we adjusted the language of several items based on item fit statistics, while the data strongly indicated the 4-point response scale was functioning well across all aspects and should be maintained.

Given the large reduction in items from the first pilot study, a second pilot study was conducted prior to the national pilot study with teachers from across the four states represented in the project as well as teachers from one additional state recommended by the project staff (five states total). Rasch results of the second pilot study reflected those of the first pilot study in relation to the effectiveness of the four-point scale in the survey. No item language adjustments were recommended as a result of the second pilot study, as all items fit well in their respective aspects. Finally, important psychometric findings led to extensive team discussions centering on the meaning of the CR construct produced several policy-, evidence-, and theory-informed item removal decisions. We removed all four items in the survey related to "race, ethnicity, culture, language and family" due to federal policy changes regarding research related to racial, ethnic, and cultural diversity. Additionally, we removed one item related to facilitating student engagement and two items related to connecting mathematics to students "everyday lives" due to psychometric and conceptual redundancy.

The national pilot study resulted in one additional item being removed from the survey due to redundancy. Across pilot studies, 23 items were removed, reducing the survey by 37.7% to 38 items. Rasch analysis indicated strong *response processes validity evidence* in alignment with Linacre's (2002a) guidelines for optimizing the effectiveness of rating scales. Each scale category had 10+ observations. All measures advanced suggesting participants with higher rating combinations produced higher measures. And outfit mean-squares (MNSQ) were all lower than 2.0, indicating appropriate randomness in the data. Across the six aspects and overall, *internal structure validity evidence* was also strong according to well-documented standards (Duncan et al., 2003; Linacre, 2002b). Rasch reliability for persons (range 0.70–0.92) and items (range 0.95–0.99) as well as separation (persons range 1.53–3.33; items range 4.39–9.21) were all acceptable or higher. No items misfit in a manner that would degrade the measure (all items < 2.0 MNSQ). And all items had positive point-biserial values suggesting they well fit within their respective aspects and the overall CR construct. Figure 2 is the complete variable map for all the items in the Curricular Reasoning Self-Assessment Survey Suite (CR-SASS) based on our pilot studies. This map identifies items that are more likely to be thought about and influence teachers' lessons (e.g., those near the bottom of the ruler) as well as those items that are more difficult for teachers to reason with (those near the top of the ruler). We also can create individual variable maps to identify the items within the six CR aspects that are more difficult and easier for teachers to reason with as they plan and implement their lessons.

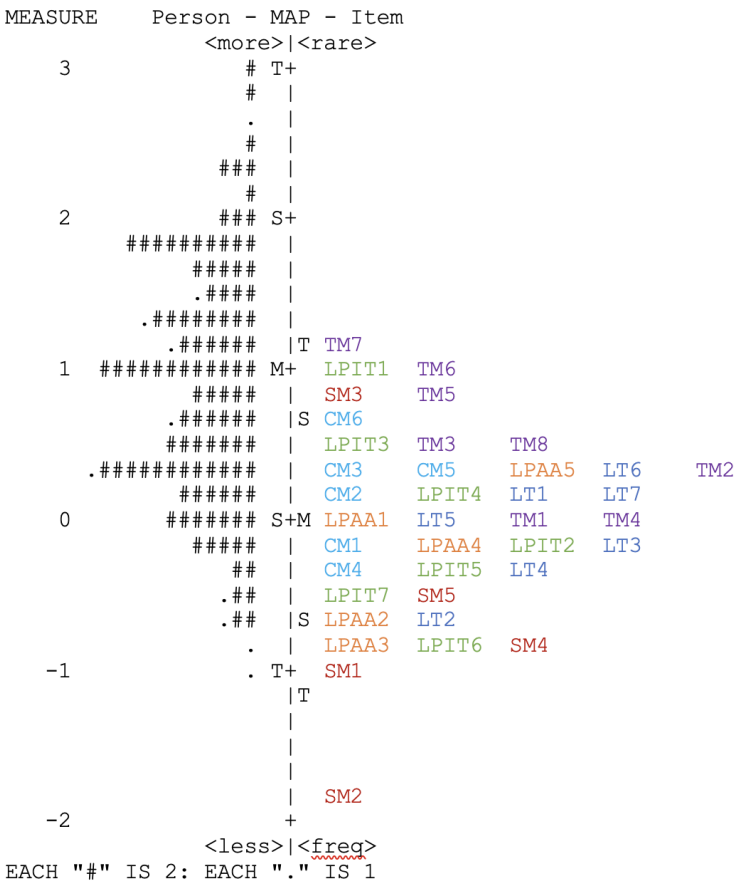


Figure 2: Variable Map with All CR-SASS Items

The Curricular Reasoning Self-Assessment Survey Suite (CR-SASS)

The resulting CR survey consists of eight sections that capture information about the lesson that teachers planned, demographic information on the teacher, and the extent to which the teacher considered various items of the six CR aspects in their instructional planning for the given lesson. In the initial section, teachers consider a mathematics lesson they planned and taught within the last two weeks. Teachers are instructed to select a lesson that was not the first or last lesson of a particular unit, as well as a lesson that encompasses the teacher's typical instructional practice. For the purposes of the survey, a lesson can last for a minimum of one day or can be a multi-day lesson. Teachers provide the title of the lesson, the grade level(s) of the students completing the lesson, the type of class (i.e., Honors/Accelerated/Gifted, General/Regular/On-Level, Co-taught/Remedial/Extra Support, or Other), the length of the class period in which the lesson will occur, the mathematical domains for the lesson, and the overall lesson duration. Teachers also provide information regarding how many years they have taught this specific lesson prior to this year and if there are other mathematics teachers who they have discussed or planned with for this lesson.

The teacher then answers a series of items for each of the six CR aspects outlined in Table 1. For purposes of space as well as ongoing efforts to finalize the survey, only a portion of the survey is included. Figure 3 below includes the survey items that teachers complete for the Learner Perspective: Intentionality of Tasks (IT) section of the survey along with the rating scale.

Learner Perspective: Intentionality of Tasks

To what extent did you think about the mathematical activities while planning and teaching this lesson this year or in past years?

I thought about:	I **did not think** about this for this lesson.	I thought about this, but it **did not influence** this lesson.	I thought about this, and it **influenced** this lesson.	I thought about this, and it **strongly influenced** this lesson.
1. **Selecting** manipulatives (fraction bars, cubes, 3-D solids, or online versions of manipulatives) to support my students' learning while working on activities.	○	○	○	○
2. **Selecting** activities to support my students in applying mathematics strategies.	○	○	○	○
3. **Modifying** activities to connect mathematics to my students' everyday lives.	○	○	○	○
4. **Modifying** activities to support my students' explorations of mathematics concepts.	○	○	○	○

5.	**Modifying** activities to meet my students' need**s** for learning mathematics.	○	○	○	○
6.	Structuring participation (independent, pairs, small groups, and whole class) in mathematics activities to support my students' learning while working on activities.	○	○	○	○
7.	Facilitating discussions about mathematics to support my students in sharing their thinking during activities.	○	○	○	○

Figure 3: Survey items from the Learner Perspective: Intentionality of Tasks (IT) section

In each CR section, teachers consider the extent to which the given items were considered and influenced the lesson under consideration. The items provide a range of considerations under each CR aspect that allows the teacher to reflect on each CR aspect in their overall instructional planning.

After the teacher completes the six sections corresponding to the CR aspects, a final section of the survey collects demographic information on the survey respondents. In addition to documenting the teacher's gender, race/ethnicity, and highest level of education, survey respondents provide the number of years of teaching experience they possess, the number of years teaching mathematics, whether or not they have a current teaching license in their grade level, the state in which the teacher works, the type of school where the teacher works (i.e., charter, private, public, other), and the geographical region in which the teacher works (i.e., rural, suburban, urban, other).

Conclusion

Prior to the development of the survey there was not a way for teachers to measure their own CR. We knew we needed to find an efficient process for providing teachers feedback on their CR. The CR survey provides teachers with a practical, formative tool to assess and strengthen their CR. By making teachers' decision-making visible, the survey encourages reflection, informs instructional adjustments, and ultimately supports richer student learning opportunities. As the first validated quantitative measure of CR, the survey extends research beyond qualitative studies and offers a scalable approach to improving mathematics instruction.

We have also found that while working with teachers involved in the cognitive interviews, these teachers reflected on their own teaching without being prompted. In some cases, teachers stated, "I have never been asked to think about this [an item] in my teaching." Therefore, the survey can provide immediate feedback to teachers without having a research person there to guide the conversation.

We think that teachers and instructional coaches will be interested in seeing and using the survey to measure teachers' CR. We also believe that once teachers use the survey, they will continue to do so in the future in ways that will promote learning and will improve their CR, which will affect their students' learning of mathematics concepts.

CR in mathematics is a practice that has only been studied qualitatively (Bostic et al., 2022; Dingman et al., 2021). Therefore, the development of quantitative measures and instruments will create broader reach in terms of practical, economical, and efficient data collection (Creswell, 2008). The survey is one part of our broader work to link research on teachers' CR with the development of tools to formatively assess teachers' CR and sharing how the tools are used by teachers in the field. By introducing the survey to classroom teachers, these teachers have the opportunity to not only learn about CR, but they also can utilize the survey to strengthen their overall curricular decision making. Our goal in developing the survey is to assist teachers in becoming aware of their CR so they may expand their reasoning, which we hypothesize can change students' opportunities to learn mathematics.

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EXPLORING EQUITY AND PEDAGOGICAL CONSIDERATIONS OF ELEMENTARY TEACHERS' MATHEMATICS CURRICULAR USE: VISION AND TRUST

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Elementary teachers navigate a multitude of curricular resources and local curricular-use policies to design and implement engaging and equitable mathematics learning experiences for their students. Drawing on interview data, we investigated how elementary teachers used available curricular resources to enact their curricular visions, specifically regarding effective instructional approaches and equitable learning opportunities. Findings highlighted the complexity of teachers' mathematics curricular decision making as they strive to meet the needs of their diverse students. The findings also revealed how vision and trust mutually reinforce one another. We discuss the relationship between available curricular resources and teachers' curricular vision and conclude by suggesting next steps to support their work in the mathematics classroom.

Keywords: Curriculum, Elementary School Education, Instructional Vision

Frameworks

We draw on Breyfogle and colleagues' (2010) curricular reasoning framework, with particular attention to curricular vision and trust. They define curricular reasoning as "the thinking processes that teachers engage in as they work with curriculum materials to plan, implement, and reflect on instruction" (p. 308) and frame these processes as the interaction among curricular knowledge, vision, and trust. Curricular vision is knowing the path of learning, specifically where students have been, where they are now, and where they need to go, along with the knowledge of how to get them there (Darling-Hammond et al., 2005). Such vision involves a sophisticated understanding of mathematical coherence—the interconnectedness of mathematical concepts and ideas—and effective teaching strategies to help students see those connections (Drake & Sherin, 2009). With such understanding, teachers make conscious, purposeful decisions about what to teach, in what order, and how to teach it for understanding. Beyond the mathematical considerations, curricular vision also involves attending to the social, emotional, and educational needs of students, both as individuals and as a community of learners, as well as understanding the social aspects of education to successfully navigate local, state, and national policies, parent and community interactions, and their own assumptions, beliefs, and values about the educational process (Darling-Hammond et al., 2005). Drake and Sherin (2009) posit that as teachers understand a curricular resource, they can decide whether they trust it to satisfy their vision, either as written or with modifications. In the process of becoming familiar with curricula and making those decisions, curricular trust and vision support and reinforce one another's development.

Relevant Literature and Purpose

Elementary teachers use a range of curricular resources to teach mathematics, including district- and school-mandated curricula and supplemental materials (Sapkota et al., 2024). Consequently, they engage in curricular reasoning when interacting with these resources, deciding which to use and how to use them (Breyfogle et al., 2010; Dingman et al., 2021; Tran & O'Connor, 2024). These interactions occur within teachers' institutional, political, and pedagogical contexts, which shape how they negotiate curriculum mandates, available resources, and pedagogical choices (Tran & O'Connor, 2024). For example, Dingman and colleagues (2021) suggested that middle school teachers continually weigh whether materials align with standards, provide coherent mathematics instruction, facilitate student interpretation, promote mathematical understanding, and fit the classroom context and content.

Because elementary teachers in the United States are grade-level generalists responsible for students' success across multiple subject areas, their curricular decision-making processes and associated curricular vision may differ from those of subject-specific secondary teachers and thus warrant further investigation. Teachers not only intentionally select resources that align with their vision, but the availability of resources also shapes how their curricular implementation unfolds. For example, Siuty and colleagues (2018) suggested a bidirectional relationship between curricular availability and teachers' beliefs, noting that "a curriculum facilitated middle school teachers' decision-making around individualizing instruction" (p. 39). Building on this research, we became interested in examining how elementary teachers' curricular vision is informed by the curricula they use and, conversely, how it is guided by available resources. In this paper, we answer the following research question: How do elementary teachers use available mathematics curricular resources to enact their curricular vision?

Methods

Participants and Data

This study is part of a larger project aimed at understanding elementary teachers' mathematics curricular decision-making processes. The data for this study came from a three-phase interview with five elementary teachers at a Midwestern school. The teachers' pseudonyms are Audrey (2nd grade), Kasie (3rd grade), Jamie (4th grade), Mary (3rd grade), and Sarah (2nd grade). The first two interviews were individual, and the third was a focus group with all five teachers. In these interviews, we asked teachers which resources they used, whether they were mandated or supplemental, who mandated them, and how much control they had over their implementation. We also asked how they established curricular coherence across multiple resources, what challenges they faced in implementing curricular resources, how they collaborated with one another, and what support they needed in the process. One teacher reported using as many as 14 curricular resources to teach mathematics, and the others used at least eight. Across teachers, the most frequently used resources included DreamBox, enVision Math, FlashToPass, IXL, Pinterest, Study Island, Super Teacher Worksheets, Teachers Pay Teachers (TPT), XtraMath, and resources they designed themselves. EnVision Math was the district's core mathematics curriculum; however, only some of the teachers reported that its use was mandated. All five teachers reported that the district mandated the use of DreamBox, and one teacher reported that the school mandated IXL. All but one of the teachers reported having "complete," "a lot," or "quite a bit" of control over their curricular use. The teachers indicated which resources they trusted or distrusted for various purposes, depending on how well aligned those resources were with their curricular vision.

Data Analysis

In the first coding cycle, we used iterative rounds of deductive and inductive analysis (Saldaña, 2016) to identify instances in which teachers discussed their mathematics curricular vision in relation to curricular trust. We began with definitions of curricular vision from the extant literature and generated a set of initial visions (deductive) that teachers might express. We then refined those codes based on what we found in the data (inductive). For example, drawing on the definition of curricular vision as knowing the path of learning—where students have been, where they are now, and where they need to go, along with the knowledge of how to get them there (Darling-Hammond et al., 2005), we created the code: "vision about effective instructional approaches for implementing resources to develop students' mathematical understanding." As we coded for instances of curricular vision, we noticed that teachers frequently discussed trusting or distrusting curricular resources in relation to their vision, such as a particular resource eliciting undesirable affective responses from students. For example, Mary shared: "If they've [students] tried all the different [curricular resource] lessons, and they're not making progress with any of them, they get frustrated." We added this to our list of vision codes. At the end of this iterative round, we identified the following six codes for vision: (1) Vision about effective instructional approaches for implementing resources to develop students' mathematical understanding, (2) Vision related to creating equitable learning opportunities—scaffolding for individual students depending on their needs, (3) Vision related to creating a coherent curricular experience—strategies for connecting mathematical ideas, (4) Vision related to using assessment to inform instruction and affect student learning, (5) Vision related to reconciling curricular mandates with their own vision, and (6) Vision related to students' affective responses. We recoded the data using these six codes, along with instances of teachers' curricular trust and distrust for each code.

As we reviewed the data, the first two vision codes emerged most frequently (i.e., effective instructional approaches and equitable learning opportunities). This finding guided our second coding cycle, in which we further unpacked the two visions into subcodes to identify the specific types of vision that teachers described in relation to their available curricular resources. We first organized the data into an Excel spreadsheet by vision and then coded inductively to determine the components of each vision code. For example, we examined the specific approaches teachers discussed for creating equitable learning opportunities and how they expressed trust or distrust in various curricular resources. After this inductive coding for each vision, the researchers met to refine the codes and finalize the list of vision subcodes.

Findings

All five teachers described how they selected and used mathematics curricular resources to support their vision of effective instructional strategies and equitable learning opportunities. In this section, we detail the teachers' responses to these visions, highlighting the often-complex nature of their curricular decisions.

Curricular Use: Vision and Trust Related to Effective Instructional Approaches

All five teachers described instructional approaches aligned with their mathematics curricular vision and supported by available curricular resources. Jamie, Mary, and Sarah noted that giving students time for individualized or independent learning allowed teachers to work with struggling learners. They agreed that IXL is a trusted curricular resource for its differentiated approach. Jamie shared, "It's [IXL] also student-paced, individualized, so they're not waiting on their classmates to do a problem or waiting on their classmates to go ahead." Mary and Jamie compared the independent options provided by IXL and DreamBox; both teachers appreciated DreamBox's rigor, as it "does try to create some independence," but agreed that this rigor also

makes it more challenging for students to work individually (i.e., they often reach out to the teacher for help). In contrast, Mary stated that "It's [IXL] not as rigorous as, say DreamBox, and so my students are more successful at doing that [IXL] independently." She added that IXL allows teachers more control over what "students need to work on," whereas "they [students] get a lot of choice with DreamBox. I wish it gave us a little more control." Sarah agreed with the other teachers about the rigor of DreamBox and added that it is "interactive and engaging." She also mentioned the support provided by individual practice resources in enVision's instructional strategy repertoire.

Closely related to independent learning, Kasie and Sarah both cited curricular resources that supported their vision of students' need for "extra practice": enVision, DreamBox, IXL, Study Island, and XtraMath. Kasie expressed appreciation for Study Island:

> It's just a website that makes worksheets for all the subjects, and that's more for practice or small groups. Sometimes, we have our 30 minutes that we're doing Study Island now. During the year, I was in charge of coming up with the curriculum that they would work with our intervention aides, and a lot of the work came from that [Study Island] website, for extra practice.

Kasie and Sarah also mentioned spiraling across the curriculum as an instructional approach within their curricular visions. Kasie shared mixed feelings about the trustworthiness of the spiraling provided by enVision, saying:

> I do like how they [enVision] do kind of spiral and add in some of that stuff that we may have already covered. But when they don't, or we haven't covered it yet, I do adapt it and take those questions away.… or if I feel like it's not something that I really need to concentrate on, or if the kids have already done really well with it, sometimes I'll adapt them and take them off so they don't have to go over that stuff and we can spend more time on the content we're currently working on.

She added that she often uses other curricular resources for spiraling, "We do add in spiraling with IXL or DreamBox, or all the other materials that we have. We do add in that spiraling, where they can come back to it again and again." Like Kasie, Sarah also mentioned using IXL to review skills from the day before.

Audrey, Kasie, and Sarah all mentioned incorporating engaging, enjoyable, contextually relevant, and/or project-based activities as part of their curricular vision. They relied on TPT and other teacher-created activities as their primary curricular resources to support this vision. Audrey and Sarah both described conducting hands-on, project-based activities on Fridays, when the math block is uninterrupted. Sarah shared:

> So, situating those math problems in a familiar context was really fun. And, the activity that we're doing on Friday, I've called it Graph in a Bag [designed by the teacher]. So, I'm just gonna put random things in a brown bag, and the tools to make a graph, and some of those are gonna be plant-related, because we've been doing a big unit on pollinators and pollination and plants. So, just getting familiar subject matter so that the math is the challenging part and not understanding the context of the problem.

Audrey described using TPT activities with multiple mathematical components and engaging contexts, such as Oreos and baseball, for her students. She expressed her enjoyment of teaching "the fun stuff," saying, "I love teaching it. That's my favorite thing to teach, and I think my students know it's my favorite because I get excited about it." Kasie trusts TPT activities as "a

way to see what students have learned and how they can apply it in different ways other than just on the workbook pages or on some type of paper-pencil assessment. So, it's just a different modality for practicing the skill."

Audrey and Kasie also expressed trust in several curricular resources for collaborative learning opportunities (i.e., Study Island, Super Teacher Worksheets, enVision), including opportunities for engagement with the whole class, small groups, and partners. Audrey mentioned opportunities for "practice with partners" in both enVision and Super Teacher Worksheets. She described facilitating whole-class discussions and sharing mathematical authority using enVision:

> So, the easiest thing for me to do is, like, let's pull out our iPads. Let's answer these questions together as a whole class, and if a question arises, you raise your hand, and we can talk it out. And, typically, I have another student answer the question. So, if someone's asking about, like, if we are doing clocks, that was a recent question, with the hands. I had one of my other students volunteer to respond to that question to help them answer it. So, that way, it's not just coming from me all the time.

She also described several challenges associated with helping students appreciate the importance of "working with each other, I'm trying to explain to them that to be successful humans, you have to be able to work with someone, no matter where they are; you have to meet that other person."

Audrey, Kasie, and Sarah expressed trust in the "Visual Learning" videos included with enVision, highlighting several features. Sarah noted the videos' "step-by-step interactive way of how to do the process," adding that "it's interactive, so students drag the symbols where they go." Audrey shared that she begins each lesson with the videos, as they "show exactly what the content is supposed to look like for graphing, collecting data, and finding resources to collect data" (this example was in the context of describing a particular lesson). Kasie also trusted the enVision videos:

> I like the videos that come along with it [enVision]. It helps to give a visual learning experience to our students, and each lesson has two different videos that each take a different approach, so I like that they teach it in different ways as well.

Like Kasie, Audrey trusted curricular resources that illustrated multiple strategies; in her case, she highlighted XtraMath, a program that promotes various strategies for fact fluency. Jamie also trusted XtraMath to develop fact fluency; in particular, she valued its progressive approach to mastering basic facts (e.g., "we're gonna work on the twos today"). Both Audrey and Mary advocated for students using what they called "paper-pencil" at times. Mary commented that "enVision does have everything digitally," but appreciated that students could also complete the "enVision pages" with "paper-pencil." Audrey agreed, saying:

> Drawing with your finger [on your iPad] is not the same as actually paper-pencil. So, especially when they're adding and subtracting and lining those things up on paper, it's easy to get sloppy. So, I have really been diligent about this is what we're doing, and we're gonna put the iPad away and have time to be here and use our blocks and use our counters and use everything that we actually have in front of us.

Mary also mentioned using manipulatives such as fraction tiles, clocks, and coins.

Four of the five teachers expressed trust not only in using a single curricular resource but also in optimizing the potential of their full set of resources, including their primary curriculum

(enVision), their mandated supplementary resource (DreamBox), and other non-mandated curricular resources available to them (e.g., FlashToPass, IXL, Study Island, Super Teacher Worksheets, TPT, XtraMath, and resources they designed themselves). Kasie shared the flexibility that access to multiple resources provides for pedagogical approaches:

> I have them get on IXL or something independent, so I can find something that is going to at least make a little more sense, or find a YouTube video, or find something that might explain it in another way that I'm not thinking of at that time. Then, also find another material that I can bring in and incorporate the next day. Again, if I have to change it on the fly, then sometimes I'll have them working independently on an app or something to give me a moment to figure it out, but usually, it's more about changing it in the coming days.

Similarly, Audrey appreciated having multiple curricular options for students: "I would say, I want seven minutes of this standard on IXL, and afterwards, you can choose. You could either get on DreamBox, or you could do FlashToPass, or you could do another IXL that you didn't get mastery."

Finally, three teachers (Mary, Kasie, and Sarah) expressed trust in the pedagogical sequence embedded within enVision lessons, highlighting the routine as a curricular strength. Kasie shared:

> We usually follow the same kind of pattern. In the book [enVision], it starts with problem solving [Solve and Share], and then we watch a video, and then we do some group problems together, then they do independent problems, and we check them. So, kind of following that same pattern. The kids know what to expect….So, my students have made that connection with math, which helps them with their routine, and they know what's coming throughout our math block.

Sarah concurred and reported using this routine, saying:

> My direct instruction and guided practice, and the online materials, there's a video called Visual Learning that does a great step-by-step interactive way of how to do the process for whatever the skill is today. So, I use that to start each lesson, and then we go to some of the workbook pages and use that as our guided practice.

Both Kasie and Sarah trusted enVision as their primary curricular routine and supplemented it with other resources as necessary. Kasie shared:

> I will start with enVision first, since that's the program that our corporation has purchased, and I'll look through all of the materials that it has on the standard. And then, if I feel like it's not enough, or it's starting too far ahead of where my students are, that's when I'll go and find other materials to start with, or find other materials to work in, but I usually start with enVision every time I start my planning.

So, in addition to the specific instructional approaches the teachers reported trusting (e.g., independent learning, spiraling, extra practice, collaborative learning, project-based activities), they also highlighted their trust in the curricular resources and the pedagogical sequences available within them.

Curricular Use: Vision and Trust Related to Creating Equitable Learning Opportunities

The teachers considered how to use mandated curricular resources and when to supplement them with other resources, depending on how effectively they could create equitable learning opportunities for their students. The teachers spoke about how curricular resources allowed them

to differentiate in several ways, how the functionality of resources enabled or did not enable the creation of equitable learning opportunities, when resources would afford them the opportunity to use and validate multiple strategies, and whether students had access to resources.

The teachers most frequently reported that their trust (or distrust) in mandated or supplemental resources hinged on whether those resources offered opportunities to differentiate instruction to meet individual students' needs. The teachers discussed differentiation in terms of comprehension, pace, and student engagement. Regarding comprehension, all five teachers indicated that digital resources, specifically IXL, enabled them to adjust content for students at different levels of mathematical understanding. For example, Mary shared,

> I use it [IXL] more for differentiation. I have a student who is working at least a grade level above in math, so on his IXL, I can set it to the fourth-grade level, and he can practice the same standards that I'm teaching, but at the fourth-grade level, so it's a little more rigorous and more set to his ability level.

In this example, Mary shared that IXL was useful for differentiating content so she could support students who were ahead of the rest of the class in terms of comprehension; however, IXL is also helpful for supporting students who are working below grade level.

The teachers also discussed differentiation in terms of pace. They indicated that they selected certain supplemental resources to meet the needs of students whose pace was different (faster or slower) than the rest of the class. Jamie trusted IXL because it aligned with her vision that students should work at their own pace: “It [IXL] is also student-paced, you know, totally individualized, so they're not waiting on their classmates to do a problem or waiting on their classmates to go ahead or things like that.”

Finally, the teachers discussed differentiation in terms of student engagement. As such, the teachers envisioned that engagement with the content for individual students might look different. Kasie shared

> So, I've been trying to find games or other activities that make it more engaging, so that they're paying attention to the lessons. I have a lot of students who have ADHD or just struggle, in general, are lower in math, and have a harder time engaging in a lesson when they get frustrated. Finding a fun way to incorporate the math has been something I've been trying to concentrate on this year.

Kasie expressed trust in supplemental resources outside the core curriculum (i.e., enVision), such as games sourced from TPT, to accommodate students with special needs and to provide “fun” activities for students who are typically less engaged.

Some of the teachers’ visions for creating equitable learning opportunities for their students extended to selecting and implementing supplemental materials or adapting the core curriculum for groups of students to better accommodate varying levels of comprehension, pace, and engagement. For example, some teachers described intentionally selecting a limited number of problems for whole-class discussion while allowing students to work independently on others, based on the anticipated rigor or difficulty for different groups. The decision to select fewer problems or to group students was based on how teachers envisioned the rigor of particular mathematics content. For example, Audrey shared

> Sometimes I feel that I neglect my high ability students because I actually have the lowest in our grade level and the highest in the exact same room, and I focus a lot of my attention on my lower-achieving students because I feel that's more of my job. I have a lot more energy

> going into them, and so, when I get to do XtraMath with these students, it's a time for them to shine and to feel successful.

In this example, Audrey grouped her students into 'higher' and 'lower' achievers and provided them with different resources for extra practice, indicating the breadth of her work as she plans and uses curricular resources for different groups of students. As we saw in these examples, teachers often trusted online resources, especially IXL, to achieve their vision for differentiation.

Beyond these strategies for supporting students through practice, teachers emphasized the importance of validating diverse strategies and representations across materials. Teachers described intentionally using multiple resources to help students recognize connections across tasks, which they believed increased students' confidence and engagement. Audrey shared, "we have manipulative bags in the back of the classroom, and we just grab them for every math lesson, just in case, because we never know what students will need." Sarah expressed her trust in enVision because it is research-based and supports the use of multiple representations:

> They're [enVision] not something that somebody created on the internet. The strategies that are included in a research textbook have been researched and shown to support student learning, support learning modalities of different students, or different representations of the ways that they can learn… Since I'm not a math research person, I should trust the research that has been done for me.

This was one of the few instances in which the teachers expressed trust in mandated resources in the context of differentiation and achieving equity. This example and Mary's quote above about her distrust of enVision show how two teachers trusted or distrusted the same resource depending on whether it aligned with their vision. From Mary's perspective, enVision did not offer students opportunities to practice at their own pace, whereas Sarah valued it for offering opportunities to use multiple representations.

Overall, deciding which supplemental resources to use and varying how they were used helped teachers scaffold instruction for their students. The most important aspects for these teachers were opportunities for students to practice at their own pace and to work on content appropriate for their comprehension levels. In all cases, teachers strived to support their students (whether by finding games or using online resources), indicating their care for students and highlighting the complex work inherent in making curricular decisions.

Conclusion and Discussion

Our findings revealed the intellectual effort elementary teachers put into decisions about curricular use, supported the notion that curricular vision and trust are mutually reinforcing, and offered insights for future investigation into how curriculum developers and school administrators can support teachers' work in the mathematics classroom. All five teachers made purposeful decisions about which available curricular resources to use and when and how to use them, in ways they believed best supported their students. In relation to the teachers' vision of effective instructional approaches, they preferred curricula that enabled them to provide time for individualized and independent learning, extra practice and review, and multimodal experiences. In relation to how the teachers envisioned equitable learning opportunities, they were most interested in the ability to differentiate instruction to meet individual students' learning needs, paying special attention to comprehension, pace, and student engagement. In line with both the teachers' vision of effective instructional approaches and equitable learning opportunities, the teachers used curricular resources that offered engaging, enjoyable, and contextually relevant

learning experiences for their students, frequently sourcing TPT to do so because it is a "go-to" resource for them. Further investigation into the impact of introducing teachers to alternative curricular resources that satisfy their visions of engaging and enjoyable learning experiences and align with differentiation practices advocated by the National Council of Teachers of Mathematics (2014), such as those that "promote mathematical reasoning and problem solving and allow multiple entry points and solution strategies" (p. 10), is a logical next step toward ensuring mathematical success for all.

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GENERALIZATION IN GHANAIAN MATHEMATICS TEXTBOOKS: A COMPARATIVE ANALYSIS USING ELLIS'S TAXONOMY

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This paper compared two Ghanaian government-approved grade seven mathematics textbook series, Best Brain and the Mathematics Association of Ghana, to identify how they support students to generalize in the Patterns and Relations strand. Using qualitative content analysis, the selected number of tasks were coded with codes informed by Ellis's (2007) Generalization Taxonomy: relating, searching, and extending. It also examines whether tasks invite direct-measure reasoning or emergent-ratio reasoning. In a closer analysis of 49 tasks, extending is the most common generalizing action in both series; however, emergent-ratio reasoning opportunities were rare, present in only seven of 49 tasks (14%). These results suggest that teachers would be advised to supplement the textbook tasks to help students move beyond direct-measure reasoning and engage in deeper forms of generalization.

Keywords: Algebra and Algebraic Thinking, Curriculum

Problem Statement

In a JoyNews report aired on December 1, 2025, the West African Examination Council's (WAEC) Head of Public Affairs, Mr. John Kapi, drew on the 2025 Chief Examiners' findings of West African Senior School Certificate Examination (WASSCE; a mandatory standard test written by all final year Senior High School [SHS] students before progressing to tertiary institutions) and highlighted a persistent concern. He noted that many Ghanaian SHS students struggled with application-focused items, especially tasks framed as "global mathematics-related problems" (JoyNews, 2025). That is, Ghanaian SHS students often have not been able to apply mathematical concepts learned to broader, more abstract, or real-world contexts. That concern means that, for many, generalization of this knowledge is a challenge. Several factors may account for this challenge, including the curriculum (Boaler, 1998), teachers' knowledge and curriculum enactment philosophies (Lambdin & Prestoewsn, 1995; Stickles, 2011), students' role during instruction (Huntley et al., 2000; Remillard et al, 2020), and textbooks (Stein et al, 2007). Research suggests that textbooks, in particular, are a dominant force in shaping the teaching and learning of mathematics globally (Stein et al., 2007). For example, Sugiarni et al. (2024) state that textbooks directly affect how teachers choose to plan and teach, how students gain access to mathematical content, and how mathematical concepts are communicated.

Mason (1996) has described generalization, which is the ability to apply skills learned in a particular context to new, different, or broader scenarios, as at the heart of mathematics. An aspect of mathematics considered foundational for mathematical competency and logical reasoning, and where generalization is central, is algebra (Kaput et al., 2017). The development of patterns and relations is an essential strand underlying algebraic thinking (Blanton & Kaput, 2011), and the generalization of patterns and relations serves as a powerful mathematical tool that allow students to extend examples to more general rules or formulas. Earlier studies have highlighted that students develop the fundamental competencies for algebraic thinking, functional thinking, and quantitative modelling when they work meaningfully with patterns (Blanton, 2008; Kaput, 2008). Yet the extent to which students' textbooks focus on

generalization instead of on procedural manipulation or rote memorization may vary. For example, Ayber and Tanisli (2017) found several gaps in textbooks that attempt to support algebraic thinking through patterns and relations.

Ellis (2007) contends that the kinds of reasoning that are promoted during instruction have a significant impact on students' capacity for generalization. These ideas lead to practical questions for Ghanaian teachers, such as what types of reasoning do our textbooks promote, and do they create sufficient opportunities to meet the national goal of improving problem solving, reasoning, and relational thinking in the new curriculum (Ministry of Education, 2020)? In Ghana, textbooks are the primary curricular material that is readily available and accessible to teachers and students. As a result, textbooks are typically central in teachers' lesson plans and lesson delivery, yet limited research has been conducted on how these textbooks engage algebra students to generalize through patterns and relations. The gap in the literature raises concerns about whether the textbooks available to these teachers and students are adequately preparing SHS students for higher-level algebraic thinking.

Without clear empirical evidence of how algebra textbooks address generalization of patterns and relations, it might be difficult for Ghanaian teachers, curriculum developers, and even policymakers to identify strengths and address weaknesses. This study sought to examine whether Ghanaian junior high school (JHS) textbooks support generalization. To do this, we selected two widely used textbooks: the *Best Brain* (BB; 2021) and the *Mathematics Association of Ghana* (MAG; 2020) textbook series. The BB *series*, though produced by a private institution, is officially approved by the National Council for Curriculum and Assessment (NaCCA), ensuring alignment with national curriculum standards. The MAG *series*, on the other hand, is developed under the guidance of the MoE and distributed nationwide across all Ghanaian JHS. This positions these textbook series to be highly influential in shaping classroom practice. The study reported here specifically analyzed how patterns and relations were presented in these textbooks for grade seven pupils. Examining these two contrasting curricula provided a comprehensive view of how both privately and publicly guided textbooks support students' development of algebraic generalization.

Analytical Framework

Ellis's (2007) Generalization Taxonomy describes the different types of generalizations students develop when reasoning algebraically. This involves the ability to "identify commonalities across cases, extending one's reasoning beyond the range in which it originated, or deriving broader results about new relationships from particular cases" (p. 444). Ellis argued that when students can engage in any of these activities, their reasoning is not limited to one particular problem or situation but extends their ideas to a wider range of phenomena, which ultimately leads to generalization. Additionally, Ellis argues that representations are a key to helping students "create meaningful generalizations" (p. 442).

The framework has three levels: *relating, searching, and extending.* Relating involves forming an association between two or more problems, situations, ideas, or mathematical objects, whereas searching includes engaging in a repeated mathematical action, such as locating a pattern, in order to identify an element of similarity. Finally, the act of extending involves expanding a pattern, relationship, or rule into a more general structure. To measure the extent of generalization, Ellis (2007) suggested two types of quantitative reasoning: direct-measures (DM) reasoning and emergent-ratio (ER) reasoning. Direct-measures reasoning is often procedural or involves recursive strategies, which include counting or computing and completing a table.

Emergent-ratio reasoning, on the other hand, involves reasoning or coordinating two quantities to form a third and is believed to support deeper generalization.

The study, therefore, was guided by the following research questions:

1. What types of mathematical representations are used in the MAG and BB textbook series to assist Ghanaian students' understanding of patterns and relations?
2. How do the MAG and BB textbook series support or not support students' development of generalization, when analyzed through direct-measure and emergent-ratio reasoning across Ellis's dimensions of relating, searching, and extending?

Methods

The data for this study consisted of all tasks within the *grade seven* Patterns and Relations strand from the two Ghanaian government-approved textbook series described earlier. A task was defined as any numbered prompt, with or without distinct subparts (a, b, c, etc.), requiring a mathematical response, including both worked examples with procedures shown and unsolved exercises. The BB *series* contained 69 tasks, and the MAG *series* contained 19 tasks, yielding a total data corpus of 88 tasks across both textbooks for that strand.

Sampling Procedure

For Research Question 1, a complete census approach was used to select all 88 tasks, which were coded for the types of representations used. This approach was adopted to ensure a comprehensive report of all the representations used in both textbook series and how frequently each representation is used in each textbook. For Research Question 2, a stratified systematic sampling technique was used to select 30 tasks from the BB *series* to help maintain a manageable yet robust corpus for in-depth analysis. However, the census approach was used again to select all 19 MAG series tasks to enable meaningful qualitative comparison.

In selecting the 30 BB tasks, tasks were first categorized into five conceptual sections (stratas) mirroring the curricular progression observed in both textbooks: (a) establishing relations, (b) developing patterns, (c) describing rules using mathematical language, (d) reasoning about patterns and far terms with algebraic rules, and (e) graphing and explaining relationships. Within each section, six tasks were selected systematically to capture the progression from early to late tasks within that conceptual phase. Selection followed a fixed protocol: the first task, the last task, and four evenly spaced tasks were selected. If a section contained six or fewer tasks, all tasks were included. This procedure ensured representation across all conceptual phases and difficulty levels, providing a balanced basis for comparing generalization support between the two series.

Data Analysis

We analyzed task prompts for opportunities for students to engage in different generalization processes and quantitative reasoning. Tasks that asked students to compare or connect across contexts and to identify similarities were coded for relating. For searching, tasks were coded for prompts that asked students to find the pattern, identify the rule, and verify if the rule holds for all rows. Extending was coded for tasks that asked students to use a pattern to find far-terms or justify that a rule works in general. To ensure reliability of the coding process, 20% of the total dataset was independently coded by a fellow Ph.D. student with expertise in mathematics education using Ellis' (2007) framework. This dual-coding process served as an external audit of the various categories that emerged. It also provided an opportunity to compare task structure, the reasoning demands, and the generalization opportunities provided or not provided by the textbooks. The comparison of codes yielded an inter-coder agreement of 82%, indicating a high

level of consistency in how the framework was applied. Areas of disagreement occurred mainly where tasks blended direct-measure and emergent-ratio reasoning. A resolution meeting ensured that the final codes reflected a shared and rigorous understanding of Ellis' (2007) framework. This collaborative approach strengthens the credibility of the findings and supports the trustworthiness of the interpretations that follow.

Findings

Research Question 1: What types of mathematical representations are used in the MAG and BB textbook series to assist Ghanaian students' understanding of patterns and relations?

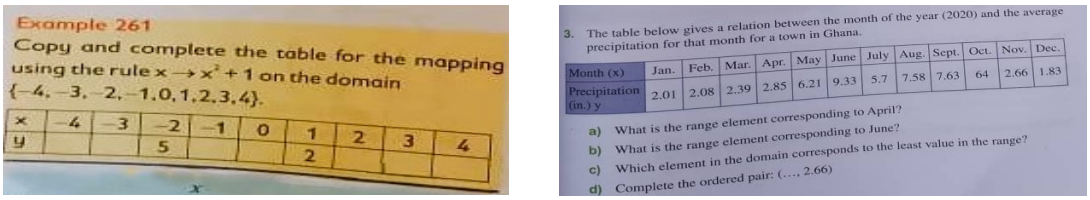
Frequency counts and percentages were used to summarize how often each representation appeared within each series. The six representation types identified in both textbook series were:

- *Words/text:* tasks framed mainly in narrative or verbal form (not just brief instructions).
- *Tables:* Any input-output tables, domain-range tables, or x-y tables
- *Algebraic expressions:* Formulas like $y = 2x + 1$, mappings like $x \rightarrow 3x - 2$
- *Pictures:* Matchstick patterns, growing shapes, diagrams
- *Arrow diagrams:* Arrow mapping structures connecting sets/values.
- *Graphs:* Coordinate axes or graphs presented in the task.

Table 1: Representations Found Across All Tasks

Representation	*BB* (69 tasks)	*MAG* (19 tasks)	Total Codes
Tables	28 (41%)	7 (37%)	35 (40%)
Algebraic	22 (32%)	8 (42%)	30 (34%)
Pictures	15 (22%)	3 (16%)	18 (20%)
Words/Text	12 (17%)	4 (21%)	16 (18%)
Graphs	6 (9%)	2 (11%)	8 (9%)
Arrow Diagrams	4 (6%)	1 (5%)	5 (6%)
Total Codes	87	25	112
Average per task	1.3	1.3	1.3

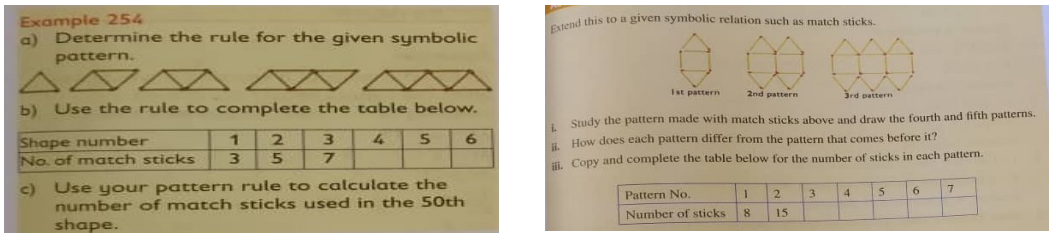
The output from Table 1 indicated that both textbook series use all six types of representations, but tables showed up most often, about 40% of tasks in both books. They were typically input-output tables showing how one quantity relates to another, like "pattern number vs. number of matchsticks" or "workers vs. lunch cost." In practice, this means students using these textbooks are frequently positioned to make sense of patterns and relations through structured input–output pairs. Algebraic expressions came second, especially in MAG, where they appeared in 42% of tasks compared to BB's 32%. However, MAG treated the algebraic rule as something students use directly to produce values, whereas BB used it as a part of a broader sequence that begins with a pattern or mapping structure, and later students are tasked to graph it. Words/text tasks showed up in 17% BB tasks and 21% in MAG tasks. These were mostly tasks like "write a pattern … and use it to calculate …," "explain the relationship," or "give real-life examples of one-to-many relations." However, a majority of the texts were also used to play a key instructional role in guiding students, yet were not coded as a word/text task. Examples of tables and algebraic tasks are presented in Figure 1 for both textbook series.



BB series ***MAG series***

Figure 1: Example of Tables, Algebra, and Texts Used for Instructions in Both Series

Pictures appeared in 22% of BB tasks but only 16% of MAG tasks. The more important difference is not just about frequency or percentage, but how the pictures are used. For example, the BB series used the matchstick pattern task, which asks students to (a) determine the rule from the symbolic pattern, (b) use the rule to complete a table, and (c) use the rule to calculate the 50th shape. This is a strong example of representational coordination: picture → table → rule → far-term reasoning. Even though a graph is not shown here, the task still uses multiple forms to guide students from visual growth to symbolic generalization. The MAG series also uses a matchstick pattern with a table, but the prompts leaned more toward continuing the pattern, drawing the next cases, and completing the table. The structure supports extension, but the coordination across representations is typically lighter and often stays close to "next steps" rather than pushing strongly toward far-term rule use. Examples of how pictures are used in both textbooks are presented in Figure 2.



BB* series** ***MAG series

Figure 2: Example of Tasks that Use Pictures in Both Textbook Series

Graphs appeared in only 9% of BB tasks and 11% of MAG tasks, indicating that students do not frequently experience tasks presented in graphs. However, most tasks required students to produce graphs rather than the graphs being used to frame the task. Arrow diagrams occurred in only 5–6% of tasks in both series. Irrespective of its frequency, the tasks showed a clear difference in purpose. For example, in the BB series, they are used mostly for numeric mappings, leading to the formation of algebraic rules. The MAG series had one numeric mapping task, and the remaining used relations in a more contextual/categorical way (e.g., pairing students with towns or matching elements in sets). Even when the language uses the term "relation," the work often reads as organizing or naming pairings rather than as building an explicit function rule. Every task averaged about 1.3 representations, meaning most tasks combine more than one form of representation.

Both textbooks gave students some level of exposure to multiple ways of thinking about patterns and relations. Tables provided concrete structure, algebra gave symbolic power, and pictures offered visual intuition. However, the heavy reliance on tables (40%) means students spend most time working with fixed input-output pairs rather than exploring dynamic relationships. BB's strength was translation and coordination, that is, students see the same pattern as a picture, table, formula, and graph. MAG's strength was algebraic fluency and table-based reasoning, that is, students more often practiced applying rules and completing structured tables efficiently.

Research Question 2: How do the MAG and BB textbook series support (or not support) students' development of generalization, when analyzed through direct-measure and emergent-ratio reasoning across Ellis's dimensions of relating, searching, and extending?

Table 2: Generalization Type by Quantitative Reasoning Lens (ER-present vs DM-only)

Generalization Type	Textbook	Sample Size	ER-present	DM-only	% of Sample
Relating	*BB*	6	1	5	20%
	MAG	7	2	5	37%
Searching	*BB*	9	1	8	30%
	MAG	3	0	3	16%
Extending	*BB*	15	2	13	50%
	MAG	9	1	8	47%
Total	*BB*	30	4 (13%)	26 (87%)	100%
	MAG	19	3 (16%)	16 (84%)	100%
Combined	Both	49	7 (14%)	42 (86%)	100%

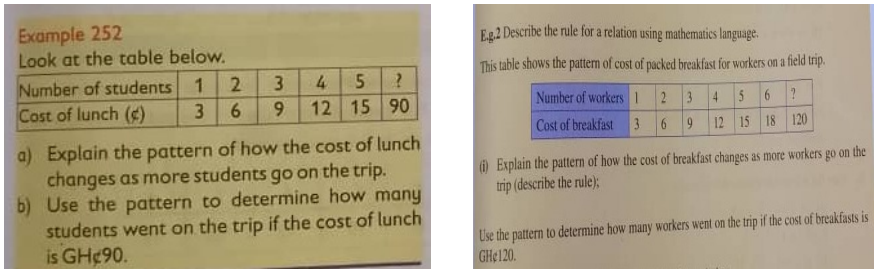
The output from Table 2 indicated that across both series, extending was the most common generalizing action. In BB, 15 of 30 tasks (50%) were coded as extending, and in MAG, nine of 19 tasks (47%) were extending. This suggested that students' opportunities to generalize in both textbooks most often take the form of continuing patterns, completing additional entries, or producing further cases beyond what is given. Many of these tasks allowed students to be successful by continuing a sequence or filling the next few values in a table, often by noticing a term-to-term change (e.g., "add three each time," "multiply by two each time"). What distinguishes the stronger extending opportunities is when a task pushes students beyond nearby continuation and toward using a generalized rule for far cases (e.g., a 50th term, unknown inputs, or a symbolic expression that applies to all terms). BB's extending set included more of these "rule-based" extensions in structure, especially when students were tasked to use expressions such as $2n + 1$ or $3x + 2$ to reach distant terms or solve for unknown inputs. MAG's extending tasks, by contrast, more often kept extension close to the given data (extend by one or two steps; complete missing entries), so most of the generalization tends to remain local.

Beyond that shared emphasis, the two series differed in how they allocated attention across the other generalization types. MAG devoted a larger share of tasks to relating (seven of 19 tasks, 37%) than BB (six of 30 tasks, 20%). In practical terms, relating in the MAG textbook series often appeared in contextual pairings where students match or connect items in the domain and range (e.g., category-to-category relations such as countries to national teams). These tasks support the idea of "relation," but many still keep the work inside one representation, rather than pushing students to translate across forms or compare across situations. In BB, on the other hand,

relating showed up when students are asked to coordinate multiple depictions of the same relationship, for example, by linking a pattern image, a table, and an algebraic rule, or by connecting a mapping diagram to a table and a graph. In that sense, when BB did include relating, it more often involved cross-representation coordination.

In addition, BB devoted a larger share of tasks to searching (nine of 30 tasks, 30%) than (three of 19 tasks, 16%). Searching tasks tested whether the same relationship holds across several pairs, rather than simply continuing the next step. In BB, many table and mapping tasks explicitly steer students toward identifying a consistent rule ("find the rule," "write the rule," "use the rule to complete the table"). Doing that required checking the relationship across multiple pairs of representations to identify what stays the same. In MAG, an example of searching tasks was for students to observe a pattern and "find out which one satisfies the relation $x \rightarrow 3x + 1$" (p. 92).

When viewed through the quantitative reasoning lens, the strongest pattern in Table 2 was the dominance of DM-only reasoning in both series. In BB, 26 of 30 tasks (87%) were DM-only, and only four tasks (13%) were ER-present. Similarly, MAG included 16 of 19 tasks (84%) coded as DM-only and three tasks (16%) coded as ER-present. Combined across both series, 42 of 49 tasks (86%) were DM-only, while only seven tasks (14%) showed ER-present opportunities. For example, in the BB *series*, "Example 252" asked students to do more than fill in missing values. Students were prompted to explain how the lunch cost changes as the number of students increases and then to determine the number of students when the total cost is GH¢90. That pairing of verbal explanation and solving for an unknown input naturally foregrounds an emergent-ratio reasoning. A closely related structure appeared in MAG's "Eg., 2" task on breakfast costs. There, the cost of breakfast in the table increases at a steady rate, per-worker, and students are asked to describe the rule and determine how many workers correspond to a total cost of GH¢120. In both cases, the tasks positioned students to treat "cost per person" as the governing relationship, making them strong examples of ER-present prompts. Examples of ER tasks for the BB and MAG textbook series are presented in Figure 3.



BB series ***MAG series***

Figure 3: Tasks that Support Emergent-Ratio Reasoning

Discussion

Ghana's new curriculum aims for students to classify, generalize, and reason relationally, but this analysis suggested that the two approved grade seven series, MAG and BB, may still position students to repeat procedures rather than build deeper structures, since most tasks remained DM-only. This aligns with Ayber and Tanisli's (2017) caution that textbooks can underemphasize generalization, and with Stein et al.'s (2007) view of textbooks as strong

mediators between curriculum aims and classroom practice. Both series included many tasks that fit Ellis's (2007) generalizing actions, relating, searching, and extending. However, most opportunities were DM-only, and few were ER-present, suggesting that generalization often became local continuation, table completion, and rule application rather than relational or functional reasoning grounded in emergent quantities (Ellis, 2007). The two books also position algebraic thinking differently. MAG leaned toward contextual relations and matching within tables or diagrams, but it often stopped short of pressing students to state and justify general rules across cases, which matters for algebraic thinking (Blanton & Kaput, 2011; Ellis, 2007). BB more often pushed students to "find the rule," use it for far terms or unknown inputs, and move across representations, which could better mediate functional thinking, yet it still rarely asks students to justify why the rule works or what stays invariant as quantities co-vary, so ER reasoning may remain limited. Taken together, the findings suggested teachers will likely need to extend tasks by pressing for justification, cross-representation connections, and give explicit attention to emergent-ratio relationships, reinforcing the idea that textbooks shape, but do not fully determine, students' opportunities to learn (Stein et al., 2007).

Limitations of the Study

The analysis was limited to the grade seven Patterns and Relations strand and to two Ghanaian government-approved textbook series, BB and MAG. It also uses a full census for representation coding (88 tasks) but a smaller in-depth sample for generalization coding, 30 tasks from BB plus all 19 MAG tasks, so the deeper claims came from that sampled corpus rather than every task in the BB *series*. Future research could look at other strands in BB and MAG and other textbooks.

Recommendation for Classroom Practice

Because DM-only tasks dominate and ER-present prompts were rare, the paper recommends that teachers who want to meet the curriculum's reasoning goals will often need to extend what the textbook tasks ask for. Teachers will have to press students to compare cases, connect tables, diagrams, and graphs, and justify why relationships hold, instead of stopping at completing tables or applying rules. This means the possibility for students to reorganize actions into deeper structures, reason with ratios, and/or articulate general rules relies heavily on teacher mediation.

Conclusion

The findings revealed that although both textbooks included a variety of representations and task structures that introduced students to patterns and relations, the depth of reasoning they elicit differs. The dominance of tables and text in both textbook series suggested that students are often positioned to read, fill, or extend given representations rather than coordinate multiple representations to uncover an invariant relationship. The findings also showed that while generalization was not absent, it is largely framed through direct-measure (DM-only) reasoning. Taken together, the results revealed that the textbooks provided a foundation for generalization but did not fully realize the constructivist and relational aspirations of Ghana's current curriculum reforms.

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MATHEMATICS CLASSROOM OBSERVATION TOOLS: PURPOSE AND USABILITY FOR TEACHERS

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There is a need in the field for classroom observation tools made for and used by teachers to reflect on and instigate improvement in their own practice. Classroom observation tools were commonly designed to evaluate instructional practices, measure implementation and growth following professional development, and compare teaching across contexts. In this paper we share results from analyzing 13 classroom observation tools for K-12 mathematics classrooms with a focus on the tools' purposes and usability for both researchers and teachers. We found that these tools generally fall into one of two categories: Tools for Teacher Reflection or Tools about Teacher Practices.

Keywords: Assessment, Professional Development, Research Methods, Teacher Educators

Over the past 30 years, many classroom observation tools were developed and used for various reasons. Often these tools are used by teacher educators, instructional coaches, and school leaders to evaluate teaching and provide feedback to improve teaching practices. Observation tools are generally developed by researchers for specific purposes (e.g., assess professional development, measure teachers' implementation of a specific practice, improve specific teaching practices). Although classroom observation tools are widely used to evaluate teaching practices, the extent to which they support teachers' reflective growth is not clear. Observation tools structured primarily for evaluation are unlikely to adequately support teacher reflection; therefore, intentionally designing tools with reflection as a primary purpose might hold more promise for improving practice.

In this paper we report on thirteen classroom observation tools developed for use in K-12 mathematics classrooms with a focus on the tools' purposes and usability for teachers and researchers. One of these observation tools is new (Roth McDuffie et al., 2026) and was designed using the AERA validity standards (2014), with the purpose of assisting middle school mathematics teachers to reflect on their own reasoning about mathematical decisions they encounter in their classroom as they plan, implement, and reflect on their lessons.

Purpose and Framework

Existing classroom observation tools were commonly designed to evaluate instructional practices, measure implementation and growth following professional development, and compare teaching across contexts. While these purposes are important for researchers, they position observation tools primarily as tools of measurement rather than tools for teacher learning. Consequently, these tools might not guide teachers through a reflective process.

Phillips et al. (2023) defines "teacher reflection as the process of teachers critically evaluating their instructional experiences to improve their practices and students' learning" (p. 636). Building on earlier conceptualizations of reflective teaching (Ur, 1999; Mathew et al.,

2017), Phillips et al. (2023) articulates a structured model for teacher reflection that includes three components: (1) collecting quality classroom data, (2) conducting a thoughtful analysis and evaluation of the data, and (3) contextualizing an interpretation of the data to inform teachers' practices. Observation tools intended to support teacher reflection can generate quality data, guide meaningful analysis, and assist teachers in constructing contextualized interpretations. With the definition of and steps for teacher reflection framing our work, we examined existing classroom observation tools and determined their usability to support teacher reflection.

Ur (1999) argues that for teachers to progress professionally, their own reflection on daily classroom events is critical. Mathew et al. (2017) states, "the complexity of teaching requires teachers to question their practices for their own professional development in order to improve and to increase learner performance" (p. 126). In our prior NSF project (Nos. 1561542, 1561554, 1561569, 1561617) we found that the post-interviews we conducted with mathematics teachers after they taught a lesson caused them to reflect on their own practice. Although our intent with the post-interview was to learn more about teachers' reasoning for their decisions, we found that teachers were deliberately reflecting on their teaching.

To better understand how existing observation tools function in practice, we consider three widely used instruments reviewed by Boston et al. (2015) as examples: the *Reform Teaching Observation Protocol* (RTOP) (Sawada et al., 2000), the *Instructional Quality Assessment* (IQA) (Boston & Wolf, 2006), and the *Mathematical Quality of Instruction Tool* (MQI) (Learning Mathematics for Teaching Project, 2011). Researchers developed each of these tools to observe and measure specific teaching practices: RTOP (Sawada et al., 2000) measures teachers' ability to implement reform-oriented instruction; IQA (Boston & Wolf, 2006) measures teachers' implementation of a cognitively challenging task and the discussion around that task; and MQI (Learning Mathematics for Teaching Project, 2011) measures the rigor and depth of the mathematics taught during a lesson. These three observation tools generate data about different practices teachers use in their classrooms, including ratings or scores based on externally defined criteria, but these tools do not structure opportunities for teachers to analyze their own instructional practices. As a result, their utility for supporting teacher reflection is limited.

Together, these tools illustrate a broader pattern: classroom observation tools are typically structured as summative measures intended for evaluation and comparison. With these tools, the observer, who is trained, determines what counts as quality instruction, applies predefined criteria, and assigns and interprets scores. While such information may inform feedback for teachers, the tools themselves do not scaffold teachers' analysis of their own practices, nor do they prompt teachers to interpret classroom events in relation to their instructional goals. As a result, reflection is not structurally supported. Alternatively, observation tools that function formatively—guiding teachers through a systematic evaluation of their instruction rather than merely assigning scores—offer the potential to support reflective practice.

Methods

We conducted a comparative document analysis of 13 classroom observation tools. The sources of our data included the tools' manuals, protocols, rubrics, websites, and publications. We treated the tools themselves as the data and used consistent criteria to categorize the tools based on design features, structure, purpose, use, and requirements.

Using web-based searches, generative AI queries, and journal articles (e.g., Boston et al., 2015), we compiled a list of 20 observation tools commonly used in mathematics classrooms. We used GenAI queries to generate an initial list of tools (e.g., "provide a list of 20 mathematics classroom observation tools," "provide links to a variety of mathematics classroom observation

tools."), and by accessing official sources, we verified every tool's relevance to our search. We complemented our AI search by conducting a literature search of relevant journals to ensure that the GenAI had not neglected important observation tools. Of the initial 20 tools, we eliminated eight. We bounded our analysis to tools used in preK-12 mathematics classrooms, and consequently, we eliminated six tools because they were primarily used in higher education classrooms. The six eliminated tools included: *The Teaching Dimensions Observation Protocol* (Hora & Ferrare, 2010), *Classroom Observation Protocol for Undergraduate STEM* (Smith et al., 2013), *Oregon Teacher Observation Protocol* (Wainwright et al., 2003), *Student Response to Instructional Practices* (DeMonbrun et al., 2017), *Classroom Discourse Observation Protocol* (Kranzfelder et al., 2019), and *Three-Dimensional Learning Observation Protocol* (Bain et al., 2020). We also eliminated *Inside the Classroom Protocol* (Weiss et al., 2003) used across K-12 math and science classrooms, because this tool's purpose is for large scale comparisons across classroom teaching in the United States, rather than focusing on individual classrooms and teachers. Finally, we eliminated *Generalized Observation and Reflection Platform* (Center for Educational Effectiveness, n.d.) because it is not an observation tool; it is a software program designed for observing classrooms that can be programmed for any given observation tool.

In addition, we included a new observation tool, the *Curricular Reasoning - Observation Assessment* (CR-OA) (Roth McDuffie et al., 2026) in our analysis. The CR-OA tool, designed using AERA (2014) validity standards and developed by a team of mathematics education researchers in collaboration with a team of statistics researchers. The CR-OA tool is part of a larger toolkit, aimed to formatively assess and provide feedback on teachers' use of curricular reasoning—"a specific form of pedagogical reasoning that teachers employ while working with curriculum materials to plan, implement and reflect on instruction" (Roth McDuffie & Mather, 2009, p. 303). The toolkit was intentionally designed to support teacher reflection while also generating meaningful, context-specific data that teachers can use to inform their instructional decisions. The purpose of the CR-OA tool is for observers to collect evidence of observable instructional practices that reflect teachers' engagement with the six curricular reasoning aspects (Dingman et al., 2021). We included the CR-OA tool in our analysis because it was purposefully designed for teachers to reflect on their reasoning as they plan and implement mathematics lessons.

Table 1 lists the 13 observation tools that were analyzed for the study. We began analysis by documenting the stated purpose, publication date, use in the field, audience, validity, reliability, whether the results were used formatively or summatively, the types of classrooms they were designed to observe, structure/theory, the type of rating scale (or lack thereof), and how observations were conducted (e.g., videotaped or live). Using the information gathered on each tool, we identified emergent categories based on how the tools were used in the mathematics education field.

Table 1: Classroom Observation Tools Analyzed Listed by Development Date

Observation Tools Analyzed
Curricular Reasoning - Observation Assessment (CR-OA, 2026)
Early Mathematics Classroom—Pre-K through 2nd grade Instrument (EMC-PK2, 2024)
The Teaching for Robust Understanding Observation Guide for Mathematics (TRU, 2016)
Mathematics Classroom Observation Protocol for Practices (MCOP2, 2015-2017)
Practical Observation Rubric To Assess Active Learning (PORTAAL, 2015)
Mathematics Scan (M-Scan, 2013)
The UTeach Observation Protocol (UTOP, 2012-2018)
Instructional Quality Assessment (IQA, 2012)
Mathematical Quality of Instruction (MQI, 2012)
The Framework For Teaching (FFT, 2007)
Math Observation Checklist and Tool: SERP 5×8 Card (SERP 5x8 Tool, 2003)
Reformed Teaching Observation Protocol (RTOP, 2000-2002)
Formerly known as "*The New Teacher Project*" (TNTP, Creation Date Unknown)

We also examined the reported or estimated training time for each tool and grouped the tools into three categories based on length of training. We defined the three categories of training as: (a) *Easy*, requiring <2 hours; (b) *Moderate*, requiring between 2 to 12 hours; and (c) *Difficult*, requiring >12 hours. We determined the time required for training in two ways. First, several observation tools have websites that describe the training, including information on the number of hours or days required to complete the training. For tools that did not have this option, we consulted articles that reported training researchers or participants on a given observation tool, including training time and sessions (e.g., Rainey et al., 2024; Rogers et al., 2020; Kane & Staiger, 2012). We selected the categories based on how the training might impact teachers' daily schedules. Training requiring <2 hours might reasonably be done outside of school hours, or during a teacher's preparation time built into their schedule. Training requiring between two and 12 hours might take 1–2 school days, perhaps requiring support from the teacher's district to coordinate the training. Training requiring more than 12 hours would likely require more than two days and would require significant effort and coordination by both teacher and the school district to be accomplished.

In addition we grouped the tools based on administrative complexity. The three categories were: (a) *Easy*, which included tools with 15 or fewer indicators and a binary rating scale (e.g., observed, not observed); (b) *Moderate*, which included tools with more than 15 indicators and/or a rating scale of three or four choices (e.g., 1–4 point rating scale); and (c) *Difficult*, which included tools with anything outside the scope of the first two categories and added more complexity (e.g., tools that require the rater to rate the observation every five minutes of a lesson).

Finally, we identified whether the tools and training materials were publicly available without requiring special access or permission. This was important as we were interested in determining whether teachers can use these tools to reflect on their teaching practices.

Results

After analyzing the 13 mathematics classroom observation tools for both purpose and usability, we identified those tools that were more ideally situated for teacher use and teacher reflection, as contrasted with other tools that were better suited for researchers, to collect and analyze data to answer research questions. Each tool we analyzed was created for a distinct purpose (e.g., MQI measures the mathematical work in a classroom (Learning Mathematics for Teaching Project, 2011)), and is used by researchers, private organizations, school administrators, coaches, and teachers to varying degrees. We classified the tools according to their usability, which included the training time, administrative complexity, and public availability. Table 2 displays the 13 tools with information about each tool's usability. Typically, the majority of tools tended to require intensive training and were administratively complex; often these qualities are associated with acceptable to strong interrater reliability when using the tool. The most usable observation tools include, in order of their overall usability, SERP 5x8 Tool (Donovan et al., 2013), CR-OA (Roth McDuffie et al., 2026), TRU Observation Tool (Schoenfeld, 2017), and PORTAAL (Eddy et al., 2015).

Table 2: Usability (Training Time, Administrative Complexity, and Public Availability)

Tools	Training Time	Administrative Complexity	Publicly Available
CR-OA	Easy	Easy	Yes
MCOP2	Easy	Difficult	Yes
SERP 5x8 Tool	Easy	Easy	Yes
TRU	Easy	Moderate	Yes
MQI	Moderate	Difficult	No
PORTAAL	Moderate	Moderate	Yes
EMC-PK2	Difficult	Moderate	No
FFT	Difficult	Difficult	No
IQA	Difficult	Difficult	No
M-Scan	Difficult	Difficult	Yes
RTOP	Difficult	Difficult	Yes
TNTP	Difficult	Moderate	Yes
UTOP	Difficult	Difficult	Yes

The 13 observation tools were sorted into three categories based on their purpose, use, and audience, as shown in Figure 1. A tool categorized as *for Teacher Reflection* (a) uses broad categories and indicators, (b) presents results narratively rather than numerically, and (c) positions the teacher as an interpreter of results. Three tools are categorized as *Tools for Teacher Reflection*. The data gathered with these tools is understandable for teachers, and the results of these tools are formative, providing notes and descriptions rather than numerical scores. These tools also had lower training requirements, which may be because the tools were used to give feedback to teachers. These three tools are different in purpose and how they present the results, yet each provides results that can be interpreted directly by the observed teachers and presents results designed to promote reflection for the observed teachers on improving their own practice.

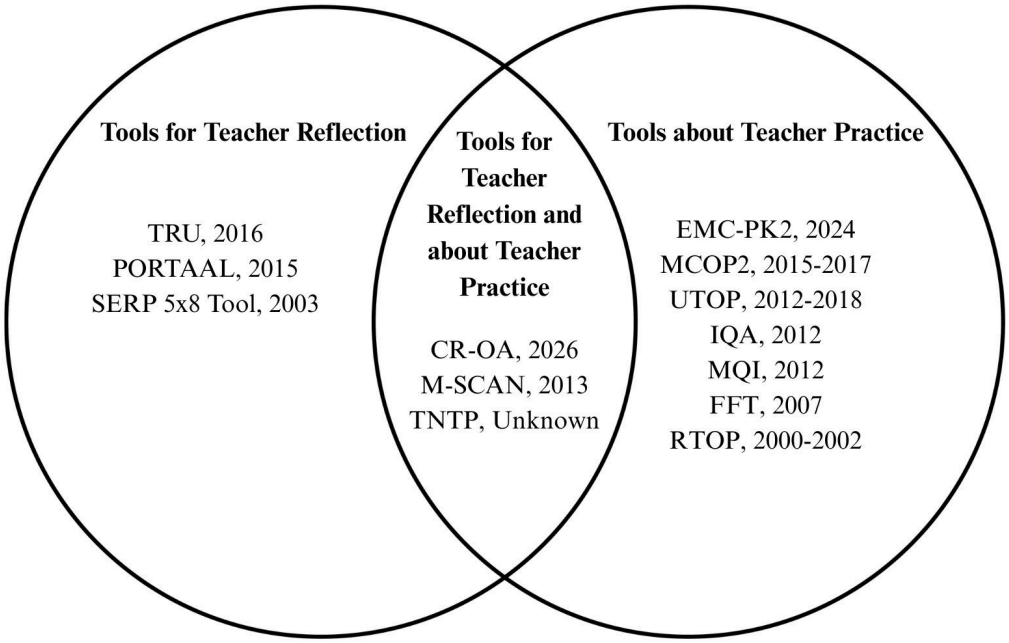


Figure 1: Categories of Observation Tools Based on Purpose, Usability, and Audience

A tool categorized as *about Teacher Practice* (a) includes a rigorous set of criteria to observe, (b) presents results as an evaluation of teacher practices, and (c) offers results that position the observer as the interpreter. Seven tools are categorized as *Tools about Teacher Practices*. These tools fulfill the first two steps of the reflection process as defined by Philips et al. (2023), collecting data and analyzing data, but they do not fulfill the third step, contextualized interpretations. These tools are used mainly by researchers and administrators. They are designed to record what teachers do in the classroom and to determine the quality of instruction, often for the purpose of evaluation or determining the effectiveness of a particular intervention. These tools take a clear normative stance about productive classroom practices, and they measure instruction according to their defined norms, often with complex rubrics and numerical scores. Using the tools and interpreting the results requires moderate to extensive training time. These tools are ideally situated for research as they provide reliable results regarding the quality of teaching. Some tools in this category are also frequently used by district and school administrators and coaches to evaluate teachers given that their results are better suited for reporting and comparison.

We classified three tools as both *Tools for Teacher Reflection* and *Tools about Teacher Practice* because the tools frame their results to be interpreted by the observed teacher and observer for quality instruction. Two tools, the M-Scan and TNTP, require extensive training, and are thus mainly used by researchers and administrators. Due to their extensive training, we viewed these tools as unlikely for teachers to use. In contrast, both the M-Scan and TNTP present results that can be understood by teachers, providing space for teacher reflection, and results can be reliably compared and rigorously analyzed by researchers, providing relevant data about teacher practices in their classrooms. The CR-OA is unique because it frames results to prompt teacher reflection and provides useful data for researchers about teacher's reasoning, while requiring less training time and offering less administrative complexity.

Conclusion

In this paper we aimed to identify classroom observation tools that provide teachers the opportunity to reflect on their teaching practices. We report the results of the analysis of 13 mathematics classroom observation tools that are currently available for teachers, educational leaders, and researchers. We found that the majority of the 13 classroom tools are not intended for teachers to reflect on their practice. Rather the purpose of these tools is to collect data about teachers' practices and often evaluate teacher practices. This suggests that many of these observation tools are developed for researchers and administrators and not for teachers. Further, we found only three of the six tools (CR-OA, SERP 5X8, TRU) that teachers might use to reflect on their practice require less than two hours of training. This creates a barrier to teachers' reflective practice. There is a need in the field to develop and validate tools that provide teachers access to systematically reflect on their teaching practice.

The 13 mathematics classroom observation tools were also limited based on the three components of teacher reflection articulated by Phillips et al. (2023). All of the tools collect quality classroom data. However, the other two components—conduct a thoughtful analysis and evaluation of the data, and contextualize an interpretation of the data—were not accessible for teachers who want to reflect on their practice.

As we have stated, the CR-OA is a new observation tool that was designed and developed using the AERA validity standards (2014). Our purpose in creating the tool was to measure teachers' curricular reasoning; however, we found that as teachers use this tool, they begin to reflect on their own reasoning for their mathematical decisions they made when teaching. This reflective practice made us wonder how many other observation tools are similar. We suggest that as new tools are developed, the field needs to place teachers' needs and reflective practices as an important part of the development of the tools. If the field of mathematics education has the goal to improve mathematics teachers' ability to teach their students, yet the tools we are developing are not helping teachers improve or reflect on their practice, it will be difficult for our tools to be useful to more than the research team who developed it.

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EXPLORING FEATURES OF GRAPHS ACROSS UNDERGRADUATE STEM TEXTBOOKS

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Given the importance of graphs in STEM, nuances in how graphs are used across STEM fields is an underexplored area. This study explores the diverse disciplinary expectations for graphical representations across STEM textbooks. We use a framework to analyze graphs based on their coordinate systems, the number and types of reference frames, and visually salient features. We note that while Cartesian graphs representing two continuous quantities are common, there is significant variability in graph use across fields. Our findings have important implications for school mathematics, including the importance of promoting covariational and multivariational reasoning to better prepare students for the specific demands of various STEM disciplines. We conclude with limitations and directions for future research.

Keywords: Undergraduate Education, Integrated STEM, Curriculum, Graphs

Graphs play a central role in STEM disciplines for modeling, analyzing, and communicating phenomena. They function as tools through which STEM professionals reason about and convey complex relationships (e.g., Meira, 1995; Roehrig et al., 2021). Consistent with this, graphical representations are pervasive in STEM textbooks and professional publications (e.g., Lemke, 1998; Paoletti et al., 2020; Roth & Bowen, 1999). Given their widespread use, supporting students in developing robust understandings of graphs is critical for preparing future STEM professionals.

Although graphs are central to STEM, findings from research (e.g., Lai et al., 2016; Ruf et al., 2024) and large-scale assessments (e.g., Nation's Report Card, 2005) suggest that U.S. students have limited opportunities to develop graphing understandings aligned with STEM disciplinary expectations. One potential explanation is that graphs differ within and across STEM fields—varying in how they are constructed, interpreted, and used (e.g., Paoletti et al., 2020). Although these disciplinary expectations likely shape how graphs are used and understood in different fields (Bowen & Roth, 2003), such differences remain insufficiently categorized in STEM education research. The overarching aim of this study is to explore similarities and differences in disciplinary expectations across STEM fields by using an existing framework that accounts for the various aspects of graphical representations. Accordingly, our research question is: *What are the similarities and differences in graphs used across STEM fields?*

In this report, we build on our on-going research by leveraging theoretical constructs that capture nuances in how students may conceive of graphs, coordinate systems (CSs) on which graphs are represented, and the reference frames (RFs) represented via the CSs. We organize these constructs along three dimensions: (a) the type of CSs, (b) the number and types of RFs, and (c) the object(s) students are expected to interpret within the graph space. Figure 1 provides a high-level overview of our framework; we provide more detail in the sections that follow.

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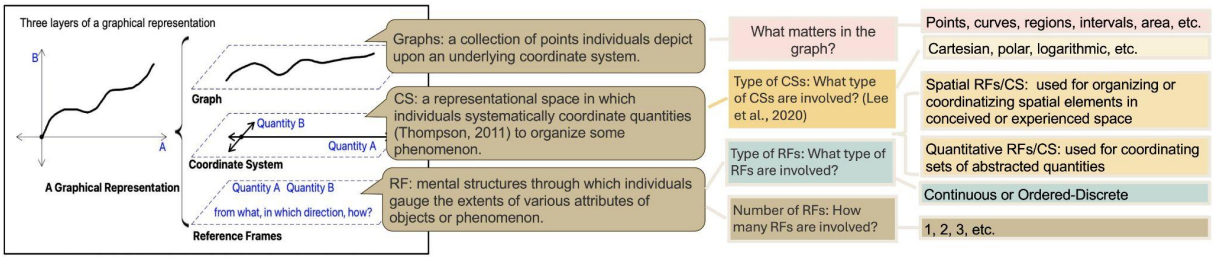


Figure 1: A Framework for Examining Graphs at a Fine-grained Level

Literature and Theory Relevant to Aspects of Graphs Across STEM Fields

In this section, we frame our discussion of literature and theory using six graphs (Figure 2) which are recreations of graphs similar to those found in textbooks across different STEM fields. We recognize that disciplinary knowledge is required to make sense of each graph and its different aspects (e.g., Altindis et al., 2024; Roth et al., 1999; Roth & Bowen, 2003; Zimmerman et al., 2023). Accordingly, we do not attempt to offer a full interpretation of each graph. Instead, we focus on supporting the reader in understanding how our framework (Figure 1) can help us identify similarities and differences across the examples that may inform instruction in undergraduate STEM courses.

Coordinate Systems

A CS is a representational space through which an individual relates quantities to make sense of a phenomenon (Paoletti et al., 2025; Thompson, 2011). Such spaces commonly involve the coordination of two quantities, though they may involve more. For example, Figures 2(a and d) represent CS that coordinate two quantities, while Figure 2f clearly involves more than two.

Across STEM disciplines, multiple forms of *conventional* CS are used. While Cartesian axes like those in Figure 2f are most commonly used in US Curricula (Lee & Guajardo, 2023; Paoletti et al., 2020), graphs using polar (or polar-like) CS (Figure 2a-b) and logarithmic scales (Figure 2d-e) are also used. Such conventions are important because researchers have shown that graphing conventions can influence students' thinking about graphs, often in unintended ways (e.g., students carry rules that apply in one conventional CS to another in which the rules do not apply; Montiel et al., 2008; Moore et al., 2014).

We distinguish between two broad types of coordinate systems: *spatial and quantitative* (Lee, 2017; Lee et al., 2020). A *spatial CS* is one through which an individual organizes elements within conceived or experienced physical space. In contrast, a *quantitative CS* emerges when an individual abstracts quantities from a phenomenon and organizes them within a representational space that differs from the original physical context. Here, the space does not mirror physical layout; rather, it provides a structure for coordinating quantities.

In some mathematical contexts, graphs may not involve clearly identifiable situational quantities (e.g., graphs representing abstract variables x and y), and not all graphs align neatly with either CS described above. Nevertheless, the distinction between spatial and quantitative CSs has proven useful in prior analyses of middle school and STEM textbooks (Lee & Guajardo, 2023; Paoletti et al., 2020), as well as in explaining differences in students' graph construction and interpretation (Knurek, 2025; Paoletti et al., 2025). Taken together, these distinctions offer a productive lens for examining the varied roles CSs play in undergraduate STEM contexts.

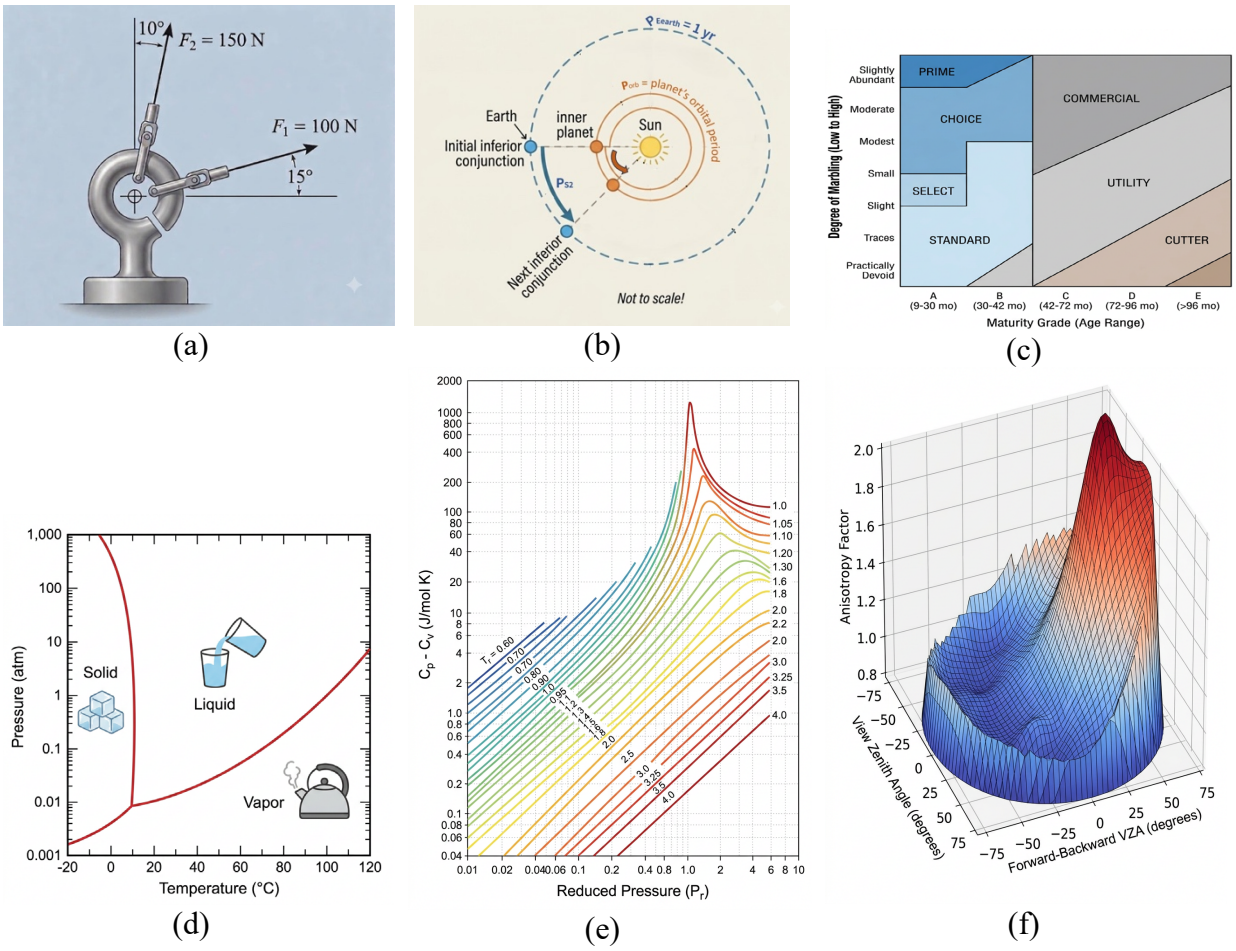


Figure 2: Recreations of Graphs from Six Textbooks: (a) Mechanic Statics (Hibbeler, 2015. p. 23), (b) Astronomy (Bennett, 2011, p. 93), (c) Agriculture (Field & Taylor, 2015, p. 120), (d) Material Science (Callister & Rethwisch, 2013, p. 297), (e) Chemical Engineering (Sinnott, 2005, p. 70), and (f) Environment (Jensen, 2013, p. 373)

Reference Frames

A CS comprises one or more RFs. RFs are mental structures through which an individual coordinates various attributes associated with objects or phenomena (Lee, 2017; Levinson, 2003). The construct of RF has played a central role in research on spatial cognition (e.g., Levinson, 2003; Taylor & Tversky, 1996), children's conceptions of space (e.g., Piaget & Inhelder, 1967; Piaget et al., 1960), and quantitative reasoning (e.g., Joshua et al., 2015; Lee et al., 2019). Across these domains, RFs are understood as cognitive structures that enable individuals to coordinate and compare attributes within a representational space.

Within a CS, RFs can be represented via quantities represented on axes as well as through other objects within the graphical space (i.e., a series of related curves (Figure 2e) or distinct regions within the graph (Figure 2c–2d)). It is common for CSs to represent two RFs. As with CSs, we distinguish between spatial and quantitative RFs, with some graphs including one (or more) spatial and one (or more) quantitative RF represented (Knurek, 2025). When students interpret graphs structured by two RFs, they may be asked to determine the value of one quantity at a specific value of another, or to reason about how two quantities vary together. Interpreting

how two quantities change together entails covariational reasoning (Thompson & Carlson, 2017).

In some contexts, CSs can represent three or more RFs. Interpreting graphs with 3 or more RFs requires coordinating multiple quantities simultaneously, which entails multivariational reasoning (Jones, 2002; Panorkou & Germia, 2023). Research suggests that learners as young as middle and high school can demonstrate such reasoning (Bagossi, 2024; Panorkou & Germia, 2023). Examining the prevalence of such graphs in undergraduate STEM courses can inform the design of curricular materials that better support students' interpretation of such representations.

Beyond considering the number of RFs represented in a graph, our research has identified distinctions in how students interpret graphs as representing continuous versus ordered-discrete RFs (Olshefke-Clark et al., 2024, accepted). In many instructional settings, students are asked to interpret graphs that represent continuously varying quantities (Figure 2e–2f). In other cases, however, graphs represent discrete but ordered regions within a graphical space (Figure 2c–2d). For example, Figure 2c describes both the degree of marbling and an animal's maturity in discrete but ordered chunks (i.e., traces, slight, small, as progressively increasing marbling). Interpreting such graphs entails attending to those regions rather than to continuously varying relationships. In such contexts, curriculum designers may expect students to reason about regions or about specific points within those regions. The extent to which undergraduate STEM curricula emphasize interpretations based on points, lines, or regions remains an open question.

Methods

To address our RQ, we followed a purposive multi-stage sampling process (Babbie, 2020) to identify commonly used textbooks across a variety of STEM fields. First, examining the STEM offerings at a variety of undergraduate institutions (e.g., UD, Texas State, Montclair State), we identified several STEM disciplines that we conjectured may provide novel insights into graph use across STEM (e.g., Agriculture; Chemical Engineering; Mechanic Statics). After identifying several STEM fields of potential interest, we used the Open Syllabus Project (OSP), which has a large-scale database aggregating curricular data from millions of syllabi.

Using OSP, we identified the three most commonly used textbooks in each STEM major, ranked by their frequency in syllabi. We then searched for digital or physical versions of these textbooks available to the research team (e.g., PDFs, versions from university libraries). Through this process, we aimed to select books that were actually used in university STEM instruction. Table 1 provides a summary of the 6 STEM textbooks we report on in this paper. In total, we identified 856 graphs, which we coded via the process described below.

Table 1: Summary of Six Selected Textbooks

STEM Discipline	Textbook Title & Authors	Published Year	Rank	# of graphs
Agriculture	Scientific Farm Animal Production: An Introduction (11th Ed.) by Field and Taylor	2015	2	48
Mechanic Statics	Engineering Mechanics: Statics (14th Ed.) by Hibbeler	2015	3	294
Chemical Engineering	Chemical Engineering Design (4th Ed.) by Sinnott	2005	2	123
Astronomy	The Cosmic Perspective (6th Ed.) by Bennett	2011	1	151

Environment	Remote Sensing of the Environment: Pearson New International Edition: An Earth Resource Perspective (2nd Ed.) by Jensen	2013	1	128
Material Science	Materials Science and Engineering: An Introduction (9th Ed.) by Callister and Rethwisch	2013	1	112

Coding

We coded each graph that represented at least two quantities in each book. We located the graph and identified the number of distinct reference frames (RFs), the types of RFs across two dimensions, the types of conventional coordinate system (CS), and the visually salient features. We describe and use examples selected from each textbook to illustrate our coding strategies.

Number of RFs (2, 3, 4+). To code for the number of RFs, we examined how many pieces of information a student would need to coordinate to potentially make sense of a graph in ways compatible with the authors' intentions. If a graph involved only two quantities and each was represented in RF, we coded it as "2 RFs." For example, Figure 2c represents the relationship between two quantities, maturity and marbling, and they serve as the horizontal axis (maturity) and the vertical axis (marbling) in a two-dimensional CS. If the representation required coordinating an additional RF beyond two axes, we increased the RF count to "3". This included cases such as: a secondary axis (i.e., two different variables are plotted against the same horizontal axis), three-dimensional surface plots (e.g., Figure 2f), graphs linked to a physical situation diagram or vector-direction frame that must be interpreted simultaneously (e.g., a schematic of the celestial sphere), or graphs where the user must coordinate the two axes with an internal family of curves representing a third quantity (e.g., Figure 2e). When the representation stacked multiple layers of coordination (for instance, an embedded physical diagram or multiple axes), we coded it as "4+," indicating four or more frames operating together.

Type of RFs: Dimension 1 (Quantitative, Spatial, Both). We then characterized the type of quantity each RF represented, either spatial, quantitative, or both. If the RF represented a quantity in a space intended to be different from the space in which the quantity should be conceived, we categorized that RF as quantitative (e.g., Figure 2c-f). If the RF was represented in a space analogous to the space in which the quantity is intended to be conceived, we coded the RF as spatial (e.g., Figure 2a-b). When a graph had both spatial and quantitative RFs (for example, motion graphs, vector diagrams with magnitude, or graphs tied closely to a pictured situation), we coded it as Both.

Types of RFs: Dimension 2 (Continuous, One or more Ordered-Discrete). For each RF we identified, we characterized whether this RF was intended to represent a continuous or ordered-discrete RF. If a graph exclusively used continuous or ordered-discrete RFs, we coded the graph as Continuous or Ordered-Discrete. As we coded graphs, we noticed that many graphs represented one (or more) continuous and one (or more) ordered-discrete RFs, leading to the code Mixed. In this report, we group Mixed and Ordered-Discrete into one category called One or More OD. We also identified several graphs in which one quantity was not continuous or ordered discrete (e.g., categorical labels such as names, colors, or types); such graphs were coded in an Other category that we do not present in the current report, as there were few such cases.

Type of conventional coordinate system (Cartesian, Logarithmic, Polar, Mix, other). Next, we classified the coordinate system itself. If the graph used perpendicular axes, we coded it as a Cartesian CS (e.g., Figure 2f). If it used a polar structure (i.e., one angle, one radial

distance), we coded the graph as having a polar CS (e.g., Figure 2b). If both axes were logarithmic, we coded the graph as having a logarithmic CS (e.g., Figure 2e). If multiple coordinate systems were used across the representation, we coded the graph as mixed (e.g., Figure 2d). Anything that did not fit the above descriptions was coded as Other.

Visually salient features. We coded the visual elements designed to stand out, as these elements often signal what the authors expect students to attend to while interpreting the graph; we note that a single graph could be coded with multiple visually salient features. For example, if a graph included discrete plotted points, we coded *Points* as visually salient (e.g., Earth in Figure 2b). If the graph included a connected curve or trend, we coded *Lines in the graph* as visually salient (e.g., the red curves in Figure 2d). We coded *Regions* as visually salient in graphs that shaded or bounded areas (the regions in Figure 2c, d). Similarly, we included codes for *Bars* (e.g., bars in a bar graph), *Vectors* (e.g., Figure 2a), *Lines to axes* (e.g., Figure 2e), and *Physical objects* (e.g., Figure 2d) as visually salient features within a graph. Figure 2d provides an example of a graph in which lines in graphs (the red curves), regions (Ice, Liquid, Vapor), and physical objects (ice cube, pitcher of water, tea kettle) were all coded as visually salient.

Results

In this section, we present the results of our coding. We first describe the number of RFs in each graph. We then characterize the graphs across the two dimensions of RFs and the conventional CSs we observed. We conclude by presenting the visually salient features.

Table 2 presents the number of RFs in each textbook. We highlight that while graphs representing 2 CS were majorities in each textbook (all over 60%), graphs representing three or more RFs were also common across all the textbooks we examined. Graphs representing 4 or more RFs were less common, but present in fields that have an explicit focus on engineering and physical sciences (Chemical engineering; Engineering Statics; Material Science).

Table 2: Number of Reference Frames Identified in Each Graph (and as Percent of Graphs in that Textbook)

Textbook	2	3	4+
Agriculture	45 (94%)	3 (6%)	-
Statics	180 (61%)	86 (29%)	28 (10%)
Chem Eng.	84 (68%)	33 (27%)	6 (5%)
Astron.	107 (71%)	44 (29%)	-
Environ.	109 (85%)	19 (15%)	-
Material Sci.	74 (66%)	22 (20%)	16 (14%)

Table 3 presents our analysis across two dimensions of the RFs present in graphs which were coded independently. With respect to types of CSs, and consistent with findings from Paoletti et al. (2020), graphs were represented on quantitative CS in the vast majority of textbooks. Mechanic Statics was an exception with all graphs represented on Spatial CSs. The Astronomy textbook included a higher mix of Quantitative (70%) and Spatial RFs (28%). With respect to continuity or ordered-discrete RFs, the majority of graphs represented continuous quantities, with the Agriculture textbook being an exception (only 31% of graphs representing continuous quantities). Hence, across textbooks, quantitative CSs representing continuous quantities are most common, but graphs on spatial CSs or graphs representing OD quantities are important for particular STEM Fields (Statics and Agriculture, respectively).

Table 3: Number of Graphs Categorized by Types of RFs and Types of Quantities (and as Percent of Graphs in that Textbook)

	Types of RFs			Types of Quantities	
Textbook	Quantitative	Spatial	Both	Continuous	One or OD
Agriculture	47 (98%)	1 (2%)		15 (31%)	28 (58%)
Statics		294 (100%)		294 (100%)	
Chem Eng.	123 (100%)			121 (98%)	2 (2%)
Astron.	106 (70%)	43 (28%)	2 (1%)	149 (99%)	2 (2%)
Environ.	124 (97%)	4 (3%)		113 (88%)	15 (12%)
Material Sci.	112 (100%)			109 (97%)	3 (3%)

For space reasons, we do not present a table showing our analysis of the types of conventional CS. We note in several books (Agriculture, Statics, Environ., and Material Science), the vast majority of graphs were represented on Cartesian coordinate systems (between 87-100%). However, in the Chemistry Engineering and Astronomy textbooks, Cartesian coordinate systems were common (52% and 48% respectively); in these texts, Logarithmic axes (35% and 19%) and mixed axes (12% and 7%) appeared often. The Astronomy text also included 24% of graphs coded as other, which consisted of 3D schematic representations of the celestial sphere that lack explicit coordinate scales or quantitative statements on variables, focusing instead on illustrating spatial geometric relationships.

Table 4 presents the visually salient features of the graphs. Recall that in this category, a graph can have more than 1 visually salient feature. With the exception of the Statics textbook, lines within a graph were visually salient in 60% or more of all graphs. In many cases, it was also common for textbooks to include lines to axes, likely to support students in interpreting the graph accurately in relation to the quantities represented on the axes. Outside of this commonality, we highlight that the textbook's subject likely influenced which features were typically visually salient. The Astronomy textbook included numerous points to represent the locations of objects (e.g., planets). Similarly, all graphs within the Statics textbook represented vectors (100%), typically shown in relation to a physical object (51%). Hence, while certain visually salient aspects of graphs were common across fields, each field's particular content likely influenced the visually salient features they included within a graph.

Table 4: Number of Graphs Coded as Having Different Types of Visually Salient Features

Text book	Points	Lines in Graph	Regions	Bars	Vectors	Lines to Axes	Physical Objects
Agric.	4 (8%)	29 (60%)	8 (17%)	15 (31%)		6 (13%)	2 (4%)
Statics					294 (100%)		149 (51%)
Chem.	11 (9%)	117 (95%)	11 (9%)			108 (88%)	8 (7%)
Astr.	110 (73%)	128 (85%)	3 (2%)	1 (1%)	2 (1%)	45 (30%)	81 (54%)
Env.	35 (27%)	108 (84%)	39 (30%)	3 (2%)	4 (3%)	54 (42%)	7 (5%)
MS.		109 (97%)	7 (6%)	3 (3%)		109 (97%)	

Discussion

Addressing our RQ, we identified similarities and differences in graphs used across STEM fields. Generally, it may feel safe to say graphs with lines representing two quantities on Cartesian CSs are the most common graphs students are likely to experience across STEM fields. The fact that such graphs are common in US curricula textbooks (e.g., Lee & Guaradjo, 2023; Paoletti et al., 2020) may lead some to conclude that we are adequately preparing students to interpret graphs across STEM fields. However, we caution that such a conclusion may miss several nuances in how graphs are used both in US curricula and across STEM fields.

First, we highlight that given the majority of graphs (implicitly or explicitly) require students to reason about the ways two quantities covary, and such reasoning is not explicitly promoted in US school mathematics (Thompson & Carlson, 2017). This finding leads us to echo the calls of others (e.g., Gantt et al., 2023; Panorkou & Provost, 2025; Thompson & Harel, 2021) for the importance of covariational reasoning for students' success in STEM fields. The findings presented here underscore how critically important covariational reasoning is across STEM fields and the need for policymakers and curricular designers to make covariational reasoning a central part of US school curricula.

Second, given that US students are rarely asked to reason covariationally, even fewer opportunities exist for them to reason about *more* than two quantities changing despite showing such an ability as early as middle and high school (Bagossi, 2024; Panorkou & Germia, 2023). Given the high prevalence of such graphs across several STEM fields, we encourage future researchers to explore the potential efficacy of supporting younger students in making sense of graphs representing three or more quantities. Such reasoning can build on the research exploring students' covariational reasoning.

Third, while most graphs represent quantitative RFs, spatial RFs played a large role in the Engineering Mechanic Statics textbook. The fact that only 1 textbook uses such graphs may lead some to conclude that we should deemphasize such graphs in school mathematics. However, we highlight that Engineering Mechanic Statics is an introductory course for numerous engineering degrees. As such, if we expect students to succeed in their engineering coursework, we should provide them opportunities to construct meanings for graphs as representing spatial RFs. Further, given the ways US school math uses both quantitative and spatial RFs interchangeably (Lee & Guaradjo, 2023), we contend that supporting all students in becoming explicitly aware of the differences between the two types of RFs would support them in developing meanings for graphs more applicable across all STEM fields.

Our analysis presents an important first step in describing ways that graphs differ within and across STEM fields. Our analysis of visually salient features within a graph highlights how disciplinary content and practices (Roth & Bowen, 2023) play a key role in what visually salient features a graph includes. Further, our framework proved useful in categorizing a variety of features of graphical representations across STEM fields. Our analysis was limited as none of the research team has disciplinary knowledge that spans various STEM fields. Future researchers may be interested in exploring the ways faculty actually use such graphs in undergraduate STEM instruction, including how they expect students to interpret or construct such graphs. Such research can support mathematics policy makers and curricular designers in updating US school mathematics to reflect the types of knowledge required for student success in STEM fields. Such curricular materials could both increase the number of future STEM professionals.

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CONTRASTING ASSESSMENTS OF HOMOGENEITY AND DIVERSITY USING PATTERN-BASED RESULTS FROM A STATEWIDE IMPLEMENTATION

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This paper centers on empirical results that highlight how plausibly distinct pattern-based items (PBIs) are in what they assess when compared with standard or legacy items. The latter are taken to include items associated with high-stakes or large-scale testing and are currently developed using item response theory (IRT) or Rasch methods. Examples of legacy items are those appearing on the State of Texas Assessments of Academic Readiness (STAAR). We highlight a test-reliability metric and expectations regarding cross-item correlations to set out the contrast. While empirically informed, this paper presents a largely theory-centered account of how the continuing enactment of tests based on a notion of ability or a latent-trait will retain aspects of ineffectiveness and inequity that have been there from the start. The contrast with PBIs may serve to highlight that there are alternatives to continuing this legacy.

Keywords: Assessment, Learning Theory, Policy, Research Methods

When students engage with a mathematical task, they bring strategies, prior knowledge, distinct forms of communication, and emerging ideas to bear. Current standardized tests typically reduce this complex of varied reasoning and forms of engagement to a binary of right versus wrong. Our paper presents an alternative assessment approach that makes visible an array of instructionally relevant distinctions. The alternative centers on the use of pattern-based items (PBIs) or pattern-based questions (PBQs). To make the case that the use of pattern-based items is an alternative, we look to place in sharp relief the contrasts with current, or what we will call "legacy," large-scale, often high-stakes, standardized assessments that currently play a significant role in public education in the United States (and, indeed, around the world).

To make this case, we draw on results from the STAAR Interim Assessment Program (IAP, 2018–2019), in which PBIs were administered at scale across grades 3 through 8 in Mathematics and grade 5 in Reading at the request of the Texas Commissioner of Education. We focus specifically on two technical metrics—Cronbach's Alpha and cross-item correlations—not to evaluate PBIs by legacy standards, but to use the contrast those metrics reveal as a lens into fundamentally different assumptions about what assessment is for. Where legacy tests are designed to maximize consistency across items, producing a single scale score meant to index a latent trait, PBIs are designed to surface the diversity of student thinking, making the reasoning students bring to their work visible and usable at scale. Critically, we argue that the technical standards used to judge legacy tests are not neutral measures of quality—they encode assumptions about the fixedness of student ability that carry equity consequences.

A Focus on Patterns

Tests like the State of Texas Assessments of Academic Readiness (STAAR)—a mandatory program covering Texas public school students in grades 3 through 8, as well as specific high

school end-of-course classes—are built on a fundamentally different logic than pattern-based assessments. Using methods drawn from item response theory (IRT) and Rasch modeling, legacy assessments are designed to produce a single, consistent score that places each student on an ability scale. The technical standards that define a "good" test in this tradition reward consistency. Items that do not align with the ability scale are removed. As a result, the instrument will tend to be optimized for ranking students rather than illuminating specific features of their understandings.

The difference between these two approaches is not merely technical. The norms embedded in standard psychometric practice—particularly the reliance on internal consistency as a measure of test quality—carry assumptions about student ability that trace their roots to the earliest days of what is now called psychometrics, or "mind measurement." As will be discussed, from early on its inception, this tradition treated intellectual capacity as a fixed, ranked quantity: something a student either had more or less of, and that a well-designed test could reliably detect. As will be discussed, Alfred Binet, whose work gave rise to the modern intelligence test, came to explicitly critique aspects of this view, insisting that intelligence was not "a fixed quantity that cannot be augmented" (1909, pp. 105-106). Yet the institutional apparatus that grew from his methods did precisely that, building tests designed to sort students into stable ability categories and building metrics that advanced tests optimized for sorting. We argue that continuing to assess student understanding through this framework reproduces both the ineffectiveness and the inequities that have characterized that tradition from the start.

An Example of a Pattern-Based Mathematics Item

Figure 1 shows an equivalent-fraction related pattern-based item that was administered at scale (N=60,517) as part of the State of Texas Assessments of Academic Readiness (STAAR) Interim Assessment Program (IAP 2018-2019). Pattern-based items were included in the STAAR IAP for grades 3 through 8 in Mathematics and grade 5 in Reading at the request of the Commissioner of Education (Stroup, 2020; Stroup et al., 2023; Stroup, 2024; Stroup, 2025). Like most of the items developed for, and then implemented on, the IAP, the alignment of this question with the Texas Essential Knowledge and Skills curricular standards was maintained by retaining essential elements of a released multiple-choice STAAR item, including, as is illustrated in this case, the use of specific language like "shaded to show equivalent fractions" (Texas Education Agency [TEA], 2018 Grade 3 p. 12).

Visual similarity to traditional or "legacy" multiple-choice or multi-select items notwithstanding, the development of the prompt and the responses is designed to reveal instructionally relevant patterns in students' understandings related to specific curricular standards (e.g., equivalent fractions). Unlike legacy multi-select questions, which also may have more than one correct response for students to endorse, pattern-based items are designed to make use of the full combinatorial space of responses.

Analogous to possible combinations of four notes on a keyboard, there are 16 possible combinations of the four responses (A–D) that are made available to identify the relevant patterns. By excluding the no-response category, we use 15 states in our analyses. These analyses include frequency plots of the 15 combinations for each item, sorted (at the request of the teachers we've worked with) from low to high according to a matching, True/False-type, partial-credit model.

The combination ABC, shown at the right in the frequency plot in Figure 1, receives full credit for this item by matching the three correct responses (the three "True" responses) and *not* selecting the one incorrect (or "False") response.

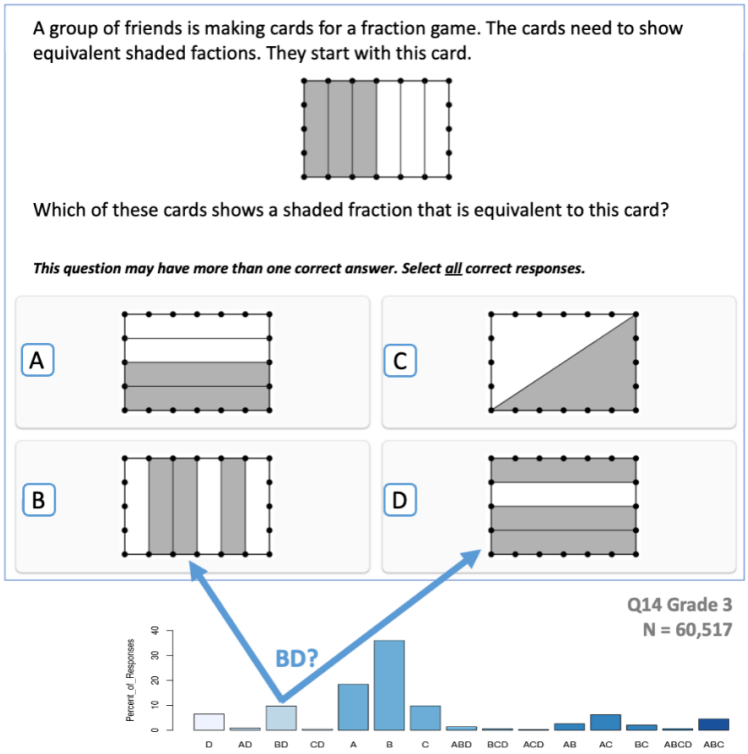


Figure 1: The Pattern of a Correct Response B Combined with an Incorrect Response D Suggests Students are "Counting Pieces."

To illustrate the importance of going beyond simply tallying how many times students selected each response (A, B, C, or D), we have highlighted the BD combination in the frequency plot. B is a shaded region that is an equivalent shaded fraction of a whole card. D, in contrast, is not an equivalent area. The question might become, what does this combination of a correct response and an incorrect response suggest about the students' understanding?

As reported in previous work (Stroup, 2020; Stroup, 2024; Stroup, 2025), the elementary teachers we have worked with have maintained that students are "counting pieces" without reference to the fraction of the whole the pieces represent. Further, they reported that counting pieces is a way of thinking about fractions that they regularly encounter in their work with students. This is precisely the kind of instructionally actionable insight that a scale score cannot produce: not that those students answered incorrectly, but that they are applying a specific, recognizable strategy that has a known relationship to the target concept and that can inform an instructional response.

Moreover, implicit in the use of scale-score reporting of assessment results is the sense that there is a consistently interpretable "something" that is being measured on that "scale" of something. This something is to be accessed through the specific use of a collection of scale-consistent items that make up a test. This focus on consistency, or what we'll call *homogeneity*, contrasts with the pattern-based approaches' focus on characterizing and engaging with the *diverse* understandings students may have regarding specific topics in the state standards, as made visible by analyzing the distinct combinations in their response selections for a given item.

Contrasting Homogeneity with Diversity Using Cronbach's Alpha and Correlations

Cronbach's Alpha (CA) is a widely deployed measure of the internal consistency of items appearing on a given test. Historically, CA (or the mathematically equivalent Kuder-Richardson 20 [KR20] test for dichotomously scored items) has been taken as a measure of a test's reliability. While we recognize this usage, we suggest a normative interpretation is implicit in the use of the word "reliability" that goes beyond what is warranted by what is a measure of internal consistency. The distinction matters partly because reliability, strictly defined, refers to whether repeated administrations of the same instrument produce consistent results. Internal consistency, by contrast, measures whether items within a single test administration align or cohere. Conflating these two properties imports an assumption that a good test must measure one coherent thing, which is precisely the assumption PBIs are designed to challenge. We certainly allow that values for CA (or KR20) approaching an upper value of "1" can be interpreted as indicating a high level of internal consistency. Rather than add the normative use of reliability, however, we suggest that the consistency associated with high CA values is a measure of the *homogeneity* of the items.

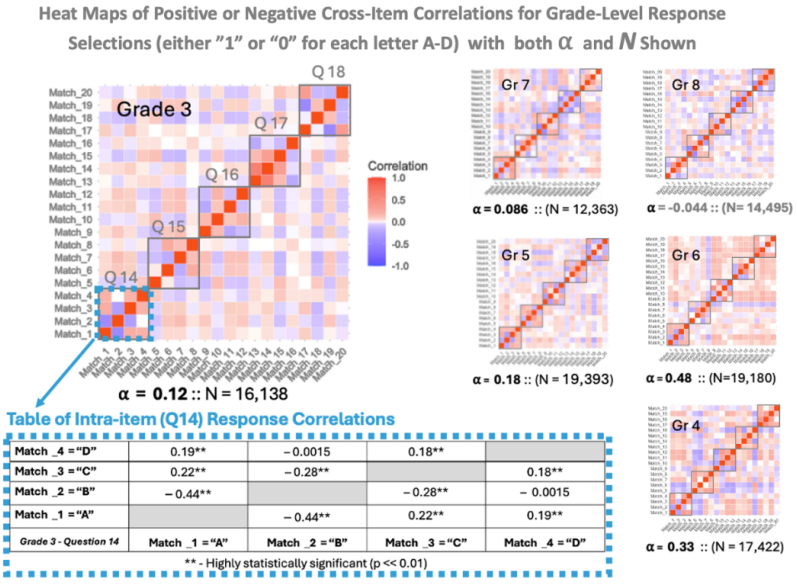
In a related way, positive correlations between results on individual items are associated with the homogeneity also associated with relatively high values for Cronbach's Alpha. The normative assumptions traditionally associated with high CA results are problematic when compared with results from analyzing responses to pattern-based items, such as the PBIs deployed on the STAAR Interim Assessment Program in Texas. The identification of distinct patterns relies on items that make visible contrasts in forms of student understanding. The ability to identify meaningful patterns relevant to instruction—e.g., the results shown above, which can inform educators that a number of students may be counting pieces rather than referring to a unit whole in determining fractional equivalence—is an important component in establishing the warrant for using pattern-based approaches to assessment.

What we might expect, then, is that measures of internal consistency, as well as cross-item correlations for response selections, would reflect this effort to specify the *diversity* of students' understandings. The values for CA would be low (or even negative). Moreover, correlations between students' response selections could be positive or negative, and, at scale, could be statistically significant ($p<0.05$) or highly statistically significant ($p<0.01$). This focus on identifying *diversity* we see as contrasting substantially with the use of *homogeneity*, as normative, in evaluating both tests and individual items.

To setup and frame the contrast with pattern-based items shown in Figure 2, it is worth first discussing the analyses of homogeneity associated with STAAR tests. The normative interpretation of relatively high CA or KR20 values is maintained in the technical digests prepared annually in Texas for the STAAR tests. Internal consistency coefficients "ranging from 0.70 to 0.79 are considered adequate, those from 0.80 to 0.89 are considered good, and those at 0.90 or above are considered excellent" (TEA, 2022, p. 31). In practice, the coefficients for the tests in each grade ranged from "0.84 to 0.92 for the primary STAAR English and STAAR Spanish assessments administered in spring 2022" (TEA, 2022, p. 31) Overall, the end-of-course tests had "reliability estimates" ranging from 0.91 to 0.94 (TEA, 2022, p. 31). Positive correlations between item scores would be expected. CA or KR20 values below 0.7 would be considered inadequate, and those below 0.5 would be considered completely unacceptable as coefficients taken to measure test homogeneity.

Using the same metrics, now framed to assess the complexity and diversity of students' understandings, the contrasts in our results for the pattern-based mathematics items used on the

IAP, shown in Figure 2, are striking. Using "1" to represent selecting a response and "0" to represent not selecting a response, as extended to 20 responses (four letter choices for each of five items), a "heat map" is shown for the correlations of each response with the other responses. The shading intensity depicts the relative strengths of positive and negative correlations. In general, the shadings and associated values for the positive and negative correlations are relatively faint, except for responses within a given item. The squares across the heat maps denote areas of intra-item correlation, with the associated shading intensities, representing higher values for both positive and negative correlations, more intense.



Match _4 = "D"	0.19**	−0.0015	0.18**	
Match _3 = "C"	0.22**	−0.28**		0.18**
Match _2 = "B"	−0.44**		−0.28**	−0.0015
Match _1 = "A"		−0.44**	0.22**	0.19**
Grade 3 - Question 14	Match _1 = "A"	Match _2 = "B"	Match _3 = "C"	Match _4 = "D"

Figure 2: Using Chronbach's Alpha and Correlations to Highlight How Pattern-based Items Highlight Contrasting Diversity of Student Responses at Scale

The actual values of the intra-item correlations for the equivalent fractions item (Figure 1) are shown in the table at the bottom of the figure. Note that the correlations are both positive and negative and that almost all these varying values are highly significant ($p<0.01$). This pattern—strong, mixed-sign correlations within items and weak correlations across items—is what one would expect from an instrument designed to capture the diversity of student thinking within each conceptual domain, while making no assumption that performance on one standard necessarily predicts performance on another. To attend to the diversity of student understandings, the contrasts between responses are intended to be most significant within each item.

The contrasting expectations regarding inter- and intra-item correlations at each grade level do, as expected, have implications for computed consistency coefficients. When the consistency coefficients are computed for the responses at each grade level (we used Cronbach's Alpha), the values are generally very low, typically well below 0.5, and in one case even slightly negative. Clearly, PBIs do not do well when assessed normatively in terms of cross-item and test-level homogeneity. This is by design. The question of whether to continue using assessments focused on homogeneity to maintain relative rankings of students, or to use assessments centered on engaging the diversity of student understandings, need not be decided by biasing the selection based on contrasting expectations for what statistically rendered comparisons can or should show.

More relevant might be which approach does a better job of assessing students in ways that provide insights into students' understandings of distinct curricular standards. Moreover, as is taken up in the next section, the choice can center on contrasting implications for equity.

Homogeneity in Relation to Equity

Susan E. Embretson in her William H. Angoff Memorial Lecture presented at Educational Testing Service in January of 2001 (Embretson, 2003) titled *The Second Century of Ability Testing: Some Predictions and Speculations* makes the case in her "overview of the last century" that the "most fundamental principles in psychometrics and testing were available by 1930" (p. 5). Then "[w]ith a few notable exceptions, the remainder of the last century of testing was devoted to applying or refining these principles" (p. 5). How does a focus on creating cross-item homogeneity draw on specific features of "ability" testing that Embretson sees as carrying forward from the early 20th century into the present?

Her account is largely historical, including the identification of Alfred Binet as a pivotal figure in the development of what were to become the fundamental principles of psychometrics. Rather than focus on low-level skill development, one of Binet's contributions "is the use of higher-order cognitive tasks to measure intelligence" (Embretson, 2003, p. 8). Referencing the works of Binet and Binet and Simon (1905, 1908, and 1911), Embretson discusses "Binet's global conceptualization of intelligence," where "empirical criteria" would be used to "select intelligence test items" (p. 8). Binet's two criteria were to be that "task performance should increase with age" and "task performance should be related to school achievement" (p. 8).

In the present, we might want to distinguish between tests of achievement and tests of intelligence or aptitude. This expectation notwithstanding, the explicit linking of what would appear to be tests of "school achievement" to intelligence or ability testing is central to understanding the development and the requirements for current scale-score reporting of test results. Embretson summarizes some of these features:

> An interesting feature of Binet's system is that items and examinees are placed on a common scale. That is, both items and examinees are referenced to mental age. Tasks were scaled for mental age from empirical data on the performance of children at various ages. Examinees were scaled for mental age by their relative success in solving the age-calibrated tasks. Due to the age-calibrations of the tasks, examinees may be compared even if they do not receive the same items. (p. 8)

Much of what is outlined is part of accepted current practices in item and test development. Homogeneity for items serves in relation to placing examinees on "a common scale." Tasks are to be "scaled," using "empirical data." as a form of "mental age" taken to be analogous to the growth associated with physical maturational, or age-referencing, traits like height (see left-most graph from the Center for Disease Control in Figure 3 [NCHS, 2000] with an overlay of a child's trajectory along an expected growth curve). And once the specific "age-calibrations" of tasks are determined, examinees do not have to "receive the same items" to be compared.

Based on the features of the growth curves, there is no expectation that one's age-referenced ability would change relative to the age referencing of other students. The sense is that normal growth would be a child remaining in their relative position until they mature (as shown in Figure 3, the curves flatten out). Relatively tall children would be expected to become relatively tall adults; less tall children would be expected to become less tall adults. The WISC IQ growth curve is included to reinforce the sense that tests of what might look like "school achievement" can, in fact, as Binet is said to have outlined, serve a "global conceptualization of intelligence."

Figure 3 is meant to summarize and illustrate the intended parallels between expected growth relative to other children for physical traits (height), the development of mental ability (now typically reported as an intelligence quotient [IQ] of mental age/actual age*100) and graphs of actual age-specific scale score results for tests from one of the largest, achievement referencing, school-aged test vendors in the United States (NWEA, 2026, p. 8).

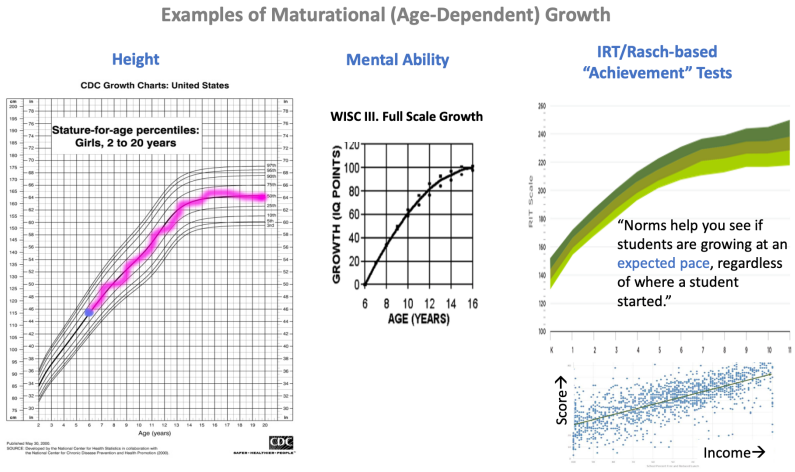


Figure 3: Similarity in Graphs of Height, IQ and "Achievement" Results.

The continuity with the "Fundamental principles in psychometrics and testing" reaching back to Binet and embraced by those wanting to treat intellectual capacity or ability as a heritable trait analogous to biological traits like height becomes clearer when the vendor states, "Norms help educators see if students are growing at an expected pace, regardless of where a student started" (NWEA, 2020). The implications for equity are demonstrated by the vendor's own analysis of scale score as a function of income (shown at the lower right of Figure 3; NWEA, 2018, p.8). Disaggregated results for other tests based on the same fundamental principles show a similar tendency to treat comparatively low (or high) outcomes for particular groups of students as normative and therefore expected.

Embretson's account of the rise of psychometrics elides the fact that Binet came to be deeply troubled by what was being done with his work in ways that fully anticipate the kind of results summarized in Figure 3. Binet observed that:

> Some recent philosophers seem to have given their moral approval to these deplorable verdicts that affirm that the intelligence of an individual is a fixed quantity, a quantity that cannot be augmented. We must protest and react against this brutal pessimism; we will try to demonstrate that it is founded on nothing. (1909, pp. 105–106)

The "brutal pessimism" that Binet identifies is centered on the idea that "intelligence" is a "fixed quantity" of an "individual."

Binet's concern notwithstanding, Embretson outlines how "the remainder of the last century of testing was devoted to applying or refining these principles" (p.5) in ways that are illustrated by Figure 3. Indeed, the educational psychologist Howard Gardner and his colleagues summarize the transformation by suggesting the "discovery of *g*," taken as "general intelligence as defined by psychometricians," and the "statistical analyses related to *g* are the most compelling findings and contributions of psychology" (1995, p.58). The contrast between possible measures of

student achievement and measures of "aptitude" now appears to have been collapsed, with achievement tests functioning as *de facto* measures of individual and, by extension, group intelligence.

Arthur Jensen (2000) embraces this possibility: "When the applicant pool is homogeneous in amount of prior schooling, a comprehensive scholastic achievement test can reflect *g* as well as a conventional paper-and-pencil IQ test" (p. 126). Jensen's point is that emphasizing the apparent "face validity" of "scholastic achievement" tests (e.g., an NWEA MAP or a STAAR math assessment made up of items that look to be about mathematics) "need not harm either construct validity or predictive validity" (p. 126) relative to measuring *g* or what Binet called the "fixed quantity" of the "intelligence of an individual" (1909, pp. 105–106).

Implications for equity are foregrounded in that Jensen "put forth his view that racial differences in intelligence were genuine, were due significantly to hereditary factors, and were not much susceptible to remediation or erasure" (Gardner et al., 1995, p. 153). Jensen and his colleague J. Philippe Rushton summarize what they see as the trajectory for a focus on hereditary factors:

> More generally, there is a need to educate the public about the true nature of individual and group differences, genetics, and evolutionary biology. Ultimately, the public must accept the pragmatic reality that some groups will be overrepresented and other groups underrepresented in various socially valued outcomes. Organizations such as the APA could play a critical role in changing the zeitgeist. (2005, p.284)

It is reasonable to be concerned that tests evaluated in terms of homogeneity may end up advancing the effort to convince "the public" of the genetic or biologically determined "reality" that "some groups" with be "overrepresented" or "underrepresented" in a widely array of "socially valued outcomes" in ways that may include representing current tests results as measures of college and career readiness.

The low sensitivity of legacy scale scores to instructional variation has been documented elsewhere (*cf.* Polikoff & Porter, 2014). Because items are selected for cross-item homogeneity and scaled to stable ability distributions, tests optimized in this way will, by design, be resistant to detecting instruction effects related to specific standards. PBIs, by contrast, are aligned with curricular standards and are designed to detect shifts in the patterns of thinking students bring to those standards, making instructional effects visible.

The equity implications of ability-based psychometrics come into sharpest relief when results are tracked over time. A system built on the assumption that relative rankings are stable will produce graphs that look like growth curves—each cohort moving upward at roughly the same rate, with gaps between groups persisting or widening. This is what Figure 3 illustrates.

Diversity and Attending to Qualitatively Distinct Understandings

Perhaps because Jean Piaget (1972) framed his work as developing a "genetic epistemology" rather than contributing to the kind of psychometrics that emerged from the work of Binet, Embretson did not take up the broader implications of Piaget's work for assessment (*cf.* Stroup & Wilensky, 2000; Stroup et al., 2011). Nonetheless, early in his career, Piaget worked with Binet as Binet was developing his approach to intelligence testing. Anticipating his development of a theory based on significant, age-related changes in children's reasoning, Piaget became attentive to how, at each stage of cognitive development, the child's thinking is qualitatively different. Without committing to specific aspects of his stage theory, pattern-based questions similarly focus on qualitatively distinct forms of reasoning (see Figure 1). Moreover, as reported in

previous work (Stroup et al., 2023), when items developed for the IAP were implemented across grade levels, the results did not follow the "growth" curves shown in Figure 3. Some results went up; others went up then down; and for one item related to volume, the partial credit scores went down from grade 5 to grade 8. Teachers had explanations for this decrease (e.g., over-dependence on formulas in later grades) in ways that are seen as related to features of instruction.

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LET'S END THE METHOD WARS AND INCITE A METHODS REBELLION!

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This paper articulates a vision for mathematics education research that moves beyond paradigmatic arguments and towards a place that grounds mathematics education scholarship in a different space using robust methodologies and methods. Utilizing innovative quantitative, qualitative, and mixed methods have substantive opportunities to explore mathematics education topics. For example, validation is one methodology that draws from quantitative, qualitative, and mixed-methods research practices (Bostic, 2026). Additionally, we argue that doctoral graduate education in STEM education must prioritize learning a variety of research methods to prepare scholars for the methodological hybridity (Edelen et al., 2026) needed for 21st century research.

Keywords: Research Methods

Historically, there has been an "ongoing tug of war" (Campbell et al., 2025, p. 348) related to methods and methodologies. Broadly speaking, quantitative research answers questions that examine "what," "to what degree," and "how" (Creswell & Guetterman, 2019; Gay et al., 2012). Qualitative research on the other hand explores 'how' and 'why' questions that reify phenomena (Creswell & Guetterman, 2019; Gay et al., 2012). Mixed-methods research blends the best of both areas in a variety of ways (Creswell & Guetterman, 2019; Gay et al., 2012) and came about as a response to the aforementioned tug of war. For decades, there have been civil methodological arguments, which stemmed from a movement from a previously dominant paradigm in mathematics education scholarship and towards a more pluralistic approach to studying mathematics education issues (Inglis & Foster, 2018). These arguments often missed a key facet: The best methods are those that effectively answer a research question (Inglis & Foster, 2018; Schoenfeld, 2016). It is precisely for this reason that this manuscript's goal is to end arguments about what methods are better and incite what we refer to as a methodological rebellion for mathematics education scholars to learn educational, and even social science, research methods more deeply.

An Abbreviated Review of Mathematics Education Scholarship Through Methods

Prior to the 1980s, quantitative research dominated mathematics education scholarship (Bostic et al., 2019; Schoenfeld, 2016). There was the start of qualitative research being published, which helped to uncover new ideas and issues that were previously underexplored. An early example of qualitative research that has helped to shape mathematics education scholarship is "The case of Benny" (Erlwanger, 1973). Piagetian and Vygotskian ideas were taken up in scholarship, as seen in *Journal of Research in Mathematics Education* (JRME) and *Educational Studies in Mathematics* (ESM) publications (Schoenfeld, 2016). Qualitative research provided scholars with new methods and approaches that led to insights absent from quantitative research. The 1990s continued the trend with more qualitative research found in mathematics education publications. The 2010s had a small but growing group of scholars engaging in mixed-methods research (Campbell et al., 2025) that developed into the robust, grounded field that it is today.

Campbell and co-authors conducted an analysis of the JRME and ESM between 2014-2023. JRME is the flagship journal for the National Council of Teachers of Mathematics thus having a large North American presence. ESM is not associated with an organization yet it was started in

Europe. The ESM sample contained 462 empirical articles whereas JRME had 125 articles. Most of the ESM articles were qualitative in nature (52%), with mixed-methods coming in second (28%), and quantitative research was found the least (20%). Said another way, 241 qualitative studies were published compared to the 91 quantitative studies an almost 3:1 ratio! JRME had a more balanced profile with 38% of the sample defined as qualitative research, 35% of the articles had quantitative research, and 27% drawing on mixed-methods research. Looking at the research methods by global region highlights a similar trend. ESM authors in Europe, North America, South America, Sub-Saharan Africa, as well as Australia and Oceania tend to use qualitative methods more frequently than quantitative methods. JRME authors portray a more balanced profile with authors from these same regions demonstrating a balance of quantitative and qualitative methods. It was rare to observe within a region when mixed methods were leveraged as much or more than quantitative and qualitative methods.

A similar scoping study was done with integrated STEM education research to give some contrast to mathematics education scholarship. This study included 64 academic journals and a total of 746 empirical articles between 2020–2025 for the sample (Edelen et al., 2026). Results indicated that 52% of the scholarship was quantitative (n=392), 32% was qualitative (n=238), and 16% was mixed methods (n=116). Much of the quantitative research used a pre-/post-test approach that can omit variance during an intervention and very few used an experimental or quasi-experimental design. Quite a bit of the qualitative studies in this sample employed interviews alongside thematic or content analysis. These approaches are good but highlight a lack of methodological diversity across the article sample, which suggests that some richness from other methodological perspectives may be lost (e.g., ethnography, grounded theory, discourse analysis, and critical pedagogy). Among mixed-methods research, surveys and achievement data were the most frequently used data collection tools. Self-report data have been shown to be problematic (Akbulut, 2025), which necessitates careful consideration.

Taken collectively, this abbreviated review signals both growth and constraint within mathematics education and, and more broadly STEM education scholarship. The field has expanded methodologically across the past five decades, moving from quantitative dominance toward greater inclusion of qualitative and, to a lesser extent, mixed-methods approaches. Yet, patterns within publication trends suggest that methodological diversity remains uneven, with certain designs and analytic traditions taken up far more frequently than others. While the knowledge base has deepened, there remains missed opportunities to capture the full complexity of phenomena. The scholarly landscape underscores the need for methodological plurality and moreover, for scholars to be prepared to engage, combine, and extend methods in ways that illuminate dimensions of mathematics education that might otherwise remain obscured.

Mathematics Education Methods Preparation in Brief

A slightly different lens on mathematics education research is to examine scholars preparation in graduate school. Such examinations may illustrate potential opportunities. There has been some scholarship on doctoral mathematics education suggesting that 75% of the recent doctoral mathematics education graduates complete no more than two quantitative research courses during graduate school (Shih et al., 2019). Typically, descriptive statistics, ANOVA, and single-variable regression are included in such courses. Experimental designs are generally explored in later courses. On the other hand, the number of qualitative courses taken by mathematics education graduates is uncertain but over 90% of the sampled doctoral graduates felt their preparation in qualitative research methods was “adequate” or “very well adequate”. It was unclear what was in those courses; potential paradigms and philosophies may be overlooked.

Integrating those findings with Edelen et al.'s (2026) prior analyses of STEM education scholarship suggest that much published qualitative work relies heavily on interviews coupled with thematic or content analysis, pointing to a narrower range of qualitative approaches than the field might warrant. This raises questions about whether perceptions of adequacy align with the depth, rigor, and methodological diversity needed to sustain and extend qualitative inquiry in mathematics education. As Shih's team wrote about doctoral graduates' methodological preparation, "This finding is worrisome, as the lack of foundation in research methods is likely to several limit research activities" (2019, p. 24).

If mathematics education scholarship seeks to build upon its extant foundation, as well as leveraging findings from fields that also conduct mathematics education research (e.g., psychology, integrated STEM education, policy, and assessment), then it is absolutely necessary that graduates are fully prepared to confidently understand at least one area, which may allow them to design, conduct, and report results from their research. And if this was not concerning enough, these findings do not report on doctoral graduates' understanding of mixed methods; an area that has grown tremendously in the last decade. For these reasons, it may be useful to reflect further on what is important in graduate education and to foster longevity of mathematics education. One viable outcome is to consider a methodological rebellion aimed at strengthening how graduates and current scholars are prepared to engage with educational research methods.

A Proposal: Let's Become Rooted in Educational Research Methods

Mathematics education scholarship is wide and deep, which is both an opportunity and a challenge. There is no possible way for a scholar to know all of it deeply; however, it is fair to expect a scholar to know at least one area deeply and to have a broad understanding of major mathematics education topics (e.g., teacher education and student learning) in some capacity. Similarly, it is not feasible to know all research methods deeply, as with mathematics education topics, a scholar could capably know quantitative, qualitative, or mixed methods deeply. Lacking expertise in one or more area opens space for collaborations with others who might be an expert in that area. Building collaborations with research methodologists helps to synergize different forms of expertise. A mathematics education researcher, working alongside a methodologist, should feel confident collaborating and be able to verify and/or understand the methods and outcomes from such work. Leaving interpretations or evaluations entirely to a methodologist may lose the richness that could only be found from a mathematics educator who can interpret results in context.

Hence, we argue that scholars in mathematics education must cultivate both depth and discernment when it comes to research methodologies and methods. From our perspective, depth anchors a scholar's intellectual identity in research; discernment allows one to engage responsibly beyond one's primary area of expertise. When collaborating with research methodologists, mathematics education scholars should not cede their epistemic, or even ontological responsibility. Rather, we should enter dialogue in which methodological rigor and disciplinary insights inform one another in a symbiotic relationship. A mathematics educator should be able to interrogate methodological choices, understand analytic decisions, and interpret findings in light of the histories, practices, and logics of mathematics classrooms. Without such a reciprocal engagement, methodological precision risks becoming detached from educational meaning in mathematics education research. Productive collaborations, then, move beyond divisions of labor to being co-constructed spaces of inquiry in which methodological expertise and mathematics education knowledge work together to generate claims that are both technically sound and educationally illuminative. Building from this stance, recent scholarship in STEM

education has begun to more explicitly name what methodological depth and discernment might be helpful for future STEM education research. Edelen and co-authors (2026) advocated for a greater understanding of the differences between methodology and method, which could foster more diversity of qualitative, quantitative, and mixed-methods approaches. A goal is to embrace the methodological hybrid nature to capture the complexities of learning, which means building greater methodological preparedness among scholars and working collaboratively. QuantCrit is one approach that is often ignored in STEM education (Young et al., 2025). Validation (Bostic, 2026) draws upon quantitative, qualitative, and mixed-methods approaches. Ethnographic work draws on interactional and historical analyses to examine how learning is co-constructed in practice and to focus on how learners make meaning visible (Walford, 2020).

Yet, this vision raises an important question: Are we, as mathematics education scholars, being prepared, preparing others, and preparing ourselves to participate meaningfully in such collaborations? More directly, do we possess the depth and discernment necessary to engage in methodological discussions as intellectual partners in the shared construction of knowledge?

Final Thoughts

One goal from this manuscript is (a) to end arguments trying to prove one research method is better and (b) to give some overview of mathematics education scholarship. A second goal is to highlight the opportunity for current scholars—especially graduate students and post-doctoral scholars—to value all methodologies and methods and to become deeply knowledgeable with the diversity and hybrid approaches to educational research. By deepening knowledge of methodologies and methods, there is potential for generating and leveraging modern approaches that effectively capture variance and/or wholly capture the essence of a phenomenon.

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CONCEPTUALIZING AND APPLYING A TASK ANALYSIS FRAMEWORK

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We applied a task analysis framework that characterized tasks according to their ambiguity and potential to make connections, in the process providing nuance with respect to mathematical activity and tools entailed by the task. We proposed four forms of mathematical activity that increased in their ambiguity and potential to make connections and created a matrix that detailed descriptions of these activities according to different types of mathematical tools. Based on middle school mathematics topics, we identified three types of tools: mathematical terminology and formulas, procedures and algorithms, and representations. We analyzed tasks from three different curriculum programs to test out the framework and to see what insights could be generated by its use. The goal is to provide specificity when analyzing tasks that has face validity for both researchers and practitioners.

Keywords: Curriculum, Mathematical Representations, Instructional Activities and Practices

We propose a framework to characterize tasks in middle school mathematics curriculum. The proposed framework builds from and expands on task analysis framework of Stein et al. (2009) that has been widely utilized to characterize mathematical tasks, especially in the US context (cf. Cai et al. 2009; Tekkumru-Kisa et al., 2020). The Stein et al. framework has four levels of task ratings, of which two are considered low cognitive demand and two are considered high cognitive demand. The low cognitive demand categories are memorization and procedures without connections, while the high cognitive demand categories are procedures with connections and doing mathematics. We argue additional nuance is needed with respect to the nature of mathematical activity and tools entailed by tasks, especially for secondary mathematics content. We describe the tenets of our framework and present results from the analyses of several curriculum programs. The potential contributions of the paper are methodological and theoretical in relation to studies of curriculum design and enactment.

Our framework builds from the perspective that curriculum materials entail two communicative functions, monologic and dialogic (Choppin et al., 2015; Choppin et al., 2022). We argue that tasks that are routine mostly adhere to the monologic communicative function while tasks that are non-routine mostly adhere to the dialogic communicative function. Non-routine tasks have the potential to generate idiosyncratic and varied approaches or ways of reasoning; these different approaches or ways of reasoning in turn afford opportunities for students to generate meaning (e.g., thinking devices) when their differences and similarities are made explicit and provide opportunities to develop new knowledge in a community. We characterize non-routine tasks largely by their ambiguity, their potential for making connections, and their potential for multiple approaches. Instead of the term cognitive demand, we use notions of ambiguity and connections to distinguish between routine and non-routine tasks. Below, we elaborate the framing around the dialogic potential of tasks and then present the framework.

Proposed Task Analysis Framework

We extrapolated two essential characteristics that distinguish low and high cognitive demand categories in the Stein et al. (2009) framework. The first essential characteristic is the ambiguity of the task, especially in terms of whether an approach or formulation is specified. The lack of

specificity of an approach invokes the need for strategic flexibility so that students must make decisions about what concepts and processes are relevant to a task (Doyle, 1983; Stein et al., 2009). The second essential characteristic is the potential for students to make connections; the two primary types of connections are (1) connections between procedures and concepts and (2) connections between different types of representations (Stein et al., 2009). The categorization we propose emphasizes these characteristics and connects them to forms of mathematical activity and mathematical tools entailed by tasks, which provides greater specificity when characterizing learning opportunities tasks afford, especially regarding secondary mathematics content.

We define mathematical activity as a process by which students engage with mathematical content represented in tasks. We propose the following four activity categories that entail increasing ambiguity and potential for connections: following established processes and recalling terminology, applying mathematics to contexts, demonstrating flexible use of tools, and generating conceptual understanding. The category following established processes and recalling terminology applies to tasks with minimal ambiguity or potential for connections in which students are expected to recall or use established procedures or terminology. The category applying mathematics to contexts applies to tasks that require students to apply or describe terminology or procedures in ways that require some interpretation of the context. The category demonstrating flexible use of tools applies to tasks that involve choices regarding terms, procedures, or tools that can be used, requiring students to make strategic decisions about an approach. The category generating conceptual understanding applies to tasks that involve developing/applying terminology or procedures in novel ways or explaining novel relationships between concepts. We use the term novel to indicate engagement with mathematical content in ways that students have not previously explored or encountered or may be unconventional.

In addition, we identified the mathematical tools that are embedded in tasks to add specificity to our descriptions of mathematical activity, with different tools entailing different ways students engage with mathematical content for each type of activity. These tools, based on middle school mathematical content, include mathematical terminology and formulas, procedures and algorithms, and representations. Identifying tools allows us to provide more fine-grained descriptions of the ways tasks influence how students engage with mathematical content. See Table 1 for a matrix of the intersection between mathematical activities and mathematical tools.

Table 1: Matrix of Mathematical Activity and Mathematical Tools

	Mathematical Activity (increasing ambiguity and potential for connections from left to right)			
Mathematical Tools	Following established processes and recalling terminology	Applying mathematics to contexts	Demonstrating flexible use of tools	Generating conceptual understanding
Mathematical terminology and formulas	Memorizing or recalling basic facts, definitions, or formulas.	Applying or explaining facts, definitions, or formulas in new contexts	Selecting or explaining facts, terminology, or formulas when multiple possibilities exist.	Developing or applying terminology or formulas in novel ways OR explaining novel relationships between concepts
Procedures and algorithms	Accurately and efficiently carrying out established procedures OR explaining how to do so.	Applying or explaining established procedures in new contexts.	Selecting or explaining procedures or algorithms when multiple possibilities exist.	Developing or applying procedures or algorithms in unconventional or novel ways.
Representations	Generating representations with a well-established method OR identifying information in a representation without having to interpret a context	Generating a representation that involves a context OR identifying information from a representation that requires interpretation of the context.	Identifying or explaining mathematical relationships from representations when multiple possibilities exist OR generating representations when multiple possibilities exist.	Developing or using representations in novel ways OR interpreting representations in unconventional or novel ways.

In Table 1, the level of ambiguity in mathematical activity increases from left to right, with the leftmost two forms of mathematical activity involving minimal ambiguity and are considered routine. By contrast, the rightmost two involving higher levels of ambiguity and are considered non-routine. We define ambiguity in terms of a lack of prescribed approaches, such as a procedure previously modelled or terminology that has been formally defined. Ambiguity entails having flexible options from which to choose in terms of tools, requiring students to make and explain choices regarding their approach. We also note that the rightmost columns can subsume some of the properties of the leftmost columns, such as when problems involve applying

mathematics to contexts in ways that involve more ambiguity than what is indicated in the applying mathematics to contexts column.

Applying the Framework

We applied the framework to characterize tasks in three different curriculum programs. Using the cells in Table 1 as a codebook, we coded at the level of task. We delineated a task when there was a distinct set of requirements from tasks that preceded or followed. Occasionally, we grouped tasks if there was a set that had a closely related set of requirements (e.g., students were expected to do the same thing or very similar thing with each task in the set). To maintain the exclusivity of the categories for mathematical activity, we required that each coder only apply a single activity code. The activity categories are what distinguishes tasks according to their ambiguity and potential for connections, which are the essential task characteristics identified in our framework as well as the Stein et al. (2009) framework. The dimension of mathematical tools allows us to create fine-grained descriptions of mathematical activities for each type of tool, providing a greater likelihood that the descriptions have face validity with researchers and practitioners. For mathematical tools, we allowed coders to apply up to two tool codes after we observed that many tasks employed multiple tools according to our descriptions. To achieve consensus, we met synchronously via Zoom to reconcile differences. We began by reconciling the mathematical activity and then reconciling the tool codes, an order that mirrored the prioritization of mathematical activity as the more important distinguishing characteristic. We iteratively revised the table over the course of reconciling the first two curriculum programs to increase clarity and comprehensiveness, leading to the version in Table 1.

Discussion

We developed and applied a curriculum framework to provide greater specification of the nature of tasks in contemporary curriculum materials at the secondary level. The framework employed a codebook (Table 1) to engage in low-inference coding of tasks so that the results would have face validity to both practitioners and researchers. The consensus process demonstrated the need for ongoing refinement of Table 1 and led to productive discussions that we hope will lead to increased reliability and confidence when applying the framework.

The mathematical activity categories provide important distinctions between mathematical processes and tools tasks require of students, especially in relation to secondary mathematical content. Ideally, the detailed descriptions in cells resonate with practitioners to inform decisions regarding curriculum programs and the kinds of mathematical experiences they envision for their students. Furthermore, the framework could serve as an impetus for mathematics educators to discuss the optimum combination of the activity/tool pairings, the extent to which tasks should involve ambiguity, and the desired level of ambiguity across a set of tasks. The analysis revealed some homogeneity between programs that may have been the result of analyzing only commercially developed programs. In future analysis, we will include at least one NSF-funded program to possibly draw distinctions between types of curriculum programs.

The framework builds on the notion of non-routine nature of tasks as a mechanism to emphasize their ambiguity and potential for connections. When tasks are ambiguous or offer multiple approaches, they provide opportunities for students to collectively negotiate the meaning of the relationships between approaches and the mathematics that makes them similar or different, which we characterize as dialogic potential. Curriculum programs that are comprised of primarily non-routine tasks are communicating that mathematics is an objective and established body of knowledge that has binary forms of understanding or performance, while programs that have a substantive number of non-routine tasks communicate that knowledge is

collectively determined by the community in which that knowledge is discussed and used. These approaches involve monologic forms of communication (e.g., transmission from one person or group to another) or dialogic forms (e.g., knowledge is created by the community that uses it, within some bounds of convention). Both approaches have their uses; the decisions educators face is what combination of these approaches is ideal or desired.

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CURRICULUM ADOPTION PROCESSES IN RURAL MATHEMATICS MIDDLE GRADES

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We interviewed 15 rural K-12 administrators or instructional leaders with responsibilities related to mathematics education in their school or district. We asked these educators about how curriculum programs were adopted in their districts to understand the rationale for selection and the processes by which curriculum programs were selected. We identified five themes: a range of stakeholders were involved in the adoption process; the criteria for adoption included state standards, internal criteria, and external rubrics; the typical length of adoption cycles was three to seven years; there were processes that informed the adoption process; and vendors had a role in the adoption process. We summarize these themes and explain their implications for learning opportunities for middle school students in rural areas.

Keywords: Curriculum, Middle School Education, Policy, Standards

Mathematics curricula exert a considerable influence on learning opportunities for students, given the role they play in teacher lesson planning and subsequently the nature and sequencing of tasks students encounter (Remillard & Heck, 2014). Consequently, the choice of a curriculum program plays an important role in students' mathematical education. Research exploring the impact of the Common Core State Standards for Mathematics (CCSSM) showed how adopted curriculum programs were a stronger influence than the CCSSM on the mathematical activities and discourse observed in middle school classrooms (Choppin et al., 2022).

Rural contexts pose unique challenges to educators relative to urban or suburban areas, given their geographic remoteness, smaller populations, and limited resources (Irvin et al., 2012; Jimerson 2005; Monk 2007). To better understand how these factors impact learning opportunities for students in rural-serving schools, we explored the curriculum adoption process in rural mathematics middle grades to better understand the rationale for choosing which curriculum programs to adopt and the processes by which the programs were chosen.

This study was part of a larger study that is exploring the landscape of rural mathematics education in grades 5 to 9. In Phase One of the study, we surveyed middle grades teachers and instructional leaders. We recruited a subset of the teachers and instructional leaders from the survey to participate in semi-structured interviews that included questions on curriculum adoption processes. In the summer and fall of 2025, we interviewed 15 instructional leaders; these interviews form the basis of the analysis for this proposal to answer the research question: How are mathematics curriculum programs selected in rural contexts?

Background and Framing

Rural Mathematics Education

Rural schools face numerous challenges due to their geographic isolation from other communities and larger urban areas (Saw & Agger, 2021). Teachers in rural mathematics classrooms frequently begin their careers with limited preparation tailored to the specific demands of rural schools and mathematics teachers, more broadly, in rural schools have fewer opportunities to attend content-specific professional development and fewer colleagues with whom to collaborate (Cady & Rearden, 2009; Dille & Røkenes, 2021; Şahin et al., 2024).

Financial challenges mean that some rural schools have reduced access to technology and STEM educational opportunities (Bouck, 2004). This negatively affects the courses available to students (Anderson & Chang, 2011), resulting in rural students having less access to highly credentialed teachers and advanced math courses (Monk 2007; Saw & Agger, 2021).

Mathematics curriculum plays a particularly important role in rural contexts, where schools often serve as central community institutions (Villa & Knutas, 2020). Curriculum can provide coherence, ensure equitable access to high-quality mathematical experiences, and create meaningful connections between mathematics and local community contexts. In rural settings, curriculum becomes a critical structure for supporting teacher capacity, sustaining instructional quality, and promoting long-term educational opportunity for students (Carrete-Marín & Domingo-Peñafiel, 2022; Carrete-Marín et al., 2024).

Curriculum Adoption Processes

Curriculum adoption processes in rural schools face unique challenges due to contextual factors such as a lack of acceptance of reform-based pedagogical practices, and standardized content across school locales (Howley & Howley, 2005). Curriculum adoption cycles can be altered due to large-scale curriculum changes, such as No Child Left Behind (Powell et al., 2009) and the CCSSM (Allensworth et al., 2021). More recently, the focus on high-quality instructional materials (HQIM) is prompting districts to consider curricular changes. These efforts are intended to ensure that all students have access to rigorous curricula and instructional practices; however, the variation in curricular resources, access to professional learning, and implementation methods can lead to inconsistent results (Polikoff, 2012).

Methods

Data were gathered from 15 rural school leaders from Ohio, Virginia, South Dakota, Georgia, Louisiana, Tennessee, Washington, and Kansas. Data were collected through semi-structured interviews that lasted approximately 45 to 60 minutes and were intended to support understanding of the affordances and challenges of mathematics education in rural spaces. Interview data were transcribed, participants were assigned pseudonyms, and the instructional leader interviews were coded at the level of turn by themes, using an inductive coding process (Vears & Gillam, 2022). Using this process on the responses to all interview questions, we generated a list of codes used to sort the interviews into topics, one of which was *curriculum adoption processes*. Given the research question of this paper, we collected all quotes related to *curriculum adoption processes* into a single document and then used a multi-phase process to generate themes (Saldaña, 2013). First, we created low-inference memos for each quote to reduce the text while preserving the main ideas. This supported the second phase, in which we grouped the low-inference memos into *sub-themes* to reflect common ideas expressed in the memos. Third, we grouped these sub-themes into larger *themes*, which we reduced to a parsimonious but comprehensive set that captured the key phenomena evident in the sub-themes.

Results

We identified five themes: (a) a range of stakeholders were involved in the adoption process; (b) the criteria for adoption included state standards, internal criteria, and external rubrics; (b) the typical length of adoption cycles was three to seven years; (d) there were processes that informed the adoption process; and (e) vendors had a role in the adoption process. Below, we report on these themes and then discuss their connections and implications.

Stakeholders Involved in the Adoption Process

Nine of the 15 participants reported that their curriculum adoption committee included a range of stakeholders, including parents, community members, instructional leaders, and teachers. Two specifically mentioned community members and parents as being involved, while another mentioned the need to attend to the community's conservative values. Three of the participants emphasized the role of teachers in the adoption process, stating that teachers were primary decision makers, while eight others mentioned teachers as playing an important role. One participant stated, "if you teach math, you get a seat at the table in determining what's best for your students," while another stated, "we definitely use teacher voices a ton in order to support what purchases we make." Four participants stated they sought School Board approval or informed the School Board of their choice. Typically, the composition of the adoption committees included teachers from each grade level, mathematics specialists, directors of curriculum, and building-level administrators.

Criteria for Adoption

The participants listed three types of criteria used to adopt curricula: (a) rubrics, generated locally or by another entity, and criteria reported by EdReports (EdReports.org, 2026); (b) state standards; and (c) internal criteria that reflected the values or instructional philosophies of the teachers. Four participants reported rubrics to evaluate programs; in three cases, the rubric was internally generated, and in one case, the participant used the Equip Rubric (Achieve, 2013). Three participants reported using EdReports to identify programs, including one who stated: "I usually go to EdReports to see what curricula are scoring where, and then I usually get samples … I usually look at the ones that score all green." Five participants reported using state standards as a criterion to select programs. One participant stated, "I think the constant driving forces are those curricular materials meeting state standards, that's the that's the biggest thing that drives those decisions." Another administrator reported that they only considered curriculum programs that were "fully approved by Virginia for alignment" with their standards. Three participants reported using internal criteria to select programs. One participant reported that differentiation was an emphasis, stating, "We have students because we are a smaller school, of varying backgrounds and varying abilities in one classroom, so making sure that the material is not a sit and get." Another stated, "We want kids to be able to manipulate things, to learn math … So that's a piece that … guides us. What does that look like for this particular company?"

Length of Adoption Cycles

Participants reported three-to-seven-year cycles. The length of the cycles added extra importance to the adoption process. One participant stated, "it's a pretty big investment for us, and we want to make sure that we're going to be happy for five years." Two participants explained that their adoption cycles were aligned with the typical period for revising state standards, with one saying that their cycle was "generally based on when the [state] standards get updated."

Processes that Informed the Adoption Process

The two processes reported by participants included a pilot period in which they tested out programs for one or more years before adopting, and an ongoing evaluation process for their curriculum programs in advance of the next adoption cycle. Five participants reported piloting two to three programs for up to two years before the adoption decision. One participant described the piloting process, "We piloted two programs … and then the pilot teachers met, and we also had some community members that came joined us to score the programs." Three participants

reported evaluating the effectiveness of their current programs between adoptions. One participant reported:

> Throughout the year, we reflect on and say, are we using these? How are we using them? When are we using them? Are they using them daily, things like that? … We start to think, you know, how did these materials prepare our students?

Another participant described ongoing needs analysis, saying that they "look at the data in regard to how our students are performing. So, what have we noticed where we're weak at standard wise? What have we noticed skill wise? Where have we noticed that that we have those gaps?"

The Role of Vendors in the Adoption Process

Two participants stated that their states provided a list of approved vendors that indicated which curriculum programs the district could consider, with one participant stating, "You have to use one of their vendors; they don't care who you choose, but it has to be one that's been vetted through their office." Four participants reported that vendors met with teachers and instructional leaders to present their programs. They reported that they met with multiple vendors in one day, either at an external site or even via Zoom. One participant described how the state educational service agency would put on "textbook fairs" where the staff could "go to one place and see three different curriculum presentations" by vendors. Another participant reported:

> We invite the reps to Zoom with our staff to get to know them. We might have those Zooms all on the same day on an in-service day, so we can compare some things, contrast some things. They kind of take us through their one-hour, one-and-a-half-hour kind of sales pitch.

A third participant reported that, "The [state-approved] vendors … come to each school. Each school has a representative, and these representatives go and have a meeting with the district supervisor and kind of hash out the pros and the cons."

Discussion

Given the unique challenges of education in rural contexts (Irvin et al., 2012; Jimerson 2005; Monk 2007), the five identified themes add to existing literature on mathematics curriculum adoption processes in rural contexts. Results from leaders in eight states show variation in curriculum adoption processes, including piloting programs before adoption, meeting with vendors, and conducting ongoing evaluation of current curriculum programs, all within their rural areas. Based on existing research that mathematics curriculum strongly influences students' learning opportunities (i.e., Remillard & Heck, 2014), the variation in processes reported suggests possible variation in potential learning outcomes for students in rural contexts. Some participants reported some state control over the process, such as creating a list of preferred or vetted vendors that restricted the choices of curriculum programs or requiring that the curriculum programs aligned with the state standards, which speaks to the power of the decisions made at the state level, knowing that adopted curriculum programs are often a stronger influence than state or national level standards (Choppin et al., 2022). In cases where stakeholders were the primary decision-makers, the values of the rural community were considered in the selection process, thereby elevating the contextual needs of those communities. Those selecting materials should be cognizant of the relationship between decision makers and the community needs, and the unique challenges in rural spaces (Howley & Howley, 2005) as the ultimate decision is consequential for student learning opportunity (Remillard & Heck, 2014).

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DISTINGUISHING SELF-CONCEPT AND SITUATIONAL INTEREST IN COLLEGE MATHEMATICS: A VALIDITY EVIDENCE STUDY

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Math self-concept and situational interest represent two central psychological processes that influence achievement-related outcomes, as well as future STEM career choices. Although prior research has developed or used survey instruments to assess these constructs and has documented various sources of validity evidence, the adaptation of such measures to new disciplinary contexts and combining the two constructs requires renewed validation efforts. As part of that process, the main study goal is to gather the following validity evidence for the two scales measuring math self-concept and situational interest in math: test content and internal structure. Results provide empirical support for sources of validity evidence, indicating that the two scales can be used to accurately measure the underlying constructs of math self-concept and situational interest in undergraduate calculus.

Keywords: Calculus, Undergraduate Education, Validity Evidence

Achievement-related outcomes in mathematics are shaped by students' conceptions of their abilities, termed as mathematics self-concept and their interest in the subject (Ding et al., 2024; Geisler et al., 2023; Nazeri, 2016; Niepel et al., 2022; Pietsch et al., 2003; Valentine et al., 2004). Interest, which can be context-based (situational interest) or long-term (individual interest), reflects both the psychological state during engagement and the likelihood of re-engagement with a given content (Hidi & Renninger, 2006). Beyond achievement, self-concept and interest in mathematics influence long-term STEM participation (Ferdinand et al., 2024; Lu et al., 2026; Parker et al., 2014; Zhou & Shirazi, 2025). These constructs frequently manifest concurrently and mutually influence one another in learning environments (Marsh et al., 2005), yet to our knowledge they are often measured separately, and few instruments examine both together in university mathematics contexts. Therefore, in this brief report, we seek to gather validity evidence for measurement models that combine these two psychological processes.

This study investigates the validity of two survey scales measuring math self-concept and situational interest in a face-to-face Calculus I course. Our research question is: What sources of validity evidence support the use of math self-concept and situational interest scales as acceptable instruments for measuring students' perceptions of their mathematics self-concept and their context-based interest in face-to-face Calculus I?

Theoretical Perspective and Prior Research

This study is part of a larger research project examining relationships among classroom climate, math self-concept, and situational interest in college Calculus I and Business Calculus courses. Guided by theories of self-concept and interest development, as well as the *Standards for Educational and Psychological Testing* (thereafter, the *Standards*; American Educational Research Association [AERA] et al., 2014), this analysis examines two sources of validity evidence, including test content and internal structure to determine whether these constructs are empirically distinguishable and accurately measured within this instructional context.

Mathematics self-concept is a well-established construct grounded in the multidimensional, hierarchical model of self-concept (Shavelson et al., 1976) and supported by research

distinguishing its cognitive and affective components (Bong & Clark, 1999; Bong & Skaalvik, 2003; Hammoudi, 2020). Within this framework, math self-concept reflects individuals' perceptions of their mathematical ability and includes an affective dimension (students' feelings about their competence) and a cognitive dimension (students' awareness and evaluation of their mathematical knowledge and skills). These perspectives inform measurement tools such as the Self-Description Questionnaires (e.g., SDQ III; Marsh & O'Neill, 1984; Rubie-Davies & Lee, 2013), PISA math self-concept scales (OECD, 2013; Ding et al., 2024; Ramazan et al., 2023), and cognitive/affective subscales (Hammoudi, 2020).

Situational interest, defined within the four-phase model of interest development, captures context-dependent engagement driven by triggered interest, maintained value, and maintained feeling (Hidi & Renninger, 2006). Linnenbrink-Garcia et al. (2010) drew from this theoretical perspective and developed a classroom-based measure assessing triggered situational interest (TSI) and two components of maintained interest, which are maintained-situational interest feeling (MSIF), and maintained-situational interest value (MSIV).

To ensure accurate score interpretations as recommended by the *Standards* (AERA et al., 2014), the present study focuses on validity evidence based on test content and internal structure. Evidence based on test content can be collected by asking experts to judge the representativeness or appropriateness of the set of items to measure the constructs in the study (Knapp & Mueller, 2010). Statistical techniques such as confirmatory factor analysis (CFA) and invariance testing provide evidence based on internal structure. Consistent with Lewis (2022), measurement invariance testing also supports equitable score interpretation by evaluating whether an instrument functions similarly across participant groups. Prior studies have gathered some of those sources of validity evidence for the math self-concept subscales (cognitive and affective) used here (Ding et al., 2024; Pietsch et al., 2003) and for the situational interest scale (Linnenbrink-Garcia et al., 2010) in either pre-tertiary educational contexts or in different courses. Therefore, combining two subscales that were developed and used in different contexts to form a new math self-concept scale will require gathering sources of validity evidence to support the specific use of the assessment instrument. Similarly, the situational interest scale was developed in domains including undergraduate psychology course, and middle and high school math class; therefore, an adapted version of the scale for use in undergraduate math courses will require collecting sources of validity evidence. The main hypothesis is that combining measures of math self-concept and situational interest in math will result in a scale that demonstrates evidence based on test content and internal structure, including support for equitable score interpretation across participant groups.

Research Methods

Participants

The study included 385 undergraduates, representing 60% of students enrolled across three coordinated Calculus I courses taught by three instructors during Fall 2025 at a research university on the West Coast of the United States. All demographic information was self-reported; the largest racial/ethnic groups were Hispanic/Latinx (29%), White/Caucasian (25%), and Asian (19%). Participants were primarily engineering majors (63%) and identified as men (61%), women (38%), or non-binary/other (1%). Students reported being continuing-generation (67%) or first-generation (32%), with the remaining 1% selecting another category. For invariance testing, gender was analyzed using binary groups (men vs. women) and generational status using first-generation vs. continuing-generation categories. Test-content evidence was obtained from eight mathematics education and educational psychology experts.

Materials and Data Analysis

For this study, math self-concept and situational interest were measured using five-point Likert-scale items (1 = *Strongly disagree* to 5 = *Strongly agree*). The math self-concept scale included cognitive items (CSC, 5 items; taken from PISA 2003/2012; Ding et al., 2024) and affective items (ASC, 3 items; Pietsch et al., 2003). The situational interest (SI) scale, adapted from Linnenbrink-Garcia et al. (2010) by replacing "year" (used in their pre-tertiary contexts) with "semester" for our college setting, was used to assess TSI, MSIF, and MSIV. Demographic information (e.g., gender, first-generation status) was also collected, given evidence of group differences in mathematics experiences (Sax et al., 2015; Snodgrass Rangel et al., 2020).

To collect test content validity evidence the expert interviews were analyzed. This process led to the identification of relevant or irrelevant items for the constructs. To gather internal structure validity evidence, confirmatory factor analyses (CFAs) tested theoretically grounded models (see Figure 1). Analyses were performed in R version 4.3.3 (R Core Team, 2024) using the *lavaan* package (Rosseel, 2012). Item-level missingness was low (0.78%-1.30%), and all indicators demonstrated acceptable distributional characteristics for robust maximum likelihood estimation (MLR; West et al., 1995). Accordingly, CFAs were conducted using MLR estimation with full information likelihood, which is appropriate for five-category Likert items and missing-at-random data (Enders, 2010; Rhemtulla et al., 2012). Because models were estimated using MLR, robust fit indices were reported for all CFA and measurement invariance models (Savalei, 2014). Data-model fit was evaluated for the four CFA models shown in Figure 1 (Models A–D). Fit-index criteria followed common recommendations for MLR estimation: CFI ≥ 0.95 (good) or ≥ 0.90 (acceptable), RMSEA ≤ 0.05 (good) or ≤ 0.08 (acceptable), and SRMR ≤ 0.08 (Hu & Bentler, 1999; Mueller & Hancock, 2019). The likelihood ratio χ^2 values, factor loadings, and factor correlations were also examined. Finally, additional internal structure validity evidence was collected through measurement invariance across gender and college generational status. Invariance testing evaluates whether the survey scales do not function differently across groups. The second-order 2-factor model was evaluated for configural (same factor model across groups), metric (equal loadings), and scalar (equal intercepts) invariance testing. Evidence of non-invariance was evaluated using same fit indices for CFAs, and changes ΔRMSEA > 0.015 (Chen, 2007) and ΔCFI > 0.01 (Cheung & Rensvold, 2002). Additional details regarding data file, participant characteristics, and R analysis code are available at: 10.17605/OSF.IO/W8SDR

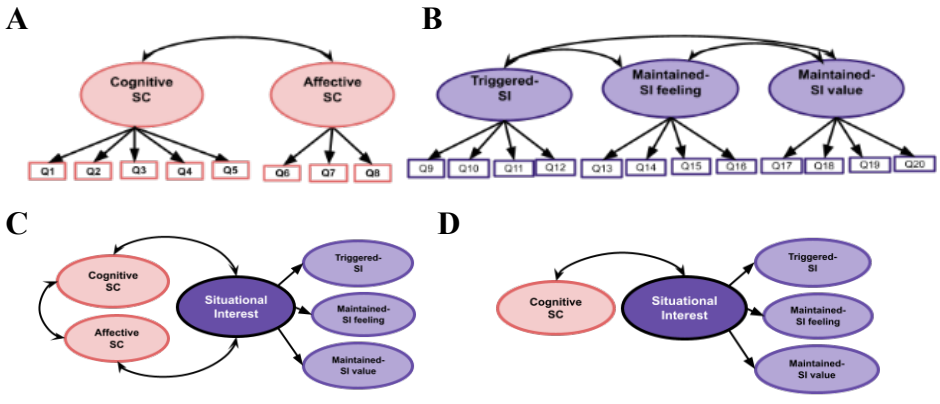


Figure 1: A) Two-factor Math Self-concept (orange), B) Three-factor Math Situational Interest (purple), C) Second-order Three-factor, and D) Second-order Two-factor Models

Results

In the process of gathering test content validity evidence, experts agreed to some extent that items reflected the intended constructs. One expert noted potential overlap between ASC and maintained feeling interest items within SI: "I would also be concerned that the Affective self-concept items will cross-load with the interest items, especially the item with "enjoy" in it".

Table 1 summarizes CFA results. The 2-factor model of math self-concept and three-factor model of situational interest demonstrated acceptable data-model fits. Of interest, the two second-order models also demonstrated good data-model fits; however, because those competing models were non-nested, model comparisons were based on descriptive fit indices rather than chi-square difference testing (Kline, 2023). The second-order model including ASC showed acceptable fit, but the strong correlation between ASC and SI (r = .81) suggested substantial overlap between the constructs. Consequently, the more parsimonious model D (CSC + SI) demonstrated the best overall fit and was retained as the preferred representation of the data.

Table 1: Data-model Fit for CFA Models with MLR Estimation

Model	Scaled χ^2	CFI	RMSEA	SRMR
A) 2-factor Math self-concept	63*	**.957**	.088	**.040**
B) 3-factor Math situational interest	134*	**.974**	**.071**	**.035**
C) Second order 3-factor model of SC & SI	364*	**.956**	**.061**	**.053**
D) Second order 2-factor model of CSC & SI	241*	**.968**	**.058**	**.050**

Note. Acceptable individual data-model fit indices are noted in bold. * p<.001

The results summarized in Table 2 indicate that configural, metric, and scalar invariance were supported across gender and college generational status groups; although the chi-square difference test was significant for the gender metric model, changes in CFI and RMSEA remained below recommended thresholds, indicating comparable item functioning across groups and supporting fair and equitable score interpretation (Cheung & Rensvold, 2002; Lewis, 2022).

Table 2: Invariance Testing by Demographic Groups for Second-order 2-factor Model

Group	Model#	$\chi^2(DF)$	CFI	RMSEA	Model Comp.	$\chi^2\ diff(df)\ p$	ΔCFI	$\Delta RMSEA$
Gender	1.Config	649 (328)	**.943**	**.069**				
	2.Metric	677 (345)	**.941**	**.069**	1&2	**28(17)** .045	**.002**	**<.001**
	3.Scalar	693 (359)	**.941**	**.067**	2&3	**14(14) .419**	**<.001**	**.001**
College generational status	4.Config	601 (328)	**.952**	**.063**				
	5.Metric	620 (345)	**.952**	**.061**	4&5	**16(17) .522**	**<.001**	**.002**
	6.Scalar	637 (359)	**.952**	**.060**	5&6	**16(14) .292**	**<.001**	**.001**

Note. Acceptable individual data-model fit indices are noted in bold. All χ^2 values were statistically significant with p<.001. The sample size for these analyses is N=377.

Discussion and Conclusions

This study gathers validity evidence supporting the use of math self-concept and situational interest scales in Calculus I. Expert review and CFA results indicated overlap between affective self-concept and situational interest, and the more parsimonious model including cognitive self-

concept and situational interest demonstrated slightly better fit than the model including both cognitive and affective self-concept. These findings suggest that cognitive self-concept remains a distinct motivational construct, whereas affective self-concept appears closely aligned with situational interest-related processes. Consequently, self-concept and situational interest may be better understood as related rather than entirely independent motivational constructs in university mathematics. Measurement invariance further supported comparable score interpretation across gender and college generational status groups.

Future work will extend this validation effort by collecting response process and relations-with-other-variables evidence in the larger research project. Future research should also examine consequences of testing by evaluating how use of the scales informs instructional decisions and interventions designed to support students' mathematics self-concept and situational interest.

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THE PSYCHOLOGY OF MEANING, PROBLEM SOLVING, AND DRILL: EDWARD THORNDIKE'S MATHEMATICS CURRICULUM 1909–1927

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Edward Lee Thorndike was a founder of educational psychology and the psychology of mathematics learning. Thorndike is understood to have emphasized drill in mathematics and to have neglected meaning and problem solving. However, primary sources reveal that Thorndike paid overt attention to both meaning and problem solving. To clarify Thorndike's views, I systematically studied his mathematics curriculum and its historical context. The results support the interpretation that Thorndike emphasized meaning and problem solving in his mathematics curriculum. He also criticized repetitive drills and designed more meaningful practice. His model of learning, connectionism, provided a scientific explanation of reasoning. These findings raise new questions about how Thorndike influenced early mathematics curriculum design.

Keywords: Curriculum, Learning Theory

Before Edward Lee Thorndike got involved in mathematics curriculum reform from 1909 to 1927, the mathematics curriculum was written primarily by mathematicians and mathematics teachers. They drew upon philosophical ideas about the mind and learning, but they did not have in their possession any rigorous model of the psychology of learning; they did not test their principles through empirical research. Thorndike was the first to develop such a model by studying animal and human behavior in problem situations. Thorndike's attention was devoted equally to matters of psychology and to matters of education, and his psychological model, connectionism, became the basis for educational psychology. Thorndike used his educational psychology to reform the mathematics curriculum. Accordingly, Thorndike is regarded as a founder of educational psychology (Berliner, 1993) as well as of the psychology of mathematics instruction (Resnick & Ford, 1981, pp. 11–13). He was incredibly influential in psychology and education in the United States and abroad.

Although Thorndike thought of himself primarily as a connectionist and criticized the narrowly construed behaviorism of John Watson (Thorndike & Herrick, 1915), mathematics educators usually classify Thorndike as both a connectionist and a behaviorist (Kilpatrick, 1992, pp. 9–10; Steffe, 2017, p. 50, note ii). Because Thorndike's connectionism explained learning via the bonds or habits formed between a situation one meets and the responses one makes to those situations, mathematics educators have had trouble identifying the place for the mind and thinking in his model. When describing Thorndike's mathematics curriculum, mathematics educators have argued that Thorndike emphasized drill at the expense of meaning and problem solving. They have situated this understanding within a narrative of curriculum reform that portrays the early twentieth century as dominated by behaviorism, reductive educational research, and a pedagogy of drill and practice suited for vocations rather than for intellectual development (Cooper-Twamley & Null, 2009; Kilpatrick, 1992; National Council of Teachers of Mathematics, 1970; Resnick & Ford, 1981; Schoenfeld, 2002; Stanic, 1986; Steffe, 2017).

Mathematics educators' existing research on Thorndike unfortunately lacks systematic study of his work. A glance at the table of contents in Thorndike's curriculum planning books reveals that he used the terms *meaning* and *problem solving* alongside *drill* to frame his discussions (Thorndike, 1921, 1922; Thorndike et al., 1923). It is therefore surprising that prior analyses

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have neglected to discuss Thorndike's views on meaning and problem solving as additional context for his views on drill.

To clarify the nature of Edward Thorndike's mathematics curriculum reforms, I studied Thorndike's mathematics textbooks and handbooks and situated them in his larger career as an educational psychologist and progressive education reformer. Here, I report on one question from that larger study: What attention did Thorndike give in his curriculum to meaning, problem solving, and drill, and how did Thorndike's choices mirror the psychological model of learning—connectionism—that he developed? To answer this question, I studied Thorndike's early career, his books on educational psychology where he spelled out his connectionism, his mathematics textbooks and his handbooks on principles of mathematics curriculum design, and the larger context of progressive education from roughly 1890 to 1930. Although Thorndike's *curriculum vitae* is vast (see Simpson et al., 1926; Woodworth, 1952), I analyzed all of Thorndike's mathematics-related sources and reviewed a generous portion of the books and articles, along with his dissertation, through which he formulated his educational psychology and science of education. My analysis followed norms of historical inquiry, including cross-referencing sources and gathering evidence for and against my interpretations as I cyclically elaborated and refined them. In what follows, I summarize Thorndike's connectionism and his psychology of meaning and problem solving. I then explain how Thorndike approached meaning, problem solving, and drill in the mathematics curriculum.

Connectionism

Before Thorndike became intimately involved with the mathematics curriculum, he spent more than a decade developing an empirical model of learning. In his landmark dissertation study, *Animal Intelligence: An Experimental Study of the Associative Processes in Animals*, and subsequent studies, Thorndike carried out an extensive program of experiments with cats, chicks, dogs, fish, and monkeys (Thorndike, 1898, 1911). Thorndike's purpose was to test hypotheses that would help him infer the degree and nature of reasoning in animals in problem situations such as wooden puzzle boxes with release mechanisms, mazes, item retrieval tasks, and perceptual discriminations. Thorndike's results revealed that it was inappropriate to imagine animal behavior being governed by logical deductions that resembled educated human reasoning. The animals did learn to succeed at their problem tasks, and they did so with increasing speed and competence. However, Thorndike ruled out imitation and logical inference as explanations for animal learning in most cases. The data were more consistent with varied trials and accidental success that gradually conditioned the animal to repeat successful behaviors. His results were devastating to philosophers who explained animal behavior anthropomorphically, and his rigorous methods showed how promising it was to test psychological principles experimentally. Accordingly, Thorndike saw his tests of animal learning as the beginnings of an educational psychology that would give deeper insight into the mysteries of human thought, action, character, and changes to these brought about by experience and education.

Thorndike worked by observing, describing, and measuring learned responses. He varied the experimental conditions to test the factors that best explained learning. He collected a vast corpus of data and interpreted it holistically. Several trends emerged. First, learning was observed in all animals, from the most primitive fish to the most sophisticated monkeys. Second, the number of things learned and the quality of the learning increased as the complexity of the organism's physiology increased. Third, most learning occurred gradually, and changes in behavior usually passed through uneven transitional phases rather than instant tipping points. This meant that most learning in nonhuman animals was through habituation rather than through

ideation or principled reasoning. Fourth, changes to behavioral patterns were caused by the effects of the animal's actions and not by mere repetition of the acts alone. Thorndike recognized that the state of mind, or attitude, or predisposition of the animal was necessary to explain the course of learning; it influenced what the animal did in a given situation and what consequences it found satisfying or annoying. Animals learned to repeat behaviors that brought satisfaction and diminish behaviors that did not. Learning was an active, not a passive, affair, and behaviors were strongly associated with both the situations that evoked them and the consequences that ensued from them in the mind of the animal (Thorndike, 1898, 1911).

After 1911, Thorndike ceased the animal experiments that defined his early career and turned decisively to the problems of human learning, a goal that had motivated his animal experiments in the first place (Thorndike, 1898, p. 2). He brought the insights he had gained from his studies with animals, but he enlarged the principles of learning to encompass the mature forms of thinking and reasoning evident in human behavior and problem solving. In bridging the gap between animal and human learning, Thorndike was careful to point out that human learning is not identical with animal learning (Thorndike, 1898, pp. 86–87), but he also avoided the common tendency among philosophers to disregard animal psychology completely. Thorndike was unsatisfied with speculative explanations of the human mind that appealed to mystical laws of rationality. He therefore started from the principles of animal learning and explored how they operated more fully and with finer sophistication in human learning. He attributed the advantages of humans to their more complex brains and their rich mental life of ideas. He traced out the origins of mental processes such as abstraction and generalization to explain the forms of thinking, reasoning, and problem solving observable in human learning (Thorndike, 1901).

In Thorndike's three-volume series, *Educational Psychology*, he summarized what was known about human nature, human learning, and the possibilities and limits of human abilities (Thorndike, 1913a, 1913b, 1914c). He reported results of experiments by himself and others across a variety of tasks and situations. Although the complete theory is too complex to describe here, a few results are of special interest. Human nature from infancy included instincts for intellectual activity and exploratory play. Perceptual stimulation was interesting, and free activity was interesting, but most interesting of all was connecting actions to their observed consequences. Perception was not a passive experience but was the outcome of learning. Humans perceived similarities and differences based on their prior learning and abstracted new qualities of things by interacting with them. Abstractions compounded on prior abstractions and led to intellectually refined perceptions of qualities and relations. Judgments across classes of experiences led to generalizations. These abilities resulted in highly complex meanings.

Thorndike paid special attention to the relationship between learned habits and problem solving. A leading question for psychologists was how individuals approached novel situations. Did they call upon their prior repertoire of habits, or did they set those habits aside and call upon some higher reasoning faculty? Thorndike was dissatisfied with accounts that invoked a mysterious reasoning faculty but did not explain it. He thought habits had been unfairly discredited. Reasoning was not hindered but enabled by prior habits, in Thorndike's view. Habits ranged from the most thoughtless repetitions under vaguely perceived circumstances to the most insightful discriminations of and judgments about the relevant parts of a situation. When facing a novel problem, individuals responded as best as they could by assimilating the situation or parts of it to meanings from situations they had faced before. They then used their habits from prior learning to get as far as they could toward the goal, using consequences and feedback to adapt their responses where necessary. New combinations and amalgamations of old habits thus led to

new habits. The subtle and adaptive nature of habits explained why they sometimes appeared to go beyond or even counteract existing habits in sophisticated reasoning. Thorndike affirmed, "Successful responses to novel data, association by similarity and purposive behavior are, however, in only apparent opposition to the fundamental laws of associative learning. Really they are beautiful examples of it" (Thorndike, 1913b, p. 46).

Armed with a sophisticated educational psychology that outpaced the informal, speculative accounts of learning that preceded him, Thorndike was prepared to suggest implications to education. He wrote about the fundamentals of human education broadly, and he used empirical research to inform school organization and curriculum planning at the broadest levels (Thorndike, 1906, 1914a, 1924, 1928).

Thorndike's Mathematics Curriculum

Thorndike's first involvement in the mathematics curriculum was in 1909 when he wrote a five-booklet series of supplemental practice problems spanning grades 2 through 5. He began to articulate principles of mathematics learning and pedagogy in about two dozen journal articles for *The Mathematics Teacher*, the *Journal of Educational Psychology*, and other journals between 1910 and 1922. He wrote an arithmetic curriculum for the elementary grades in 1917 and for Jr. High arithmetic and algebra in 1925. He repurposed his algebra material in a book for high school in 1927. In 1921–1923, he published professional handbooks for teachers and curriculum writers that spelled out the underlying principles and considerations that informed his mathematics curriculum design. These handbooks, *The New Methods in Arithmetic*, *The Psychology of Arithmetic* and *The Psychology of Algebra*, were the first of their kind. Never had a psychologist so thoroughly commented on the school mathematics curriculum and been so directly involved with writing it. Thorndike used problem samples from his own books to draw out and explain the underlying theoretical principles of curriculum design. In what ways did Thorndike treat meaning, problem solving, and drill in these mathematics curriculum materials?

Meaning

Thorndike argued that students develop mathematical meanings from "what they *do with*" facts rather than by "what they *are told about* them" (Thorndike et al., 1923, p. 242). Accordingly, Thorndike criticized lessons that began with tedious definitions of unfamiliar concepts followed by application problems. Teachers of those lessons made two errors: they assumed that students could make sense of abstract definitions ahead of experiences with the actual concepts involved, and they posed application problems that students could not profitably engage with until the meanings had already been mastered. Student needs, in Thorndike's view, had been entirely neglected. Thorndike preferred instead to design problems that brought insight into the underlying principles. After students had accumulated sufficient experience with the relevant concepts, the teacher or textbook would provide students with useful terms and definitions. The subtle qualities and relations behind concepts would be learned gradually. Thorndike did not expect students to master a concept at the first learning, no matter how thorough or complete a teacher's explanation or definition was. He observed that students typically began with crude meanings and refined them as they confronted sets of diverse and instructive problems (Thorndike, 1921, pp. 225–229; Thorndike et al., 1923, pp. 240–242).

Thorndike chose problems carefully to ensure students had opportunities to develop useful meanings. A strong arithmetic curriculum was based on a robust meaning of number (Thorndike, 1922, pp. 2–8). Philosophers prior to Thorndike had articulated four meanings that Thorndike drew from. The series meaning, promoted by Daniel E. Phillips, was to know that *one* was the name for the first number, after which came *two*, then *three*, and so on, counting as high as

possible. Phillips thought this meaning arose from the serial and rhythmic experiences of early life, such as breathing, long before cardinality was understood (Phillips, 1897). The collection meaning was to know that a number represented the amount or cardinality of a set. The ratio meaning, promoted ardently by William W. Speer (1897) and James A. McLellan and John Dewey (1895), was to think of number as measurement in a multiplicative relationship with a designated unit, the entity connoting 1. A final meaning came to be known as the relational meaning. Its method, first advanced by Grube in Germany (Seeley, 1888), was to put each number in all possible relations with other numbers through addition, subtraction, multiplication, and division, advancing systematically from each number to the next in instruction. Thorndike's attitude was that each meaning of number was important and that all the meanings should be cultivated in a harmonious and useful way. He criticized pedagogical programs that cultivated number knowledge too narrowly through a single meaning in isolation from the others. He also criticized programs that were unnecessarily formal and systematic, instead emphasizing a diversity of quantitative problems that were sequenced according to natural psychological development in children. For instance, he bemoaned the monotonous and systematic drills on small numbers in the Grube method. He also advised against teaching exclusively from the ratio meaning, despite its merits, because children more instinctively grasped counting at the outset and developed ratio thinking gradually thereafter (Thorndike, 1922, pp. 5–8, 207–208). Thorndike was not branding a proprietary method of instruction in number with a catchy name and a rigid formula; he was instead showing how wise coordination of principles tempered the radical, extreme curriculum programs at the time and led to more balanced growth in number knowledge.

Thorndike likewise articulated the worthwhile algebraic meanings for students to develop. In a traditional algebra course, students were taught simply to solve for the variable in an equation. Thorndike was concerned that this treatment could become too mechanical and constitute mental gymnastics if not accompanied by a relational meaning. He explained, "The understanding of equations as the expression of relations, goes straight to the heart of all applied mathematics, showing the formula and equation as the story of a rule or law which certain events in nature follow or approximate; it introduces the most important idea of mathematics, that of quantitative dependence or functionality" (Thorndike et al., 1923, p. 107). Thorndike made curriculum recommendations to help students reach the latter meaning more effectively. He criticized the routines teachers used that crippled students' ability to think about algebra relationally. He distinguished between *particular equations*, such as those used to solve unknown-value problems, and *general equations*, which expressed general relations among changing quantities. To cultivate understanding of the latter, he had students use relationships such as "Length in feet = (length in inches) ÷ 12", evaluate relationships at many different values of the variables, plot relations on a cartesian graph, trace with chalk how the y-values varied together with the x-values, study proportional, linear, and nonlinear relationships, and interpret multi-variable relationships by reasoning about how changes to one variable influence another (Thorndike et al., 1923, pp. 105–108, 121–131, 252–255).

Problem Solving

Thorndike explained that problem solving was a desirable condition for mathematics learning (Thorndike, 1914b, p. 146). Invoking the authority of John Dewey, Thorndike espoused the view that students be "active seekers" working on problems that satisfied "some real need" rather than being "led along blindfold as it were by the teacher and textbook, not knowing where they are going or why they are going there" (Thorndike, 1922, p. 266). Thorndike wished for students to

work under a problem attitude throughout all the problems and exercises in the curriculum. A problem attitude meant that students took interest in and saw value in the problems they solved, and they used genuine forms of reasoning that resembled the kinds of reasoning they might use in comparable situations in everyday life. Thorndike criticized remote and unrealistic problems, replacing them with more sensible and appealing scenarios. He criticized rigid book organization by chapter and topic such that all problems using a common method were grouped together. After shuffling and regrouping problems more organically, Thorndike hoped to promote greater thought and discrimination in the methods selected (Thorndike, 1922; Thorndike et al., 1923).

Problem solving meant facing new problems and trying to work them out before being given a technique or method. Thorndike maintained that "other things being equal, it is more educative to solve a problem as an original" and that "pupils who are able to solve a certain type of problem as an original should certainly be excused from training in a routine method of solving it" (Thorndike et al., 1923, pp. 150–151). Even if students did not arrive at the solution independently, their preliminary efforts were worthwhile to their total learning. Thus, Thorndike tended to introduce the standard arithmetic algorithms by first allowing students an opportunity to solve the problem with a simpler method they already knew. They could use their results to verify that the new algorithm worked as expected. Although Thorndike did not require students to derive and justify standard algorithms from first principles, he took care to ensure that students understood and appreciated the problem at hand and recognized the computational service done for them by the algorithm. He had students verify algorithms using their prior methods and number sense so that they did not put their trust blindly in an outside authority. Where possible, Thorndike crafted additional problems that helped students see into the whys behind the standard algorithms (Thorndike, 1922, pp. 60–74, 81–83, 115–117, 266–277). Furthermore, he chose to "give no rules for rules' sake" (Thorndike, 1921, p. 234). If students solved problems effectively without formal rules and procedures, there was no need for the rules, Thorndike insisted.

Thorndike used an inductive teaching approach rather than a deductive teaching approach. The inductive approach lessened the importance of teacher explanations and increased the importance of problems designed to teach students through their own efforts. Students were meant to gain insight into quantitative principles by working directly with problems that gave rise to those principles. The teacher was to act as a guide, using questions to help the students express their reasoning and make corrections until the problem was solved. Deductive explanations and proofs were positioned near the end, rather than the beginning, of the teaching sequence, helping students acquire the vocabulary to talk about the things they'd learned and to generalize the patterns and principles they observed earlier (Thorndike, 1906, pp. 147–164; 1921, pp. 37–49, 147; 1922, pp. 74, 115–117).

To get the most value from problem solving, Thorndike believed that mathematics teachers needed to be more intentional and selective with the problems given. Before Thorndike had carried out the experiments that informed his connectionist psychology, mathematics teachers subscribed to a view known as faculty psychology. They believed that training in mathematics strengthened a few core mental faculties, such as perception, judgment, and reasoning. The outcome of such training was mental discipline, and mathematics, along with Latin and Greek, was thought to be superior at cultivating mental discipline. When carried to the extreme, this view justified mathematics teachers in giving indiscriminate exercises. Meaningless and computationally tedious problems were perpetuated by the mere force of tradition. As a response to these conditions, Thorndike carried out experiments that refuted faculty psychology. Thorndike did not discredit transfer, but he sobered expectations regarding its extent and

pressured teachers and curriculum authors to plan programs of study that were useful and worth learning about. Thorndike called for more attention to useful fundamentals and less emphasis on tedious, elaborate, specialized computations and procedures with large numbers and complex algebraic polynomials. Thorndike was critical of word problems that he viewed as bogus or contrived. Students would lose respect for arithmetic and algebra if they were fed a steady diet of such problems, Thorndike believed. Thorndike discussed the uses of intellectual or puzzle problems and problems that required students to reason about algebraic relationships through nonroutine solution methods. Finally, Thorndike pioneered tests that were intended to measure students' general ability to reason and solve problems so that the disciplinary worth of mathematics courses and other courses could be better defined, observed, and measured (Thorndike, 1921, 1922; Thorndike et al., 1923).

Drill

The term *drill* was not new when Thorndike used it to describe mathematics teaching. It was a widely circulated term with multiple meanings. As Joseph Mayer Rice's sweeping surveys of American classrooms between 1892 and 1893 showed, repetitive drill and recitation was commonplace (Cremin, 1961, p. 5; Cuban, 1984). However, some pedagogies framed drill for more instructive purposes. For instance, the mathematics professors and school principals who penned the mathematics recommendations in the 1893 report by the Committee of Ten used the term *drill* for targeted or intentional practice aimed at cultivating particular ideas and abilities ("Report of the Committee on Secondary School Studies," 1893, pp. 111–112).

Thorndike criticized rote drill and devoted a lot of effort to clarifying the nature of effective drill in the mathematics curriculum. He suggested that drill be more meaningful, more authentic, more intellectually varied, less repetitive, and less mindless. Thorndike's typical attitude was expressed well in the following quotation.

> Another means of stimulating the essential interest in quantitative thinking itself is the arrangement of the work so that real arithmetical thinking is encouraged more than mere imitation and assiduity. This means the avoidance of long series of applied problems all of one type to be solved in the same way, the avoidance of miscellaneous series and review series which are almost verbatim repetitions of past problems, and in general the avoidance of excessive repetition of any one problem-situation. Stimulation to real arithmetical thinking is weak when a whole day's problem work requires no choice of methods, or when a review simply repeats without any step of organization or progress, or when a pupil meets a situation . . . for the five-hundredth time. (Thorndike, 1922, p. 225)

Thorndike insisted that the quality, not the quantity, of practice was the thing to improve (Thorndike et al., 1923, p. 334). He used variety to stimulate thinking. He devised questions that allowed students the freedom to think of examples, to prove things, and to choose methods strategically based on the unique affordances of problems. Variety did not mean randomness without purpose. Thorndike chose numbers and question formats intentionally to create the experiences that best suited his aims. Special exercises involving patterns and relationships were used to help students discover principles through counting, relating operations to one another, working with place value and multi-digit algorithms, extending whole-number knowledge to fractions and decimals, and so on. He gave students enough practice to make their habits sure and reliable, but he did not want those habits to become automatic in the sense of unconscious or without awareness. He measured the distributions of problems in common textbooks and provided loose rules of thumb to make learning more complete and economical.

Thorndike's criticisms of repetitive drills were sensible given that his connectionism explained learning through the effects of one's acts in light of a purpose rather than through the mere repetition of those acts. Because other psychologists had supposed repetition explained learning, Thorndike carried out experiments to test whether repetition alone brought about changes to behavior patterns over time. These tests gave strong evidence to the contrary. "So far as I can now see, the repetition of a situation in and of itself has no selective power," Thorndike concluded (Thorndike, 1931, p. 14).

Thorndike believed that units of learning, which he frequently referred to as bonds, were more effective when they were interrelated and organized into a total intellectual system. He therefore advised that programs of practice be organized to cultivate insight into relationships and principles. He did not wish for number facts to be learned in isolation and remembered independently. Rather, he hoped for prior knowledge to facilitate the acquisition of new knowledge (Thorndike, 1922, pp. 137–139).

Implications and Future Questions

The foregoing evidence suggests that Thorndike's views on meaning, problem solving, and drill in the mathematics curriculum were more sophisticated than has been supposed. Likewise, connectionism was a more fruitful basis for reforming the mathematics curriculum than has been supposed. Thorndike explained thinking, perception, abstraction, generalization, and reasoning about novel problem situations through his connectionism. He used these insights to propose a mathematics pedagogy that, although imperfect, thoroughly critiqued traditional mathematics curriculum materials and placed enormous pressure for sweeping improvements. Thorndike can profitably be understood as a reformer who centered the student rather than the teacher in mathematics curriculum design and who ambitiously attempted to make mathematics more interesting, useful, and rewarding to learn. The fact that so many of Thorndike's mathematics curriculum principles and psychological insights fell into obscurity and went unnoticed in our field for the better part of a century is worthy of contemplation. To advance as a field, it is important to understand our history and disciplinary foundations, account for the forces that influence curriculum reform, and assess the progress and setbacks that we have attained by our efforts. Thorndike's case suggests that there is unrecognized work to be done along these lines.

The findings of this study raise new questions about Thorndike's influence on the mathematics curriculum between 1909–1927 and thereafter. What did students and critics have to say about Thorndike's mathematics curriculum reforms? How did Thorndike's connectionism dictate the terms of the conversation about mathematics teaching and learning? What alternatives to connectionism arose, and why? How did Thorndike's larger program of educational research shape the role of mathematics in curriculum policymaking and school organization? Thorndike's case is remarkable considering how central he was during the modernization of American schooling and how much his reputation suffered during later periods of reform. There is much to learn about the formation of mathematics education as a community during the progressive years from about 1890 to 1950.

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EVALUATING INTRODUCTORY STATISTICS TEXTBOOKS FOR ALIGNMENT WITH GAISE RECOMMENDATIONS

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The Guidelines for Assessment and Instruction in Statistics Education (GAISE) college reports (Aliaga et al., 2005; Carver et al., 2016) call for the use of real, contextualized data, active learning strategies, and appropriate technology in introductory statistics courses. Many undergraduate students leave introductory statistics harboring misconceptions (Garfield, 2002). Undergraduates rely on the textbook as their main resource, using it for content review, worked examples, and practice problems (Zieffler et al., 2013). Despite the role of texts in shaping the enacted curriculum, they remain an under-examined resource; most studies to date have used the 2005 GAISE recommendations as an analytic instrument (e.g., Dunn et al., 2017). This study evaluated the alignment of descriptive statistics chapters of three introductory statistics textbooks (henceforth referred to as text/texts) with selected 2016 GAISE recommendations.

This content analysis study examined end-of-chapter tasks in the descriptive statistics chapters of three texts that claim GAISE alignment (De Veaux et al., 2022; Moore et al., 2021; Triola, 2021). A single task (including all subparts) was the unit of analysis. We focused on the GAISE (2016) principles that can be meaningfully assessed in a text: *investigative statistical thinking (G1)*, *integrating real data with context and purpose (G3)*, and *using technology to explore and analyze data (G5)*. Tasks were coded for *G1* and *G3* on a 0-2 ordinal scale indicating *minimal*, *partial*, and *robust* alignment. For *G5*, we only coded tasks where technology use was either explicitly required or clearly implied. We reached 89.9% intercoder reliability before coding all data and discussing discrepancies.

Table 1 summarizes the findings (M, P, and R are minimal, partial, and robust). Across all three texts, procedural tasks substantially outnumbered tasks that promoted investigative statistical thinking, and technology was required relatively infrequently. Triola (2021) emphasizes procedural fluency and minimal integration of technology; De Veaux et al. (2022) offer a more balanced approach with opportunities for robust reasoning; while Moore et al. (2021) prioritize engagement with real data and offer moderate engagement with technology.

Table 1: GAISE Recommendation Alignment by Text

GAISE	Triola (2021)				De Veaux et al. (2022)				Moore et al. (2021)			
Recommendation	N	M	P	R	N	M	P	R	N	M	P	R
G1	202	68%	26%	6%	204	48%	38%	15%	83	65%	25%	10%
G3	202	35%	63%	2%	204	29%	66%	5%	83	14%	82%	4%
G5	37	56%	41%	4%	34	50%	29%	21%	33	24%	61%	15%

Our findings suggest that publisher claims of GAISE alignment do not necessarily translate into robust opportunities for students to engage in investigative statistical thinking, meaningful use of technology, or critical engagement with real data.

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POSITIONING MATHEMATICS AS DECISION MAKING IN MIDDLE SCHOOL ENGINEERING DESIGN STEM LAB

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Mathematics instruction was embedded within a Project-Based Learning (PBL) unit integrating the Engineering Design Process (EDP) to strengthen meaningful connections between mathematics and engineering. Rather than treating mathematics as a separate subject, students engaged in mathematical reasoning as they defined problems, generated solutions, tested prototypes, and redesigned based on evidence. Within iterative EDP cycles, students used measurement, proportional reasoning, spatial reasoning, scaling, and data analysis to evaluate constraints and optimize designs, supporting mathematical modeling and evidence-based argumentation (National Council of Teachers of Mathematics [NCTM], 2014; Schoenfeld, 2014).

This study investigated how positioning mathematics as a decision-making tool within iterative PBL/EDP cycles influenced students' mathematical reasoning in a middle school STEM laboratory. Using a mixed-methods action research design (Creswell & Plano Clark, 2018), data were collected over two academic years from approximately 120–150 seventh- and eighth-grade students enrolled in a STEM/STEAM elective at a Title I middle school. Quantitative and qualitative data were collected concurrently, analyzed separately, and integrated during interpretation.

Analysis revealed four key episodes illustrating shifts in students' mathematical reasoning. First, *Constraint-Driven Quantification* emerged as students moved from intuitive judgments (e.g., "this looks stronger") to using measurements and proportional reasoning when required to meet design constraints such as load limits or filtration rates. Second, *Iteration and Covariational Reasoning* developed across redesign cycles as students coordinated changes in design variables with changes in performance outcomes. Rather than attributing results to chance, students analyzed relationships between quantities, such as how base width or material thickness affected load capacity or flow rate, demonstrating emerging covariational reasoning (Thompson & Carlson, 2017).

Third, *Representational Reasoning and Modeling* evolved as students progressed from informal sketches to labeled diagrams, tables, and graphs. These representations enabled students to track relationships among quantities, interpret results, refine designs, and communicate evidence-based mathematical arguments (Lesh & Doerr, 2003). Finally, *Reasoning as Participation* reflected shifts in students' identities as mathematical thinkers. Student reflections suggested that mathematics became a shared resource for explaining and predicting outcomes rather than a teacher-imposed requirement. Statements such as "the numbers show what works" indicated increasing ownership of quantitative reasoning.

Findings suggest that engineering design constraints create authentic modeling demands that make mathematical reasoning necessary and meaningful. By positioning mathematics as a tool for decision making, PBL/EDP environments can support students' development of quantitative, covariational, and representational reasoning while fostering productive mathematical identities.

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ADAPTING A NOTICING OF ARGUMENTATION FRAMEWORK FOR ELEMENTARY INTEGRATED STEM CONTEXTS

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Mathematical argumentation is central to meaningful mathematics learning (Stylianides, 2007), and what teachers attend to when selecting and planning tasks shapes whether and how argumentation is supported in their classrooms (Ayalon & Hershkowitz, 2018). Yet research on noticing of argumentation has focused almost exclusively on secondary mathematics. Ayalon (2024) developed a model specifying what teachers should notice when students construct arguments, interpret the factors shaping argumentation, and respond with instructional alternatives, but it was built for secondary teachers analyzing written student work from classrooms other than their own. No existing framework addresses how elementary teachers notice mathematical argumentation in integrated STEM environments, where coding artifacts, robot behaviors, and digital representations create new sites for reasoning (Shumway et al., 2023). This poster presents a framework adapted for this context.

We adapted Ayalon's (2024) model to account for three contextual shifts: (a) elementary students' argumentation tends to be intuitive and empirical rather than formal and deductive (Knox & Kontorovich, 2023); (b) video-stimulated recall interviews (Lyle, 2003) capture teachers narrating perceptions of their own instruction rather than producing written analyses of others' classrooms; and (c) integrated STEM environments introduce technology-mediated evidence that may not be recognized as mathematical justification by teachers who view mathematics and coding as separate activities. Drawing on Louie et al.'s (2021) argument that framing precedes and shapes other forms of noticing, we added framing as a fourth component. Pilot data from a third-grade teacher implementing a coding-infused area measurement unit motivated this addition; her framing of student engagement in mathematics and coding as sequential rather than integrated organized her entire noticing pattern. Table 1 presents an overview of the revised framework. We used binary assessment designations (present/absent) to replace Ayalon's numbered levels across all components, with richer narrative variation. Interpreting sub-factors were streamlined from 19 to 10 through principled merging of overlapping constructs and addition of sub-factors grounded in pilot evidence.

Table 1: Framework Overview

Component	Focus	Dimensions/Factors
Framing	How the teacher conceptualizes the integrated activity and the role of mathematics within it	3 Dimensions
Attending	What the teacher notices about students' mathematical reasoning	2 dimensions; 7 sub-dimensions
Interpreting	How the teacher explains what shaped the reasoning observed	4 factors, 10 sub-factors
Responding	What instructional alternatives the teacher proposes	Quality and target category analysis

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SEQUENCING MATHEMATICAL CONTENT OF THE INDIVIDUAL

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Keywords: Curriculum, Cognition, Instructional Activities and Practices

Curriculum content and its sequencing, referred to as scope-and-sequence herein, often falls short of accounting for how learners "make sense" (Maker; 1986; Coker et al., 2023; Graves, 2023; Segedinac et al., 2018). This poster presents a method for the learner to construct their own scope-and-sequence of mathematical content. Sensemaking of the individual is accounted for, herein, by visually sequencing the subsequent problems posed by the learner on their way to solving an initial problem. I adopt the practice of Barabé (2023) who diagrammatically shows the subsequent problems posed by a collective of learners around a routine mathematics task. I apply Barabé's diagrammatic technique to the sensemaking of an individual learner to answer the question: *How might the scope-and-sequence of mathematics curriculum account for sensemaking of the individual learner?*

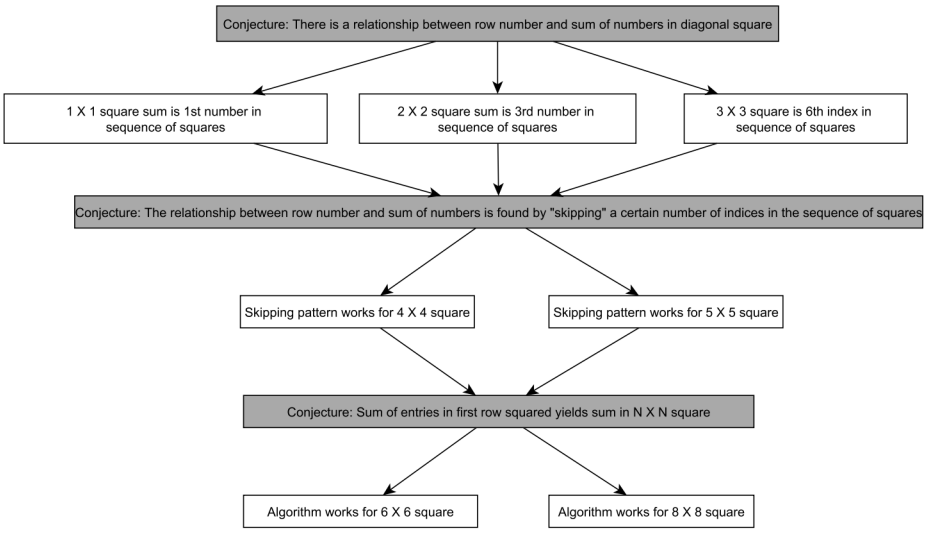


Figure 1 : Diagram Showing Mathematical Sensemaking of a Student (Cifarelli & Cai, 2005)

This poster presentation follows the paradigm of *enactivism* where cognition is considered skillful know-how (Thompson, 2007). This view differs from predominant theories of cognition that are representational, looking at cognition as knowing-what (Thompson, 2007). Enactivists consider *sensemaking* as evaluation undergone by an individual when in a situation unfamiliar to them (Thompson, 2022). I apply here Barabé's diagrammatic technique to excerpts from the transcript of Cifarelli and Cai (2005). The transcript is of a mathematics graduate student, Sarah, who talks-aloud as she solves a problem she has posed. I have analyzed and visualized the precedence relationships between mathematical problems posed by Sarah (Figure 1). This poster describes how the result is a scope-and-sequence that accounts for her sensemaking and how the described method lends itself to future research of mathematical sensemaking.

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EARLY ALGEBRA, ALGEBRAIC THINKING, AND FUNCTION

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

CHALLENGES ASSOCIATED WITH FACTORING QUADRATICS: CHARACTERIZING STUDENT STRATEGIES WHEN USING ALGEBRA TILES

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While researchers continue to advocate for the use of concrete manipulatives to support students in understanding a range of mathematical topics, limited research, especially at the secondary level, has illustrated exactly how such tools enhance learning in the classroom. To fill this gap, we investigated how high school students leveraged algebra tiles to explore quadratic factoring after a week-long unit based on their use. Analysis resulted in a framework of ways students employ algebra tiles to overcome challenges associated with fluency and cognitive load, and ultimately reason more flexibly about factoring. The role of manipulatives and productive struggle is discussed.

Keywords: Mathematical Representations, Algebra and Algebraic Thinking, High School Education

Introduction

Manipulatives have long been used as a tool in supporting students to learn and explore mathematical topics. Cognitively Guided Instruction classrooms have shown that young students are capable of insightful thinking as they use blocks to learn to count and model mathematical problems involving a range of operations (Carpenter, et al., 2014). Manipulatives support mathematical learning by anchoring abstract reasoning in physical interaction as well as providing immediate feedback when exploring new ideas. Students can visualize, review, and correct their steps, enhancing reflection and error detection (Tran et al., 2017). Moreover, the bodily movements and gestures associated with manipulatives provide opportunities to practice mathematical ideas during interaction with concrete or digital objects (Bakker & Van der Kolk, 2019).

While a considerable body of research has explored the use of manipulatives in mathematics classrooms, much of the work has concentrated on assessing performance, comparing outcomes with their use relative to traditional symbol-based methods. Results of such studies have had mixed conclusions (Carbonneau et al., 2013), indicating that the use of concrete objects does not guarantee deeper understanding and can even hinder students' mathematical learning. No doubt the pedagogy associated with how students interact with these tools shapes what they take away from their use. For example, teachers might fail to help students bridge the mathematical meaning between the physical and symbolic representations and instead treat manipulatives as trivial, leading to an instrumental understanding of rote-based procedures (Sarama & Clements, 2016).

In addition to emphasizing comparative outcomes, most research around manipulatives has focused on younger students as they grapple with early operations. As such, there is a critical need to explore how learners, especially those in middle and high school, use such tools to leverage their preexisting abilities, build internal representations, and develop problem-solving strategies, an area where empirical qualitative studies are scarce. The goal of this study is to fill this gap. Specifically, we aim to answer the research question: *In what ways do students use algebra tiles to support their ability to factor quadratics?*

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Literature Review

Embodied Learning and Representations

While results of empirical studies involving manipulatives have been mixed, researchers have detailed various theoretical arguments for their use. Theories of embedded and embodied cognition suggest that knowledge transfer requires physical actions to internalize abstract concepts (Pouw et al., 2014). In fact, neuroscience has found that learners continue to use the somatosensory part of the brain which serves to represent physical objects, even when they cease to use their fingers in calculations (Boaler et al., 2016). Such results highlight the important role of embodied learning. Additionally, learning is enhanced by experiencing ideas through multiple modalities. In particular, the abstract nature of mathematics means students only have access to concepts through different representations (National Council of Teachers of Mathematics [NCTM], 2014). As such, a deeper understanding is achieved by engaging with topics through different representations. Han and Black (2011) found that physical simulations with concrete objects form multimodal representations, helping students understand concepts more quickly and transfer knowledge to applications, providing empirical evidence of their benefit.

Manipulative Structure

The nature of manipulatives can also influence learning and engagement. Martin and Schwartz (2005) found that non-descript (bland) manipulatives, such as blocks, foster better transfer of fractions than pre-structured (perceptually rich) materials with a narrow interpretation, such as pie wedges. Although perceptually rich manipulatives may increase interest and activate prior knowledge, surface details can distract from key mathematical concepts and do not necessarily improve performance (Kaminski, et al., 2009). Notably, Carbonneau et al. (2020) demonstrated that bland manipulatives were not only more effective than perceptually rich manipulatives in supporting conceptual understanding, but also contributed to other attributes. They showed that bland manipulatives promoted greater academic perseverance by increasing students' likelihood of trying multiple strategies and persisting longer in problem solving.

Cognitive Offloading

When tasks are too complicated or perceived too complicated for working memory, humans look for ways to use the environment to overcome limits with short term memory, what is referred to as cognitive offloading. From this perspective, another affordance of manipulatives is their ability to help students offload cognitive demand to support the coordination of various problem-solving processes. Manipulatives act as an external cognitive support, scaffolding working memory, and organizing thought processes. For example, when problems require multiple tasks, students can offload different steps of the process to prevent the overload of multiple ideas, helping them to organize tasks and solve problems more accurately and efficiently. According to Risko and Gilbert (2016), the choice to offload can be intentional, using tools, or subconscious through gesturing. Various studies have shown that hand gestures and body movements unconsciously serve to reduce cognitive demands (Goldin-Meadow et al., 2001; Novack et al., 2014; Tran et al., 2017). However, while manipulatives can support learning by lowering cognitive demand and organizing thought, over-reliance on offloading can hinder internalization and metacognitive development (Martin & Schwarz, 2005; Risko & Gilbert, 2016). Therefore, instruction with manipulatives should guide students to gradually internalize mathematical processes beyond manipulatives by fostering sensorimotor interactions that are

mathematically meaningful, rather than relying solely on their physical concreteness (Pouw et al., 2014; Sarama & Clements, 2016).

Factoring with Algebra Tiles

Factoring is the process of transforming an algebraic expression from an additive structure to a multiplicative structure. However, learning to meaningfully factor an expression can be challenging. In most cases, the process cannot be determined directly but requires carrying out the distributive property and identifying patterns to undo this process, a method often referred to as guess and check. While teachers often tell students these number combinations, algebra tiles can be used to explore and develop fluency with such patterns. Since the multiplicative form consists of two expressions multiplied together, they can be interpreted as the dimensions of a rectangle with the resulting product the area (Caglayan, 2013; Lischka & Stephens, 2020). Hence, the goal when utilizing algebra tiles is to arrange the additive pieces into such a rectangle.

For example, the expression $x^2 + 11x + 24$ in additive form can be modeled by algebra tiles, consisting of a single (x by x) tile (or square), 11 (x by 1) tiles (or planks) and 24 (1 by 1) tiles (or blocks) (See Figure 1). To form a rectangle, we start with the square in the top left. We then either distribute the 11 planks on adjacent sides of the square in such a way that the rectangle remaining can be filled in with the 24 blocks or create different rectangles out of the 24 blocks so that the sides align with/add up to the 11 planks. This latter approach is what is typically provided to students when they are told to find two numbers that multiply to 24 and add to 11.

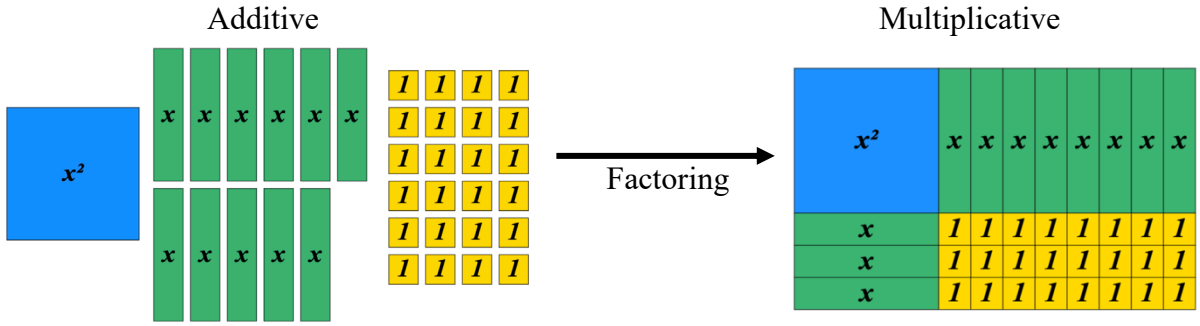


Figure 1: Algebra Tiles Representing the Factoring Process of Transforming the Additive Structure of $x^2 + 11x + 24$ to the Multiplicative Structure of $(x + 3)(x + 8)$

Research on the use of algebra tiles in classrooms is limited. However, results in the few examples indicate subtle, yet positive results associated with their use. For example, in a clinical study, while Carbonneau et al. (2020) found no difference in undergraduate students' ability to complete the square when instruction involved algebra tiles compared to symbolic approaches, algebra tiles did lead to students persevering much more, as they attempted to redo problems almost twice as frequently when they experienced difficulty. In a classroom context, Altizer (1977) found no statistical significance in the scores of pre-algebra students when factoring with or without the use of tiles during instruction, but did not that manipulative students demonstrated longer retention, presumably as a result of visualizing tile-based actions. While most students in both groups struggled to connect factorization as the inverse of multiplication, interview results indicated some manipulative students developed this understanding through imagery built from the physical interactions with the tiles.

Methods

Context

This study took place in a predominantly White (73.7%), rural US high school in the Southeast, with poverty rates equivalent to that of the state. Data collection followed three days of 90-minute instruction, led by the first author, where students were introduced to factoring quadratics using algebra tiles to explore associated patterns. In an effort to capture a range of thinking, two 9th grade Algebra I classrooms, one Regular and one Honors, were chosen. On day 1 (90 minute instruction), students were shown how base 10 blocks could model whole number multiplication using an area model which was then extended to binomial multiplication using algebra tiles. On the second day, students worked in groups, investigating how to factor monic quadratics $x^2 + bx + c$ using the tiles with the goal of generalizing the pattern of finding the combination of numbers adding to term b and multiplying to term c. Instruction was exploratory in nature and organized around group work with students asked to verbalize and discuss their ways of thinking.

Data Collection

To explore the different ways students used the algebra tiles when factoring, on the last day of instruction, nine students (six enrolled in Honors and three in Regular) participated in 30-minute, individual semi-structured interviews (Ginsburg, 1997) which were videotaped. The sample included five boys and four girls, with a racial composition that mirrored that of the school and two classes: six White, two Hispanic, and one Black student. The students were asked to factor two different problems: $x^2 + 12x + 20$ (factorable) and $x^2 + 10x + 20$ (nonfactorable) using the method of their choice and then $x^2 + 11x + 30$ specifically without tiles. Any time students asserted a quadratic was not factorable, they were asked to explain their rationale.

Data Analysis

To systematically examine the solution methods and reasoning afforded by algebra tiles, we began by identifying the different approaches students used as they explored factors and organized these data into addition- and multiplication-related methods. This revealed three cognitive challenges associated with factoring. Documenting these challenges helped us understand how and why students invoked these different resources, exposing nuances in students' reasoning. We then returned to the data and coded each student strategy according to the tools they used and how they used them to navigate these challenges. Using a constant comparison method (Strauss & Corbin, 1994), we looked across different students to identify both similarities and differences in their approaches. This method allowed us to construct a broader framework consisting of three distinct categories of approaches, characterizing the full range of ways students leveraged resources to make progress. In addition, to quantify the degree to which algebra tiles supported perseverance, similar to Carbonneau (2020), we recorded how many trials per question each student attempted to find the factors or determine it was not factorable as well as the average amount of time students engaged with each problem.

Results

Two interconnected findings resulted from this study. The first is the notable challenge factoring monic quadratics presented students and the second is the variety of ways students leveraged algebra tiles to overcome these challenges. Below we outline three cognitive hurdles involved in the factoring process that emerged and then describe the different strategies algebra tiles afforded students in reasoning through these different components.

Challenges Associated with Factoring

Factoring a monic quadratic, $x^2 + bx + c$, requires finding a number pair that adds to b and multiplies to c. One surprising result of this study was how demanding this task was for students. Analysis of student solutions revealed three different cognitive elements involved in this process: 1) fluency with addition and multiplication, 2) coordination of these two operations, and 3) a structured system of exploration/testing. While decomposing factoring into these various subcomponents serves predominantly to illuminate the different resources students brought to bear as each strategy students used was linked to one or more of these challenges, these details also provide insight into the challenges of factoring.

Fluency with addition and multiplication. Students must be sufficiently fluent with addition and multiplication to find two numbers that either add to b or multiply to c. Such an understanding goes beyond carrying out these operations. Direct calculations on two numbers provided can be supported with counting methods as the exact parameters are known. Usually, teachers tell students to start with factors of c, since there are fewer and therefore more efficient than finding two numbers that add to b. However, this approach is quite challenging as it requires students to simply "know their multiplication facts." Blocks can support this process as students can move them around to check which combinations create rectangles, but such a resource is not available in most classrooms.

Coordination of two operations. Students also need to coordinate the multiplication and addition of different number pairs, simultaneously identifying a factor pair and checking its sum or finding an additive pair and checking its product. Often students are asked to do this mentally, which can be quite taxing, especially for those who need the support of counting methods. To ease the cognitive demand, students will write down factor pairs so they can then devote their resources to adding them.

Structured system of exploration/testing. Finally, students need an overall method of keeping track of combinations they have checked and determining the next number pair to examine. Without a structured system, they can check certain combinations but are unable to determine if they have exhausted all possibilities. To overcome this challenge, students are often taught to list all possible factor pairs.

Factoring Strategies Employed with Algebra Tiles

While factoring proved to be quite challenging for almost every student interviewed, all of the participants were able to use the algebra tiles differently to enhance their reasoning in ways that helped address each of the identified cognitive hurdles. The sheer diversity of strategies employed highlight the benefits of such a tool. Depending on the arithmetic fluency students possessed, some relied heavily on the tiles, while others chose more numerical methods. Through our analysis, we identified three categories of approaches, with the first category consisting of three sub-categories (see Table 1). The first group, Tile-Dependents, were characterized by their heavy reliance on algebra tiles. All five students in this category modeled every possibility they explored with tiles, some even recreating the blocks on paper when asked to factor without the tiles. The subcategories emerged as students used separate methods to structure their exploration and keep track of different additive combinations. The second category, Numerical, consisted of two students who relied consistently on symbolic approaches, carrying out and coordinating the addition and multiplication operations mentally. The two students in the third category, Flexible, employed a range of resources, depending on the nature of the problem. While they both clearly possessed the ability to use numerical approaches, at times they leveraged the tiles depending on the nature and challenge of the problem.

Table 1: Strategies to Factoring

Categories (Students)		Description	Average Time	Average Attempts
Tile-Dependent	Unstructured exploration without verification (2)	Errors stem from limited exploration; No verification of errors	1:43	2.6
	Unstructured exploration with verification (1)	Limited structure to exploration; use of alternative resources to check work	3:10	3.5
	Algorithmic exploration (2)	Use algorithm to structure-offload exploration	3:28	6.2
Numerical	(2)	Prioritize mental calculations	1:08	2.4
Flexible	(2)	Leverage a variety of resources	1:12	2.8

Note. Each sub-category in Tile-Dependent group consisted of 1 student from a regular algebra class, while the remaining students are from the honors class.

Tile Dependent

Students in this category relied heavily on the tiles, using them to essentially carry out the operation of addition by counting out the number of planks to match the linear coefficient b and then arranging them to find different additive combinations. Highlighting the degree to which they had offloaded the calculations onto the tiles, they repeatedly needed to count the number of planks distributed into each group as they explored different combinations. However, offloading addition onto the tiles supported them in overcoming challenges with *fluency* and lowered the cognitive demand associated with *coordinati*on, as it allowed them to focus only on multiplication. Nonetheless, they still used different resources to carry out multiplication, as two students were able to calculate the product mentally while the other three leveraged the tiles in different ways. Some used the arranged planks to scaffold skip counting, while others did not carry out multiplication, but tested combinations by placing the unit tiles out or assessing their viability by estimating the spatial area. The necessity of the tools became even more apparent when students were asked to factor without the tiles. Several used their fingers to find additive pairs; others continued to multiply by skip counting, and some recreated and manipulated images of the tiles on paper. As such, while students in the first group were quite reliant on the tiles, they clearly gained critical benefits from their use, leveraging them to explore and make connections on their own, despite lacking strong computational skills.

Unstructured exploration, without verification. The first sub-category included students who lacked a structured exploration strategy, resulting in inconsistent success. They were not exhaustive in their search for factors and struggled to keep track of their work. For example, both students mistakenly determined that $x^2 + 12x + 20$ was not factorable after trying three combinations. Moreover, when errors did emerge, they failed to use alternative approaches to check their work. Both made errors attempting to factor $x^2 + 10x + 20$, one with the tiles and one with calculations, resulting in incorrect number combinations which neither verified. Although the students in this sub-category spent less time and made fewer attempts per problem than others in the Tile-Dependent group, they still showed emerging perseverance working through multiple combinations.

Unstructured exploration, with verification. Similar to students above, the one student in this category did not follow an ordered structure while checking possibilities but was more exhaustive in his approach. While inefficient, he was able to find factors for all expressions using

a range of approaches to help check his work. For example, after incorrectly using only nine planks to factor $x^2 + 12x + 20$, he was able to find his mistake with a final pass. Averaging over three minutes per problem, algebra tiles seemed to support perseverance, providing a tool to not only explore combinations, but check his work and verify his answers.

Algorithmic. Students in this category, not only used the algebra tiles to carry out the addition, but also offloaded the processes of organizing their exploration onto the tiles by starting with a particular distribution of additive factor and moving exactly one plank to check each subsequent possibility. For example, to determine that $x^2 + 10x + 20$ had no factors, one student laid out 5 and 5 planks, then moved to 6 and 4, then 7 and 3, and so on. By using this algorithm, he knew that when working with 7 and 3, all previous factors had been attempted, hence storing the information in the tiles. Offloading this amount of cognitive load onto the tiles allowed these students to operate at a higher capacity than normal and complete problems with a high success rate. Evidence for perseverance is most prominent in this group, as their entire method was based on a structured exploration using the planks to model all possible additive combinations. Consequently, they spent more time and engaged in more attempts than any other group.

Numerical

Students in this group distinguished themselves by mentally carrying out and coordinating both addition and multiplication. Opting for mental work, they did not need to offload any of this work or cognitive demand onto the tiles or paper to support their *fluency* with the two operations or *coordinate* between them. Notably, this group of students sometimes found additive pairs and checked their multiplicative values, while other times found multiplicative factors and checked their sum. For example, when asked how he explored different combinations, a student clarified that "I go more adding, and then I go to the multiplication" showing that he preferred to find additive factors and then multiply them out. However, when working through problems, he seemed to prioritize multiplication, whispering factors of the c term, rather than additives to the b term. Although these students chose to work mentally, it was clear that conceptualizing factoring as rectangular area was an image that supported their thinking. Finally, while the students did not engage in long sustained explorations due to their arithmetic fluency, the algebra tiles aided perseverance by allowing them to explore both additive and multiplicative approaches.

Flexible

The last group consisted of two students who flexibly leveraged different resources. Depending on the nature of the problem, they explored combinations mentally like group 2, used tiles similar to group 1, and even offloaded work onto paper. For example, one student who quickly found the factors mentally to $x^2 + 12x + 20$ decided to offload her work when factoring $x^2 + 11x + 30$, starting first with additive combinations of 11 before shifting to multiplicative factors of 30 (see Figure 2). In another example, while initially believing $x^2 + 10x + 20$ factored, stating incorrectly that "6 times 4 equals 20," she was able to find her mistake by using the tiles to work through different combinations and convince herself it indeed did not factor. Similarly, the other student in this category invoked the tiles to work through different additive combinations of 10 to determine that $x^2 + 10x + 20$ did not factor. Stating each possibility out loud, she moved the longs correspondingly, seemingly to support her in keeping track and *structuring* her exploration as well as offloading some of the work involved in *coordinating* the two operations.

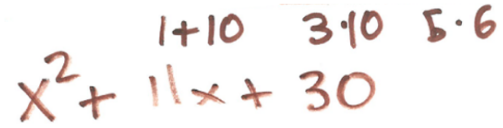


Figure 2: Symbolic Offloading of Mental Calculations

As highlighted above, while both students in this category possessed the fluency to explore different combinations mentally, they chose to use tiles when they needed support, especially in convincing themselves expressions were not factorable. To determine a quadratic is not factorable requires a more exhaustive approach, one that can be facilitated by the tiles. While students in the Numerical group correctly stated that quadratics were not factorable, they did so without providing a comprehensible argument. In contrast, the two students in the Flexible category were able to draw on and connect between more resources and representations, resulting in more sophisticated answers. The tiles also supported their perseverance. Similar to the Numerical group, they did not spend much time nor make many attempts working with factorable quadratics. However, they did leverage the tiles to methodically work through all combinations and make a convincing argument in cases that were not factorable.

Conclusion

Factoring is a difficult operation, requiring students to simultaneously carry out and coordinate two different operations. Moreover, to successfully test viable options, students must do so repeatedly and systematically. Results from this study characterize the complexities of this process and highlight distinct ways algebra tiles support overcoming these different challenges. The three different cognitive tasks we identified that compose factoring, *fluency*, *coordination*, and *structure*, help clarify why this process is difficult. In addition, the three categories of strategies identified provide a learning hierarchy of how students, depending on the strengths and weaknesses they possess, can leverage algebra tiles to progressively overcome these challenges.

The variation in approaches students employed highlights the autonomy afforded by algebra tiles and provides evidence of how these tools support perseverance. As noted in the results, students employed a range of strategies involving different modalities, including gesturing, paper, mental arithmetic. Regardless of the level of fluency students possessed, algebra tiles offered a powerful resource to persevere through each of the different challenges identified. With a specific, achievable goal of completing a rectangle, algebra tiles provide a different way for students to reach their goal. Moreover, they provide a concrete tool to explore and get immediate, actionable feedback about the accuracy of their thinking. This benefitted the Tiles-Dependent students to a great extent, as they tested and in some cases exhausted possible combinations. In contrast, students in Numerical and Flexible categories primarily used the tiles to confirm or disprove factorability. Regardless, the tiles seemed to enhance students' ability to persevere, supporting them in probing possible solutions as well as finding mistakes they had made.

Educators emphasize the importance of perseverance but often treat it as purely an individual trait. We forget that students not only need an environment that fosters such an orientation, but also tools and methods that allow repeated problem solving. Without such resources, students have limited to no means to persevere. Struggle is only productive if students possess resources to formulate and test new ideas. In contrast to traditional methods to teaching factoring where

students must rely on simply their own fluency with abstract symbols, access to algebra tiles significantly expands the approaches afforded students to factor. Moreover, using tiles to investigate patterns normalizes trial and error which inherently supports perseverance.

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EXPLORING MIDDLE SCHOOL STUDENTS' INTERPRETATION OF GRAPHS REPRESENTING THREE QUANTITIES

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This study explores how middle school students interpret graphs representing three quantities within a two-dimensional coordinate system. While mathematics curricula typically focus on two quantities, STEM disciplines frequently require interpreting graphical representations related to more than two quantities. Using conceptual analysis and a qualitative case study of a seventh-grade student, the researchers modeled his reasoning as he engaged with a faucet task, coordinating hot/cold knob turns and water output. Findings show the student successfully leveraged multivariational reasoning and gestures to coordinate three quantities simultaneously. This work provides evidence of the existence of middle school students' ability to reason about novel graphs that represent three quantities, suggesting the potential for more sophisticated graphical curricula in middle school mathematics education.

Keywords: Mathematical Representation, Multivariational Reasoning, Middle School Education

Graphs are fundamental tools for modeling and analyzing complex situations in STEM disciplines (Knorr-Cetina & Amann, 1990; Roehrig et al., 2021; Tabachneck-Schijf et al., 1997). STEM professionals often construct and interpret graphs that simultaneously represent three or more quantities. For example, meteorologists use a psychometric diagram (Figure 2a) to represent temperature, absolute humidity, and relative humidity by using lines and curves in two-dimensional space. Interpreting such graphs entails multivariational reasoning (Jones, 2022), of which students from sixth grade (Panorkou & Germia, 2023), high school (Bagossi, 2024), and college (Jones, 2022) are capable. However, neither multivariational reasoning nor graphs representing three or more quantities are common features of U.S. math curricula (e.g., Lee & Guajardo, 2023). Further, we are unaware of studies exploring how younger students (i.e., middle school) may be able to leverage their multivariational reasoning in graphical representations.

A growing body of research highlights that students as young as middle school can leverage their reasoning about quantities to develop meanings for graphs as representing two related quantities (Ellis et al., 2015; Johnson, 2012; Paoletti et al., 2023, 2024). Further, at least some middle school students are capable of constructing more than two quantities within a situation and coordinating them in complex ways (Paoletti et al., 2025; Panorkou & Germia, 2023). Hence, we conjectured that while students can develop quantitative meanings through the act of graphing, providing them with the opportunity to first establish robust meanings for a situation involving three (or more) quantities better positions them to interpret graphs as representing

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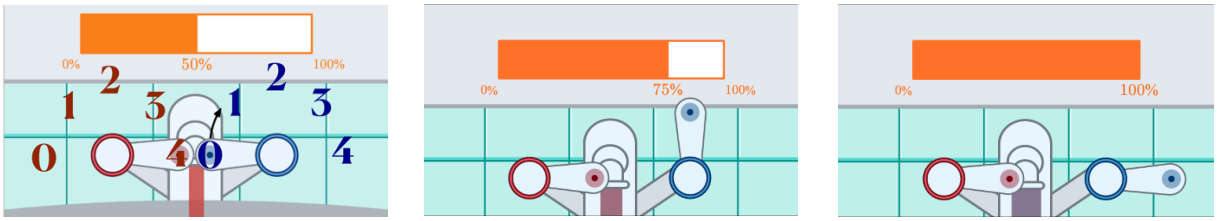
these quantities, which aligns with research suggesting that deep domain knowledge helps sophisticated interpretation of graphs (Roth & Bowen, 2001).

In this report, we address the research question: *How might a middle school student interpret graphs representing three quantities within a two-dimensional (2D) coordinate system (CS)?* To answer the question, we present a case study (Yin, 2018) of one seventh-grade student's activity as he interpreted multiple points and lines in graphs that we intended to represent three quantities in a 2D CS. Before presenting our findings, we first briefly summarize relevant literature and theoretical constructs that frame our analysis, and then we describe our methodological approach and the context in which the task was posed. We conclude with a discussion of the implications of this work and directions for future research.

Literature Review and Theoretical Perspective: Graphs and Quantities

Graphs in school mathematics and STEM disciplines are intended to represent quantities. We adopt Steffe, Thompson, and colleagues' (e.g., Smith & Thompson, 2008; Steffe, 1991) characterization of quantities as conceptual entities that an individual constructs to organize their experiential reality (von Glasersfeld, 1995). A quantity is "someone's conceptualization of an object such that it has an attribute that could be measured" (Thompson & Carlson, 2017, p. 425). A conception of a quantity entails (at least) three interrelated components: a reference object, an orientation or directionality, and an implicit or explicit means of gauging "how much" of an attribute is present (Joshua et al., 2015; Lee, 2017, 2024; Lee et al., 2019).

To provide an example relevant to this study, consider the faucet situation in Figure 1 in which each knob can rotate from pointing left (all the way off) to pointing right (all the way on) (as shown in Figures 1a and 1c). After they describe the directionality of the turn (i.e., from off to on is positive), we tasked students with considering a way to measure each knob turn by defining a full knob turn (from left to right) as comprised of four quarter knob turns (*QKT*) (see the added numbers in Figure 1a for a recreation of how the students imagined these values).



(a) 4 Hot QKT, 0 Cold QKT: 50% water output

(b) 4 Hot QKT, 2 Cold QKT: 75% water output

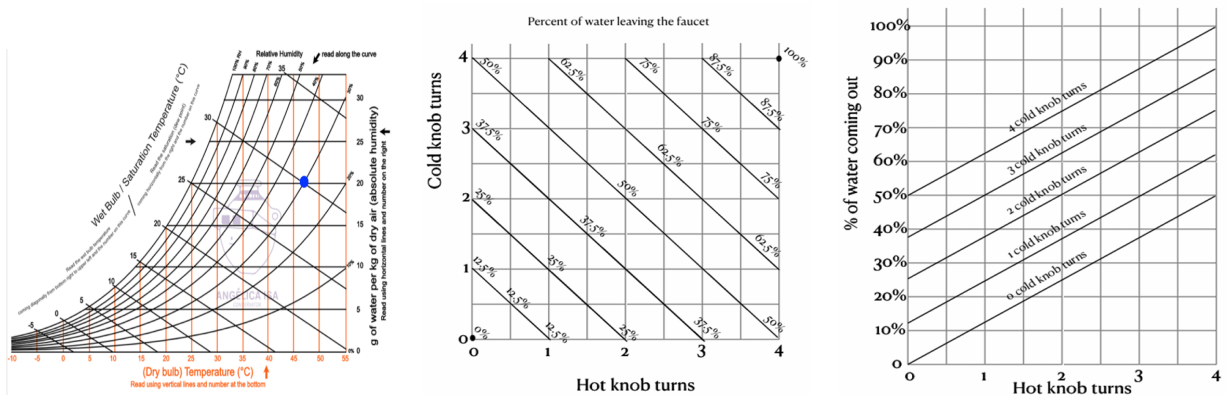
(c) 4 Hot QKT, 4 Cold QKT: 100% water output

Figure 1: A Faucet Task Showing How the Amount of Hot (the Left One) and Cold QKT (the Right One) Combine to Determine the Percentage of Total Water

Most graphs in school mathematics represent two quantities in a 2D Cartesian CS (Paoletti et al., 2020; Lee & Guajardo, 2023). Although graphs are often intended to represent two changing quantities, in this report, we focus on graphs like those in Figure 2, where any given coordinate point can represent three non-varying quantities at a particular moment or state (Moore et al., 2025). Interpreting such graphs can entail multivariational reasoning (Jones, 2022; Panorkou & Germia, 2023), which refers to coordinating three or more quantities. Panorkou and Germia (2023) examined sixth-grade students' multivariational reasoning, showing how sixth-grade students can interpret and compare situations involving three related quantities. In their study,

the students did not always focus on simultaneous continuous changes. In some instances, students exhibited forms of multivariational reasoning that entailed considering discrete changes or comparing different states of the system rather than envisioning a single dynamic flow. To exemplify multivariational reasoning, consider the quantity of the total percentage of possible water (hereafter, *percent of total water*) leaving the faucet in Figure 1. Students can conceive of this quantity as a linear combination of the amount of QKT (e.g., in Figure 1b, there is 75% of total water leaving the faucet, as there is 4 Hot QKT contributing 50% of the total water and 2 Cold QKT contributing 25% of the total water). By coordinating the amounts of two independent quantities (Hot QKT, Cold QKT) in relation to a third quantity (percent of total water), students engage in multivariational reasoning.

Relatedly, Bagossi (2024) investigated high school students' reasoning about varying quantities when interpreting a psychometric diagram (Figure 2a) intended to represent temperature, absolute humidity, and relative humidity in a 2D CS. Points on the graph can be interpreted as simultaneously representing the magnitudes of three quantities in a state. Of relevance to this report, we asked students to construct or interpret points in graphs like those shown in Figure 2 (b and c), which we intended to represent the three aforementioned quantities in the faucet task (Figure 1). In Figure 2b, Hot QKT and Cold QKT serve as the horizontal and vertical axes of the CS, and the percentage of total water leaving the faucet is represented via parallel diagonal lines moving down-and-to-the-right. In Figure 2c, Hot QKT and the percentage of total water serve as the horizontal and vertical axes of the CS, and the amount of Cold QKT is represented via parallel diagonal lines moving up-and-to-the-right. Importantly, if given values of two quantities, either graph could be used to find a value of the third quantity. In the next section, we present our methods, describing the research design, participants, and data analysis.



(a) (b) (c)

Figure 2: (a) A Psychrometric Graph from Bagossi (2024), (b) and (c) Graphs Showing the Relationship among Three Quantities: The Amount of Hot and Cold QKT, and the Percentage of Total Water Leaving the Faucet

Methods

To address our research question, we present a qualitative case study (Yin, 2018) to describe how a seventh-grade student constructed meanings for graphs (Figure 2b and 2c) representing three quantities in the faucet context shown in Figure 1. We collected the data during the second year of a two-year teaching experiment (Steffe & Thompson, 2000) with two students (with self-selected pseudonyms), Lil Man Man (LM) and Michael Jameson (MJ) (see Table 1); for

brevity's sake, in this report, we only focus on LM's activity. Focusing on a single student allowed us to conduct a fine-grained investigation of his real-time cognitive shifts. In Year 1, the students had worked with the teacher-researcher (hereafter TR, the second author) in sixth grade to develop meanings for graphs that related to two quantities in various contexts (see Paoletti et al., 2023 for examples).

Table 1: An Overview of the Teaching Experiment Sessions over Two Years

Session	Participants	High-level Goal
1–5	LM, MJ	Construct the relationship between two quantities (Base length and Side length) and describe them via graphs within the context of playpens.
6–7	LM, MJ	Construct and reason about three quantities (Hot knob turns, Cold knob turns, and total percentage of water coming out) within the faucet task in Desmos.
8a 8b 9	LM MJ LM, MJ	Make sense of the meanings for two graphs (Figure 2b and 2c; Figure 2b was introduced first) and connect them with specific situations within the context of the faucet task.
10–12	LM, MJ	Extend to relative percentage problems in the faucet context.

Aligned with exploratory teaching (Steffe & Thompson, 2000), in Year 2, we explored how to extend LM and MJ's meanings for graphs as representing two quantities to other situations and contexts, especially contexts involving three quantities. In this report, we focus on Sessions 6–9 with LM and MJ (typically paired, but occasionally individually) within the faucet context (Figure 1). Each session lasted about 40 minutes. During each session, the TR helped the students recall the previous sessions, posed tasks we designed, and asked follow-up prompts to explore the students' reasoning. We used two cameras and screen-recording software to capture students' utterances, hand motions, written work, and computer activity.

Analysis

To analyze the data, we built models of LM's mathematics via conceptual analysis, which entails "building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 60). We began by watching the videos to identify instances where we inferred that LM coordinated more than two quantities within the CSs to answer the prompts posed by the TR. We drew on our observations of LM's gestures and utterances during these moments with a particular focus on the ways that he connected points and lines in the 2D CSs to his quantitative meanings for the three focal quantities in the faucet task. Using these instances, we generated tentative models of his reasoning processes with multiple quantities. We then returned to the data with these models in mind, examining and refining the viability of our explanations. In the results, we describe our model of LM's reasoning activities when coordinating three related quantities in 2D CSs. We mainly presented episodes from Session 8a, where LM initially exhibited meanings for the given graphs.

Results

Across Sessions 6–7, and prior to engaging with the graphs in Figure 2(b/c), LM had opportunities to conceive of Hot QKT and Cold QKT as quantities varying from zero to four (as shown in Figure 1a). Further, he understood that each knob contributed 0–50% of the total water.

In Session 8a, although the TR had introduced Figure 2b to LM, when he initially asked LM to determine one quantity when given the other two, LM would typically try to rely on his quantitative meanings for the situation to determine the third value. For example, when the TR asked LM to determine the percent of total water coming out with two Hot QKT and one Cold QKT, LM mentally calculated that two Hot QKT contributed to 25% and one Cold QKT contributed 12.5%, and added these together to get the correct value without using the graph. Hence, LM understood the situation as entailing three interrelated quantities (i.e., multivariationally).

We now provide episodes showing how LM made sense of the diagonal lines and used the given graphs (Figure 2b) to coordinate the three quantities he constructed via his multivariational reasoning. To avoid LM using his aforementioned calculational strategy, the TR opted to ask LM to determine a percentage without integer knob turns (i.e., 1.5 Hot QKT and 3 Cold QKT). Hoping to explore LM's interpretation of the novel graph, the TR also marked a point on the graph representing this situation ([1] in Figure 3a). Addressing the TR's prompt, LM looked at the graph for 20 seconds and guessed, "I think it's going to be like 56." Explaining this answer, LM put his right hand on the graph, with his first finger pointing at the 50% line and his middle finger pointing at the 62.5% line ([2] in Figure 3a). He pressed his left hand to his forehead and frowned, "I don't know how to explain it, but like…" Then, after several seconds of silence, he suddenly looked up and asked, "Wait. Are you asking what this line [*moving his finger to represent a line across the marked point, ([3] in Figure 3b)*] would be?" With affirmation from the TR, LM drew a line on the graph ([4] in Figure 3b) approximately parallel to the other fixed percentage lines. He looked at the graph and thought for 12 seconds, finally saying it would be around 56.2 or 56.3%.

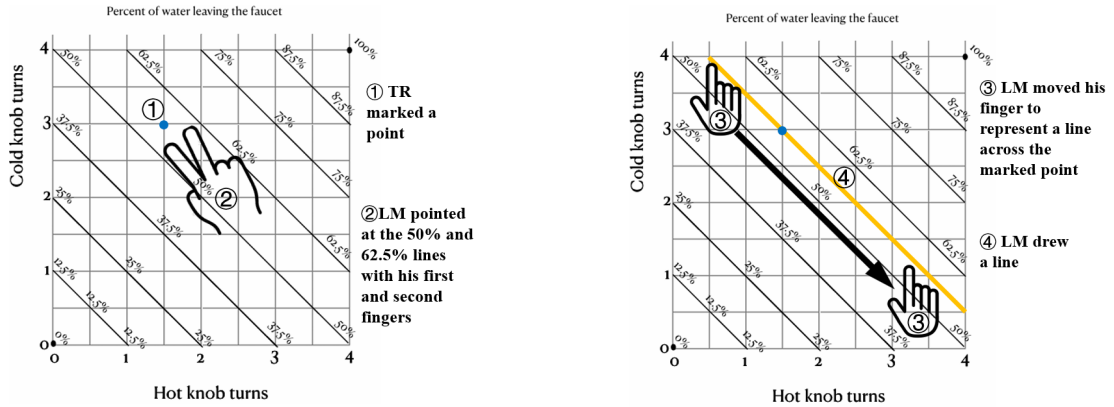


Figure 3: LM's Hand Motions Determining the Percentage of Total Water with 1.5 Hot QKT and 3 Cold QKT

We inferred that as LM engaged in effortful sensemaking of the diagonal lines related to the TR's added point, he initially approximated that the percent was halfway between 50% and 62.5% ("it's going to be like 56"). Although he initially was unsure how to articulate how he obtained this value, he eventually drew a line parallel to the 50% and 62.5% lines, indicating that this new line represented around 56.2% or 56.3%. Addressing a different task, LM drew another line representing 40% of water coming out of the faucet and identified several pairs of QKT that would result in 40% of water. Hence, LM provided evidence that he established the diagonal

lines as representing the total percent of water, but had not yet coordinated the other two quantities, Hot and Cold QKT.

The above situations entailed a focus on how to conceive the diagonal lines as representing a third quantity, and the TR also wanted to explore how LM might coordinate three quantities simultaneously in Figure 2b. Addressing the prompt, determining one QKT when given another QKT and the percentage of total water, LM again engaged in effortful sensemaking as he coordinated three quantities in a 2D CS. In particular, the TR asked LM to find out how many Hot QKT would correspond with one Cold QKT and 37.5% of the total water. Responding to this, LM looked at the graph and thought for 12 seconds before hesitantly saying, "three". As he explained his value, he realized an inconsistency, revising it immediately. He said, "I was seeing one [*pointing as in (1) in Figure 4a*] here. And then I was seeing like whatever [*motioning rightward to cross the 37.5% line as in (2) in Figure 4a*] passed here [*stopped and looked at the graph*]. Oh, two." He further explained his new answer:

> Because it is 37.5 *[using two hands and pointed at both intercepts of the 37.5% line at (0, 3) and (3, 0); (3) in Figure 4b].* It is this line over here. And then I see one *[using his left hand to point at 1 cold knob turn at (0,1) on the left axis; (4) in Figure 4b]* and two *[using his right hand to point at 2 hot knob turns at (2, 0) on the bottom axis; (5) in Figure 4b],* and then up *[moved his right hand horizontally to intersect the fixed 37.5% line; (6) in Figure 4c],* and then *[moved his left hand vertically to meet his right hand at the intersection point (2, 1) on the 37.5% line; (7) in Figure 4c]* and then meet there."

From LM's words and motions, we infer he was able to coordinate an amount of Cold QKT ([1] in Figure 4a) and a fixed percent of total water ([3] in Figure 4b) to determine the number of Hot QKT ([5] in Figure 4b). Later, LM used similar motions to solve another problem in which the percentage was not clearly represented on the graph (i.e., determining how many Cold QKT there are for 2.5 Hot QKT and 70% of total water). Hence, we inferred that LM had constructed meanings that supported him in coordinating three quantities in this 2D CS in a different way.

Given the relative ease with which LM interpreted the graph in Figure 2b, TR was curious if he would have similar success interpreting the graph in Figure 2c. As with Figure 2b, LM initially needed some time to examine the novel representation. The TR asked LM to determine how many Cold QKT would correspond to one Hot QKT and 40% of the total water. LM pointed at the one Hot QKT on the horizontal axis and moved slightly upward. He looked at the graph for eight seconds, saying, "Three [*pause for three seconds*] I think it is two, actually [*pointed at the graph*]." LM was unsure how to justify his value, so the TR asked him to mark the point he referred to on the graph (blue [leftmost] point in Figure 4d). Looking at this point, LM revised his answer again, "a little bit more than two." At that moment, the TR inferred that LM used the fact that the blue point was slightly above the line representing two Cold QKT to determine his answer. To explore this possibility, he provided additional prompts.

Next, the TR asked LM to determine the number of Hot QKT when given one Cold QKT and 30% of the total water. Addressing this, LM used his right hand to point at the one Cold QKT line and moved along with it ([1] in Figure 4d), and his left hand was pointing at 30% of total water on the vertical axis ([2] in Figure 4d). He pointed at the intersection point of the horizontal 30% line and the one Cold QKT line ([3] in Figure 4d), thinking quietly for 10 seconds. TR then drew a green dot at the point LM had indicated (Figure 4d) and asked LM if he could use this point to determine the answer. LM looked at the point, then gazed downwards from this location to the horizontal axis, saying, "Oh, it is 1.5 hot turns." We infer that LM determined the point by coordinating an amount of Cold QKT (represented by the diagonal lines) and a given percent of

total water (represented on the vertical axis) to determine the amount of Hot QKT (represented on the horizontal axis).

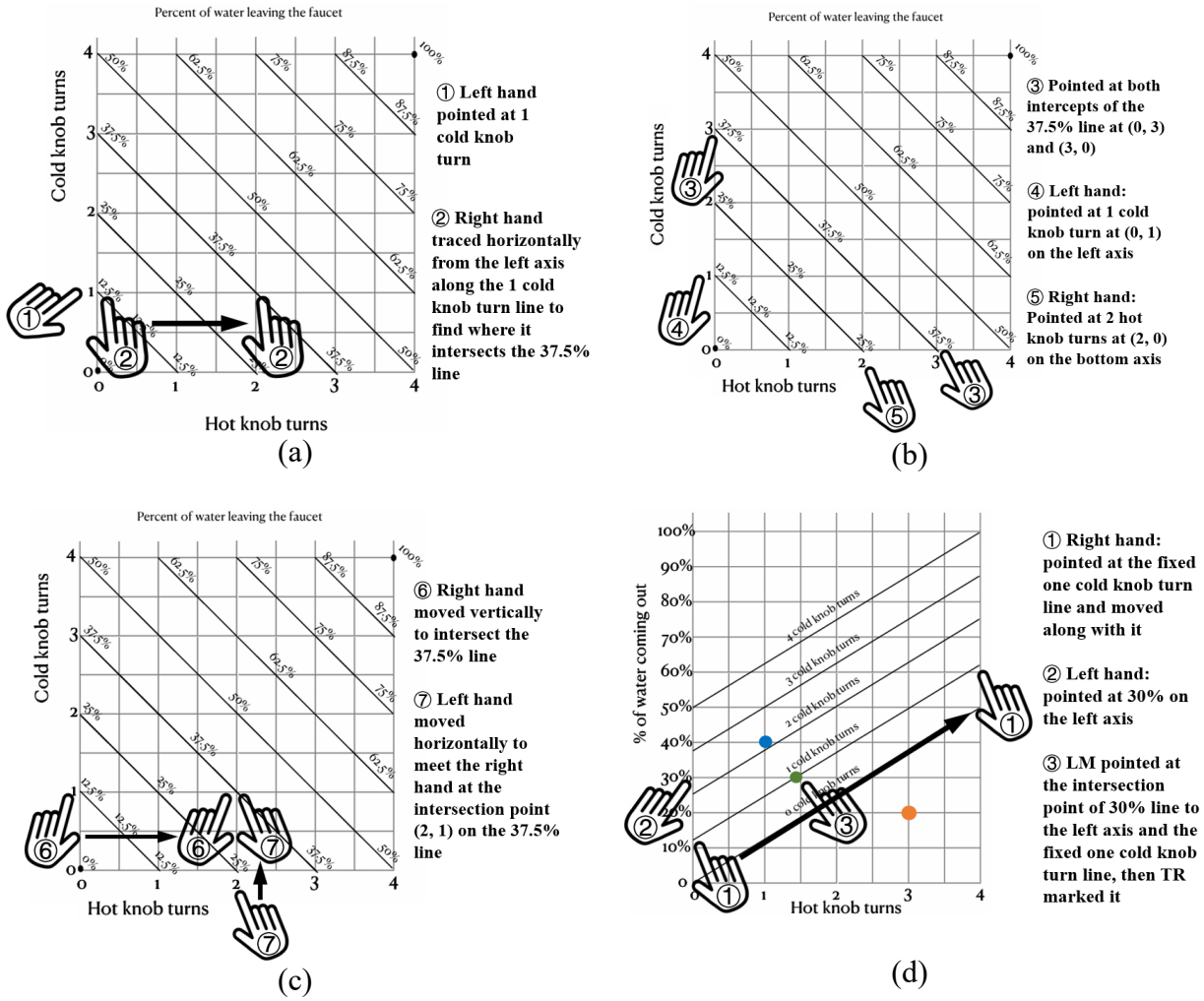


Figure 4: (a), (b), and (c) LM's Hand Motions Determining the Amount of Hot QKT with One Cold QKT and 37.5% of Total Water on Figure 2b; (d) Some Marked Points and LM's Hand Motions Determining the Amount of Hot QKT with One Cold QKT and 30% of Total Water on Figure 2c

The TR decided to see how LM would reason about a situation that was not possible within the context, but which could be marked with a point in the graph (i.e., 3 Hot QKT and 20% of total water). After hearing the prompt, LM immediately marked the point (orange (rightmost) point in Figure 4d). He quickly said, "Wait. Would it be a negative number? [*TR asked why?*] Because it is underneath the zero cold turns." We infer that LM maintained an awareness of the situational quantities he understood from his multivariational reasoning while leveraging his novel meanings for interpreting the 2D graph representing three quantities. By arguing that the answer would have to be a negative number of Cold QKT, LM provided evidence that he understood the graph as being constrained by the situational quantities. Hence, we infer that,

collectively, LM engaged in effortful sensemaking as he connected his multivariational reasoning about the situation to novel graphs representing three quantities in 2D CSs.

Discussion

Although U.S. mathematics curricula typically only ask students to reason about two quantities in 2D CSs (Lee & Guajardo, 2023), the results of this paper provide an existence proof that at least some middle school students can develop the ability to reason about three (or more) quantities in a graphical representation. Building on the work exploring multivariational reasoning with younger students (Panorkou & Germia, 2023), our findings suggest that for LM, establishing these quantitative meanings first was a critical foundation for his subsequent ability to interpret graphs as representing multiple situational quantities (Roth & Bowen, 2001). For example, LM initially relied on mental calculations based on his meanings for how each QKT contributed to the percent of total water. However, when faced with situations where mental arithmetic became difficult, he transitioned to using the novel graphical representations to determine the third quantity based on the given two quantities. However, his activity addressing the last prompt provides evidence that he remained aware of the situational quantities throughout his activity when interpreting the graphical representations.

Gestures played an important role in LM's reasoning with the graphical representations. His hand motions provided the research team with clear, observable evidence of his mental coordination processes (e.g., Figures 3 and 4). For example, in his activity in Figure 3a, he used his hands to physically model the interval required to identify the third quantity. In other prompts, he used two hands simultaneously to coordinate two given quantities to find the other (e.g., Figures 4a–c, 4b). In these cases, LM would use each hand to represent one of the two of the given quantities; by coordinating these two values in the graph, he was able to determine the value of the third quantity. These gestures were the primary vehicle through which we observed LM negotiate the relationship between the situational quantities and the graphical space.

Although we mainly focused on how LM interpreted points and lines representing quantities in a moment or state (Moore et al., 2025) that involved three quantities in a 2D CS, we note his later activity showed that reasoning about three varying quantities may be accessible to him (as in Bagossi's (2024) study). We call for future research to explore how middle school students might build on their ability to reason multivariationally about three (or more) quantities (Panorkou & Germia, 2023) to interpret graphs like those in Figure 2 as representing changing quantities in 2D CSs.

In this paper, we provided an existence proof that middle school students can reason about three quantities in a 2D CS. Such graphs are not uncommon in STEM fields (e.g., Figure 2a). Given that LM and MJ were the first two middle school students we engaged with such graphs, we wonder the extent to which middle school students are developmentally ready for more sophisticated reasoning about graphs than they are currently given opportunities to explore. That is, we conjecture that there is untapped potential for middle school students, who have been given opportunities to construct and coordinate varying quantities (Thompson & Carlson, 2017), to make sense of more complex graphs than they experience in their school experiences (Lee & Guajardo, 2023). We believe this study can serve as a starting point to create curricular tasks that leverage powerful forms of reasoning to support students in developing meanings for graphs in ways that are applicable across STEM fields.

Acknowledgements

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DISPLAYED GRAPHS, CONCEIVED GRAPHS, AND HOLISTIC SHAPE THINKING

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Research on students' and teachers' graphical reasoning has identified static (graphical) shape thinking (SGST) and emergent (graphical) shape thinking (EGST) as distinct meanings. SGST refers to reasoning about a graph as an object in-and-of-itself, essentially treating a graph as a piece of wire. EGST refers to reasoning about a graph as emerging from the covariation of two quantities. In this report, I extend the shape thinking construct to include holistic (graphical) shape thinking (HGST). An individual who engages in holistic shape thinking assimilates a graph statically while anticipating its emergence. I illustrate HGST and discuss its implications for abstracting a relationship that consists of a collection of displayed graphs.

Keywords: Graphing, Mathematical Representations, Cognition, Precalculus

Moore and Thompson (Moore, 2021; Moore & Thompson, 2015) introduced *static (graphical) shape thinking* (*SGST*) and *emergent (graphical) shape thinking* (*EGST*) to capture distinct student meanings for relationships and their graphs. SGST involves conceiving a graph as if it is essentially a malleable piece of wire (i.e., *graph-as-wire*). A student engaging in SGST assimilates a graph in terms of perceptual cues and figurative properties of shape, such as a perceived direction of drawing the graph. EGST involves conceiving a graph simultaneously in terms of what is made (i.e., a trace entailing corresponding values) and how it is made (i.e., a sustained image of quantities having covaried). A student engaging in EGST assimilates a graph as a trace in progress that is born from images and coordination of covarying quantities. A completed graph is a record of the emergent correspondence between covarying quantities (Carlson et al., 2002; Frank, 2017; Saldanha & Thompson, 1998; Thompson & Carlson, 2017).

Researchers have used SGST and EGST to characterize important differences in student reasoning across a variety of STEM contexts (e.g., Altindis, 2025; Bowe et al., 2025; Moore et al., 2019; Paoletti et al., 2023; Paoletti et al., 2024; Rodriguez et al., 2019; Tasova, 2021; Tasova & Moore, 2025; Zimmerman et al., 2024). Researchers have also used the constructs to analyze curricular materials and classroom discourse (e.g., Kimber, 2024; Lee et al., 2020; Moore, Wood, et al., 2024; Tallman et al., 2021). Other researchers have incorporated shape thinking to explore task design (e.g., Johnson et al., 2020; Margolis, 2025; Paoletti, 2019). Despite the usefulness of SGST and EGST, Moore and Thompson left potential relationships between SGST and EGST unspecified. Incorporating aspects of SGST and EGST, I introduce a third form of shape thinking—*holistic* (*graphical*) *shape thinking* (*HGST*). HGST involves an individual assimilating a graph in ways that are, on the surface, consistent with SGST. A student engaging in HGST additionally *anticipates* that any perceived properties have emergent entailments. Figurative aspects of the graph are persistently understood to be a consequence of quantities' covariation that can be reconstructed if necessary. In what follows, I provide terminology to aid in differentiating between various researcher and student conceptions of graphs. I then introduce HGST and describe how it clarifies students' actions when making sense of a graph.

Some Terminology for Discussing "The Graph"

I have used "graph" without clarifying for whose meaning I am referencing. Moving forward, I use "graph," "the graph," "function," or their variants using the terms in Table 1.

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Table 1: Phrases to Distinguish among Meanings of "the graph"

Phrase	Meaning	To Whom?
Function's Graph	$(u, v)\mid u \in D_w$, $v = w(u)$ where D_w is w's domain.	Researcher
Displayed Graph	A researcher's meaning for a function's graph as presented within conventions of a particular coordinate system.	Researcher
Student's Displayed Graph	A student's meaning for what the researcher takes to be a displayed graph.	Student
Student's Conceived Graph	A student's conception of properties that are common to all displayed graphs constituting their conceived graph.	Student

Function's graph refers to a researcher's conception of coupling quantities' values sans a specific coordinate system. *Displayed graph* refers to a researcher's understanding of a function's graph as displayed within the conventions of a particular coordinate system. Speaking of a function's graph as distinct from a displayed graph enables characterizing a function's mathematical properties common to all displayed graphs of that function. An example is the function's graph defined by $f(x) = 3x$, $-\infty < x < \infty$. The function f's graph appears as a line when displayed in rectangular coordinates and as a spiral when displayed in polar coordinates. Both displayed graphs are produced by a trace so that the output value of f varies at a constant rate of change of three with respect to f's argument; this quantitative relationship is a property of the function's graph and hence of all displayed graphs. Figurative properties can differ across displayed graphs as influenced by the coordinate systems' conventions. Another example is the sine function's graph: $\{(u, v)\mid -\infty < u < \infty, v = \sin(u)\}$. The sine function's graph has the property that v increases by decreasing amounts as u increases by equal successive amounts for $0 \le u \le \pi/2$. **Error! Reference source not found.** presents six displayed graphs of the sine function, each representing the interplay between the sine function's graph and the conventions of the coordinate system in which its graph is displayed. **Error! Reference source not found.**e uses two displayed graphs to show that changes in the sine function's output value decrease with equal successive increases in its argument. Each displayed graph entails the same quantitative relationship but differs in its figurative properties.

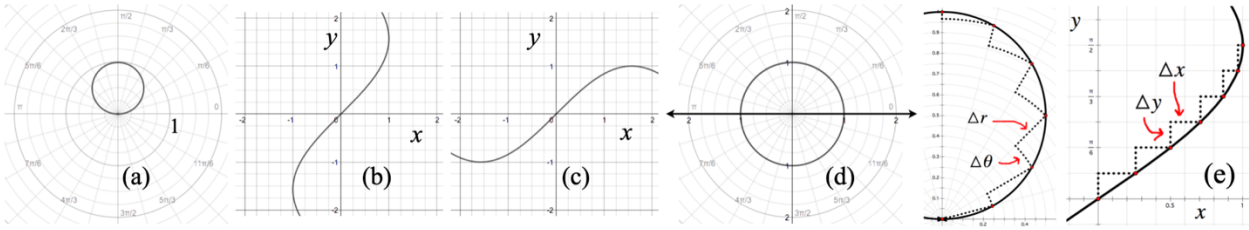


Figure 1: Six Displayed Graphs of $(u, v)\mid -\infty < u < \infty$, v = sin($u$): (a) $r = \sin(\theta)$ in Polar Coordinates; (b-c) $x = \sin(y)$ and y = sin(x) in Rectangular Coordinates; (d) $y = \sin(\theta)$ in a Combination of a Polar and Rectangular Coordinates; (e) Two Graphs Annotated to Identify v Increasing by Decreasing Amounts as u Increases by Equal Successive Amounts

Student's displayed graph refers to a student's understanding of what the researcher perceives to be a displayed graph. I use **Error! Reference source not found.** to illustrate the

advantage of distinguishing between a researcher's displayed graph and student's displayed graph. I conceive **Error! Reference source not found.**a-e to be displayed graphs created by the covariation defined by $v = \sin(u)$. A student can conceive each displayed graph to be created by that covariation, thus establishing a set of displayed graphs they understand as "sine graphs" (Moore et al., 2013). Alternatively, a student might assign each displayed graph a unique association between perceived shape and name. In such a case, the student may only establish **Error! Reference source not found.**c as "the sine graph" (Moore, 2021). To that student, the other displayed graphs could not be "the sine graph" because they are not the shape associated with sine. The student's displayed graph **Error! Reference source not found.**c is a shape associated with a name.

I use *student's conceived graph* to mean that a student has constructed a meaning that entails (potentially mathematical) properties that are invariant among a set of displayed graphs. Returning to the previous examples, it would be accurate to say the latter student's conceived graph for "sine" was a learned shape, and their set of displayed graphs was technically a single displayed graph that was the known shape. On the other hand, a student who conceives **Error! Reference source not found.**a-e to be displayed graphs created by the covariation defined by $v = \sin(u)$ has constructed a conceived graph for "sine" that is an invariant property of covariation. The student's set of displayed graphs is any displayed graph that enables representing the invariant property.

Further drawing attention to the need to differentiate between a researcher's and student's realities, I note that a "function's graph" and "displayed graph" are shorthand for a *researcher's function's graph* and a *researcher's displayed graph*, respectively. Also, a "student's displayed graph" and a "student's conceived graph" are technically shorthand for a *researcher model of a student's displayed graph* and a *researcher model of a student's conceived graph*, respectively. These technicalities acknowledge and honor students' autonomy and their independent mathematical realities by making it clear that a researcher cannot escape her or his personal reality (Hackenberg, 2010; Hackenberg et al., 2024).

Holistic (Graphical) Shape Thinking

In introducing SGST as distinct from EGST, Moore and Thompson (2015) noted that images of covariation are absent from *in-the-moment* understandings established via SGST. As an example, SGST could involve understanding the displayed graph in Figurea as having a greater slope than that in Figureb because it is more vertical or steeper. The reasoner might use this perceptual association to *infer* that over some interval of x, y increases by more in Figurea than in Figureb (e.g., the steeper the line, the more y increases). More generally, the perceptual properties constructed via SGST can *imply* learned information about quantities' values and generalized relationships (e.g., formulas and pairs or tables of values), but images of quantities and their relationships are not organic to the *in-the-moment* assimilation of a displayed graph.

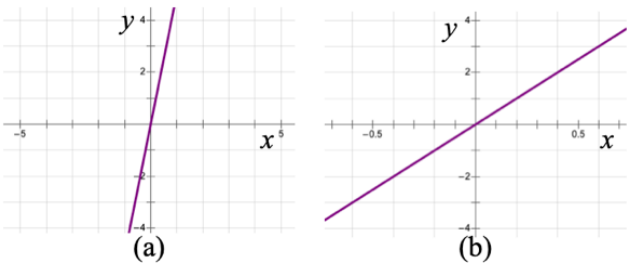


Figure 2: Two Displayed Graphs

In addition to an image of quantities and their relationships being absent from *in-the-moment* understandings of a displayed graph, SGST is also absent of anticipating a graph as an emergent trace of covariation (Moore, 2021). Consider an alternative case in which emergent schemes of covariation are in the space of implications established via reasoning that resembles SGST. That is, consider a student who assimilates a displayed graph based on perceptual or figurative properties but anticipates those properties are a consequence of an emergent trace of covariation. For instance, having inferred that Figurea has a greater slope than that in Figureb because it is steeper, the individual anticipates that this is a consequence of the magnitude change along the *y*-axis being greater in Figurea than in Figureb for an equivalent magnitude change along the *x*-axes. Here, the individual anticipates the displayed graphs as emergent traces of covariation to which *all inferences* about quantities and their relationships are tied. I term such reasoning *holistic graphical shape thinking* (*HGST*), which involves a student assimilating a displayed graph as if it is an object in-and-of-itself (i.e., SGST) while persistently anticipating an emergent trace of covariation of which all properties are a consequence (i.e., EGST).

A student thinking holistically assimilates or recalls displayed graphs "all at once" on the basis of perceived shape and learned facts, but they do so with the persistent understanding that displayed graphs are visual records of emergent schemes. Shape and associated facts are thus understood as inseparable from a displayed graph's emergence and the individual can reconstruct a graph's emergence if necessary. For instance, upon encountering a novel coordinate system or experiencing confusion, the student might choose to enact those schemes. Or the individual might enact emergent schemes to further explore the relationship conveyed by a displayed graph. For instance, to further compare Figurea and Figureb, the individual might conceive each displayed graph in terms of the quantities' covariation along the axes and recognize the rate of change of y with respect to x is equivalent. To further illustrate HGST, I use Rico's response to the task of graphing $\theta = \sin(r)$ in the polar coordinate system. I suggest the reader attempt to construct a displayed graph prior to reading more or viewing the solution (see Figure 5 at the end of the paper).

Rico was a former high school mathematics teacher engaged in an empirical study. A colleague and I had determined he had not previously associated this formula with a displayed polar graph. We used $\theta = \sin(r)$ because to envision a displayed graph, one must accommodate to two departures from customary actions. The first departure is that, customarily, θ (a point's angular direction coordinate) is a function's argument and r (a point's radial coordinate) is a function's output value. This is reversed in $\theta = \sin(r)$. The second departure is that an argument to sine is customarily defined as an angle measure, but r is a distance in the PCS. One must therefore translate a value of r from being a polar distance to being an angle measure (Figure 3a) and then envision the sine of that angle measure (Figure 3b) to produce a direction (another angle measure) as the function's output value (Figure 3c). As Stevens (2026) noted, this process necessitates uniting those two values into a pair that determines a coordinate point on the function's displayed graph determines a point on the function's displayed graph (Figure 3d).

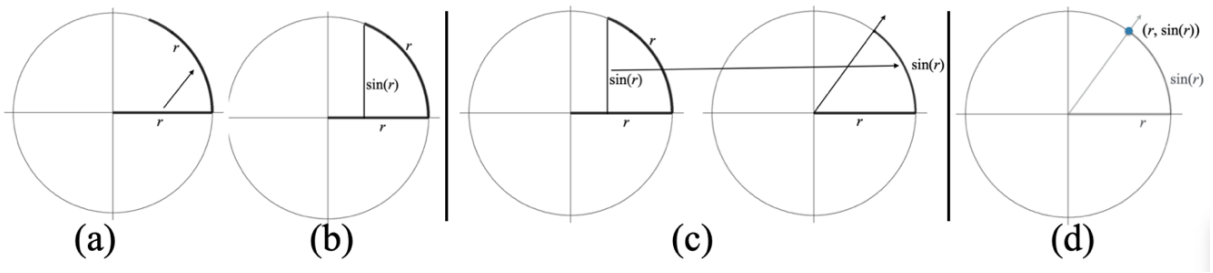


Figure 3: (a) A Radial Distance Becomes an Arc, (b) Envisioning sin(*r*) for a Value of *r*, (c) sin(*r*) Becomes a Direction and Arc, (d) Uniting the Values of *r* and sin(*r*). See http://bit.ly/RicosGraph for an animated version of this process

To construct the graph of $\theta = \sin(r)$ displayed in polar coordinates, one must keep these processes intact as one also envisions the behavior of sin(*r*) as the value of *r* varies. Rico did all of this. Table 2 summarizes the phases of Rico's construction of a displayed graph for $\theta = \sin(r)$.

Table 2: Summary of Rico's Construction of a Displayed Graph for $\theta = \sin(r)$

Phase	Summarizing Rico's Statements	Commentary
1	*r* is distance from origin. θ is angle measure. I must interpret *r* as an arc length. *r* starts at 0 and increases in length.	Rico envisions the mental actions depicted in Figure 3a.
2	I need a reference point. As *r* increases, vertical displacement (*i.e., sin(r)*) increases. I envision a spiral kinda like this (*see Figure 4a*).	Rico coordinates actions depicted in Figure 3a–d while imagining that the value of *r* increases.
3	I know how *r* and θ are related in Cartesian coordinates (*see Figure 4b*). I know as *r* increases, angle measure increases because how the sine function behaves. I know that when $r = \pi/2$ the angle measure (*i.e.,* θ) will be largest (*i.e.,* 1).	Rico's sketched Cartesian graph reminds him of sine's covariation. This recalled shape, for Rico, symbolizes emergent covariational properties. He uses this to anticipate sketching a polar graph that will not exceed a polar angle of 1 because at $r = \pi/2$, sin(*r*) is 1 and sin(*r*) then decreases in value.
4	As that distance (*i.e., r*) increases, the angle measure (*i.e.,* θ) increases toward $\pi/2$ (*see Figure 4d*).	Rico uses the anticipated covariational relationship to coordinate θ and *r* in the polar coordinate system.
5	As that distance (*i.e., r*) increases [*beyond π/2*], the angle measure (*i.e.,* θ) will decrease. Something like this (*see Figure 4e*).	Rico continues to use the anticipated covariation to covary values of *r* and values of θ according to the relational structure depicted in Figure 3a–d.
6	So as I get further from the origin, that's corresponding with an increasing angle measure (*i.e.,* θ) up to one radian. That happens when $r = \pi/2$. As *r* gets larger, the angle measure (*i.e.,* θ) starts to decrease, until it stops decreasing when θ is –1 [*and the cycle repeats*] (*see Figure 4f*).	Rico has constructed an image of covariation of *r* and θ. His image relies on a recalled shape with anticipated emergent properties (sine behavior) that he dis-embed from Cartesian coordinates and re-embeds emergently in polar coordinates.

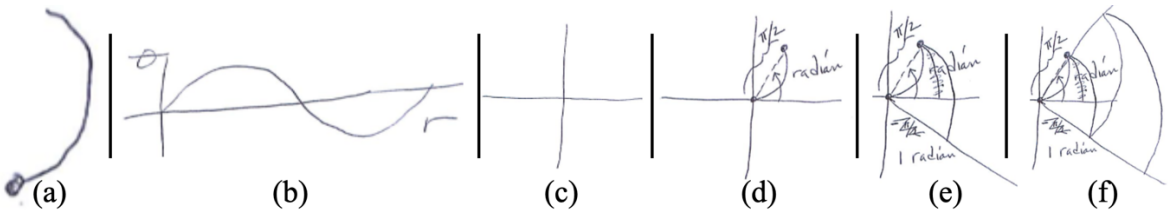


Figure 4: Rico's Progression of Work

I underscore two aspects of Rico's reasoning. First, Rico's construction of the graph displayed in polar coordinates involved a persistent focus on the emergence of covarying quantities. When identifying points, it was for the purpose of partitioning intervals of covariation. Second, Rico drew the Cartesian graph all-at-once from memory. This is consistent with SGST on the surface, but Rico's recalled shape symbolized emergent properties associated with sine that he was able to use to construct the polar graph. Such actions are indicative of HGST. In summary, he coordinated his static image of a Cartesian sine graph with an anticipated covariational relationship to reason emergently and construct a displayed graph of $\theta = \sin(r)$ so that it had the same emergent properties. Rico's actions illustrate that HGST is productive for recalling previously constructed displayed graphs in ways that are flexible for novel displayed graphs.

Indexical Reasoning and Accommodation

There is an important difference between HGST and EGST. With respect to a student inferring information about quantities and their relationships via HGST, the *in-the-moment* associations with quantities and their relationships are indexical: a particular shape or properties of shape *imply* something about how the displayed graph emerged through quantities' covariation. Recalling a displayed graph holistically enabled Rico to anticipate covariational properties, but he did not enact them *until* he constructed a novel displayed graph; EGST involves *in-the-moment enactments* of covariation operations so that quantities and their covariation are organic to a student's displayed graph; a student thinking emergently constructs quantities and they operate on them. On the other hand, HGST is such that quantities and their covariation including how the displayed graph emerges through that covariation are implications of a student conceiving a displayed graph. HGST enabled Rico to recall a displayed graph and then enact its emergent entailments in a novel setting, illustrating the blend of SGST and EGST.

When compared to SGST, me and my colleagues' work suggest EGST and HGST are more productive for students accommodating to novel experiences HGST and SGST each refer to students assimilating displayed graphs on the basis of previously experienced shapes and learned associations. In the case of HGST, these learned associations have emergent entailments. In the case of SGST, these learned associations act strictly as pointers between memorized shapes and learned facts in the form of function rules, numerical patterns, etc. These quantitative associations are not derived from a displayed graph's emergence. Hence, students who are constrained to SGST hold meanings suited for displayed graphs consisting of shapes and associations that have been previously experienced and learned. SGST requires maintaining the representational system in which the shape associations were learned; shapes typically change when representational conventions are changed. For instance, Rico's recalled shape for the sine relationship is based on a particular orientation of variables and positive values. If his meaning

for “the sine graph” was constrained to that static recollection, some other meaning would be necessary for accommodating to a novel graphing experience that departs from the context in which that shape association had been constructed.

The differences between HGST and SGST provide an important distinction to students’ graphical resources (Rodriguez & Jones, 2024). In the moment of enactment, both HGST and SGST appear as similar graphical resources (i.e., steepness as rate with Figure). However, they differ in their space of implications and, hence, their generativity beyond *their in-the-moment* enactment; that enactment is not indicative of the same cognitive structure. SGST’s implicative space consists of learned associations and facts, even when there are generalized properties between values in them. HGST’s implicative space can entail the same learned associations and facts, but those are understood as a consequence of the schemes of covariation organic to a displayed graph’s emergence. Because schemes and images of covariation are generative (Moore et al., 2013; Thompson, 2011), HGST and its emergent entailments thus provide resources suited to unpacking in novel graphing experiences. They also promote understanding when those resources are relevant (Moore et al., 2022; Moore et al., 2013; Moore, Stevens, et al., 2024). Relatedly, and taking on an Action-Process-Object-Schemas framing (Breidenbach et al., 1992), HGST might form a productive object-level meaning, with process- and action-level meanings being rooted in the anticipation and enactment of covariational schemes, respectively. By enacting EGST, a student can constitute a trace in any coordinate system orientation as representative of the same covariational relationship. Returning to steepness as rate, thinking holistically and emergently supports understanding that such a resource is constrained to Cartesian conventions, but that the underlying emergent schemes of covariation are not.

Shape Thinking and Conceived Graphs

Each shape thinking form characterizes the meanings that constitute students’ displayed graphs. The shape thinking forms are also useful when describing students’ conceived graphs. Recall that a student’s conceived graph is a meaning that entails (potentially mathematical) properties that they conceive as invariant among a set of displayed graphs. No matter shape thinking form, in some instances a student’s meaning is constrained to a particular displayed graph and thus entangled with their understanding of a specific coordinate system. I do not use “constrained” with a negative connotation. A student’s initial graphing experiences are necessarily tied to a specific coordinate system and orientation, and there is little reason to expect such a student to anticipate displayed graphs with equivalent properties in alternative coordinate systems. Hence, I use “constrained” to mean that a student’s conceived graph *is* a displayed graph so that it does not occur to the student that particular features or attributes (whether static, emergent, or holistic) could be displayed in another coordinate system or orientation. For instance, had Rico never experienced the polar coordinate system, his conceived graph may have been constrained to that which he recalled via HGST.

Regardless of form, a student’s conceived graph could also be such that its elements are not constrained to a particular displayed graph. In such a case, a student’s conceived graph consists of an equivalence class of displayed graphs formed by the properties understood as invariant among them. With respect to SGST, a student’s conceived graph could be constituted by a collection of named graph-as-wires, figurative cues and features that each shape maintains, and the anticipation of graph-as-wires in other coordinate systems that maintain compatible figurative cues and features. To illustrate, via reasoning statically, a student could construct a conceived graph that is constituted by figurative features common to each shape in **Error! Reference source not found.**. For instance, the student could perceive the drawing of each

displayed graph in terms of repeating the same shape (e.g., repeatedly drawing a wave from left-to-right or repeatedly drawing a circle counterclockwise). The figurative action of repeatedly drawing a pattern constitutes the student's conceived graph. Furthermore, the student could anticipate that displayed graphs in other not yet experienced coordinate systems and orientations would also consist of that figurative action.

In the case EGST, a student's conceived graph is constituted by a covariational relationship that each trace is understood as representing within its respective coordinate system. The student might also anticipate the existence of emergent traces in other coordinate systems and orientations that capture an equivalent covariational relationship within those coordinate constraints. To illustrate, a student could construct a conceived graph that is constituted by the collection of emergent traces in **Error! Reference source not found.** and a covariational relationship common to each shape. The student could understand each trace in terms of some quantity A's value having an increase to 1 as quantity B's value increases from 0 to $\pi/2$. Moreover, the student could understand each trace in terms of quantity A's increase having a decreasing magnitude for successive equal increases in quantity B over that same interval of values. If not constrained to particular displayed graphs, the student anticipates that displayed graphs in other (possibly not yet experienced) coordinate systems and orientations maintain this covariational relationship. The student understands that a displayed graph is always the interplay between their conceived graph (i.e., a covariational relationship) and the structure of the corresponding coordinate system.

With respect to HGST, a student's conceived graph could be constituted by a collection of shapes that the student treats as if they are objects in-and-of-themselves. The student could perceive perceptual cues and visual features that each shape maintains and anticipate that shapes exist in other coordinate systems that maintain compatible features. The student's conceived graph *also* entails a covariational relationship they can enact emergently to produce each associated shape. The student could also anticipate that shapes exist in other coordinate systems that could be produced by thinking emergently within that coordinate system. Another feature of HGST could be that if a student's conceived graph involves generalized perceptual features, the student anticipates these features are a consequence of covariational properties represented within respective coordinate systems. To illustrate this latter point, a student's conceived graph $(u, v)| -\infty < u < \infty, v = \sin(u)$ could entail anticipating that symmetry must occur no matter the coordinate system: the displayed graphs in **Error! Reference source not found.** have symmetry with respect to $\theta = \pi/2$, $x = \pi/2$, $y = \pi/2$, and $\theta = \pi/2$, respectively. The student would also understand that these features of symmetry are a product of v covarying in the same manner with respect to u increasing *or* decreasing from $u = \pi/2$, thus establishing covariational symmetry with respect to $u = \pi/2$.

Closing

Moore and Thompson (Moore, 2021; Moore & Thompson, 2015) left unclear whether a student may use SGST and EGST in-tandem. I identify a productive form of reasoning is that which appears to be SGST but technically has emergent entailments. HGST is not simply the combination of SGST and EGST. SGST involves figurative and thematic properties of displayed graphs and does not capture meanings that include emergent entailments. HGST resembles SGST in-the-moment of assimilation, but assimilation via HGST involves anticipating its emergent entailments. These entailments enable accommodating to novel displayed graphs through enacting mental operations that maintain covariational properties. HGST positions figurative properties strictly as consequences of quantities' covariation. Moving forward, an area

of inquiry is how students construct and leverage meanings consistent with HGST, particularly as they abstract a conceived graph from a displayed graph collection. It is also important to understand learning environments that support SGST, EGST, or HGST. With researchers illustrating the short- and long-term importances of covariational reasoning, it is imperative that we understand how to support EGST (e.g., Paoletti et al., 2023; Tasova & Moore, 2025) and HGST so that students' graphical meanings and resources include emergent entailments.

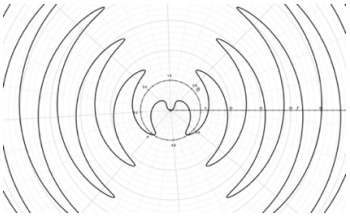


Figure 5: A Fifth Displayed Graph of $(u, v)| -\infty < u < \infty, v = \sin(u), \theta = \sin(r)$

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HOW MIDDLE SCHOOL STUDENTS INTERPRET GRAPHS IN A ROBOTICS CONTEXT

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This study examines how sixth-grade students interpreted distance–time graphs while programming Sphero robots in a robotics-based algebra task. Using the constructs of figurative and operative thinking, I analyzed classroom episodes to describe students' emerging meanings for graphs and unit rate. Findings show three patterns: (1) iconic translation (graph as a path of motion), (2) blended reasoning that coordinated quantities while preserving graph shape, and (3) quantitatively coherent reinterpretations that matched distance–time values but obscured the slope–speed relationship. Whole-class discussion supported students in identifying unit rate from graphs and connecting speed and slope across representations

Keywords: Algebra and Algebraic Thinking, Mathematical Representations

The ability to construct and interpret graphs is central to middle school mathematics and foundational for later learning in algebra and STEM disciplines (Carlson et al., 2002; Thompson & Carlson, 2017). Graphs support reasoning about functional relationships, rate of change, and covariation (Carlson et al., 2022; Ellis, 2011; Johnson, 2015). However, research shows that many students interpret graphs primarily as visual shapes or pictures of situation rather than as representations of quantitative relationships (Clement, 1989; Johnson et al., 2020; Monk, 1992; Moore & Thompson, 2015). Such interpretations limit students' ability to reason about rate, invariance, and underlying structure (Frank, 2016; Moore et al., 2014; Tasova & Moore; 2020).

From a quantitative and covariational reasoning perspective, productive graphing involves coordinating two varying quantities and conceiving a graph as a trace emerging from their covariation (Moore & Thompson, 2015; Moore, 2021; Saldanha & Thompson, 1998). Yet instruction often emphasizes symbolic procedures or static representations, providing limited opportunities for students to experience graphs as dynamically generated objects (Frank, 2016; Moore, Silverman, et al., 2019; Paoletti et al., 2022). As a result, students may foreground perceptual shape features over quantitative coordination (Moore & Thompson, 2015; Moore, Stevens, et al., 2019).

Programmable robots, such as Sphero, offer an embodied context in which motion is measurable and directly programmable (see https://www.youtube.com/watch?v=f21VLjQwtoY). When students code a robot's speed and duration, they engage with distance, time, and rate dynamically. Such environments may support students in developing productive and powerful ways of graphing.

This study investigates how middle school students interpret distance–time graphs in a robotics-based instructional setting, with particular attention to how figurative and operative aspects of thought emerge and interact during the activity. By examining students' classroom reasoning as they program and analyze robot motion, this study extends prior work on students' graphical meanings (Frank, 2017; Moore & Thompson, 2015; Paoletti et al., 2023; Tasova & Moore, 2025) into an embodied robotics context.

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Theoretical Perspective

Radical Constructivism and Instructional Design

This study is grounded in radical constructivism (von Glasersfeld, 1995), which views knowledge as constructed rather than transmitted. Learning involves reorganizing prior meanings through cycles of action, reflection, and adaptation. In classrooms, such reorganization is often triggered by perturbations—moments when students' expectations conflict with outcomes encountered through tasks, tools, or discourse. From this perspective, instructional design aims not to deliver correct explanations but to create environments that provoke reflection and reconstruction of students' mathematical ideas. Tasks, tools, and discourse are designed to surface students' existing meanings and create opportunities to reorganize them.

Didactic Objects

To conceptualize instructional design, this study draws on Thompson's (2002) notion of a *didactic object*—a deliberately designed "thing to talk about" that supports reflective mathematical discourse. An object is not inherently didactic; it becomes so when used to foreground specific mathematical relationships and operations. Its effectiveness lies not in its physical or digital form, but in the mental actions it invites students to enact (Bowers et al., 2011). Designing a didactic object therefore requires identifying the mental operations students must coordinate to interpret a situation productively. In this study, programmable robots and graphs function as didactic objects when used to organize classroom discourse around distance, time, rate, and covariation. The robot's motion provides a shared object of reflection through which students can examine quantitative relationships.

Figurative and Operative Thinking

To analyze students' reasoning within this environment, this study draws on the distinction between figurative and operative thinking (Moore, Stevens et al., 2019; 2024; Thompson, 1985). Figurative thinking foregrounds perceptual features and imagery. In graphing contexts, students may attend primarily to shape, direction, or visual resemblance, engaging in iconic translations that map a situation directly onto a graph (Monk, 1992). Such reasoning privileges appearance over quantitative and covariational reasoning.

Operative thinking, in contrast, foregrounds coordinated mental actions. A graph is conceived as a trace emerging from the covariation of quantities (e.g., emergent shape thinking; Moore, 2021). From this perspective, slope represents a relationship between changes, and graphical shape is understood as a consequence of those coordinated changes.

In this study, the distinction between figurative and operative thinking serves as an analytic lens for examining how students interpret graphs in a robotics-based setting. Specifically, I investigate how engagement with programmable robots surfaces students' figurative and operative interpretations and how these interpretations interact with, and sometimes reorganize into, more operative, quantitatively grounded reasoning.

Methods

Participants and Setting

Participants were 87 sixth-grade students enrolled in a four-day Summer STEM Camp at a southwestern public university. Students attended one 75-minute robotics-mathematics session per day in four class periods (approximately 20–24 students each). Although students had prior exposure to coordinate graphs and proportional relationships, most had not previously worked with programmable robots. This study focuses on a proportional distance–time graphing task implemented on Day 1.

Data Collection

All lessons were video recorded using three cameras: one focused on the teacher (the author), one on a focal group of four students, and one capturing whole-class activity, including student discussions, gestures, robot motion, and written work. Students' written responses and coding sequences were also collected. The multi-angle recordings allowed me to examine not only students' verbal explanations, but also their embodied interactions with the robots, gestures toward graphs, and revisions of code during the "code–watch–revise" cycles.

Data Analysis

Analysis proceeded in three phases. First, selected small-group discussions were transcribed and reviewed alongside video to preserve gestures, intonation, and interactions with robots. Second, episodes were identified in which students interpreted graphs to determine motion, discussed robot placement or starting conditions, determined speed from graphs, or experienced disagreement about what the graph represented. These moments were selected because they made students' graphing meanings visible.

Third, episodes were analyzed using figurative and operative thinking as analytic lenses. We examined whether students foregrounded perceptual features (e.g., shape-matching, iconic translation) or coordinated quantities such as distance and time (e.g., identifying speed as distance per second). Particular attention was given to moments of tension or perturbation. Rather than quantifying frequencies, the analysis characterizes qualitatively distinct ways of thinking that emerged and how instructional interactions supported or constrained them.

The Robotics Graphing Task

Students began with a 10–15-minute warm-up in which they experimented with a single "roll" command that specifies heading, speed (cm/s), and duration (seconds; see Figure1). I briefly elicit students' ideas about what each parameter controls before they begin experimenting. This initial exploration was intended to familiarize students with the robot interface while also establishing key quantitative relationships among speed, time, and distance through embodied activity.

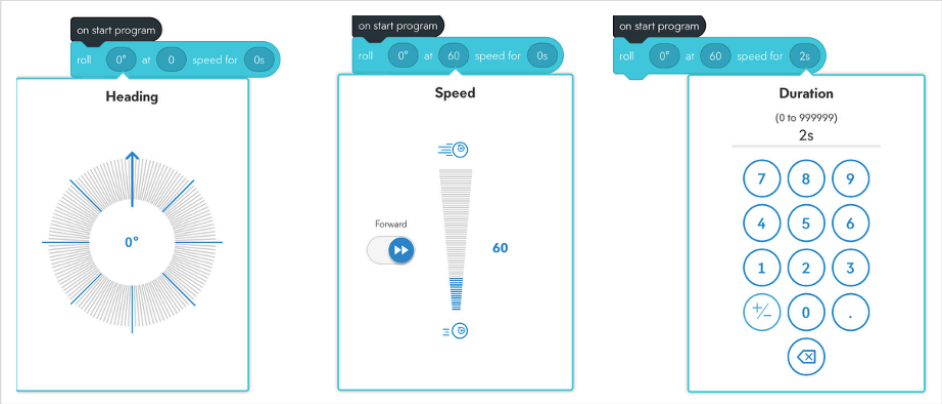


Figure 1: The Roll Block in the Sphero Edu App, Showing Controls for Variables

The focal task was situated in a beach context (Figure 2a). Students were given a distance–time graph containing two proportional linear functions (Robot A and Robot B; Figure 2b). Working in groups of four (two pairs), each pair programmed one robot to match one of the lines. Robots were required to move straight toward the water and to start simultaneously, so that students could directly compare the motions produced by their programs.

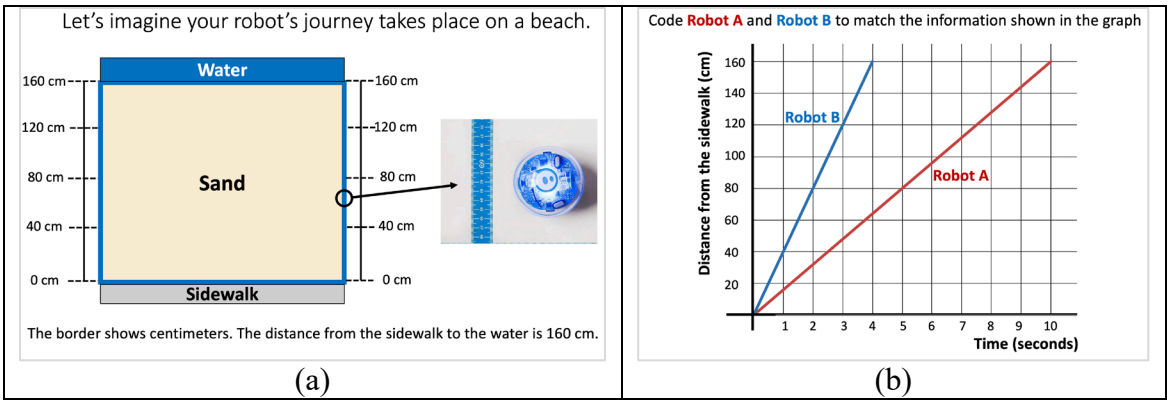


Figure 2: (a) Floor Setup for the Robotics Graphing Task Showing the Beach Context, (b) Distance–Time Graph Used by Students to Program and Compare Robot A and Robot B

Students engaged in iterative "code–watch–revise" cycles: they wrote code, observed robot motion, compared the motion to the graph, and revised accordingly. When both robots were released at the same time, differences in their motions became publicly visible, creating natural opportunities for comparison and discussion. Figure 5 shows an example of a correct solution.

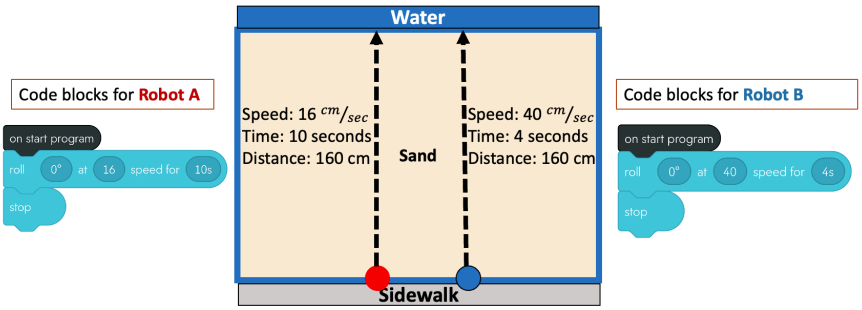


Figure 3: Correct Motion Paths and Code Blocks for Robots A and B

The robot–graph pairing functioned as a didactic object (Thompson, 2002). The graph itself was not inherently didactic; rather, it became didactic through the classroom conversations it supported and the specific mathematical relationships it helped bring into focus. Robot motion also provided immediate, non-evaluative feedback. When a programmed motion did not align with the graph, discrepancies between expectation and outcome created perturbations that anchored discussion around quantitative relationships.

The task was designed with two instructional intentions.

Goal 1: Support students in interpreting graphs as representations of covariation. A central goal of the task was to support students in interpreting graphs as records of two quantities varying simultaneously (Moore, 2021), rather than as pictures of situation (Clement, 1989) or spatial paths (Johnson et al., 2020). The design constraints (straight movement and simultaneous start) intentionally disrupted shape-matching interpretations and created occasions for students to confront what, exactly, a distance–time graph is representing. Classroom discussion focused students' attention on identifying the quantities represented on the axes and interpreting individual points as simultaneous values of distance and time. For example, students were asked:

At 3 seconds, how far is the robot from the sidewalk? Could the robot be in different physical locations and still be the same distance from the sidewalk? If the graph starts at (0,0), what does that tell us about where the robot begins? Such questions were designed to foreground quantitative relationships and point-wise interpretation rather than iconic interpretations.

Goal 2: Support students in identifying unit rate from the graph. A second goal was to support students in identifying unit rate (speed) directly from the graph and relating it to slope. Because students had to determine a speed value that would allow the robot to travel a given distance in a specified amount of time, they were pressed to interpret the graph quantitatively rather than rely only on trial-and-error. Some students initially adjusted speed values until the robot arrived at the water at the correct time. When this occurred, instructional questions redirected attention to the graph: *How much distance does the robot travel in 1 second? Can you find a point where time equals 1? What does that y-value tell you? Can you choose any point on the line and use its coordinates to find the speed? If you divide distance by time, what does that number represent in this situation?* Through such prompts in small groups, students were supported in interpreting speed as distance per second and relating that quantity to the constant of proportionality represented by the graph (i.e., [1, *r*] where *r* is the unit rate).

Results

Figurative and Operative Thinking in the Robotics Task

Analysis of classroom episodes revealed three recurring ways students initially approached the Robotics Graphing Task. These approaches illustrate varying combinations of figurative and operative aspects of thought.

Iconic translation: Purely figurative thinking. A subset of students interpreted the graph as a picture of the robot's physical path (Figure 4a). Rather than attending to the quantities represented on the axes, these students attempted to make the robot trace the shape of the line. For example, one group examined the slanted line on the graph and immediately positioned their robot diagonally across the sand, saying, "It [*referring to the robot*] should go like that." This reflects figurative reasoning in which perceptual shape guided interpretation.

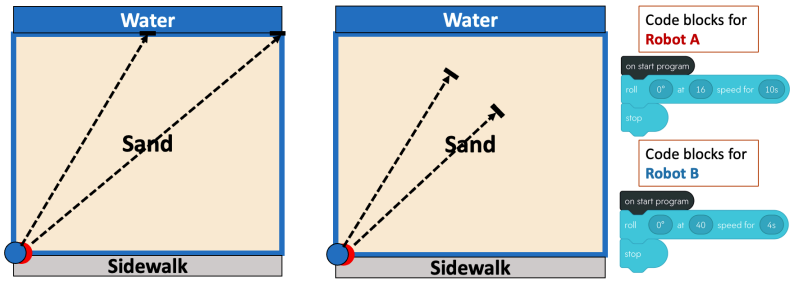


Figure 4: (a) Iconic Translation: Robots Tracing the Visual Shape of the Graph; (b) Coordinating Quantities While Preserving the Graph's Shape in the Robot's Motion

After observing this pattern in several groups during the first period, I added an explicit constraint to the task: the robots should move *straight* from the sidewalk to the water. This constraint, combined with the requirement that both robots start simultaneously, created immediate tension. When reminded that the robot must move straight toward the water, one student responded, "But the graph is not straight," providing evidence of an iconic translation.

Students who attempted to match the shape of the graph spatially on the floor also encountered tension when both robots could not occupy the same starting location. Some groups resolved this tension by taking turns, despite the instruction to start simultaneously. However, this moment opened space for discussion about what the origin (0,0) represents: not a physical corner of the sand, but zero seconds and zero distance from the sidewalk. These tensions became opportunities to discuss what the graph actually represents—not the physical path of the robot on the sand, but a relationship between time and distance from the sidewalk.

Mixed reasoning: Coordinating quantities while preserving shape. A second group of students correctly identified information from the graph—such as the time it takes each robot to reach the water from the sidewalk—yet still attempted to preserve the graph's visual shape in the robot's physical motion (Figure 4b). For instance, one student stated, "This one [Robot B] takes 4 seconds. … this one [Robot A] 10 seconds." However, they positioned the robots at an angle to mimic the shape of the lines. After testing, the robots did not reach the water at the intended time, which surprised these students. One student said, "we used 40 but it didn't work" referring to the speed value they had identified from the graph for Robot B.

Here, students coordinated time, distance, and speed (operative elements) while simultaneously relying on shape-matching (figurative elements). Because the diagonal motion increased the actual distance traveled, the robots failed to reach the water within the specified time. I did not anticipate this form of reasoning—simultaneously quantitative and perceptual—in the original conceptual analysis. It illustrates that figurative and operative thinking are not mutually exclusive; students may blend them within the *same* activity.

To resolve this perturbation, one student suggested, "more speed," indicating a shift toward adjusting the quantitative relationship. They implemented a trial-and-error strategy to determine a speed value that would allow the robot to reach the water in 4 or 10 seconds. I elaborate on this approach in the following section.

Mixed reasoning: Coordinating distance–time information while preserving shape. A third pattern emerged in which students simultaneously matched the visual shape of the graph and satisfied the time and distance information shown on the graph, but in a way that altered the intended quantitative relationship (Figure 5a). Like the second group, these students positioned the robot diagonally so that its physical path resembled the shape of the graph. However, unlike the second group, they ensured in their first attempt that the robot reached the water at the correct time (e.g., 4 seconds or 10 seconds, see the code blocks in Figure 5a).

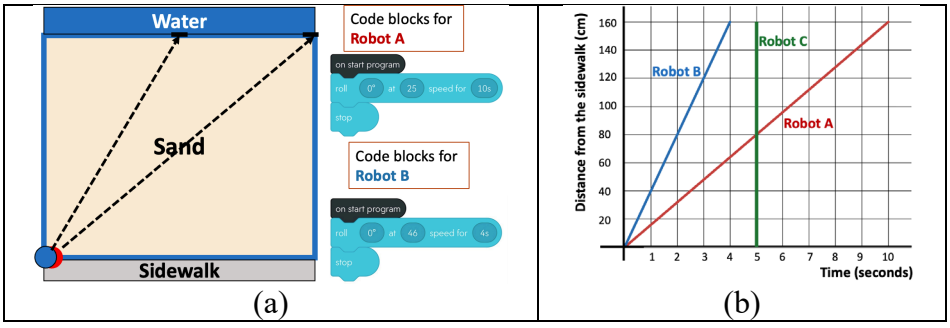


Figure 5: (a) Coordinating Distance-Time Information While preserving the Graph's Shape in the Robot's Motion; (b) An Additional Graph Introduced to Further Probe Student Thinking

Because the diagonal path was longer than the straight distance from the sidewalk to the water, students increased the robot's speed to compensate for the additional distance traveled. As a result, two things were simultaneously true: (i) at each second, the robot's distance from the sidewalk matched the y-value of the graph, and (ii) the robot arrived at the water at the correct time. However, the robot's actual path length differed from what was intended, and therefore its speed differed from the slope (unit rate) represented by the graph.

In this way, students satisfied the information displayed in the graph while preserving its visual shape in the robot's motion. The graph's quantitative information was respected locally (distance from the sidewalk over time), yet the global relationship between slope and speed was obscured. This pattern reveals a blended form of figurative and operative reasoning. Students coordinated time and distance (operative elements) yet continued to rely on shape-matching interpretations (figurative elements).

For this group of students, perturbing their reasoning proved more challenging because their response was mathematically valid and coherent within their reinterpretation of the situation. After observing this pattern across multiple class periods, I introduced a new graph on the same coordinate plane (see Figure 5b) to further probe students' thinking. The goal was to help students recognize that the shape of the graph alone does not determine the robot's physical motion.

When shown a vertical line at five seconds, students offered competing interpretations. One student said, "It [Robot C] goes straight [*gesturing a direct path from sidewalk to the water*]." When asked, "How long does the robot move?" the student responded, "Five seconds," but appeared unsure. After a brief pause, he added, "It doesn't say… it's like same time." Another student interjected, "It can't be [inaudible] at the same time." Here, operative reasoning emerged spontaneously. Some students recognized that a vertical line on a distance–time graph implies multiple distances at the same time, which is physically impossible. This discussion further highlighted the distinction between perceiving a graph as a shape and understanding it as a representation of relationships between quantities.

Unit Rate and Graphs

Whereas the first section focused on the interplay of figurative and operative thinking, this section highlights how students reasoned about unit rate within the task.

Some students initially relied on trial-and-error strategies when determining the speed of the robots: "Try 30." "No, too slow." "Try 45." Rather than correcting these attempts, I posed questions to redirect attention to the graph. For example, I asked, "How far does it [Robot B] go in one second?" A student responded while looking at the graph, "At one second… it's 40." When asked, "What does that tell you?" the student replied, "here [*pointing to the 40 cm mark on the floor*] in one second."

In several groups, students began locating the point where time equals 1 and reading the corresponding distance. Others selected a point and divided distance by time (e.g., 120 divided by 3 is 40, which is the value of the speed in centimeter per second).

Following small-group activity, I brought the class together to consolidate key mathematical ideas. The goal of this discussion was to connect different strategies students had used to determine the robot's speed (i.e., unit rate) from the graph. I projected student work illustrating two common approaches (Figure 6): (i) identifying the speed by reading the distance at one second and (ii) selecting a point on the line and dividing distance by time. I asked students to compare these strategies: "How are these two methods the same?" and "What does each one tell

us about the robots?" Through this comparison, students began to see that both strategies reveal the same quantity: distance per second (e.g., "it goes 40 every second").

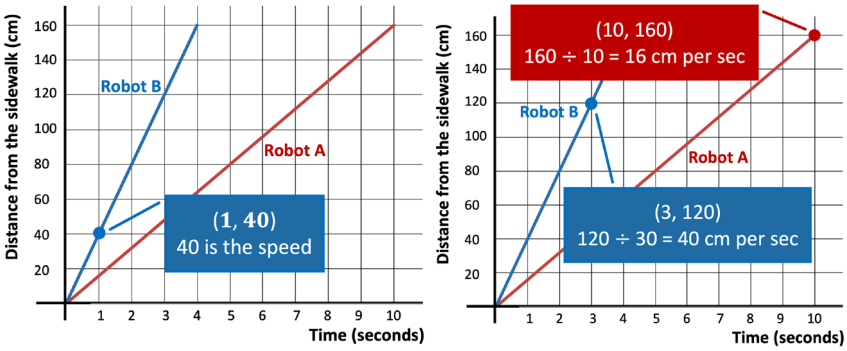


Figure 6: Student Work Highlighting Two Common Approaches for Determining Speed from Proportional Graphs

I then introduced slope triangles as a third strategy. Drawing vertical and horizontal changes on each graph, I asked students to compute their ratio and see how that compared to the robot's speed. This discussion made the connection between speed and slope explicit: both represent the ratio of vertical change to horizontal change. For sixth-grade students, this discussion was intended as an early introduction to slope. For students with prior exposure, it reinforced slope as a rate of change.

Discussion and Conclusion

This study examined how middle school students engaged in figurative and operative aspects of thought while participating in a robotics-based graphing task. With respect to Goal 1, findings show that many students initially interpreted graphs iconically, treating them as pictures of motion rather than representations of covarying quantities. The task constraints—requiring straight movement and simultaneous starts—surfaced these interpretations and created productive tensions. A key contribution of this study is the illustration of blended reasoning: students coordinated time and distance while simultaneously preserving the graph's visual shape in the robot's motion. Although prior research has documented iconic translations and quantitative interpretations separately, fewer studies have illustrated this kind of simultaneous blending of figurative and operative aspects of thought within the same activity. Here, students maintained locally correct distance–time reasoning while reorganizing the physical situation in ways that obscured the intended unit-rate/slope relationship.

With respect to Goal 2, although many students began with iconic or trial-and-error strategies, several articulated quantitative interpretations by the end of the lesson and identified unit rate directly from the graph. Whole-class discussions helped students connect reading the value at one second, dividing distance by time, and using slope triangles. These moments suggest that the robotics environment supported connections between speed, unit rate, and slope. At the same time, this study does not claim that students developed deep or durable understandings of these relationships within a single lesson. Such understandings require sustained opportunities for reflection and practice. Moreover, because of space limitations, this paper focused primarily on moments when figurative and operative meanings co-occurred, rather than tracing individual shifts in detail—an important direction for future research.

Several limitations should be acknowledged. This study was conducted in a whole-classroom setting during a summer camp. While this context allowed observation of a wide range of student reasoning within a short period of time, it did not permit the same depth of individual analysis possible in one-on-one or small-group teaching experiments. Although classroom discussions often culminated in apparent consensus, shared verbal agreement does not guarantee shared meaning. I did not collect systematic individual-level evidence to determine the extent to which each student internalized the targeted quantitative interpretations of graphs or unit rate.

Future research could strengthen this line of inquiry by incorporating individual written reflections, such as asking students to independently explain the story a graph tells or to write code based solely on a graph without testing it physically. Such artifacts would provide insight into the internal coherence of students' reasoning. In addition, implementing pre- and post-assessments could help document changes in students' graphing meanings over time. Finally, conducting individual or small-group teaching experiments using the Robotics Graphing Task would allow for the construction of detailed models of student thinking and support iterative refinement of the task as a didactic object.

Acknowledgments

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FROM EXPLORATORY WORK WITH MANIPULATIVES TO THE DESIGN OF DIDACTIC SEQUENCES ON QUADRATIC EQUATIONS

DE LA EXPLORACIÓN CON MANIPULABLES AL DISEÑO DE SECUENCIAS DIDÁCTICAS SOBRE ECUACIONES CUADRÁTICAS

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An analysis is presented on the potential use of algebra tiles in university level algebra courses for students in economics and administration. Based on the identification of common difficulties in solving quadratic equation problems, an exploratory sequence of introductory algebra activities using algebra tiles is implemented to examine whether the favorable results reported at previous educational levels can be extended to higher education. The exploration es analyzed from an instrumentalist perspective, and hypotheses are proposed for instructional design with an emphasis on the instrumental genesis of algebra tiles.

Keywords: Algebra and Algebraic Thinking, Cognition, Design Experiments, Undergraduate Education

Introduction

Quadratic equations are part of students' educational trajectory from basic education through university. However, having encountered them previously does not guarantee a deep understanding of the concept or the ability to use them meaningfully in specific disciplinary contexts.

In university Algebra courses for Economics and Business Administration programs, it is common to find students who recognize the algebraic procedures associated with this type of equation but experience difficulties in interpreting them, representing them graphically, or relating them to problems involving cost, revenue, or profit analysis. In many cases, quadratic equations appear as a curricular requirement rather than as a tool for solving real situations in the administrative field.

Based on the identification of this issue, it is evident that algebraic difficulties developed during secondary education (ages 12–15) persist among university students, as reported by several authors, including Didiş (2023), O'Connor and Norton (2022), and López and colleagues (2016). In particular, the emphasis on algorithmic methods, such as the quadratic formula for solving the general equation $ax^2 + b\,x + c = 0$ or the strategy of completing the square, reveals that students often struggle to formulate equations, interpret their solutions, and especially understand the different types of solutions and how the elements of an algorithmic process relate to the original problem or situation. Therefore, this project proposes the use of concrete materials, specifically algebra tiles, as support for an experimental teaching approach aimed at addressing learning difficulties associated with solving quadratic equations in Economics and Business Administration contexts.

With this overall goal, and as part of the preliminary analysis conducted to generate instructional design principles, we present here the results of an exploratory study carried out with the specific objective of confirming students' difficulties in solving algebraic equation problems and assessing the feasibility of implementing instruction supported using algebra tiles with undergraduate students.

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Theoretical Framework

This study is grounded in the Theory of Instrumental Genesis and Instrumental Orchestration developed by Luc Trouche (2004). From this perspective, an artifact such as algebra tiles does not automatically become a learning instrument simply because it is present in the classroom. For this transformation to occur, students must develop utilization schemes that integrate action, meaning, and intention. The analysis focuses primarily on Instrumental Genesis in terms of two interrelated processes. On the one hand, the artifact influences the student's activity (instrumentation); on the other hand, the student adapts, reorganizes, and reinterprets the artifact according to their purposes (instrumentalization). This process is not immediate; rather, it is gradually constructed through interaction, reflection, and discussion about how the material is used. To describe the development of instrumental genesis, we also draw on the distinction between *usage schemes* and *instrumented action schemes*. Usage schemes refer to the actions and strategies that a user develops to handle an artifact and transform it into an instrument. These schemes may be individual or collective and evolve through practice. They are oriented toward the basic management of the artifact and include the actions and knowledge required to operate its fundamental functions (Trouche, 2004).

Instrumented action schemes are schemes that combine mathematical knowledge with the use of an artifact to solve problems. In other words, they involve not only knowing how to operate the tool but also integrating it into the process of mathematical reasoning. These schemes are more complex and are oriented toward the accomplishment of specific tasks using the artifact. They require a deeper integration of knowledge, as they are based on how the user employs the artifact to carry out a particular activity.

Therefore, the preliminary analysis presented here focused on observing the types of schemes that participating students developed while using algebra tiles to represent and solve quadratic equations.

On the other hand, the possibilities for instruction are analyzed in terms of Instrumental Orchestration. Within this approach and based on the identification of the characteristics of instrumental genesis, potential teaching strategies that could be implemented to foster instrumental genesis in the mathematics classroom are examined, as suggested by previous studies conducted by Tasayco (2021), Manrique (2020), and Jiménez and Mejía (2015).

In this project, we are interested in identifying which instructional strategies could be implemented to support the teaching of quadratic equations using artifacts such as algebra tiles.

Methodology

We worked with a group of first-semester students enrolled in the bachelor's degree in business administration, implementing two activities adapted from a textbook based on the use of algebra tiles (Wah & Picciotto, 1994). In the first activity, students were expected to recognize, through direct manipulation, how each tile represents a unit, a variable, or a quadratic term, and to establish relationships between these concrete pieces and the algebraic expressions that are traditionally written on the board. Rather than memorizing equivalences, the aim was for students to understand the meaning of each representation and how they are connected to one another.

In the second activity, students were invited to build on the work completed in the first activity and use the algebra tiles to construct the corresponding equation. The intention was for them to visualize how a quadratic expression emerges from the multiplication of two linear expressions and based on that construction, to use the material as a tool for analyzing and solving the equation (Figure 1).

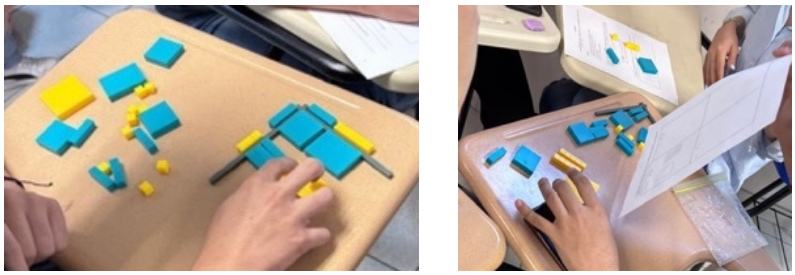


Figure 1: Students Carrying Out the Binomial Multiplication Activity

The adaptation of the instructional materials included the design of structured sessions that allowed students to actively interact with the manipulatives and digital tools, promoting the collective construction of meaning. During the implementation, classroom dynamics emphasized collaborative work and guided discussion, so that interactions among teams encouraged the comparison of strategies, the validation of results, and joint reflection on different ways of approaching the proposed problems.

Based on the implementation of the activities, the classroom sessions were analyzed to identify the difficulties involved in systematizing the expected uses of algebra tiles for formulating and solving equations.

Instrumental Orchestration is characterized by three fundamental concepts: didactical configuration, mode of exploitation, and didactical performance.

Didactical configuration refers to the structure of the learning environment, that is, the way in which tools, students, and tasks are organized. In this project, this structure involves collaborative work with algebra tiles and digital exploration using GeoGebra, focused on contextualized problems related to Business Administration.

Mode of exploitation concerns the way in which the teacher implements the task, including providing instructions, posing questions, and engaging students in the learning process. Rather than intervening directly in the completion of the task, the teacher supports students through guiding questions that encourage reflection without explicitly solving the activities for them.

Finally, didactical performance refers to the teacher's actual practice during the lesson. It takes place throughout the intervention as the teacher responds to unexpected events, develops new strategies, and facilitates student participation. At this stage, the teacher's flexibility becomes evident, as well as their ability to guide the process of Instrumental Genesis according to the emerging needs of the group.

One of the instrumental orchestrations selected was the screen-board linkage orchestration. It should be noted that this type of orchestration represents a classroom management strategy in which the teacher orchestrates, guides, and coordinates the connections among students, technological resources, and traditional instructional media.

General Description of the Exploratory Study

When carrying out the adapted activities, the algebra tiles function as a manipulable model that allows students to physically organize the terms x^2, x, and the constants, while establishing connections between concrete and symbolic representations. This process supports the development of modeling and algebraic reasoning skills through the identification of factors and the structuring of procedures (Picciotto, 1990). The tangible outcome of the activity is the correct

construction of a quadratic expression from the combination of two linear expressions, demonstrating that algebra tiles serve not only as a visual aid but also as a representation of the underlying algebraic structure of a quadratic equation.

The usage scheme includes implicit rules such as grouping tiles of the same type, forming rectangles or squares to represent products, and arranging the tiles in ways that reflect the structure of the equation. This scheme functions as an internalized set of instructions, guiding students in how to manipulate the tiles meaningfully before developing their own strategies. In the first activity, it was observed that some students were able to identify the relationship between the tiles representing the unit, the variable, and the squared term, establishing correspondences between the concrete objects and their symbolic representations. However, during the development of the tasks, a recurring difficulty emerged related to *elementary operations*, particularly the addition and multiplication of like terms. Some students experienced difficulties when performing basic calculations or when establishing equivalences between the concrete representation and the corresponding symbolic expression. The second activity focused on constructing basic configurations with algebra tiles to represent products of linear expressions. Students worked in teams. The purpose of this exercise was for them to identify the structure of quadratic expressions generated by the multiplication of two linear expressions and to verbalize their observations regarding the relationships between the concrete construction and its corresponding symbolic expression. At least two teams were able to explain, in their own words, the meaning of the representations they had constructed with the algebra tiles, as can be seen in Figure 2.

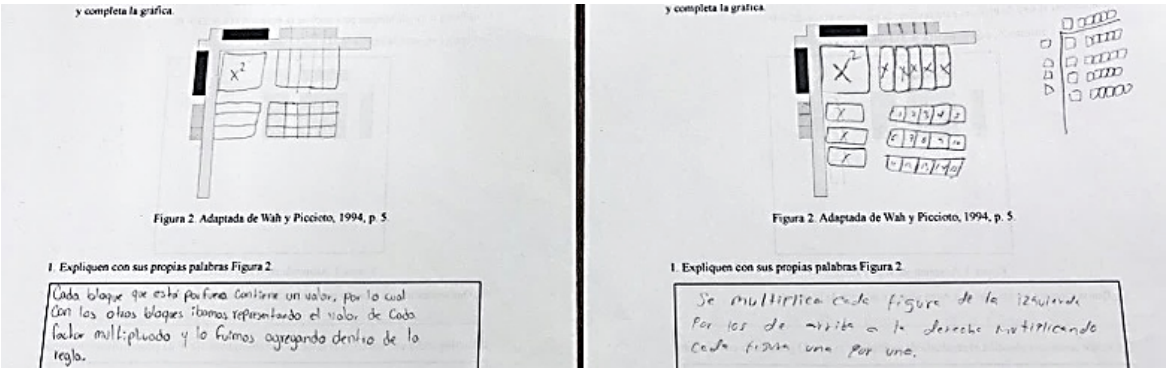


Figure 2: Responses from Two Student Teams to Activity 2

Among the most relevant aspects that emerged during the exploratory implementation were the students' active and collaborative participation, their interest in manipulating the material, and their willingness to discuss and compare procedures. However, difficulties in elementary operations also became evident, particularly in the simplification of algebraic expressions.

Final Considerations

Unlike primary and secondary education contexts, where algebra tiles are typically used to support the initial understanding of algebraic concepts and the development of basic operational skills, in this proposal manipulable materials are incorporated into a higher education setting with different purposes. Their use is intended to support mathematical modeling processes related to situations encountered in the fields of Economics and Business Administration, promoting the interpretation, analysis, and validation of results in contextualized problems.

In this sense, the manipulable material is neither an end in itself nor a strategy for introducing elementary algebra. Rather, it serves as a pedagogical resource that supports the construction of meaning around quadratic equations and their application in professional contexts, enabling students to establish connections among concrete, algebraic, and contextual representations that are relevant to their university education.

It is worth noting that, the exploratory experience revealed that the use of algebra tiles transformed the usual classroom dynamics. While working with the material, students did not simply carry out procedures; they began discussing how to organize the tiles, questioning the results they obtained, and explaining their decisions to their peers. On several occasions, disagreements emerged that required them to revisit and revise their constructions. However, it became evident that specific strategies are needed to promote the development of both usage schemes and instrumented action schemes. In this regard, the project will draw on the taxonomy proposed by Paul Drijvers et al. (2010, p. 998).

Regarding the design hypotheses, we propose that the development of usage schemes can begin through activities involving binomial multiplication, quadratic equations, and their solution using the "completing the square" procedure. Furthermore, we hypothesize that the construction of an instrumented action scheme for factorization could be fostered based on these usage schemes, supported by figural strategies for completing squares.

Rather than confirming that the manipulative material automatically "improves" learning, what was observed is that it creates a space in which algebraic relationships can be discussed through concrete representations before being formalized symbolically. This observation has led us to design four instructional sequences aimed at providing continuity and greater depth to what began as an exploratory experience.

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TYPES OF INTERNAL STRUGGLES WHEN SOLVING A GROWING PATTERN TASK

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Students' internal struggles that are not visible to teachers when solving a growing pattern task were explored. Seven 11th-grade students solved a pattern problem by thinking aloud. Data in the form of recorded interviews and students' work were analyzed by using open coding. Task-based clinical interviews documented the most frequent struggles when solving a growing pattern task. The most repeated struggle was writing a quadratic equation. The other struggles were knowledge of variables, generalization for growing patterns, drawing the figures, use of multiple representations, and reading the full question.

Keywords: Internal Struggle, Pattern Task, Clinical Interviews

Pattern tasks, such as table tiling or growing patterns, place significant cognitive demands on students, requiring them to recognize patterns, coordinate visual and numerical information, and generalize relationships across stages (Hiebert & Grouws, 2007; Beatty et al., 2013). These demands often give rise to struggle, both externally, as evidenced in errors or requests for clarification, and internally, as uncertainty or difficulty in forming rules (Warshauer, 2015). Wilkie (2022) argued that figural pattern generalizations are cognitively demanding for students, and noticing the functional relationship and symbolizing requires meaningful development. Guided by the framework of productive struggle, which views learning as developing understanding through sustained sense-making with challenging tasks (Hiebert & Grouws, 2007; National Council of Teachers of Mathematics [NCTM], 2014), this study informed the design of the patterning task and the analysis of students' difficulties in generalizing patterns and formulating quadratic equations. Building on the literature, this paper focuses on answering the research question: *How do students experience internal struggles when solving a growing pattern generalization task during task-based clinical interviews?*

Research Settings and Participants

Seven 11th-grade students, four males and three females, attending a public charter high school in the central-south United States, participated in this study. Each student has previously completed Algebra 1 and Geometry. Before the interviews, students covered the linear and quadratic functions in the Algebra 2 course. The researcher and the teacher identified the students based on classroom observations. Students who showed signs of struggle during class or who could not progress on the tasks were selected for an interview. The participants remained anonymous.

Methods, Materials, Data Collection, and Analysis

A task-based clinical interview (Clement, 2000; Goldin, 1997) was used to provide a detailed picture of participants' internal struggles and individual differences. During the interview, students sat down with the researcher, who set up and explained the task and asked them questions to probe their thinking as they worked. Participants had access to calculators, manipulatives, and other classroom materials. Participants were asked to vocalize the words in

their minds while working on the task. The table tiles task (Mathematics Assessment Resource Service [MARS], 2015) was used for the interview (see Figure 1).

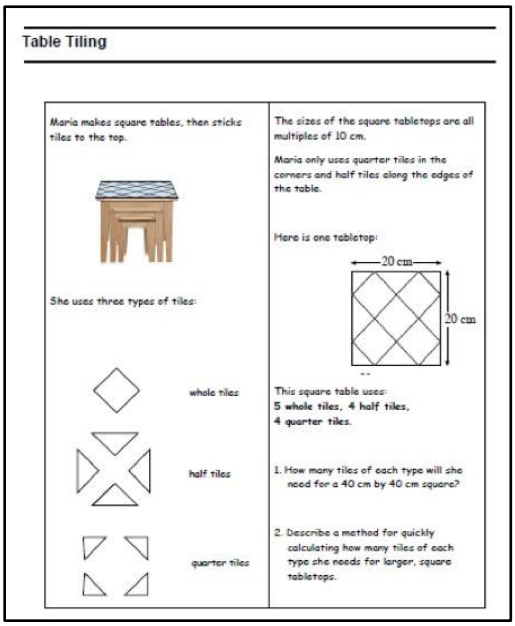


Figure 1: Table Tile Task (Adapted from MARS, 2015)

The first part of the task aims to evaluate whether students can draw a 40-by-40 square using the given figure as a reference and identify the number of each tile type. The second problem was designed to gather data on students' awareness of examining data, identifying patterns, and generalizing those patterns using multiple representations. Participants were also asked to describe which part of the task was most challenging. The seven clinical interviews lasted approximately one hour. The data consist of transcribed video and audio recordings from seven interviews, along with participants' written data.

The data were analyzed using content analysis to yield a concise yet broad description of students' struggles. Interview transcripts were analyzed using *open coding* (Corbin & Strauss, 2014). The interview data were analyzed in three stages suggested by Creswell (2015): preparing the data by transcribing, reducing the data into themes through coding, and representing the data. The authors noted that patterns are identified through data familiarization, data coding, and theme development and revision. After creating the initial codes from the interview transcripts, an Excel file was used to organize and document extracts and their corresponding codes, and to support line-by-line coding of the students' clinical interviews. In the next step, themes were developed by identifying significant patterns of meaning.

Preliminary Results

These qualitative data focused on seven high school students working through the table-tiling task. The task-based video-recorded clinical interviews were analyzed, and 115 codes were created from the video recording transcripts. Struggle codes were merged into main themes (see Table 1). Confusion, overthinking, and difficulty identifying half and quarter tiles were classified into the "other" theme. In this paper, we focus specifically on the most frequent struggle identified among participants: writing a quadratic equation.

Table 1: The Distribution of the Codes for Struggle

Struggles	Number of Codes	Frequency
Writing a quadratic equation	42	37%
Knowledge of variables	19	17%
Generalization for growing patterns	16	14%
Drawing the figures	16	14%
Use of multiple representations	9	8%
Read the full question	9	8%
Other	4	3%

Writing a Quadratic Equation

The categories in Table 1 were developed through iterative qualitative coding and represent the primary observable difficulty in a student response. As shown in Table 1, the most frequent struggle theme was writing quadratic equations. The table tiles question asked for a method to determine the number of tiles for larger square tabletops. Students needed to create an equation to determine the number of each tile type given the tabletop size. For the number of quarter tiles, students were expected to obtain a constant function; for the number of half tiles, a linear function; and for the number of whole tiles, a quadratic function. All students easily found the number of quarter tiles because it was constant. All seven participants constructed a table to represent the pattern. Six participants used the slope-intercept form to generate a linear equation representing the number of half tiles. While Student 1 (a typical response) made a connection between a geometric pattern and an algebraic equation, Student 7 constructed two separate tables: one in which x denoted the figure number and another in which x denoted the length of the figure (see Figure 2).

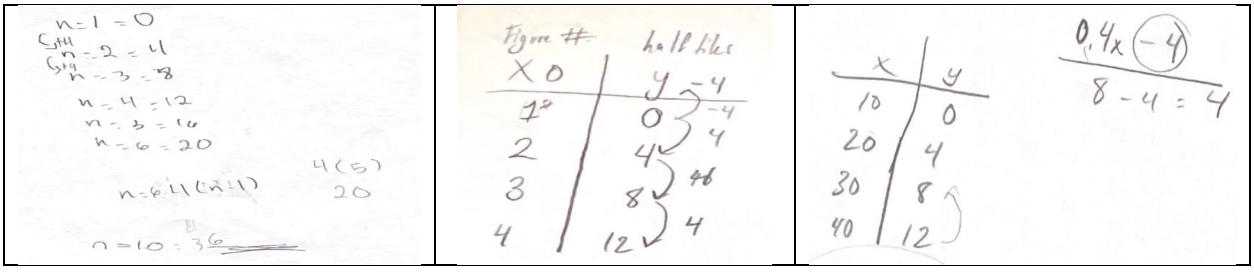


Figure 2: Student 1's Solution (Left) and Student 7's Solution (Middle and Right)

On the other hand, all seven participants struggled to write the equation for the whole tiles, which is a quadratic equation. Only Student 1 got the algebraic equation for the whole tiles without using a calculator. Student 1 made a connection between a geometric pattern and an algebraic equation. Four participants used the TI-Nspire calculator to determine the equation for the whole tiles by entering the table values. However, they did not initially recognize that the relationship required a quadratic equation; they only realized this after the calculator generated a quadratic model. Students stated that writing the equation for whole tiles was the most challenging part of the task. Figure 3 presents an excerpt from the clinical interview that highlights the difficulties students encountered in determining the quadratic equation for the

number of whole tiles. Student 3 created a table and sketched a graph for the whole tiles, attempted to identify a pattern, and used three forms of quadratic equations; however, he was unable to determine how to construct the equation (see Figure 3). Students experienced difficulty identifying the appropriate type of equation.

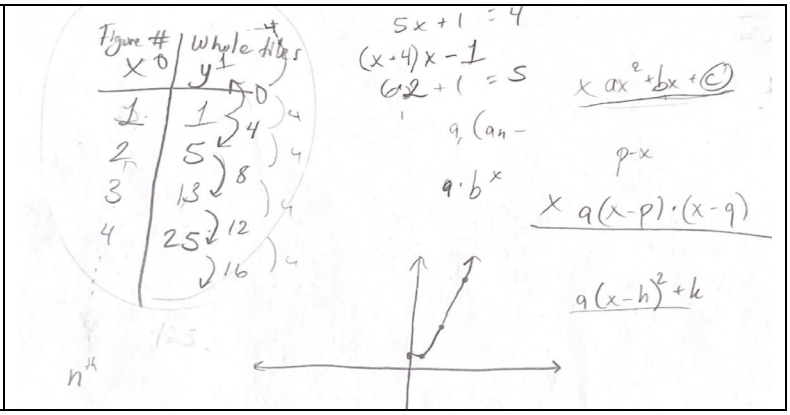
Student 3: "The most challenging part for me is finding the equation for the whole tiles. Since it is a quadratic equation, it feels unfamiliar. We cannot determine the vertex; we do not know the zeros, the values of *a*, *b*, and *c*, or the *y*-intercept. These three equations do not help."	

Figure 3: An Excerpt from Student 3's Interview (Left) and Student 3's Solution (Right)

After constructing the table, all participants recognized that a linear function described the half tiles. However, none explicitly stated that a quadratic function was needed for the whole tiles. Some students also assumed that if the relationship was not linear, it might be exponential, limiting their reasoning to only two options: linear or exponential.

Conclusion and Implications

Participants in the interviews demonstrated that writing a quadratic equation to generalize a growing pattern was the most challenging part of a high-demanding problem-solving task. One of the main issues in identifying the quadratic pattern was students' inability to connect the geometric features of the shape—such as how it was constructed—to how that structure could inform the development of an equation. Quadratics are the first non-linear polynomials that students find difficult, and they struggle to understand what makes a relationship quadratic (Suominen, 2018). This finding aligns with Orton et al.'s (1999) study, in which participants were able to translate the values into a table but were not successful in finding the equation. The analysis revealed that students experience significant difficulty in generalizing patterns, particularly when the underlying relationship is non-linear. These findings may help teachers support students' understanding of quadratic functions through tasks that connect geometric and algebraic representations. A limitation of this study is its small, context-specific sample. Future research could examine how individual and collaborative settings influence students' ability to generalize patterns and formulate equations.

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MIDDLE SCHOOL STUDENTS' CONSTRUCTION AND COMPREHENSION OF MATHEMATICAL VARIABLES

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This pilot study investigates the ways middle school students understand mathematical variables in the contexts of numeric solving, graphing, and using equations. Three students participated in a series of task-based interviews designed to highlight various components of their understanding of variable. A framework was developed throughout the interviews to classify a student's variable understanding as either fixed, chunky, complete, or as a combination of two of the categories. The results indicate that a student's understanding of variables within equations and within graphing are often different. Additionally, a student's use of equations may be the strongest indicator of their understanding of variables across all contexts.

Keywords: Algebra and Algebraic Thinking, Middle School Education

Across mathematics, variable plays a central role, and thus it is important that students have a full and accurate understanding of what a variable is. This understanding can begin developing as early as elementary school but is addressed more in the middle grades. How a middle-school student conceptualizes variable affects the way they construct general middle-school mathematics and thus can have a lasting impact on their experience in mathematics as a whole (Bush & Karp, 2013; Hackenberg & Lee, 2015; Lee, 2018; Smith & Thompson, 2008). Although research has investigated how students handle the concept of variation and covariation more generally (Castillo-Garsow et. al., 2013; Thompson & Carlson, 2017), there is no universally agreed-upon definition of a variable within mathematics, making it difficult to determine a student's understanding of variable (Ely & Adams, 2012). More research is needed to better help students develop a robust understanding of variable. As such, in this pilot study, we iteratively refined a framework of students' variable understanding based on their approaches to tasks involving variables. More specifically, we were seeking to answer the question: In what ways does a student's approach to solving for numeric values, graphing relations, and using equations indicate their level of variable understanding?

Theoretical Framework

Approach

Student understanding of mathematical concepts will be viewed through a Neo-Piagetian framework, in that an individual's conceptualizations are based on mental objects and operations (Castillo-Garsow et al., 2013; Johnson, 2016). Patterns in student explanations and behaviors are indicative of their underlying reasoning and conceptualizations. The researchers generalized these patterns and used them to iteratively refine a framework for variable understanding.

Defining Quantitative Reasoning

Quantity will be defined as a conceptualized object that is measurable in some way (Thompson & Carlson, 2017; Hackenberg et. al., 2021). A quantity defined in this way is a cognitive object that lives in the mind of the individual. From this definition, we define quantitative reasoning as the conceptualization of a certain situation in terms of quantities and the relations between them (Thompson & Carlson, 2017). A student's understanding of variable is affected by their quantitative reasoning because a variable is a type of quantity.

Defining Variable

Thompson and Carlson (2017) provide three ways in which students have been observed to understand variables. The first of these is the understanding of a variable as a fixed quantity with no variance. This will henceforward be referred to as *fixed understanding*.

Two other student understandings of variable Thompson and Carlson note are a) a value that changes based on context, and b) a quantity's value that varies within a given context. Castillo-Garsow and colleagues (2013) make this division more distinct by considering how a student imagines change. While thinking about a variable changing, students can construct chunks or units of values and then imagine change as being the shift from one chunk to another. When constructing these chunks, the student is not imagining variation or active change within the chunk. Rather, they focus on the discrete points on the edge of each constructed chunk. Castillo-Garsow and colleagues call this chunky thinking. In contrast, the ability to think about variation without chunks is called smooth thinking, because the values are seen as continually changing. For the purposes of this research, chunky and smooth thinking will be respectively referred to as *chunky understanding* and *complete understanding*.

Although these three understandings encapsulate most students' understanding of variables, the definitions tend to imply a variable does not vary with respect to anything else. However, when given solely one variable, most students insert time as a secondary variable the initial variable is changing with respect to. Thus, despite not all researchers having this perspective, all variation will be understood to be a form of covariation for the purposes of this research. Covariation is the concept that a quantity's value varies as another quantity varies (Thompson & Carlson, 2017). Ely and Adams (2012) include covariation as something that distinguishes a variable from an unknown or a placeholder. Note this idea of a varying quantity aligns closely with complete understanding of variable described above.

The full definition of variable for this research results from combining complete understanding of variable and covariation. We define a mathematical variable to be something that stands for a generic element of a large set of values that varies alongside another variable and takes on values as the other variable continually changes. As explained above, each participant's understanding of variables will be classified as either fixed, chunky, or complete.

Table 1: Initial Framework Created Prior to Data Collection

Category	Variable Understanding	Example
Fixed	A variable represents a single number. A variable's value does not change in a given context.	When illustrating a covariational relationship, the student draws a number line or plots a single point.
Chunky	A variable can represent multiple values. A variable's value can change in intervals or chunks.	When illustrating a covariation relationship, the student plots multiple points, then connects them.
Complete	A variable can represent multiple values. A variable's value can change in a smooth or continuous way.	When illustrating a covariational relationship, the student can sketch an accurate graph without plotting points.

Initial Framework

The threefold division of variable understanding (fixed, chunky, and complete) was used to construct a framework. The purpose of this framework was to classify students' understandings

of variable based on their approach to tasks. A simplified version of the initial framework developed from a review of literature can be found in Table 1.

Methods

Data Collection

Participants were middle school students from private schools in the Pacific Northwest United States. Three students were screened to assess their current understanding of variables and arithmetic reasoning (as part of a larger study). All participants had completed at least grade 6 math and at most pre-algebra. As a result, some mathematical topics covered in the interviews were novel for the students, providing the opportunity to see their initial thoughts and conceptions. Table 2 displays the results of the screening for each student. Note that all three students were placed at both chunky and complete understanding for solving. At this stage of the study, it was hypothesized that a student would have either fixed or not fixed variable understanding for solving.

Table 2: Screening Results for the Participating Students

Student	Solving	Graphing	Equation	Underlying Reasoning
S1	Chunky/Complete	Complete	Chunky	Chunky
S2	Chunky/Complete	Fixed	Fixed	Chunky
S3	Chunky/Complete	Complete	Fixed	Chunky

After the screening, all three students were invited and agreed to participate in a full set of task-based semi-structured clinical interviews (Clement, 2000). Interviews lasted between 30 and 60 minutes and involved the researchers and an individual student, totaling six interviews per student, with one task per interview. One researcher acted as the primary teacher-researcher, with the other as a research observer. Tasks were completed on a screen-recorded iPad, and three of the tasks involved an animation the students could interact with on the iPad. All interviews were also video- and audio-recorded to note students' gestures and explanations. Three types of tasks were used, focusing on either generating an equation, graphing a given equation, or numerical solving. For each type of task, there was a linear and non-linear relationship involved. The order in which the tasks were presented to the students alternated between the equation, graphing, and solving types, while also alternating between linear and non-linear (Table 3). It was structured in this way to minimize the impact of students' previous work on their approach to each new task.

Table 3: Task Descriptions

Task	Type	Animation	Equation given	Relation
T1	Equation	Yes	No	Linear
T2	Graphing	No	Yes	Quadratic
T3	Solving	No	No	Linear
T4	Equation	Yes	No	Exponential
T5	Graphing	No	Yes	Linear
T6	Solving	Yes	No	Exponential

Data Analysis

Several rounds of analysis were conducted in this study. Immediately following each interview, together the researchers noted their initial thoughts on what level of variable understanding the student displayed during the interview. These analyses served as a gauge for the researchers' understanding of the framework as they became exposed to more data. The primary researcher did small-scale data analysis between any two interviews for a student. During this analysis, little comparison was made between different tasks and students, primarily focusing on a student's work within one task.

After conducting all interviews, a large-scale retrospective analysis was performed by the primary researcher, focusing on one student at a time across all tasks. The recordings of each interview for a student were reviewed closely by noting indicators of specific levels of variable understanding. For example, say a student creates a graph by solving for two points, plotting them on a pair of axes, and then drawing a line to connect the points. By plotting the two points and then connecting them after, the student visually constructed a chunk and acknowledged intermediate values after constructing it. Based on the framework shown in Table 1, this would be an indicator of chunky variable understanding for graphing.

Once all the tasks for the student were fully reviewed, comparisons were made across all the tasks for each student, and an overall placement within the framework was selected. This process was done for all three students. When this was completed, the three students' attributed levels of variable understanding were compared to check for inconsistencies and to note any patterns across the tasks.

Results

Throughout data collection, the initial framework (Table 1) was revised to include more specific indicators in a student's numeric solving, graphing of relations, and use of equations. As this adaptation continued, the three levels of variable understanding became distinguished by two main features: determinate versus indeterminate, and not covariational versus covariational. This aligns with the definition of variable provided by Ely & Adams (2012). These features can be found in the simplified version of the full framework displayed in Tables 4-6. Because fixed understanding inherently lacks the conception of covariation, note that this framework cannot fully capture nuances within fixed understanding. Included below are the results of the large-scale retrospective analysis. The three participants will be referred to under the pseudonyms of Addie, Jim, and Duncan. The students were placed at either fixed, chunky, or complete variable understanding for solving, graphing, use of equations, and underlying reasoning.

Solving

Whenever a student was determining a particular numeric value based on other known values, this was considered numeric solving. Numeric solving did not align well with the three-fold division of variable understanding because it reduces a variable to a determinate value. For the purposes of this report, solving will not be discussed further.

Graphing

Behaviors surrounding the creation or interpretation of a graph were considered indicators of variable understanding while graphing (see Table 4). Figure 1 displays written work from each student when graphing in Task 4. In this task, students were given a scenario where a growing redwood tree starts as half of a foot tall and triples in height every two days. The students were asked to graph their understanding of the scenario.

Table 4: Framework for Graphing

Variable Understanding	Feature of Variable Understanding	Behavioral Indicators
Fixed	Determinate Not Covariational	A student plots one or two values (that are known) on either a number line or on a graph. The student treats each plotted value as an independent object.
Chunky	Determinate Covariational	A student plots multiple values that are known on a graph and then connects the points with a line or curve, acknowledging there are more points. When plotting values, they tend to be evenly spaced out. The student can articulate why the graph is shaped a certain way.
Complete	Indeterminate Covariational	A student is able to plot an accurate sketch of an entire relation without plotting more than one known (solved) value. The student can articulate why the graph is shaped a certain way.

Addie's graph Jim's graph Duncan's graph

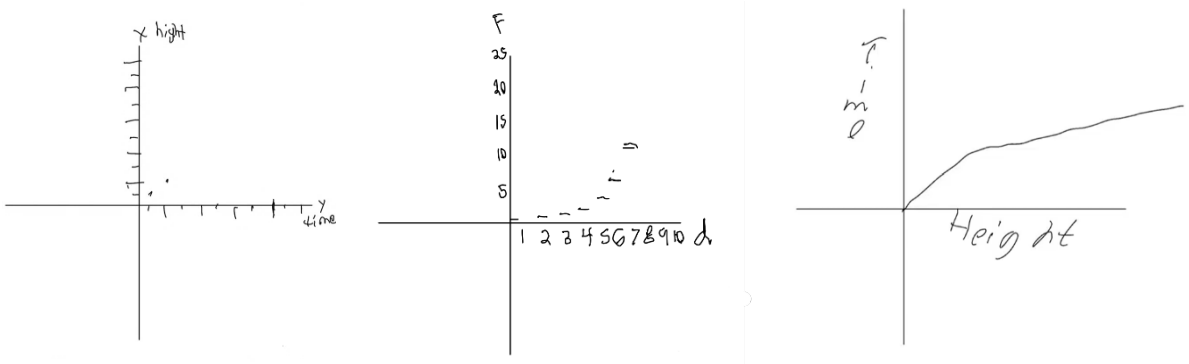


Figure 1: Student Graphs from Task 4

Addie demonstrated a combination of fixed and chunky variable understanding while graphing. When a task did not have time as a variable, she had a more fixed understanding. In these tasks, she initially only plotted one point whose value she had already solved for. After plotting more points, she saw each point as independent of the others, indicating limited covariational thinking. When a task did have time as a variable (the tasks with animations), she had a more chunky understanding. Her ability to sketch the graph's shape and describe its end behavior indicates that she thought about covariation in these contexts. However, she remained focused on the plotted points and previously solved values, treating the values as determinate.

Jim appeared more comfortable with graphs and demonstrated a chunky variable understanding while graphing. For four of the six tasks, he plotted points easily and without much thought, often referring to each point as an "answer," which is a determinate description. When asked about other points besides the ones he drew, he frequently talked about splitting sections with midpoints and forming additional chunks of equal size.

On the two tasks where an equation was given rather than student-generated, Duncan also displayed a chunky understanding. For these tasks, the graph was a display of "the answers" to the equation, just as Jim explained. However, unlike Jim, Duncan displayed a complete understanding of variable while graphing when an equation was *not* given (see Figure 1). He easily sketched a graph relating two quantities and did not focus on specific numbers, indicating he saw the variables as indeterminate. Once he created a graph, he clearly explained how it demonstrated the relation between multiple quantities, revealing his covariational thinking.

Equation

When a student was generating or interpreting an equation, the associated behaviors were considered indicators of variable understanding while using equations (see Table 5).

Table 5: Framework for Equation

Variable Understanding	Feature of Variable Understanding	Behavioral Indicators
Fixed	Determinate Not Covariational	A student interprets a variable in an equation as something that gives or stands for an "answer". When asked to create an equation, they construct an equality exclusively with numbers. When given an equation, a student cannot interpret how two variables are related.
Chunky	Determinate Covariational	A student interprets a variable in an equation as something that gives or stands for an "answer". When asked to create an equation, they construct an expression with at least one variable, treating some or all of them as placeholders for known values. When given an equation, a student can interpret how two variables are related.
Complete	Indeterminate Covariational	A student interprets an equation as something that relates two changing quantities. When asked to create an equation, they construct an expression with two or more variables. When given an equation, a student can interpret how two variables are related. A student can isolate a variable without dropping operations.

Across all tasks, Addie displayed a fixed understanding of variable when working with equations. Although she often verbally described a pattern between two quantities, she did not generalize it or create an equation. Any expression she created involved only numeric values or had a single unknown, which are very strong indicators for determinate thinking. When given an equation, she often had difficulties interpreting what it was representing, usually focusing on a few specific values for each variable involved. This shows she was unable to see the covariation in the equation.

In contrast, Jim and Duncan both displayed chunky understanding of variables when working with equations. Throughout all the tasks, they treated equations as a source of an answer or solution, reducing the variables to a determinate value. When given an equation, they articulated dependence and some covariation between the variables. However, their tendency to plug in numbers to "check" the equations is a counter-indicator of complete variable understanding. It is

evident that both Jim and Duncan tended to see variables in an equation as standing for either known numbers or solvable unknowns. An explicit instance of this is from Duncan when talking about the equation $\frac{1}{3}b = a^2$ in Task 2.

> Duncan: So this is kind of like a formula. So there's variables, and you can put in different numbers to make, to try and make it like, solvable. So, what I'm thinking, when I answered your question, was we like, put a number, for b.

This instance shows Duncan thinking of b and variables in general as standing for a number and not as something that varies. This is a very deterministic understanding of variable and is a strong counter-indicator of complete variable understanding when using equations.

Underlying Reasoning

An indicator was attributed to a student's underlying reasoning when the student explained their thinking in a more general sense or made connections between different parts of the task (such as connecting solved numeric values with a graph). A student's underlying reasoning impacts all aspects of the student's approach to a task and is not limited to either solving, graphing, or using equations. See Table 6 for a simplified version of this part of the framework.

Table 6: Framework for Underlying Reasoning

Variable Understanding	Feature of Variable Understanding	Behavioral Indicators
Fixed	Determinate Not Covariational	A student understands that a variable can stand for a known number or an unknown number that can be determined. When presented with two varying quantities, a student understands them to be unrelated to each other.
Chunky	Determinate Covariational	A student understands that a variable can stand for a known number or an unknown number that can be determined. A student understands that a variable is related to or dependent upon another variable.
Complete	Indeterminate Covariational	A student understands that a variable can stand for a generic element of a set and can change. A student understands that a variable is related to or dependent upon another variable.

Addie demonstrated a combination of fixed and chunky understanding of variable in her underlying reasoning. Throughout all the tasks, she was highly reliant on numbers and was unable to generalize patterns, indicating determinate thinking and counter-indicating complete variable understanding. In some instances, she struggled to conceptualize the idea of an unknown and articulate covariation between two related quantities, which are counter-indicators of chunky understanding. Below is an instance in Task 1 where Addie explains why there are other values for x and y that she did not graph. In this task, students were given the scenario of a steady-growing plant that starts as 3 inches tall and is 9 inches tall after three days. x and y were the respective symbols chosen by Addie to represent height and number of days.

> Researcher: How many different numbers could those [x and y have]?

Addie: Oh, a lot because you just keep going up... Um, yeah, you could go up forever because you'd just be adding two and adding one.

When referring to "adding two and adding one," Addie is referring to adding two to the height and one to the number of days, indicating she saw the two values as being separate. She gave similar explanations throughout this task. However, there were instances where she could articulate some covariation. These behaviors were not consistent enough across all tasks for her to be placed fully in fixed or chunky variable understanding.

Jim and Duncan were more consistent in their underlying reasoning throughout the tasks and both demonstrated a chunky understanding of variables. As mentioned previously, both were very focused on getting an "answer" or "solution" throughout the tasks. Although they occasionally talked abstractly, they ultimately always fell back onto a determinate understanding of variable when they encountered a contradiction. Duncan was more comfortable with the idea of covariation than Jim, allowing him to describe relations with accuracy and additional detail. However, similarly to Jim, Duncan consistently described and treated variables as something that you put a number in for, which is a good summary of chunky understanding of variable.

Discussion

Although this pilot study has a small sample size, comparing each student's variable understanding placement across the categories of graphing, equation, and underlying reasoning provides insight into what impacts a student's understanding of variable. It is notable that each student's variable understanding while graphing and using equations is not always the same. This implies a student who has a fuller understanding of variables in graphing may struggle to translate that understanding to using equations, and vice versa. Further, these observations indicate a student may be able to graph or use equations with variables well, while not truly understanding what a variable is. Since this observation has emerged in multiple studies (Carlson et al., 2002), it raises questions about what procedure might best gauge a student's variable understanding. Another noticeable pattern in the results is that, across all tasks, each student's variable understanding while using equations never exceeded their variable understanding in their underlying reasoning. This implies a student's comfortability with using equations may be a strong indicator of their true understanding of variable. It is worth noting that variable understanding while graphing did not display the same pattern.

During analysis, it was observed that the students' variable understanding shifted depending on the given task's attributes. For example, Addie displayed chunky understanding more when time was a variable, and Duncan displayed complete understanding more when not given an equation. Previous research has also shown that context can largely impact a student's interpretation of a scenario (Izsák, 2011). Although task attributes were not the primary focus of this study, these findings ought to be investigated in the future.

Considering the small number of participants in this pilot study, further research ought to be done regarding the relation of equations, graphing, and students' underlying reasoning with variables, as it may have implications on methods of assessment and tracking of student understanding. Additionally, the framework created in this study ought to be tested with a larger variety to students for further revisions.

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DESCUBRIMIENTO GUIADO DE ALGUNAS CARACTERÍSTICAS DE LA FUNCIÓN CUADRÁTICA

GUIDED DISCOVERY OF SOME CHARACTERISTICS OF THE QUADRATIC FUNCTION

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Se presentan los resultados de un estudio cualitativo cuyo objetivo fue analizar cómo los futuros ingenieros descubren de manera guiada la influencia de los parámetros de la función cuadrática en su gráfica, a partir de la realización de actividades de una secuencia didáctica. El estudio está basado en la teoría de Educación Matemática Realista y contó con la participación de treinta estudiantes de diferentes carreras de Ingeniería. Se diseñaron tres actividades que están contextualizadas dentro de situaciones de la ingeniería Civil: canal hidráulico abierto, carga parabólica que recibe una liga en voladizo y puente Juscelino Kubitschek de Brasilia. Los resultados muestran cómo fue el proceso de matematización de los estudiantes al pasar por los cuatro niveles de comprensión: situacional, referencial, general y formal.

Palabras clave: Función cuadrática, Educación matemática realista, Ingeniería civil

Introducción

La función cuadrática es un tema importante en distintas áreas de la vida cotidiana, su aplicación es amplia en la mecánica, la economía, la ingeniería entre otros. Por ejemplo, en el caso de la Ingeniería Civil, los cálculos estructurales, cantidad de materiales y los costos de trabajo de puentes colgantes o canales de agua con frecuencia provienen de las parábolas que los representan. Sin embargo, varios investigadores mencionan que los alumnos tienen dificultades en el aprendizaje de este tema relacionadas con: articulación de registros gráficos y algebraicos (Endara, 2021); lenguaje algebraico (Villada, 2013); vinculación de la parábola con sus diferentes formas de notación: estándar, canónica y factorizada (Díaz et al., 2015; Villada, 2013).

Dado que la función cuadrática es un tema importante para distintas áreas, y existen dificultades de comprensión, es necesario cambiar la forma mecánica de enseñarla.

El propósito de esta contribución es presentar los resultados de una investigación desarrollada con alumnos de la carrera Ingeniería Civil, basada en Educación Matemática Realista (EMR), para quienes se desarrolló una secuencia didáctica que buscó propiciar el descubrimiento guiado de la influencia de los parámetros de la función cuadrática en la apertura de la parábola, su concavidad, desplazamientos verticales y horizontales.

La pregunta de investigación es ¿cómo evolucionan los niveles de comprensión—situacional, referencial, general y formal—durante el proceso de matematización del grupo de estudiantes al realizar las actividades de una secuencia didáctica basada en EMR del área de ingeniería en el marco de la función cuadrática?

Marco teórico

El estudio está basado en la teoría de la EMR propuesta por Freudental. Una de las principales ideas de la EMR es la matematización, la cuál puede ser de dos tipos: horizontal y vertical. En la matematización horizontal se traduce el problema del mundo real a un modelo matemático y se conectan los lenguajes natural, simbólico y formal. Y en la matematización vertical, el aprendizaje de los alumnos se traslada a lo largo del mundo simbólico. El proceso de matematización de un objeto matemático se desarrolla según cuatro niveles de comprensión: situacional, referencial, general y formal (Bressan et al., 2005).

El diseño de las actividades se da con base en seis principios: 1) de niveles, 2) de interacción, 3) de actividad, 4) de reinvención, 5) de interconexión, 6) de realidad (Bressan et al., 2005).

Metodología

La investigación fue de tipo cualitativa. Participaron treinta alumnos de las diferentes carreras de Ingeniería de una universidad pública. Sin embargo, sólo se describen los resultados de siete de ellos de Ingeniería Civil, por ser representativos de lo ocurrido en el aula.

Con base en los seis principios de diseño señalados por Bressan et al. (2005) se elaboraron tres actividades para promover la matematización. Debido a la extensión de esta contribución se presentan sólo los resultados de las dos primeras actividades.

La primera actividad está relacionada con un canal hidráulico abierto con una sección transversal de forma parabólica. Esta actividad se diseñó con el objetivo de que los alumnos descubrieran la influencia de los parámetros de la función cuadrática en la apertura de parábola, su concavidad y el desplazamiento vertical de la gráfica. En la segunda actividad se aborda una situación relacionada con una viga que se encuentra en cantiléver que recibe una carga distribuida a lo largo de este elemento. En la ingeniería civil, el ejemplo más sencillo de esta aplicación es un balcón. Esta segunda actividad se diseñó con el objetivo de que los estudiantes descubrieran cuáles parámetros determinaban los desplazamientos horizontales y verticales de la parábola. Y la tercera actividad está relacionada con el puente colgante Juscelino Kubitschek, en Brasilia. A partir de las características dadas en el problema, el estudiante debía proponer la función que describe a cada uno de los arcos.

Los instrumentos de recolección de datos fueron grabaciones de audio y hojas de trabajo. Estos se usaron para el análisis del proceso de matematización y para identificar la evolución de los niveles de comprensión—situacional, referencial, general y formal—alcanzados por los estudiantes.

Para el análisis del proceso de matematización se utilizaron los siguientes criterios de clasificación de los niveles de comprensión: 1) Nivel situacional. Los alumnos identifican visualmente una parábola en los objetos de la vida cotidiana. 2) Nivel referencial. Graficando las funciones cuadráticas con diferentes coeficientes propuestos por el docente, los alumnos comparan y analizan las gráficas de las funciones dadas. 3) Nivel general. Los alumnos proponen coeficientes de las funciones cuadráticas cuyas gráficas satisfacen características dadas. 4) Nivel formal. Los alumnos proponen las funciones cuadráticas que definen las parábolas, a partir de información obtenida de situaciones realistas.

Resultados

Enseguida se presenta el análisis de los resultados de la implementación de las dos actividades de la secuencia didáctica guiada, el proceso de matematización de los estudiantes y la evolución en sus niveles de comprensión.

La primera actividad

Los alumnos usando su intuición, y apoyados en las imágenes incluidas en la actividad (Figura 1), relacionaron el canal hidráulico con su modelo matemático $f(x) = ax^2$. La respuesta fue codificada como nivel situacional.

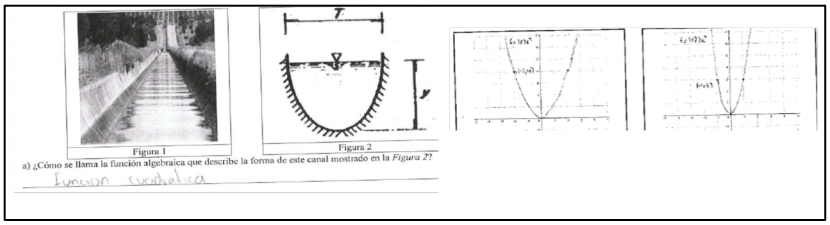


Figura 1: Respuesta de nivel situacional en la Actividad 1

Manipulando el valor de coeficiente a exploraron la influencia de éste en la apertura y concavidad de la gráfica, estas respuestas fueron codificadas como nivel referencial. Luego hicieron predicciones para determinar los valores del coeficiente a para los cuales la función $f(x) = ax^2$ se expande, se contrae o tiene cambio de la concavidad, estas respuestas fueron codificadas como nivel general de comprensión. En la Figura 2 se observa cómo los alumnos propusieron los parámetros para los cuales la gráfica tiene cambio de apertura y desplazamiento vertical indicado.

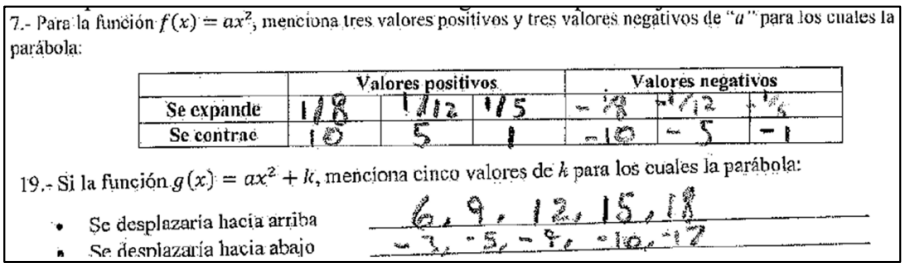


Figura 2: Respuesta de nivel general en la Actividad 1

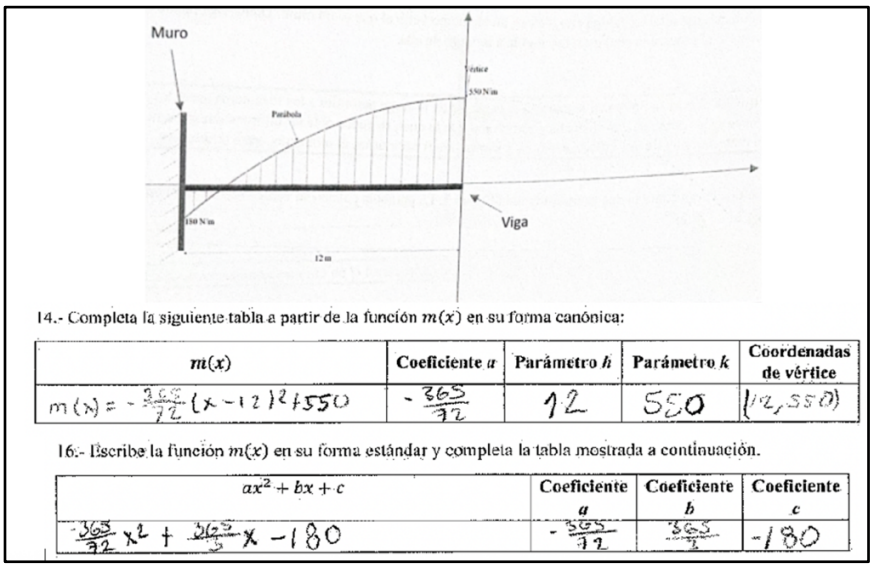
Finalmente, con base en los patrones identificados, establecieron que si el parámetro a pertenece a los intervalos $(-\infty, -1)$ o $(1, \infty)$ la apertura de la parábola es más ancha en comparación con los valores de a del intervalo $(-1,0) \cup (0,1)$; además, que la parábola cóncava hacia arriba para valores positivos de a y hacia abajo para los negativos. Estas respuestas fueron codificadas como nivel formal. Debido a que no hubo explicación previa por parte del profesor, el procedimiento de los alumnos se interpreta como descubrimiento guiado de estas características de la función.

Análogamente para la función de la forma $f(x) = ax^2 + k$, propuesta por el docente durante esta Actividad 1, descubrieron cómo los valores del coeficiente k, provocan desplazamientos verticales en la gráfica.

La segunda actividad

De manera similar a como se realizó en la primera actividad, en esta actividad relacionada con una viga en voladizo la cual recibe una carga parabólica, se introdujo el coeficiente h de la función cuadrática $f(x) = a(x - h)^2 + k$, el cual permite desplazar la parábola a la izquierda o

derecha. Los estudiantes graficaron funciones con diferentes valores del parámetro, analizaron, compararon y llegaron a conclusiones matemáticas. Finalmente, escribieron la función de acuerdo con las características de la carga parabólica propuesta en la Actividad 2 (Figura 3), además escribieron su respuesta en la forma canónica y estándar, lo que permitió a interpretar esta respuesta como nivel formal de comprensión.



14.- Completa la siguiente tabla a partir de la función $m(x)$ en su forma canónica:

$m(x)$	Coeficiente a	Parámetro h	Parámetro k	Coordenadas de vértice
$m(x) = -\frac{365}{72}(x-12)^2+550$	$-\frac{365}{72}$	12	550	(12, 550)

16.- Escribe la función $m(x)$ en su forma estándar y completa la tabla mostrada a continuación.

ax^2+bx+c	Coeficiente a	Coeficiente b	Coeficiente c
$\frac{365}{72}x^2+\frac{365}{3}x-180$	$-\frac{365}{72}$	$\frac{365}{3}$	-180

Figura 3: Respuesta de nivel formal en la Actividad 2

Reflexiones finales

Los resultados muestran que, mediante la realización de las actividades de la secuencia didáctica, los estudiantes analizaron el efecto de los valores de los parámetros de las funciones cuadráticas en las modificaciones de las gráficas, buscaron regularidades y comprendieron el comportamiento de la parábola producido al modificar los coeficientes de la misma. Además, exhibieron los cuatro niveles de comprensión (Bressan et al., 2005). Las preguntas incluidas en la secuencia didáctica ayudaron a los alumnos moverse a lo largo del mundo simbólico y desarrollar la matematización vertical. A su vez, el nuevo conocimiento les permitió generar predicciones y trasladarlas a las situaciones realistas propuestas, lo cual promovió la matematización horizontal.

La implementación de las actividades ofreció a los alumnos la oportunidad de reinventar matemáticas y alcanzar el nivel formal de acuerdo con el enfoque de EMR.

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WHEN ALGEBRAIC ERRORS REFLECT WEAK PRACTICES RATHER THAN MISCONCEPTIONS: EVIDENCE FROM A CLINICAL STUDY

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Keywords: Algebra and Algebraic Thinking, Learning Trajectories and Progressions, Mathematical Representations, Learning Theory

Students' difficulties with basic algebraic ideas, especially equality, relational reasoning, and variables, are well documented in the mathematics education research literature (Booth & Koedinger, 2008; Kieran, 1981; MacGregor & Stacey, 1997; Sfard, 1991; Sfard & Linchevski, 1994). These difficulties are often interpreted as misconceptions, such as treating the equal sign as an instruction to compute rather than as a statement of equivalence (Kieran, 1992; Knuth et al., 2005; Stephens et al., 2013) or interpreting letters as labels for objects rather than as symbols representing numerical quantities (Clement, 1982; Küchemann, 1978). In this study, however, closer analysis suggests that some *classic* algebraic errors reflect weak representational practices, even when the student demonstrates a robust semantic understanding of the underlying quantitative relationship. This poster presents episodes in which an incorrectly written expression co-occurred with evidence of a strong conceptual understanding, suggesting that some error patterns may be more productively explained as weaknesses in representational practice rather than evidence of conceptual misunderstanding.

Theoretical Framework

This study is situated within Piaget's (1964) theory of cognitive development, which conceptualizes learning as the reorganization of prior knowledge structures through processes of assimilation and accommodation.

Methods

A Grade 9 high school Algebra 1 student participated in three clinical teaching sessions designed to explore his current understanding of equality, variables, and linear relationships. As an interview method, clinical teaching is designed as a "powerful vehicle for entering the child's mind" (Ginsburg, 1997, p. ix) to uncover the underlying structures of thought that guide their interpretations, strategies, misconceptions, and emerging understandings.

Results and Implications

This clinical study challenges the assumption that algebraic errors necessarily indicate misconceptions. As illustrated in the Student–Professor task (see Figure 1), the student initially wrote $S \times 6 = P$, consistent with Clement's (1982) findings. However, after further engagement with the question, he quickly revised his response to $P \times 6 = S$, explaining that "however many professors there are, times 6 equals the [number] of students."

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

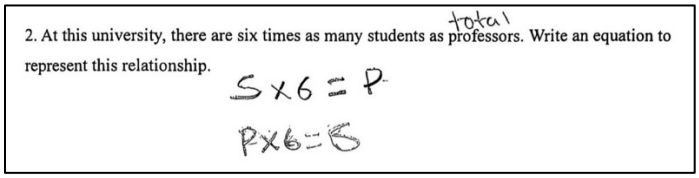


Figure 1: Evidence from the Student-Professor Task

Although the initial notation reflected a word-order matching strategy based on the linguistic structure of the statement, his explanation demonstrated a correct understanding of the multiplicative relationship. Thus, from a Piagetian perspective, the student's self-correction can be viewed as a process of cognitive reorganization in which existing arithmetic schemes are accommodated to support more advanced algebraic relationships. Similar patterns appeared in tasks involving equality and variables, where the student defined the equal sign operationally yet successfully interpreted non-canonical equations.

The findings from this study imply that instructional responses should distinguish between conceptual deficits and representational habits. Clinical teaching proved valuable in uncovering latent relational and structural understanding that may remain invisible in traditional assessments.

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EQUITY AND JUSTICE

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DEVELOPING REFUSALS TO LOGICS OF EXTRACTION AND REMOVAL

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We amplify and extend a critique made by Leanne Betasamosake Simpson regarding a problematic similarity between the underlying logic of "abstraction" and the logics of extraction and removal that perpetuate many oppressive cultural phenomena. Locating ourselves at the site of the emergent definitions for Computational Thinking (CT), we similarly problematize the underlying logics of the common presentations of definitions for "abstraction" and "debugging." We offer alternative definitions for these and develop a refusal through diffractive inquiry involving narratives crafted with intention to amplify frequencies of relationality (and attenuate frequencies of coloniality). We briefly discuss implications for teaching.

Keywords: Debugging, Abstraction, Computational Thinking, Refusal

In this theoretical research report, we share some of our work in reimagining definitions for Computational Thinking (CT) that we intend to be less resonant with broader societal deficit narratives and colonial logics. Our goal is to develop attunement to ways in which computer science education can become more resonant with relationally-oriented ethico-onto-epistemologies (Chen, 2024; Paton & Sinclair, 2024). Using the examples of *abstraction* and *debugging*, we suggest that a relational understanding of CT concepts is possible—and may even better characterize the practices of computer scientists.

We position this article as diffractive inquiry (Barad, 2014; Murris & Bozalek, 2019) that investigates the resonances between two vignettes—which we have intentionally crafted to act as narratives (Rosiek & Snyder, 2020) that express our understanding of disciplinary relationships. Through this diffraction, we show how conventional definitions of debugging and abstraction in CT leave out relations among different kinds of beings and materials which are central to practices of disciplinary knowledge-creation.

Our goal with these relational definitions is to open space to account for the many dialogical and interconnected relationships in which disciplinary practices are embedded, and to move away from underlying colonial logics if possible. We motivate our inquiry with a discussion positioning iterative critical re-readings of disciplinary practices as necessary for enacting daily practice of justice, and subtle-yet-substantive acts of refusal to colonial logics. In our first vignette, we share an account of the many relationships that were inherent to Edward Lorenz and his research team during the early 1960s as they developed an abstract understanding of what is now known as "sensitivity to initial conditions." In our second vignette, we share an example of two teachers and a facilitator engaged in debugging within our designed learning environment.

This work emerged from an NSF grant through the STEM+C funding stream. This project consisted of the development and implementation of a professional learning approach that would

introduce teachers to CT, support their understanding of the mathematics and science they teach, and support the integration of CT into their math and science instruction.

Philosophical Perspective: Critical Re-readings and Refusals

Scholarship in STEM Education has increasingly begun to emphasize the importance of expecting and respecting multiple ways of knowing (Rosebery et al., 2010; Warren & Rosebery, 2011) and building dignity-oriented frameworks for theorizing learning and assessment (Kayumova et al., 2025; Kim & Saplan, 2024). One of the starting places for our research team has been the perspective that "theories and theoretical frameworks manifest onto-epistemological beliefs and identities and are therefore lenses through which we can see, analyze, and build new visions and justice-oriented worlds" (Kayumova & Dou, 2022, p. 1113).

Like many other scholars, our work involves "critical re-reading[s]" (Warren et al., 2020, p. 278) of STEM disciplines in order to look closer "at the ways disciplinary practices are narrated, framed, and enacted" (p. 279). The theoretical frameworks that we draw from position our ways of knowing and being as inherently and inevitably constrained and enabled by a material agency of inquiry and methodology (Garroutte & Westcott, 2013; Gouvea, 2024). Building from prior work, our perspective is the following:

> Recurring inequities, deficit-based discourses, and racialization of youth from nondominant communities are intertwined with larger knowledge systems in science education. These are rooted in (settler) colonial and racialized histories, which manifest themselves in our current thinking about what it means to be a science person and do science. (Kayumova & Dou, 2022, p. 1100)

We have thus been attempting to work towards "changing/ transforming science structures and norms by *forging new asset-based and relational justice oriented spaces*" (Kayumova & Dou, 2022, p. 1113).

A Critique and Form of Refusal

There is a critique regarding the concept of abstraction that we believe deserves to be more widely considered by the STEM Education community. Nishnaabeg scholar Leanne Betasamosake Simpson (2017) writes:

> Western thinkers get [abstraction] confused with extraction … Extraction is a cornerstone of capitalism, colonialism, and settler colonialism. It's stealing. It's taking something, whether it's a process, an object, a gift, or a person, out of the relationships that give it meaning, and placing it in a nonrelational context for the purposes of accumulation. (p. 201)

Since the concept of abstraction runs so deeply through STEM, we are alarmed by the idea that the logics that perpetuate epistemological erasure (Andreotti et al., 2011; Turkle & Papert, 1990) are legitimized (and even encouraged) by STEM disciplines when abstraction is conflated with extraction. Eve Tuck and K. Wayne Yang recommend (2014) that, "*Refusals* are needed to counter narratives and images arising (becoming-claims) in social science research that diminish personhood or sovereignty, or rehumiliate when circulated" (Tuck & Yang, 2014, p. 811, emphasis added). We question whether this resonance with colonial logics is necessary, if there could be another way to enact and embody the disciplinarity of STEM, and if it is possible to develop forms of refusal.

Simpson describes an Indigenous approach towards *abstraction* as a practice of "shifting the relationality to change meaning or to illuminate a different meaning" (Simpson, 2017, p. 201).

We notice that this counter-framing of abstraction still describes disciplinary practices of Computer Science and Mathematics.

We are thus interested to find similar counter-framings for other CT concepts. However, we recognize a tension of hybridizing knowledges in ways that inevitably lead to further assimilation and erasure (Ahmed, 2016). We tentatively (Ahenakew, 2016) present our work with a firm disclaimer that our proposed counter-definitions may still be problematic, and may still be much too subtle. Additionally, our goal of creating alternative definitions should *not* be thought of as a "replacement" of dominant definitions. In fact, functionally "equivalent" definitions would surely reproduce equivalent problematic resonances and impacts of harm. We instead share our theoretical ideas to highlight the importance of *intentionality* in our enactments of STEM—so that we may pay better attention to what they help re-produce. We see Simpson's reformulation of the concept of abstraction as a window to an alternative and incommensurable way of being "abstract." Looking through this window, the previous problematic resonances with colonial logics do not appear to be necessary to tolerate.

Motivation

Crucially, our search for refusal is motivated by noticing a reciprocal implication of the critique: If daily enactments of disciplinary practices can potentially resonate with deficit discourses and colonial logics, and contribute to impacts of harm; then attuned counter-enactments of disciplinary practices can contribute to emergent impacts of care, and daily practices of justice (Everett, 2020). That is, if subtle disciplinary discourses can potentially contribute to forms of harm, then subtle counter-framings should also be theoretically capable of contributing to forms of care and repair.

Gloria Ladson-Billings (2015) argues that we need to attend to injustice by shifting from "justice as theory to justice as practice." Further, Sakeena Everett (2020) argues that "we need to adopt a paradigm that understands *justice as a daily practice*" (p. 102). Restorative justice practices begin with careful listening for impacts and implications—similar to the daily practices of reflective scientists and attentive teachers. Drawing from Ladson-Billings (2015) and Winn (2018), Everett (2020) describes justice as praxis as "a *daily theory-informed reflective practice*—a theoretical and pedagogical paradigm committed to seeing the full humanity of others and making things right" (p. 102).

Given any curriculum or lesson plan, a teacher inevitably shapes how students are enabled or constrained to engage with disciplinary concepts. Expert teachers are expertly attuned to noticing and counteracting subtle ways that the learning environment might be reinforcing narratives that harm students (such as, discourses of gender or cultural deficits). Our motivation to problematize and reimagine the impact of CT concepts is strengthened by the implication that it is possible to enact at least some changes—even if a teacher cannot entirely change their curriculum. Our reimagined definition of debugging and Simpson's formulation of abstraction can be enacted as themes that teachers emphasize or simply ideas that teachers begin to internalize or think-with for their own practice.

Implications of Dominant Framings of Abstraction and Debugging

In this section we elaborate on some of the hidden implications of dominant framings of abstraction and debugging in CT. We hope that by bringing these entailments to the fore, the potential harm of these definitions becomes more tangible.

Abstraction as "Decontextualizing and Manipulating Symbols"

We notice that the dominant framing for what it means to practice "abstraction" in mathematics curricula (e.g., Common Core State Standards for Mathematics, 2010) and

Computer Science (Lafuente Martínez et al., 2022; Rijke et al., 2018) often implies that disciplinary expertise entails becoming an arbiter of what is important and what is discardable in a process of decontextualization. The Standards for Mathematical Practice, for example, describe reasoning abstractly and quantitatively in the following way:

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize—to abstract* a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, *without necessarily attending to their referents*—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. (Common Core State Standards for Mathematics, 2010)

Despite valuing reflective questioning to some degree, this framing implicitly (but undoubtedly) positions disciplinary proficiency as proficiency with deciding when it is "necessary" or not to remove the context from a given situation and extract symbols to "manipulate." Notice some subtle implications of the previous description: (a) The act of *decontextualizing* is framed as an *ability* that one can simply be either good or bad at. (b) There is no distinction made between *discarding* context, *responsibly choosing relevant* contexts, and *failing to notice* contexts. And (c) in choosing/creating representations, one unavoidably *stops* seeing certain context, which turns the question of *who* stops seeing *what* context *crucial* for abstraction practices—a question absent from this definition.

Debugging as "Finding and Fixing Bugs"

"Debugging" is a key component of CT, and is conventionally defined as a systematic approach to finding and fixing bugs in the system (Grover & Pea, 2013; Weintrop et al., 2016; Wing, 2006). We can again attune to subtle yet real entailments of this definition. This framing of debugging implies that: (1) There are *intrinsic* bugs—intrinsic wrong things—in a system that one needs to find and fix; (2) "becoming a computational thinker" entails being good at finding what's "wrong;" and (3) there is the hidden assumption that we all have the same understanding of what "the system" is supposed to *do* and of what it means to "fix" bugs in the system.

The Implied-Default Relational Ethics

We are sympathetic to the reaction that this might not initially seem very alarming. But we feel an urgency to take seriously the potential implication that simply to become enculturated with the logics that perpetuate the STEM disciplines may implicitly *require* you to become comfortable absolving colonial logics that perpetuate harm.

Scholarship emphasizing the relational aspects of CT (Dickes & Farris, 2019; Sengupta et al., 2018; Silvis et al., 2022; Sohr et al., 2023) helps us to notice the following in discourses of CT and other STEM disciplines: The *absence* of a central tenant of care within current discourses of computing and mathematics *simultaneously mobilizes* a complementary disciplinary ethic in STEM disciplines. Maintaining a distance between discourses of STEM and discourses of care inherently establishes a default ethic of ambivalence to relational impacts such as harm or care. Without ethics of relational care built into the definitions of what it means to practice disciplinarity, ignorance and ambivalence are instead left behind as a default disciplinary ethics.

To express how one can infuse relations of care in abstraction and debugging practices, we now shift the relationality within our narration of two vignettes that involve these CT practices. This shift shows that it is possible to construct disciplinary ethics of care that inhere in CT practices.

Vignette A: Edward Lorenz Engaged in a Moment of Disciplinary Listening

This vignette is an account of the early mathematical development of Chaos Theory. Specifically, Edward Lorenz can be understood to be in many different material-dialogic (Hetherington et al., 2018) conversations, and listening to many more-than-human agents (Krishnamoorthy, 2023). For Lorenz and his team of researchers (Sokol, 2019), the development of the abstract property of "sensitivity to initial conditions" centrally involved paying close attention to the subtleties of relationships between the particular computing machine, the system of differential equations, and human desires to better understand emerging relationships. Before 1961, Edward Lorenz was using a system of 12 differential equations and an early computer to numerically simulate weather patterns. He describes deciding to reexamine the machine's output:

> I stopped the computer, typed in a line of numbers that it had printed out a while earlier, and set it running again. I went down the hall for a cup of coffee and returned after about an hour, during which time the computer had simulated about two months of weather. (1993, p. 134)

The new numbers were wildly different from the old numbers, and Lorenz initially suspected a faulty vacuum tube or some problem with the machine he was using (Lorenz, 1963a). Inquiring further, Lorenz (1993) found that, "the new values at first repeated the old ones, but soon afterward … more or less steadily doubled in size every four days [of simulated data] or so, until all resemblance with the original output disappeared" (p. 134). The computer used six digits of precision, but its readout only showed three. So his re-entry of data from the readout introduced a small perturbation to the trajectory of the simulation. Lorenz realized that he was seeing the round-off errors steadily being amplified and creating significantly new patterns of behavior.

This property of "sensitivity to initial conditions" is now understood to be quite strange: Imagine the dynamics of dust blowing around the inside of a box with wind on the inside—the dust moves but it stays in the box. It is possible to have a bounded system of differential equations (describing the wind and dynamics of the dust) such that—no matter how close two pieces of dust started next to each other—it would be impossible to guarantee that those two pieces would remain nearby to each other in a way that you could estimate. This is strange. If the trajectory of a piece of dust corresponded to the time-evolution of a weather system, for instance, you might expect that modeling the behavior of a "nearby" estimate would help you make a weather prediction. You might expect that you could make a prediction for any later time if your estimate is accurate enough. But with a system with this chaotic property of sensitivity to initial conditions, no such estimates are possible.

Lorenz noticed this phenomenon by attuning to the difference between the original numerical predictions and the predictions after his accidental perturbation (chopping the tail of the decimal expansion). In a way, the computer was speaking back to the research team in a new and unexpected way, and they chose to listen and engage. Lorenz then followed a suggestion from Barry Saltzman to study a simplified system (Lorenz, 1963a) with only three nonlinear differential equations that seemed to exhibit a similar sensitivity to initial conditions. Lorenz's choice of a system of three equations could only possibly involve attempting to capture some essence of behavior that he *felt* with the previous twelve. Remember: this property was not yet formalized, and the field of Chaos Theory had not yet been started. He had no framework to be sure that it would exhibit the same abstract property.

Lorenz's process of choosing a system of three equations from the previous twelve was a process of intentional listening to different agents and attempting to understand the interconnected relationships. Their abstraction of the concept of "sensitivity to initial conditions"

could not possibly be a removal of context because his team was in the very middle of *creating* that particular ontological innovation (diSessa & Cobb, 2004).

By 1975, this new emergent middle ground between order and randomness was coined chaos (Li & Yorke, 1975). Writing in Transactions of the New York Academy of Sciences, Lorenz (1963b) ended his conference paper humorously with this dry paragraph:

> When the instability of a uniform flow with respect to infinitesimal perturbations was first suggested as an explanation for the presence of cyclones and anticyclones in the atmosphere, the idea was not universally accepted. One meteorologist remarked that if the theory were correct, one flap of a sea gull's wings would be enough to alter the course of the weather forever. The controversy has not yet been settled, but the most recent evidence seems to favor the sea gulls. (p. 431)

Eventually, this metaphor evolved into what is now called the butterfly effect. Lorenz's presentation at the 139th meeting for the American Association for the Advancement of Science (1972) was famously titled, "Predictability; Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?"

Lorenz could not have done this computational modeling work without the computer present. But his exploration depended on being able to hear what his simulation communicated back to him. And in this case, the output of the simulated computational model was one of the voices that directed the conversation into new and generative territory. A Bakhtinian (1986) analysis of what Lorenz "listened" to in this interaction might at the very least consider the computer's responses as a kind of distorted echo of his own programming. With a perspective of the social and distributed aspects of that cognitive system (Hutchins, 1995), we might additionally conceptualize the computer as a refined artifact of multiple human conversations and understanding. The computer was carefully designed, built, tested, adjusted, and shipped to "speak" in a particular way; and, in this sense, the computer represents the cultivated outcome of many accumulated voices in material-discursive (Barad, 2003) conversations.

When it happened that the computer spoke to Lorenz in a way that he did not expect, its voice emanated from a sedimentation of culture—much as the way our real voices (and all utterances) come from internal echoes of cultural practices within our own minds (Wertsch, 1991). So Lorenz's scientific contributions emerged as the product of many relationships and conversations with human and more-than-human agents. He attuned his listening (with the help of the computer) to what different equations and systems of equations would say to him as he attempted to deeply understand these many relationships.

But Lorenz's relationships with the computer and systems of equations were not the only disciplinary relationship supporting this work. Margaret Hamilton and Ellen Fetter worked (consecutively) as technical assistants for Lorenz, and their work would be worthy of coauthorship by today's contemporary publication norms (Sokol, 2019). Writing for Quanta Magazine, Joshua Sokol interviewed both Hamilton and Fetter to revisibilize their entanglement with Lorenz's work. Sokol (2019) quotes Hamilton as saying, "He loved that computer. And he made me feel the same way about it." Sokol relays anecdotes from Hamilton:

> She recalls being out at a party at three or four a.m., realizing that the LGP-30 wasn't set to produce results by the next morning, and rushing over with a few friends to start it up. Another time, frustrated by all the things that had to be done to make another run after fixing an error, she devised a way to bypass the computer's clunky debugging process. To Lorenz's delight, Hamilton would take the paper tape that fed the machine, roll it out the length of the

hallway, and edit the binary code with a sharp pencil. "I'd poke holes for ones, and I'd cover up with Scotch tape the others," she said. "He just got a kick out of it."

Lorenz first mentioned the sensitivity to initial conditions of their weather model at a conference talk in November of 1960—meaning that Hamilton would have been running the calculations where this piece of chaos theory was first noticed. Hamilton moved to another project in the summer of 1961, and trained Fetter as her replacement. Margaret Hamilton eventually led the team for the onboard flight software for the Apollo project and also the team that wrote software for the first US space station (Sokol, 2019). Hamilton is very highly regarded in the history of computing, and she is credited with coining the term "software engineering."

Vignette B: Debugging within a Professional Learning Environment for Teachers

Our own research project involved developing and implementing a professional development program for teachers to both learn about computational thinking and develop their own lessons incorporating computational thinking practices into their curriculum. We videorecorded approximately 90 hours of professional development over two separate year-long sequences. Each year, participating teachers attended a week-long workshop over the summer, monthly professional development sessions during the school year, and monthly one-on-one coaching sessions with the workshop instructors with feedback on their lesson planning.

Debugging is an underrepresented topic in K-12 classrooms (Kafai et al., 2020), and there is a gap in the literature examining exactly how debugging emerges for K-12 teachers and students. So one aspect of our data analysis was to search for episodes of debugging that happened during the professional development sessions. We identified such a moment of debugging during one of the monthly professional development sessions, and we noticed that it was novel in the sense that the episode did not quite fit the standard narrative of debugging. Through repeated video review and event reconstruction consistent with Interaction Analysis (Jordan & Henderson, 1995), we developed transcriptions and analytical memos to describe the interactions that inhered in the teachers' and instructor's debugging process.

Two of our teachers (Mary and Julie) and the workshop instructor (Jarod) found themselves debugging a system involving a small mobile robot programmed in a block-based coding environment. The robot has a sensor on its "chest" that uses a calibrated frequency of light to measure the distance to an object in front of it. The group was attempting to program this robot so that it would check whether something was blocking its way, and then either continue ahead or turn around and move in the opposite direction. Julie's code was not making the robot behave how they wanted. It continued ahead without turning around when it should have. Jarod then changed the code. The robot's behavior running the code was to turn around without continuing ahead when it should. The group questioned whether having the robot on the floor was affecting the response of the sensor and causing the robot to behave how they observed. The group then put the robot on the table and ran the code while Jarod held his arms in front of the sensor. The robot continued ahead into Jarod's arms. But the same code then caused the robot to repeatedly turn around when nothing was in front of the sensor. Jarod questioned whether his previous understanding of the code syntax was correct, and the group changed the code back to Julie's original version. The robot then responded in the way they had intended to program it.

Developing a Relational Definition of Debugging

Our teachers and facilitator wanted the robot to do something, and at first the robot was not doing it. They spent time trying to understand and adjust. And then the robot did what they had

hoped. So, we can confidently say that whatever happened in their time understanding and adjusting is fair to characterize as "debugging." However, we find it interesting that this debugging event is difficult to reconcile as a systematic search for errors. Instead, the group examined the ways that all the elements of the system were coming together. In this example, the "debugging" done by the participants involved *coming to understand the nuances of the relationships among elements of the system.*

We are resistant to identify any one "bug" that they needed to identify and fix. Julie's original code was logically correct, and her understanding of the conditional statements was correct. The sensor was not broken; it was working correctly—the materiality of the sensor just is the way that it is. Jarod's temporary understanding of the syntax of the coding environment could be considered wrong. But that error did not cause the problem, so we are resistant to calling that "the bug." Their reasonable expectations for how the sensor would react with the robot on the floor could possibly be identified as the "bug." But we are resistant to characterizing their debugging process as systematically "fixing" that expectation.

To understand how the sensor, code, and surface were interacting, the group engaged in a creative process of imagining what could be happening to the agents in the system. We notice that this framing of debugging is much more relational, and less resonant with deficit-discourses problematized earlier.

Debugging can then be understood as a relational activity of listening closely to many interconnected relationships and coming to understand relationships in the system in richer ways.

Describing debugging in this way—as a process of paying close attention in order to understand the subtle ways that many things interrelate and influence each other—fits this scenario just as well as other dominant narratives of debugging.

With respect to Vignette A, consider the initial moment of confusion that Lorenz mentions as the genesis of their scientific journey. Lorenz and Hamilton had programmed the computer to do something, and at first, it was not doing it. They then certainly spent time systematically trying to understand the problem. Lorenz had said that the first assumption that came to mind to explain the errors was a faulty vacuum tube (a consumable part of early computing technology). Although finicky by today's standards, there was no bug or faulty tube inside the machine.

Like the teachers, before Lorenz and Hamilton could get the computer to respond in a way they expected, they needed to adjust both their expectations as well as their code. And they needed to understand how all elements of the system came together. It seems fair to call this process debugging since the computer was not acting as they expected, and they went on a systematic journey to uncover the problem. But this journey does not easily fit the dominant definition of debugging: (1) there were no intrinsic errors to be found and fix, and even more, (2) what the system was "supposed to do" had to be re-imagined. We are dissatisfied to call this a failed or abandoned search for errors because their search produced a whole new field of research, a fantastic story of a mathematical eureka moment, and a result so culturally impactful as to reach far beyond mathematics into the zeitgeist for decades.

So on one hand, this event for Lorenz and Hamilton initiated some kind of systematic search for errors. Yet on the other hand, it did not terminate for years and did not succeed in finding or removing any errors. And the event involving the teachers and facilitator in our study also involved a search for errors in the system that included their code, the environment, and their thinking. Both events involved trying to debug a system so that it behaves as expected, neither event fits the dominant definition of debugging, and both events can be characterized according to our relational definition.

In this article, we chose to look deeply at the relationality and potential relationality of debugging and abstraction because we envision relational redefinitions such as these ones to hold potential for teachers to enact daily theory-informed reflective practices of justice in their own contexts and ways, no matter what curriculum they find themselves in the middle of.

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DISRUPTING DEFICIT-FRAMED INSTRUCTIONAL PRACTICES FOR MULTILINGUAL LEARNERS IN MATH: TENSIONS AND OPPORTUNITIES

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In this study, we engaged 24 elementary preservice teachers (PSTs) in a series of discussions aimed at addressing common deficit framings of multilingual learners and research-practice implementation gaps. Through analysis of these discussions, we found that PSTs problematized deficit-framed instruction practices while also using deficit framings of MLLs themselves. We also noted areas of tension for PSTs when they considered engaging MLLs using asset-framed instructional practices.

Keywords: Preservice Teacher Education; Equity, Inclusion, and Diversity

Introduction

Deficit narratives about multilingual learners' (MLLs') ability to engage in rich mathematics are all too common in K-12 schools (de Araujo, et al., 2016; Shapiro, 2014), with statements like "She does not have the language skills to do word problems" (Gustaveson & Cross Francis, 2025). These discourses promote linguistic hegemony by prioritizing English development and positioning multilingualism as a deficit (Wiley & Lukes, 1996; Endo & Reece-Miller, 2010) while obscuring the repertoire of cultural and linguistic assets MLLs already have (Yosso, 2005). Meanwhile, teachers' limited access to research findings (Sato & Loewen, 2019) creates a persistent gap in the implementation of evidence-based practices in classrooms (Korthagen, 2017). This is specifically true for implementation of rich practices when working with MLLs. For example, preservice teachers (PSTs) have been found to lower the cognitive demand of math tasks as a practice for supporting MLLs (de Araujo, et al., 2021). Some MLLs receive individualized instruction through specialized programs where teachers are trained in evidence-based practices, but the separate instruction and labeling of students can lead to stigmas from their general education classroom teacher (Kanno, et al., 2024). Therefore, creating opportunities for teachers to learn evidence-based approaches for working with MLLs in the general education classroom is needed to close the implementation gap.

These challenges together call for more efforts in teacher education to: (i) develop math teachers' awareness of deficit narratives about MLLs, and (ii) prepare teachers to better enact evidence-based practices with MLLs in mathematics. In our study, we engaged 24 elementary PSTs enrolled in a math methods course in three discussions to support these aims. In the first discussion, PSTs unpacked a sample deficit narrative about MLLs; in the second, they addressed lingering questions from the first; and in the third, they considered research-based practices for working with MLLs. Using the transcripts from the three discussions, we explored the following research questions: What did PSTs name as potential asset-oriented practices for working with MLLs and language development in math? What tensions around practices for working with MLLs and language development in math came up in the discussions?

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Conceptual Framing

Shifting to an Asset Frame

Since the inception of public school, deficit labels like "slower learner" and "limited English proficiency" have been used to describe students' competence in math, focusing on what students lack and describing students as problems to be fixed (Deschenes et al., 2001). These labels reflect broader systemic narratives, or master narratives, that view ability as a fixed trait (Larina & Markina, 2020; Dweck, 2006) and privilege the use of the dominant language in schools (i.e., English in the United States; Wiley, 2014). Deficit systemic narratives can also serve to blame students' families and culture for students' success outcomes in math (e.g., standardized test scores), rather than looking at the role students' access to opportunity plays in those outcomes (Ladson-Billings, 2007; Milner, 2012). Because of the prevalence of these systemic narratives, deficit discourses are common in schools and in research (Adiredja & Louie, 2020). For example, Jones & Marchant (2021) found that the majority of research in mathematics education describing work with students from la Raza (i.e., Hispanic, Latin@) used deficit frames to contextualize their studies. Therefore, even if teachers have access to research about what the field puts forward as best practices, they may still be framed in deficit ways. These deficit frames have tangible negative impacts on MLLs. First, deficit ideas about multilingualism influence the types of opportunities MLLs receive to learn math (de Araujo et al., 2021). Second, students are often aware of their teachers' expectations for them which can influence their feelings of belonging in the classroom space (Shapiro, 2014).

To counter deficit frames, scholars have called for the use of asset frames that focus on students' strengths (Gustaveson & Cross Francis, 2025; López, 2023). This framing acknowledges students' funds of knowledge (Moll, et al., 1992) and community cultural and linguistic capital (Yosso, 2005), in order to view multilingualism as an asset rather than a problem to be solved. Additionally, asset-oriented frameworks embrace growth models of competence, understanding ability as ever evolving through experience (Dweck, 2006). An important part of asset-framing is for teachers to consider how to create opportunities for MLLs that help them to grow in their mathematical thinking by building from students' existing knowledge. However, shifting toward an asset frame is not without its complexities or pitfalls. Scholars warn against superficial views of culture and language, that treat all students from a group as the same and fail to engage in systemic change (Ladson-Billings, 2014). For example, overemphasizing growth models of competence without improving students' access to opportunity can slip into grit ideologies which can unfairly blame students' effort instead of addressing systemic barriers (Golden, 2017; Gorski, 2019). Additionally, an overfocus on the assets of students' home language without providing opportunities for English development could leave students underprepared for context-based math problems in English-only contexts. Therefore, it is important to consider possible tensions as teachers engage in taking on asset framings of student competence.

Research as a Resource for Supporting Instruction

Drawing strong connections between theory and practice has been a focus of education researchers for decades (e.g., Darling-Hammond et al., 2020; Dewey, 1904). These connections have been intensified by the growing body of scientific inquiries into teaching and learning, as well as their significant impact on educational practice and policy (Darling-Hammond et al., 2020; McKenney, 2018). In this regard, educational research findings have become a significant resource to support teachers in making evidence-based instructional decisions (Willingham, 2017; Yu & Cross Francis, 2025). The importance of using research findings as a resource for

guiding teachers' instruction has both theoretical and empirical foundations. In his original work on teacher knowledge, Shulman (1987) foregrounded educational theory and research as one of the four major sources for advancing teachers' knowledge and supporting the implementation of evidence-based practices. Specifically, Shulman highlighted the unique value of educational scholarship in enriching teachers' visions of what constitutes good teaching or what students can achieve given the most appropriate intervention and scaffolding. Empirical studies suggest that such visions, enhanced by educational scholarship, can further shape how teachers understand the goal of teaching specific content, thereby influencing how they design, implement, and reflect on the lesson (Bransford et al., 2005; Lohse-Bossenz et al., 2015; Yu et al., 2025).

The unique value of educational research becomes more evident in supporting PSTs in developing knowledge about practices that have not been well established in classrooms. Studies have shown that, despite methods classes shifting their beliefs, PSTs still struggle to align their practice with those beliefs (e.g., Philipp, 2007). These struggles may stem from a limited view of instruction within established classroom routines (Cheng et al., 2012) or difficulties in realizing their beliefs in specific contexts (Yu & Cross Francis, 2025). In terms of teaching multilingual learners in math, challenges do not only exist in changing teachers' perceptions of MLLs (e.g., deficit-oriented to asset-oriented), but also in enacting practices proven to be effective in providing equitable learning opportunities for MLLs. Educational researchers have engaged in efforts to dismantle the myth that mathematics is free from language (e.g., the word problem) (e.g., Arizmendi et al., 2021). Additionally, there has been an empirical focus on identifying practices that support MLLs in learning math; practices that have been verified as effective in classrooms (e.g., using multiple representations to address linguistic diversity) (Sharma & Sharma, 2023). This information can expose PSTs to evidence-based approaches that challenge and complement the current practices they witnessed in their field placements.

Methods

Participants and Context

This study is situated in a graduate-level mathematics methods course for elementary PSTs. The major goals of this class are to build PSTs' productive beliefs about mathematics teaching and learning, and to equip them with tools and knowledge to teach in equitable ways. While enrolled in the methods class, PSTs also visited elementary field placement classrooms weekly. Participants in this study were recruited in a voluntary manner which led to a total of 24 PSTs.

Supporting MLLs was a major topic in the methods class, as the current practices for teaching MLLs in PSTs' field placements were not perceived as asset oriented. Two teaching assistants in the methods class specifically designed discussions to (1) support PSTs to combat the existing deficit-oriented framing as well as (2) build their goals and knowledge for supporting MLLs in evidence-based ways. The first, narrative discussion asked PSTs to unpack a hypothetical teacher statement related to MLLs: "*She doesn't have the language skills to do word problems. I'm giving her easier work that she already knows how to do, so she feels successful*." They were prompted to describe the underlying assumptions, potential impact, and language practices they could use in their own classrooms (Gustaveson & Cross Francis, 2025). The second narrative discussion was designed around their follow-up questions to the first discussion, such as "how do we make language support more humanizing?" In the third discussion on related research, we shared a summary of research findings on evidence-based practices for MLLs. PSTs were prompted to read the summary and reflect on how the findings resonated with their current experiences as student teachers in field placements and how they might inform their

future teaching. Table 1 presents the artifacts presented to PSTs, along with the discussion questions that guided conversations.

Table 1: Discussion Artifacts and Discussion Questions

Discussion Artifact	Discussion Questions
Deficit Narrative A common deficit narrative: "She doesn't have the language skills to do word problems. I'm giving her easier work that she already knows how to do, so she feels successful."	What are the potential assumptions being made in this statement? What might be the impact of this statement? What implications does this discussion have for your own teaching practice?
Lingering Questions A list of questions PSTs brainstormed: What is the context behind "language skills"? What about students who don't have any home language? If we ask another student to help, how do we ensure the student helping does more "guiding" rather than "doing" the activity for the other student? How can it be more humanizing/bring in language assets?	What lingering questions do you have from the deficit narrative discussion? What are your thoughts on these questions?
Research-Based Practices A short synthesis passage on research-based practices: "Classrooms in US public schools have experienced an increase in language diversity in the past 20 years. In 2017 Fall, English Language Learners (ELL) made up 10.1% of the school-age population, with nearly 75% of these children having Spanish as their first language (U.S. Department of Education, 2020). It is commonly thought that math is free from language, however, additional vocabulary learning is required for learners to become proficient with the language of math. However, this can be minimal. For example, "plus," "sum," "add," "altogether," "increase," "in all," and "more than" could all be interchangeably used for one single concept "+" (Veel, 1999; Wolf & Leon, 2009). This common belief may lead to language skills being confounded with students' math abilities, especially for math word problems where language processing places additional cognitive demands on students (Arizmendi et al., 2021). Alt et al. (2014) found that when language–heavy math tasks were reduced in language load, ELLs performed as well as their monolingual peers. Sharma & Sharma (2023) identified the four successful interventions for ELLs in multilingual mathematics classrooms, including dual language programs (e.g., incorporating marginalized cultural content and heritage language in classroom mainstream), curriculum integration (e.g., asking kids to create, define, and reflect on integers using the number line), teacher professional development (e.g., teachers learning and using research to challenge common myths like "math person" and developing growth mindset), and cognitively focused interventions (e.g., asking students to "think aloud" while solving a problem and building their skills in monitoring their thinking)."	Are there any interventions differentiated to the needs of multilingual learners in your placement classrooms? How are they similar or different from any of the research findings? Reflect on the interventions you identified. How well do you think it works in supporting multilingual learners' learning? Think ahead, what would you like to try in your future classrooms with multilingual learners?

Data and Analysis

Data included transcripts of each of the three discussions at both the whole-group and small-group levels (~4 students). All three activities include the whole-group discussion transcript, while only the Narrative Talk discussion includes small-group discussion transcripts. To answer

research question one, we explored the kinds of practices PSTs hope to use in their classrooms and how they frame them. We began by highlighting any instructional practices that PSTs named in relation to incorporating language development in math in the transcripts for the 3 discussions. We considered instructional practices to be the concrete actions teachers take in the classroom as part of their daily teaching. For example, PSTs described "giving students a translated copy of the material," an action they would take in their teaching to specifically support MLLs' language comprehension. We then considered how PSTs framed these practices when bringing them up in discussion. We initially planned to label the practices PSTs suggested as asset-oriented, deficit-oriented, or unclear/neutral. As we attempted to label the list of highlighted instructional practices, we noticed that PSTs sometimes talked about practices of other teachers, attempting to problematize them rather than suggest their use. We decide to label the practices as: "suggesting practices (asset frame)," "suggesting practices (deficit frame)," "suggesting practices (neutral/unclear frame)," and "problematizing practices" (see Table 2 for code descriptions). In our findings, we describe the types of practices PSTs mentioned and look at the percentage of practices that fall under these four labels.

Table 2: Codebook

Code	Description
Suggesting practices	PSTs suggest a practice that could be used when working with MLLs or to develop language in math)
Asset frame	The practice named is specifically described as honoring student assets or responds to a question about honoring assets.
Deficit frame	The practice named is described as a way to address or fix a perceived deficit in students (e.g., not speaking English or not comprehending).
Unclear/neutral frame	The practice named could use an asset-oriented or deficit-oriented, but there is not enough evidence to say.
Problematizing practices	A practice is named as problematic for being deficit-oriented or for having a negative impact on MLLs.

For research question two, we considered the tensions PSTs encounter when attempting to teach MLLs in asset-oriented ways. We began by identifying areas where PSTs' perspectives on teaching MLLs seem to diverge, as these may indicate where PSTs are grappling with how best to teach multilingual students. We descriptively coded moments in the discussion transcripts where PSTs' perspectives seemed to diverge from us or each other indicated by instances where they shared differing opinions from a peer's idea, asked questions about each other's ideas, or pushed back on the material presented by the instructors. We then looked at the portions of each discussion where there were multiple instances of divergence that might indicate central tensions to building asset frames of MLLs. In the findings we describe three vignettes that were characterized by multiple instances of divergence.

Findings

RQ 1: What did PSTs Name as Potential Asset-Oriented Practices for Working with MLLs and Language Development in Math?

We found 58 instances where PSTs mentioned instructional practices related to language in math (see Figure 1 for code frequency). They described a range of practices, including using manipulatives or visual images, creating a flip chart with key terms, translating materials, using

intentional partnering, using small group instruction, breaking down word problems with the whole class, and teaching explicit word problem comprehension strategies. In 21 of the instances where PSTs named instructional practices, they were problematizing the practices of other teachers. Both Justine's and Alice's experiences provide examples of PSTs problematizing practices. In the narrative discussion, Justine reflected on the impact of assigning some students easier work, which would lead students to compare themselves to their peers. She explained that if a teacher used this practice, then "[students] will give up wanting to learn... they will just realize, 'oh, I'm behind and everybody else.'" In the research talk, Alice described a practice from her field placement that she thought was problematic. She shared, "the way that [our math curriculum] works, they introduced numeric values of 4 plus 4. And students are supposed to master that before they do the word problems. That didn't work well for our [MLLs]." She continued to explain that the assessments used in her field placement drew heavily on word problems. She found it problematic that students would be given numeric problems in their daily work, but be required to complete word problems on assessments, thereby problematizing not teaching word problems but using them to assess understanding of math concepts. Both examples show how PSTs found some practices to be unsupportive of language development and were not practices they wanted to use.

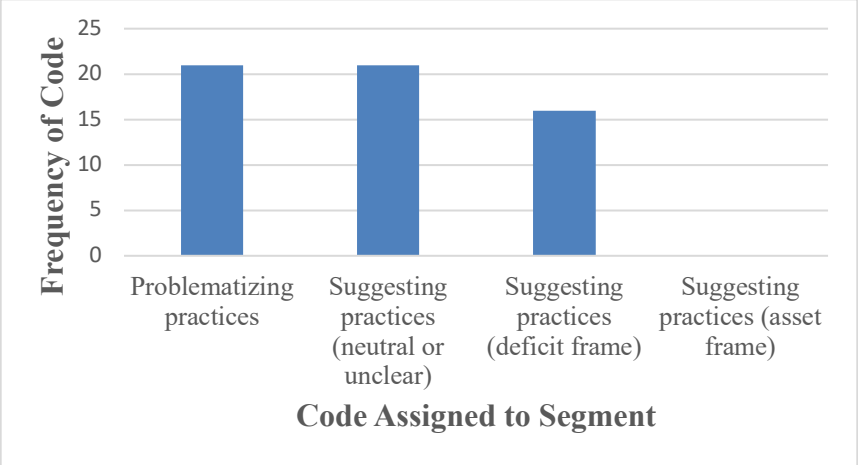


Figure 1: Frequency of Segment Codes

There were 37 instances of PSTs suggesting practices they would use—21 instances framing students in neutral or unclear ways and 16 framing students in deficit ways. Examples of unclear or neutrally-framed practices included when Maggie described a practice of "putting English in Spanish and Spanish in English," without elaborating on whether this was to support a deficit (i.e., not speaking English proficiently) or to build on an asset (i.e., drawing on their linguistic repertoire). Anna described offering read-alouds as an option for all students when she shared, "we'll be like 'if you need any problem read out loud for you like we'll just go read it out for you and then you can ask questions, too.'" This was also marked as unclear/neutral because she did not frame it as addressing a deficit (i.e., fixing their comprehension) or an asset (i.e., giving students opportunities to make meaning for themselves).

Practices framed around students' deficits ranged from subtle to more explicit. Ollie shared that she thought an important practice was to separate MLLs who speak the same language at home. She said, "I think seating charts are very vital. Because there's one student [who is fluent

in both languages], and I don't want students to feel like they have to have the burden to be a translator." While she was attempting to be considerate of the workload of the student who could translate for students still learning English, she also subtly described the student who needed to be translated for as a burden. In other words, not speaking English well enough is a deficit, and it is the teacher's role to ensure students are not held responsible for that deficit (i.e., creating a seating chart). Callie more explicitly described deficits when she explained that her MLLs were not receiving support in her field placement classroom. She shared, "I come from a place of understanding and frustration about this [statement] mainly because we do have you have two kids who are so low in their English language knowledge because they are English language learners. And because we've had such a big class, my teacher hasn't had time to like sit with her and like sound out each word, give her the meaning for each word." She described MLLs as "low" (i.e., deficit frame) while suggesting that they would benefit from a teacher sounding out words (i.e., a teaching practice). In another, more explicit example, Gina shared the practice of creating a chart of key terms and applying a deficit frame to MLLs. She said, "If you think this kid is not linguistically up to snuff… then maybe they should have like a little flip chart of key terms."

Overall, PSTs attempted to problematize classroom practices in about one-third of the practices they discussed. In the other two-thirds of the practices discussed, PSTs suggested instructional practices that framed MLLs in deficit or unclear ways. Notably, no practices were shared using asset framing of MLLs (i.e., none of the practices named explicitly noted how PSTs were noticing or leveraging students' linguistic, cultural, or mathematical assets).

RQ 2: What Tensions Around Practices for Working with MLLs and Language Development in Math Came Up in the Discussions?

There were three notable tensions around practice that came up in each of the three discussions. In the narrative discussion, the instructor asked PSTs to identify the deficit in a common discourse about language. Initially, the PSTs noted that the deficit stemmed from the teacher not believing in students' mathematical ability. Carter shared,

> You don't want to simplify [the math] to the point that they're losing the skills that they need to be building up. You can't just simplify the task and say, "oh, well, since you can't read English, I'll just be loose [with expectations]."

She challenged teachers' assumptions that mathematical ability is the same as language ability and argued that teachers shouldn't lower their expectations for MLLs because of language. Tina agreed with this idea, arguing that there was a language barrier for the student, but that the teacher should "talk about the actual math problem versus the comprehension." This seemed to be the general consensus of the cohort, that the student does have language deficits but that the teacher saying the discourse was making unfair deficit judgments about their math ability, until Dillan shook his head and interjected, "to think about language skills as an absence of it assumes that these students don't already have a primary language that they're successful in, which is where the deficit problem comes up." His argument was that seeing MLLs' English proficiency as a deficit was actually at the heart of the statement, similar to the statements of his peers. The key tension for PSTs was whether assumptions were being made about MLLs' ability to do math because of language *or* whether being an MLL was being seen inherently a deficit (*or* both).

In the lingering questions discussion, the instructor shared a strategy for highlighting students' linguistic assets, inviting PSTs to teach words in multiple languages as they worked through a word problem featuring strawberries. She called on a Korean PST to share the word

for strawberry in Korean. The instructor intended to show how mathematics can be a space for including language beyond the language of instruction, and how it can support students in making sense of word problems in multiple languages. Gina raised her hand to ask a question about this practice saying, "In another class we learned that it is not ok to ask Students of Color to share their experiences because it is 'tokenizing.' Would this not be an example of tokenizing?" Several other PSTs chimed in, "no," and "that's not what she meant here." Others looked uncertain. This moment highlights a second tension PSTs experienced when reflecting on asset-framed practices—whether attempting to honor home languages and multilingualism by inviting languages into the classroom can be tokenizing.

The research discussion sparked the most debate of the three discussions around the question: should MLLs be asked to do word problems? As PSTs read the research summary and considered what research says about working with language in math, a specific line initiated debate. It said, "Alt et al. (2014) found that when language–heavy math tasks were reduced in language load, ELLs performed as well as their monolingual peers." This line was included in the research summary to debunk any ideas that MLLs are not as mathematical as their peers, which PSTs had already problematized in the narrative discussion. However, some PSTs interpreted this to mean that if teachers do not give MLLs language-heavy math tasks then they will be more successful. PSTs generally did not know how to handle this idea. Alice argued that it wasn't fair that MLLs were assessed using word problems, saying, "I know they can do it, but then on the exit ticket, we're having to give them a 0 or 1 because they can't read it." Responding to this idea, Devin asked the instructor, "on their exit tickets—until they're able to comprehend word problems as a whole—would you recommend just having the math already written?" She wondered whether it might be fairer to assess MLL's operations solely with numbers, without reading. Gina did not like the idea of simplifying word problems for MLLs. She described her concern that it might hinder their future development sharing,

> At some point when their comprehension is up to snuff, they'll get thrown into these more difficult questions—word problems that will require more out of them, and they might not be able to meet that level that they're being put into.

Carter wondered, "wouldn't translating be a better option than just simplifying the language?" The tension in this discussion centered on whether using word problems would be harmful or beneficial to MLLs' mathematical development.

Discussion

Despite explicit work in the math methods course to build PSTs' asset-orientations towards MLLs, their framing of instructional practices and surfaced tensions point to a continued need to intentionally name and highlight strategies that recognize and build on students' existing strengths. Our findings showed that PSTs were beginning to problematize practices that viewed MLLs' math ability in deficit ways, and they could name a range of practices to use with MLLs. However, they were not yet explicitly describing practices that used an asset frame of MLLs. Additionally, several tensions arose when attempting to shift from deficit to asset frames of MLLs' competence. First, there was a tension about whether deficit frames about MLLs involved making assumptions about ability because of language, or assumptions that speaking languages other than English was a deficit. Second, PSTs worried that attempting to bring home language into the math classroom could sometimes be tokenizing. Third, PSTs debated whether using word problems was beneficial or harmful for MLLs.

These findings have implications for both teacher education and research. When creating opportunities for PSTs to interrogate deficit discourses and to use asset frames to select instructional practices for MLLs, it is important that teacher educators are explicit in naming practices using an asset frame. This work must attend to both deficits about innate ability (Larina & Markina, 2020; Dweck, 2006) and to assumptions of linguistic hegemony (Wiley, 2014). This includes teaching PSTs strategies that make opportunities for MLLs to access mathematics without reducing the cognitive demand. Furthermore, teacher educators must foreground students' linguistic capital (Yosso, 2005) and explain how practices can build from this capital. At the same time, the work is not merely engaging PSTs in discussions for one semester, but rather an ongoing practice of considering asset-oriented ways to work with MLLs that moves beyond being superficial or tokenizing (Ladson-Billings, 2014). Additionally, if we want to close the implementation gap in evidence-based practices while also framing MLLs in an asset-based way, we need more research that presents MLLs' knowledge in asset-oriented ways and includes practices that build on their strengths (Jones & Marchant, 2021).

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EXPLORING BLACK EDUCATORS' PERSPECTIVES OF CULTURALLY RELEVANT PEDAGOGY WITH BLACK STUDENTS

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This research study explored the perspectives of Black educators (elementary teachers, a math coach, and school administrators) on the benefits, support, and challenges of implementing culturally relevant pedagogy and its three tenets (academic success, cultural competence, and critical consciousness) when engaging Black students during math instruction. The perspectives from the Black educators provided (a) practical approaches of CRP to support and enhance their Black students' learning and doing of mathematics, (b) instructional support provided to support the math teaching practices of CRP, and (c) the challenges when implementing culturally relevant pedagogy. The study concludes with implications to address the challenges experienced by elementary teachers when implementing CRP during mathematics instruction and potential support that can be provided by math coaches and school administrators.

Keywords: Culturally Relevant Pedagogy, Black Students, Mathematics, Equity

In public elementary classrooms in the U.S., educators are encouraged to implement asset-based pedagogies to disrupt the current systems of inequity and injustice experienced by Black students. Some of these asset-based pedagogies include: culturally relevant pedagogy (CRP); culturally responsive teaching (CResT); and culturally sustaining pedagogy (CSP). Educational researchers continue to examine how these asset-based pedagogies support Black students' learning, promote educational equity, and disrupt systemic inequities that have negatively affected Black students when learning and doing mathematics (Alim & Paris, 2017; Alim et al., 2020; Gay, 2000, 2018; Ladson-Billings, 1995a, 1995b, 2014, 2021; Martin, 2012; Matthews et al., 2022; Paris, 2012, 2021; Paris & Alim, 2014, 2017; Zavala & Aguirre, 2024). However, the perspectives of Black educators' implementation of asset-based pedagogies to support and enhance the mathematical learning and doing of Black students have been underinvestigated.

In connecting the teaching, learning, and doing of mathematics to be relevant to Black students' culture, the importance of culture must be recognized, celebrated, nurtured, and sustained inside the classroom. Ladson-Billings (1997) noted that culture "refers to the deep structures of knowing, understanding, acting, and being in the world" (p. 700). Tillman (2002) further defined culture as "a group's individual and collective ways of thinking, believing, and knowing, which includes their shared experiences, consciousness, skills, values, forms of expression, social institutions, and behaviors" (p. 2). Recognizing the importance of Black students' culture inside the classroom disrupts existing notions that often describe mathematics education as culture-free (Bishop, 1988; Joseph, 2021). Leonard and Martin (2013) move this notion further by articulating how Black students' culture plays a vital role in their learning and doing of mathematics in U.S. classrooms.

Theoretical Framework

The study is grounded in the work of culturally relevant pedagogy. The research of CRP is first explained, followed by its three tenets that provide a description of teaching and learning practices that must be enacted when engaging Black students during mathematics instruction.

Culturally Relevant Pedagogy

The theorization of CRP came into existence to foster support and enhance the educational experiences of Black students in the U.S. school system. Although there have been countless research studies on the lack of educational opportunities and inequities faced by Black students, the theorization of CRP exemplified teaching and learning practices that disrupt systemic practices and inequities experienced by Black students. Ladson-Billings (1995a) asserted that CRP is a "pedagogy of opposition that is committed to collective empowerment" (p. 160). Ladson-Billings (1995a, 1995b, 2014, 2021) further asserted that CRP is grounded in three tenets: academic success, cultural competence, and critical consciousness. Academic success emphasizes the need for equitable learning for all Black students to achieve their full potential as learners inside and outside the classroom. Cultural competence ensures Black students' cultural identities and backgrounds are incorporated throughout different aspects of their learning and school experiences. Critical consciousness enables Black students to use their learning to examine and critique inequitable practices in order to create change and take action. In welcoming and remixing the work of CRP, Ladson-Billings (2014) articulated that CSP comes in where the beat drops of CRP toward "recreat[ing] instructional spaces to ensure that consistently marginalized students are repositioned into a place of normativity" (p. 76). Ladson-Billings (2014) further asserted that the work of CSP builds on CRP, which holds firm to the notion that students must be represented as subjects and not objects inside the classroom.

Methods

This research study incorporated a qualitative inquiry design (Creswell & Poth, 2025) with Zoom online interviews to collect data from Black elementary teachers, a Black math coach, and Black school administrators on their perspectives of CRP to support and enhance the teaching and learning of mathematics with Black students. This study was guided by the question: *What are the perspectives of Black educators on implementing Culturally Relevant Pedagogy to support and enhance Black students' learning of mathematics?* From this guiding question, two specific research questions were created.

Research Questions

1. What math teaching practices do Black educators (elementary teachers, math coaches, and school administrators) discuss when describing and explaining their implementation of Culturally Relevant Pedagogy with Black students?
2. What types of support do Black math coaches and Black school administrators provide to teachers implementing Culturally Relevant Pedagogy in mathematics instruction with Black students?

Sample Population and Locations

The research participants consisted of Black educators (three Black elementary teachers, one Black math coach, and two Black school administrators) in both urban and suburban school districts located in a Midwestern and Southern state of the U.S. The selection of the participants (see Table 1) encompassed Creswell & Poth's (2025) purposeful sampling to ensure the Black educators had prior knowledge of CRP and its implementation during mathematics instruction with Black students. The participants were selected based on their race (Black); in addition to completing a narrative Qualtrics survey to explain their prior knowledge and learning of CRP, instructional practices of CRP in mathematics with Black students, and professional learning/support on implementing CRP. The selection criteria also required participants to teach

or work with Black students in the classrooms or school building, with an emphasis on mathematics education.

Table 1: Description of the Research Participants

Pseudonyms	Gender	Highest Education	Race	Education Experiences (Years)	Role	School Demographics and Location
Ms. Clarke	F	M.Ed.	Black	21	K-2 Math Teacher & Interventionist	Midwestern Urban
Mr. Mason	M	B.Ed.	Black	9	4th Grade Teacher	Midwestern Urban
Ms. Johnson	F	M.Ed.	Black	20	4th Grade Teacher	Southern Suburban
Ms. Brown	F	M.Ed.	Black	17	Instructional Math Coach	Southern Suburban
Ms. Dixon	F	M.Ed.	Black	20	School Administrator	Southern Suburban
Dr. Francis	F	Ed.D.	Black	14	School Administrator	Southern Urban

Note. F = female; M = male; M.Ed. = Master of Education; B.Ed. = Bachelor of Education; Ed.D. = Doctor of Education

Data Collection

The data for this study were collected through one-on-one semi-structured recorded Zoom interviews with Black educators. One-on-one online Zoom interviews were used instead of in-person interviews due to the location of the selected educators (i.e., Midwestern and Southern states) in the U.S. The interview questions were designed and grouped using the guiding question and specific research questions. For example, all participants answered interview questions relating to the guiding question and research question one. However, in contrast, only the Black math coach and Black school administrators answered interview questions relating to research question two.

Data Analysis

The interview data were transcribed using Zoom transcription software. To ensure accuracy in the transcription, an additional review of transcribed data was done by relisting and rewatching the recorded video interviews to ensure the data from each interview transcript correlated with the information shared by each participant. A deductive analysis (Galman, 2018; Tisdell et al., 2025) using in vivo coding (Saldaña, 2016) was used to code the transcribed data based on the three tenets of CRP—academic success, cultural competence, and critical consciousness (Ladson-Billings, 1995a). Based on the description of each CRP tenet, the Black educators' math teaching practices were coded by categorizing their responses under each tenet. For example, Mr. Mason expressed how he set high expectations for all his Black students to demonstrate their mathematical learning abilities. This response was coded under the CRP tenet of academic success. In addition, specific words and sentences were used to code the math teaching practices. The support that was provided by the Black math coach and Black school

administrators was also coded deductively based on the following categories: (a) general instructional support; (b) instructional support to implement the three tenets of CRP; and (c) examples of professional learning support.

In bridging the theory-to-practice gap experienced by teachers when implementing CRP during mathematics instruction, second-cycle coding was conducted using an inductive analysis (Galman, 2018; Tisdell et al., 2025) by applying the focused coding strategy (Saldaña, 2016) to identify categories and sub-categories of culturally relevant math teaching practices that would have been missed during deductive analysis. To identify categories and sub-categories, the interview transcript of each Black educator was re-read to develop sub-codes of mathematics teaching practices explained by the Black elementary teachers. These sub-codes were further categorized into five themes. Based on these themes, specific CRP math practices (sub-codes) were shared to support classroom teachers in implementing the CRP tenets in mathematics classrooms. The incorporation of inter-rater reliability was carried out by two colleagues in a qualitative data analysis group. These colleagues read through the interview transcripts and checked the codes and sub-codes that were identified.

Findings

The elementary teachers, math coach, and school administrators described and explained various benefits of CRP when engaging Black students in mathematics instruction. A prominent theme in the described benefits when implementing CRP focused on addressing the concern of inequity in mathematics classrooms by ensuring math teaching practices incorporate Black students' identity, language, culture, background, and community. The Black educators' explanations targeted the CRP tenet of academic success, along with connections to developing Black students' cultural competence and critical consciousness when learning and doing mathematics.

Section 1: Black Educators' Perspectives on the Benefits of Culturally Relevant (Math) Pedagogy

Ms. Johnson explained that through the implementation of CRP, her Black students see their cultural backgrounds represented during mathematics instruction. She also adapts the school curriculum to cater to the learning needs of her Black students.

Ms. Brown shared that this form of pedagogy [CRP] "builds their [Black students] confidence and toward seeing themselves as mathematicians."

Ms. Dixon expressed that CRP disrupts inequitable practices inside classrooms. Dr. Francis explained that CRP ensures Black students are held accountable during math instruction. She further explained that CRP affirms the notion that they are "capable and brilliant" at solving math problems.

CRP Math Teaching Practices

The three CRP tenets (academic success, cultural competence, and critical consciousness) were operationalized to identify the math teaching practices of the Black educators. For each tenet, the Black educators' perspectives and examples were reported based on their specific implementation of CRP. The areas of strength and growth for each CRP tenet were also discussed.

Academic success. Ms. Johnson explained how she incorporates sports, music, family gatherings, and other activities into mathematics scenarios that are relevant and real-life to her Black students' lives and communities. This further allows her Black students to utilize their unique ways of solving math problems and develop a mathematical identity in her Black students

to enhance their academic success. Ms. Johnson also noted how she introduces her Black students to Black mathematicians and Black scientists to learn about their history and contributions to further develop their mathematical identity as people who look like them.

Ms. Brown described and explained the importance of "relevant" in culturally relevant pedagogy by sharing that math lessons must be relevant to the current time and culture of the students. She also explained that the mathematics classroom must be inclusive and build on Black students' funds of knowledge and prior learning experiences. Ms. Brown emphasized the importance of connecting mathematics with literacy instruction by using a text that talks about the cultural history of mathematics (e.g., multiplication) while also connecting the lesson to real-life problems.

Ms. Dixon noted that Black students' background knowledge, culture, and lived experiences must be included in all lessons. She shared that: "[Black] students can learn or acquire valuable math skills that can translate to all content areas and life—one of those is just the ability to grapple during productive struggle and the ability to question and think critically." Dr. Francis, however, explained how her teachers develop their Black students' academic success by engaging them in project-based math activities.

Cultural competence. Mr. Mason explained how he builds relationships with his Black students by learning about their culture and background, and utilizing that knowledge when planning and teaching math instruction. He explained: "Relationships are my big thing—to kind of have that support and that background of students, so I know more about them. They know more about me, so we can make references to things. I understand their languages and the things that they're into." Ms. Johnson explained that in developing her Black students' cultural competence, she ensures they utilize their cultural backgrounds in math lessons to support their learning and engagement. Ms. Johnson also explained how she adapts the curriculum to ensure her Black students' culture is represented in the math content they are learning.

Ms. Brown provided an example that distinguished the practice of cultural competence from multicultural education. She noted: "When we talk about, I know a lot of teachers think culturally relevant pedagogy is just making sure all students' cultures are represented in the classroom. But that's more so multicultural education, but when we're talking about culturally relevant, that's talking and thinking about their Black students' current culture and ensuring they learn more about it."

From an administrator's stance, Ms. Dixon shared that implementing cultural competence requires classroom teachers to "see and know [their] Black students and be able to leverage their cultural and linguistic experiences in instruction."

Critical consciousness. Mr. Mason explained how he connects his math lesson to his Black students' community, so that they can learn more about the community. Ms. Clarke shared that she incorporates the teaching of Black history into her math lessons for students to learn more about their historical backgrounds. She also shared: "We introduce and expose our scholars to the [Black] history; we expose them to current events going on in the Black community. We expose them to the good and the bad as well." Ms. Johnson's explanation for developing her Black students' critical consciousness includes the incorporation of Black mathematicians and future careers that use math to raise students' awareness during mathematics instruction.

Although the Black elementary teachers incorporated their Black students' community, Black history, and Black mathematicians into their math lessons to develop their critical consciousness, based on the CRP literature, further extension might be impactful for students to use their mathematical knowledge to critique structural inequity and take action. I question,

however, how the terminology of critical consciousness, as used in the CRP description, is interpreted the same by educators during mathematics instruction.

The Black math coach and Black school administrators were asked the question: How do the math teachers under your leadership support Black students to develop and apply critical consciousness when learning math? I was unable to code the responses provided by the Black math coach and Black school administrator as examples in developing Black students' critical consciousness. For example, Ms. Dixon expressed that it was an area they are working on.

Teaching Practices for Culturally Relevant (Math) Pedagogy

To further investigate the Black educators' math teaching practices to develop their Black students' critical consciousness, the interview data were reanalyzed inductively to identify math teaching practices that could have been missed during the first cycle of deductive coding. The findings revealed approaches along with culturally relevant math teaching practices (see Table 2) that were shared by the Black elementary teachers, Black math coach, and Black school administrators. Five themes emerged relating to culturally relevant math teaching practices: (a) inclusive learning environment; (b) learning of mathematics; (c) mathematical identity; (d) relevant mathematics instruction and cultural connection; and (e) CRP guiding principles for math.

Table 2: Teaching Practices for Culturally Relevant (Math) Pedagogy

Approaches	Culturally Relevant Math Teaching Practices
Inclusive Learning Environment	(a) Including students in the learning process using their strengths; (b) Utilizing students' culture, race, language, community, and background knowledge and experiences as a form of inclusion in the learning process
Learning and Doing Mathematics	(a) Developing math confidence in Black students; (b) Teaching Black students how to think when solving math problems by including their background and cultural knowledge; (c) Identifying math competence in Black students and encouraging them to use it; and (d) Creating real-world math connections and applications
Mathematical Identity	(a) Developing Black students' mathematical identity; (b) Utilizing the math practice standards; (c) Engaging in collaborative math discourse; (d) Challenging Black students' thinking when learning math; (e) Developing a mathematical identity in Black students - participating in critical thinking; (f) Connecting out-of-school experiences to in-school mathematics
Relevant Math Instruction & Cultural Connection	(a) Developing the knowledge of your students; (b) Knowing your students—knowing their interest in math, identifying their strengths and areas of growth in math; (c) Connecting math lessons to current learning (e.g., gaming, technology, current interests, real life); (d) connecting students' learning to the current culture; (e) modifying the curriculum to students' cultures and background knowledge

CRP Guiding Principles for Math	(a) Unpacking the learning target; (b) Utilizing academic language; (c) Providing access to all students in learning mathematics (Gutiérrez, 2012 – access & equity); (d) Providing multiple entries into the lesson for different learning levels; (e) Providing scaffolds that are responsive to all students' culture and learning needs; (f) Creating classroom culture for students to feel empowered; Building a teamwork and collaboration learning environment

Note. CRP = Culturally Relevant Pedagogy

Section 2: Instructional Support and Challenges for Implementing Culturally Relevant (Math) Pedagogy

Ms. Brown described how she provides book chapters and research articles that explain CRP and equity in mathematics teaching and learning. Ms. Brown also expressed the usage of the coaching cycle, which provides yearlong support to teachers on implementing CRP when teaching mathematics with Black students. As an administrator, Ms. Dixon explained that she allows her teachers to engage in diversity and racial equity training in the form of professional development to support their racially and linguistically diverse students.

A constant theme across Black educators' responses was that the curricula that are given by the school district for teachers to use during instruction do not cater to the cultural, racial, linguistic, and community backgrounds of their Black students. For example, Ms. Johnson argued that although she adapts the curriculum and redesigns her math teaching practices to be culturally relevant to their Black students, she explained that on state and district standardized assessments, the math questions and tasks do not connect to her Black students' cultural, racial, linguistic, and community backgrounds. Similarly, Ms. Dixon explained that: "all curricula are culturally relevant, but culturally relevant to the dominant White culture in U.S. classrooms." She further explained that classroom teachers with Black students must search for additional resources to address the cultural mismatch and content of the curricula they are required to teach.

Conclusion & Implications

The findings of this study add to the existing research on culturally relevant pedagogy, but importantly, the perspectives of Black educators when teaching and working with Black students in mathematics education. All six Black educators provided a thoughtful explanation of the culturally relevant math practices to cater to the academic, social, cultural, and sociopolitical needs of Black students. The Black educators expressed that the learning of mathematics must not operate in a vacuum that only facilitates the teaching and learning of mathematics based on the curriculum, but must also connect to Black students' cultures, backgrounds, identities, ways of knowing and doing, and real-life experiences and situations. The shift from the traditional approach to teaching mathematics to the implementation of CRP and CSP is to ensure Black students can learn and apply mathematics to solve problems inside and outside of the classroom. The implementation of CRP addresses the curricula mismatch, which at times fails to connect learning materials to the cultural, linguistic, and community background of Black students.

In implementing culturally relevant pedagogy in mathematics education, the importance of relevance was emphasized by Ms. Brown to ensure math instruction connected to current time, culture, and real-life experiences and situations of Black students. Both Ms. Johnson and Ms. Brown shared the role the curriculum plays in Black students' learning, motivation, and engagement during math instruction. Ms. Johnson expressed that through curriculum adaptation,

her Black students can see themselves represented, along with their cultures and backgrounds, in the math lesson. This math teaching practice by Ms. Johnson further develops the math confidence in her Black students, which was also alluded to by Ms. Brown. Adding to these approaches to support and enhance Black students' learning and doing mathematics, Ms. Dixon emphasized the notion that equitable math instruction must be implemented to disrupt the practices of providing Black students with remedial and below-grade-level math instruction.

To address the challenges often experienced by teachers to implementation CRP inside the classroom, I recommend professional learning in the form of communities of practice (Wenger, 2000; Wenger et al., 2002) within a grade-level and school-wide setting. This will allow educators to unpack theoretical and practical teaching and learning approaches of culturally relevant and sustaining pedagogies when teaching mathematics. Additionally, administrators (and teachers) must ensure they are updated with the current research literature (theoretical and practical) to further support their learning and instructional practices of their teachers when engaging Black students during mathematics instruction.

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KNOWLEDGE PRODUCTION IN PIECES: ANALYZING EPISTEMIC NICHES IN MATHEMATICS EDUCATION RESEARCH

PRODUCCIÓN DE CONOCIMIENTO EN TROZOS: ANÁLISIS DE NICHOS EPISTÉMICOS EN LA INVESTIGACIÓN EN EDUCACIÓN MATEMÁTICA

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This theoretical paper builds on Reys, Reys, and Shih's (2021) concept of epistemic niches in mathematics education by advocating analyses that interrogate the power relations shaping knowledge production. Adopting a mereological perspective—examining how parts constitute the disciplinary whole—three dangers of niche formation are discussed: epistemic anchoring (theoretical rigidity), severe partitioning (mutual unintelligibility), and ghettoization (systematic marginalization). Using equity-focused scholarship as a case study, the commentary argues that these dynamics reproduce epistemic injustice, particularly affecting scholars of color. The commentary calls for a critical meta-epistemic agenda: sustained reflexivity about whose knowledge counts, under what conditions niches achieve legitimacy, and how mathematics education research can address equity while tolerating epistemic injustice in its own structures.

Keywords: Doctoral Education; Equity, Inclusion, and Diversity

Research in mathematics education in the United States issues from a loose collection of "jagged, uneven pieces"—a fragmented constellation of discrete yet overlapping epistemic niches (Lawrence-Lightfoot, 1983; Silver & Kilpatrick, 1994). These niches, stitched together as institutional networks of universities, school systems, research institutes, funding agencies, and political actors, constitute the terrain on which academic knowledge about mathematics teaching and learning is produced, validated, and circulated. Like most academic fields, mathematics education has evolved into an assemblage of communities of practice (Wenger, 1998), organized around core areas such as learning, teaching, and curriculum development.

The history of knowledge production in mathematics education is fundamentally a story of these niches—their formation, evolution, institutionalization, and sometimes their dissolution. Many are familiar by name: the School Mathematics Study Group (SMSG), the National Council of Teachers of Mathematics (NCTM), QUASAR, the Algebra Project, NSF Centers for Learning and Teaching, TODOS: Mathematics for ALL. These represent relatively independent sub-specializations, each with distinct intellectual genealogies, methodological commitments, and epistemological orientations.

As the field evolves, what Kilpatrick (2013) calls "critical reflexivity" becomes increasingly necessary. We need ongoing accounting of these sub-specializations: their progenitors and interlocutors, their purposes and agendas, their contributions over time, and crucially, the conditions under which they emerge, persist, or disappear. This is not a call for surveillance but for a critical, meta-epistemic agenda: a concerted way of knowing about our knowing in mathematics education.

Analyzing Epistemic Niches: A Comment on Reys, Reys, and Shih (2021)

Reys, Reys, and Shih's (2021) contribution extends a valuable longitudinal tradition of surveying doctoral programs in mathematics education. Their mapping of specialized doctoral

programs provides important descriptive information about the contemporary landscape. However, their analysis leaves critical dimensions of epistemic niche formation unexplored. By treating doctoral programs as the primary unit in a frequency-based, descriptive analysis, the results sidestep fundamental questions about power, access, and epistemic injustice, risking naturalizing the current configuration as simply normal development without interrogating the complex processes by which specialized niches arise and the barriers they navigate toward institutionalization.

While doctoral programs offer certain affordances as a unit of analysis—they are institutionalized, relatively stable, and documentable—this choice carries significant limitations. As Gray and Lowery (1996) suggest, the question of what constitutes the appropriate unit becomes paramount. If epistemic niches are fundamentally about people and knowledge production rather than institutional structures, then focusing exclusively on programs may obscure as much as it reveals. Deeper layers of analysis—who enters and what knowledges are produced from known programs—would be needed.

Treating programs as the primary unit tends to render doctoral students as passive actors. As Pallas (2001) argues, this perspective "views students as relatively passive participants in the socialization process" and "assumes that students' personal and social origins, and hence their personal epistemologies, are largely irrelevant to research preparation—a particularly dangerous assumption in view of the changing composition of today's graduate students" (p. 7). This overlooks how prospective scholars choose their programs, navigate participation within them, and how their epistemic purposes influence their trajectories and contributions.

Most consequentially, focusing on doctoral programs without interrogating their epistemological and ideological orientations obscures how power operates in doctoral socialization. This approach "does not make problematic whose skills and values are internalized through the socialization process" (Pallas, p. 7). This silence regarding power becomes particularly acute when we consider persistent underrepresentation of scholars of color. As Young (2001) observes, "despite their growing presence in doctoral programs, people of color remain underrepresented among the contributors to knowledge production, with critical consequences for the nature and range of perspectives, understandings, and paradigms that inform this work" (p. 4).

Epistemic Niches and the Problem of Equity

The focus on epistemic niches becomes particularly urgent when we consider equity—not merely as a research topic but equity in the structure of knowledge production itself. What Berenstain, Dotson, Paredes, Ruíz, and Silva (2021) observe about colonial epistemologies applies to academic fields: "the injustice of epistemic injustice is not the exception but the rule... The injustice goes all the way down in the systems that produce it—and it does so, for the most part, by design" (p. 3).

The "Equity Niche" as Case Study

The emergence of an "equity niche" in mathematics education—focused on access, representation, identity, and justice—provides an instructive case. Beginning around the 1980s and gaining momentum through the 1990s, researchers and practitioners organized scholarship around questions of equity, challenging the field's tendency to treat differential outcomes as incidental.

Two prominent examples illustrate different approaches: QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) and the Algebra Project. QUASAR, funded by the Ford Foundation and based at University of Pittsburgh, represented a

university-based research initiative focused on improving mathematics instruction in middle schools serving economically disadvantaged students. The Algebra Project, founded by Dr. Robert P. Moses, emerged from community organizing traditions and positioned mathematical literacy explicitly as a civil right.

Both contributed significantly to knowledge about equity yet occupied different positions within the field's epistemic landscape. QUASAR operated largely within mainstream research paradigms and institutional structures. The Algebra Project, while influential and well-supported, often occupied a more marginal position—celebrated for its grassroots approach but sometimes viewed as less "research-oriented," despite producing important knowledge about mathematics teaching and learning. This differential positioning reveals something crucial: the question is not simply whether equity-focused work exists, but whose equity work, grounded in which epistemological and methodological traditions, achieves recognition and institutional support.

The concentration of equity-focused scholarship into a particular niche carries risks that Gray and Lowery (1996) term "ghettoization"—the cordoning off of specific concerns into specialized subfields that can then be ignored by the mainstream. Bullock (2014) cautioned about this emergent danger. When equity becomes a niche rather than a cross-cutting concern, problematic dynamics emerge: the broader field can claim to address equity by pointing to equity-focused scholars and programs, without fundamentally examining how its own structures, methods, and theories may reproduce inequity.

Epistemic Enclosure: Anchoring, Severe Partitioning, and Ghettoization

The metaphor of the niche borrows from ecology and carries connotations of natural fit and adaptive specialization. Yet this ecological framing risks obscuring the deeply social and political processes through which epistemic niches form, persist, and sometimes calcify. To understand these risks, we must distinguish between productive specialization—the necessary division of intellectual labor—and epistemic enclosure: the hardening of boundaries that prevents dialogue, limits circulation of ideas, and reproduces hierarchies of epistemic authority. Three interconnected dangers merit attention: epistemic anchoring, severe partitioning, and ghettoization.

Epistemic Anchoring Practices

Wilson (1973) observed that "the easiest and most prudent maintenance strategy is to develop autonomy—that is, a distinctive area of competence, a clearly demarcated and exclusively served clientele or membership, and undisputed jurisdiction over a function, service, goal, or cause" (as cited by Gray and Lowery, 1997, p. 28). While autonomy enables focused inquiry, it can also produce epistemic anchoring—the tendency of niches to become so firmly fixed to particular theoretical commitments, methodological approaches, or topical concerns that they cannot adapt to new questions or integrate insights from other communities.

Epistemic anchoring manifests when certain niches become synonymous with specific theoretical frameworks; scholars working within these niches may struggle to publish work diverging from established theoretical vocabularies. Graduate students socialized within anchored niches learn implicit rules about what constitutes legitimate scholarship—negotiating "the meaning of what counts as knowledge through interactions with others in the same community, as well as through exposure to reifications (e.g., books and articles), which are often interpreted in local terms" (Pallas, 2001, p. 9).

The danger is not specialization but rigidity. When niches become anchored, they risk paradigmatic insularity—the inability to recognize anomalies or integrate revolutionary insights because doing so would require abandoning commitments that define the niche's identity. This

reproduction occurs through doctoral education: students inherit not only intellectual tools but also boundaries of legitimate inquiry, learning to think within the niche rather than critically about it, creating epistemic path dependency.

Severe Partitioning and Balkanization

Beyond individual niche insularity lies severe partitioning: the fragmentation into mutually incomprehensible or hostile camps—what political scientists call balkanization. Pallas (2001) articulates this danger: "If educational researchers cannot understand and engage with one another, both within and across at least some educational research communities, the enterprise is doomed to failure. Thus, to prevent recurring epistemological single-mindedness, educational researchers will need to engage with multiple epistemological perspectives to the point that members of different communities can understand one another, despite, or perhaps through, their differences" (pp. 6–7).

In mathematics education, we observe partitioning along multiple fault lines: quantitative versus qualitative researchers, cognitive versus sociocultural theorists, learning versus teaching scholars, curriculum versus equity focus. The "math wars" represent the most severe and hostile partitioning in recent history. These divisions aren't inherently problematic—different questions require different approaches. Danger emerges when niches become so partitioned that scholars cannot engage meaningfully across boundaries.

Severe partitioning produces pathologies. It enables epistemic provincialism—assuming one's niche possesses the correct or most important approach. It facilitates mutual unintelligibility when niches develop highly specialized vocabularies without sustained dialogue. Most troublingly, it enables niches to ignore work challenging prevailing assumptions, providing plausible deniability: "That's not my area; that's what the equity people study." Balkanization becomes particularly dangerous when it maps onto ideological or identity-based differences, losing the field's capacity for productive disagreement.

Epistemic Ghettoization

The third and most consequential danger is epistemic ghettoization—the process by which certain niches become marked as marginal, applied, or less rigorous, effectively quarantining their insights from the mainstream. Unlike anchoring (internal rigidity) or partitioning (mutual separation), ghettoization is fundamentally about power and hierarchy, describing how some niches achieve epistemic authority while others are relegated to subordinate status.

The spatial metaphor is deliberate. Just as urban ghettos result from deliberate policies of segregation and disinvestment—not natural sorting—epistemic ghettos emerge from structured processes determining whose knowledge counts, whose questions matter, and whose methods are legitimate. The social conditions of broader society—segregation, gentrification, neoliberalism, intergenerational wealth hoarding, white flight—are not merely analogies but actual material and ideological forces shaping which scholars access elite institutions, sustained funding, and positions of influence.

Ghettoization manifests in how different research is valued and cited. Work focused on equity, particularly employing critical frameworks or community-based methods, often gets coded as "advocacy" rather than research, as "political" rather than scholarly. This coding justifies its exclusion from theoretical conversations shaping the field's core commitments. Equity-focused scholars may be celebrated for their "important work" while simultaneously passed over for editorial positions, theory-building invitations, or major awards.

This hierarchical coding has profound consequences for scholars of color. The problem is not merely numerical underrepresentation—it is that epistemological orientations and research

questions that scholars of color often bring are systematically devalued. When the equity niche becomes ghettoized, scholars whose lived experiences and intellectual commitments orient them toward questions of justice find their work marginalized, regardless of rigor or insight.

The danger extends beyond individual scholars to the field's epistemic health. When perspectives from marginalized niches cannot circulate freely, when their orientations cannot shape core theoretical debates, the field's knowledge becomes impoverished. We lose what standpoint theorists call "epistemic privilege"—the insight that "in a socially stratified society, different social positions yield distinctive epistemic positions, and some are better than others" (Siegel, 2006, p. 8). By ghettoizing equity-focused work, mathematics education cuts itself off from perspectives most able to illuminate how the field itself reproduces inequality.

The Compounding Effects

These three dangers—anchoring, partitioning, and ghettoization—reinforce one another. Epistemic anchoring increases partitioning likelihood, as rigid niches struggle to communicate. Partitioning facilitates ghettoization by allowing dominant niches to ignore marginalized ones. Ghettoization reinforces anchoring within marginalized niches, as scholars develop increasingly specialized languages to sustain work in hostile environments.

The result is an academic field that, despite genuine commitments to equity among many scholars, reproduces epistemic hierarchy through its structure. Doctoral students learn which questions are serious, which methods rigorous, which theories foundational. These lessons get encoded in hiring decisions, promotion processes, editorial practices, and funding priorities. Niches calcify into structures determining whose knowledge counts.

As Berenstain et al. (2021) remind us, this is not accidental: "the injustice goes all the way down in the systems that produce it—and it does so, for the most part, by design" (p. 3). The current configuration of epistemic niches in mathematics education emerged through structured, political decisions about funding, institutional prestige, methodological legitimacy, and theoretical authority. Addressing niche formation dangers requires more than encouraging dialogue or celebrating diversity—it requires confronting power relations determining which niches thrive and which struggle.

Toward Critical Reflexivity: Power and Accountability

If these dangers represent genuine threats to mathematics education's epistemic health, what might a constructive response look like? This is not a call for eliminating niches—specialization serves important functions. Nor is this conjuring false unity papering over genuine epistemological differences. Rather, this proposes a critical, meta-epistemic agenda: sustained, reflexive engagement with how we produce knowledge about mathematics education, who participates in that production, and whose interests current arrangements serve.

Make Power Visible

As Dotson (2018) argues, "What generates an assessment of an epistemological orientation as 'problematic' concerns the operative orienting variables of the analysis in question, e.g., what generates 'oughts,' including epistemic 'oughts'" (p. 130). We must interrogate not merely what different niches study or how, but why certain approaches achieve legitimacy while others are marginalized. This requires examining material conditions of knowledge production: Which institutions house which niches? Which receive sustained funding? Whose graduates obtain elite university positions? Which journals are "top tier," and which niches are represented on their editorial boards?

Making power visible means acknowledging that epistemic hierarchies are not natural outcomes of intellectual merit but products of historical and ongoing processes of inclusion and

exclusion. When quantitative methods are treated as inherently more rigorous than qualitative ones, when cognitive frameworks are deemed more fundamental than sociocultural ones, when research ignoring race and power is coded as "objective" while research centering them is coded as "political"—these are exercises of epistemic power requiring examination.

Epistemic Accountability

Niches should be held accountable not merely for quality of work produced within their paradigms, but for engagement with questions and perspectives challenging those paradigms. This means creating structures and incentives for cross-niche dialogue beyond superficial acknowledgment, insisting that scholars engage genuinely with perspectives different from their own.

Epistemic accountability extends to doctoral education. Mathematics education doctoral programs should be evaluated not only on graduation rates and job placements, but on whether they prepare students for epistemological diversity. As Pallas (2001) argues, preventing "recurring epistemological single-mindedness" requires that "educational researchers engage with multiple epistemological perspectives to the point that members of different communities can understand one another" (pp. 6–7). Doctoral programs should explicitly prepare students to work across epistemic boundaries, translate between theoretical languages, and recognize value in approaches different from their own.

Conclusion: Toward Ways of Knowing About Our Knowing

This commentary necessarily leaves much undone but hopefully stimulates dialogue. We need comprehensive functional analyses mapping how different epistemic niches operate—their funding sources, citation networks, reproduction through doctoral education, relationships to policy and practice. Such analyses would require substantial empirical work examining who produces knowledge in mathematics education, under what conditions, with what resources and constraints, and with what consequences.

The question of equity in knowledge production—not as research topic but in the structures through which knowledge is produced and validated—requires sustained engagement. Underrepresentation of scholars of color is not merely demographic but reflects and reproduces epistemic injustice. Dismantling these systems requires more than good intentions or procedural reforms; it demands fundamental rethinking of what counts as knowledge and who is authorized to produce it.

Silver and Kilpatrick's (1994) observation remains apt: what we see in the field, we see in ourselves. The fragmentation, mutual ignorance, limited awareness beyond our own plots—these are not merely organizational problems but epistemic ones. They reflect and produce a field that cannot fully account for its own knowledge production, that cannot critically examine conditions under which certain questions become central while others are marginalized, certain methods become standard while others deemed suspect, and certain scholars achieve influence while others struggle for recognition.

A critical, meta-epistemic agenda—toward ways of knowing about our knowing—is not a luxury but a necessity. Without sustained attention to how epistemic niches form, function, and relate to one another, mathematics education risks reproducing through its knowledge production structures the very inequities that many scholars study in mathematics classrooms. The field cannot adequately address equity in mathematics education while tolerating epistemic injustice in mathematics education research. By expanding the concept of niche to encompass critical analysis of power, attending to dangers alongside possibilities, and insisting that epistemic justice is integral to epistemic health, we may evolve the field toward critical reflexivity. The

epistemic niches we have, the epistemic niches we need, and the relationships among them—these are not settled questions but ongoing challenges demanding continued attention, critique, and collective commitment to a more just and epistemically robust field.

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WHEN ASKING PARENTS HELPS OR NOT: SES DIFFERENCES IN PARENTAL MATH HELP-SEEKING AND ACHIEVEMENT

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This study examined elementary students' parental math help-seeking and its links to achievement across socioeconomic contexts. Open-ended responses were coded into six themes, including no hesitation and five barriers. Low SES students more often cited parental knowledge limits and emotional concerns. An exact test using dichotomized measure of hesitation showed significant group differences. Despite a shared tendency not to hesitate in seeking help, the high SES group demonstrated higher achievement than its low SES counterpart. Moreover, tutoring was associated with achievement differences only within the low SES group. Findings suggest help-seeking is context dependent and shaped by home resources, emotional climate, and access to alternative supports. Implications include strengthening emotionally safe family interactions, supporting parental math competence, and expanding school-based pathways for help.

Keywords: Elementary School Education; Equity, Inclusion, and Diversity; Communication

Purpose of the Study

Learning in mathematics emerges not only from formal instruction but also from the everyday interactions, expectations, and resources that surround children at home. Sociocultural perspectives emphasize that thinking is mediated by tools, language, and relationships (Vygotsky, 1978). In this view, the home is a consequential site where meanings about mathematics are negotiated and where identities as learners are formed (Anderson & Gold, 2006). Within this setting, parents are often the first adults to whom children turn when they encounter difficulty or curiosity.

Help-seeking is central to self-regulated learning (Karabenick & Newman, 2006), yet research on parental involvement has often treated requests for help as uniformly beneficial and has relied primarily on reports from adults rather than from children (Zhang et al., 2024). Less is known about how children themselves understand the decision to ask parents for help with mathematics, and how these understandings vary by SES. South Korea provides a compelling context where academic expectations are high, families differ widely in access to educational resources, and a substantial market for private supports coexists with public schooling (Kim et al., 2010). In such a context, children's help-seeking decisions are likely to be shaped by both structural conditions and relational dynamics at home.

Help-seeking is not merely a function of personal confidence or motivation but is embedded within sociocultural environments. Attending to these broader influences is essential for understanding variation in help-seeking behaviors and for shifting the discussion beyond individual responsibility toward the structural conditions that shape students' fair opportunity to learn. This perspective motivates attention to the home as a consequential learning setting and to socioeconomic differences in resources and emotional climate that condition when asking for help is feasible.

This study takes an exploratory approach to children's perceptions of seeking parental help in mathematics. Open-ended survey responses from South Korean elementary students are examined to identify how children describe the barriers and affordances of seeking

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parental assistance with mathematics and these descriptions are situated in relation to SES. The aim is to clarify the phenomena that children themselves report, to map how these reports cluster into interpretable themes, and to outline how such themes can inform subsequent inquiry into connections between help-seeking, instructional opportunity, and achievement.

Accordingly, we ask two questions. First, how do students describe the barriers to seeking parental help in mathematics, and how do these descriptions organize into coherent categories? Second, how are students' parental help-seeking tendencies related to mathematics achievement within and across SES groups, and what patterns emerge when considering access to external supports such as private tutoring?

Theoretical Background

The model of the parent involvement process (Hoover-Dempsey & Sandler, 1997; Hoover-Dempsey et al., 2005) specify how parents become involved in their children's learning and how that involvement influences student outcomes across a series of levels (Whitaker, 2018). Our focus is on level one, which identifies three categories of motivators (i.e., personal, contextual, and parents' life context) that shape how parents engage with their children's learning.

First, personal motivators concern what parents believe about their role and their efficacy for supporting learning. Role construction captures parents' understandings of what they should do in relation to schoolwork, while efficacy reflects confidence that their efforts can make a difference in the child's success (Bandura, 1997; Hoover-Dempsey & Sandler, 1997). Second, contextual motivators arise from invitations by children, schools, and teachers (Hoover-Dempsey & Sandler, 1997). For example, the invitation from children signal that parental participation is valued (Baumrind, 1991). Third, parents' life context motivators refer to the practical conditions that enable or constrain involvement, including available time and energy, knowledge and skills, and access to social capital (Horvat et al., 2003; Hoover-Dempsey & Sandler, 1997).

Centering level one is warranted in the present study because these motivators are foundational for understanding parental participation in elementary mathematics. Extending the framework to the perspective of the child further clarifies the dynamics at home. Students appraise whether parents can assist with grade level mathematics, anticipate the emotional climate that help seeking may elicit, and consider the availability of alternative supports when parental assistance is limited. Framed this way, the model allows a nuanced analysis of how beliefs, invitations, and life context interact, and how those interactions may differ by SES.

A complementary foundation comes from Vygotskian sociocultural theory. Sociocultural perspectives conceptualize learning as participation in culturally organized activity mediated by tools, language, and social interaction (Vygotsky, 1978). From this standpoint, the home is not a passive backdrop but an active site where mathematical meanings are negotiated and where learner have opportunities to receive assistance from more knowledgeable others.

Taken together, these perspectives treat help seeking from parents as a social practice shaped by both structure and relationship. The present study integrates the two strands by centering student reports of perceived barriers, comparing their distribution across socioeconomic groups, and relating these perceptions to mathematics performance within groups.

Methods

This study employed a mixed-methods design. Both content analysis and statistical analysis were conducted to compare the similarities and differences between mathematical help-seeking behavior and achievement between high and low SES groups.

Mode of Inquiry

Qualitative analysis. Grounded in the Hoover-Dempsey and Sandler model of the parent involvement process (1997), qualitative coding proceeded through a combined deductive and inductive sequence. Provisional categories were derived a priori to reflect core level one motivators in the model and were refined through iterative review into a six-category codebook. The focal open response prompt asked students whether they ever hesitate to ask a parent for help when they need assistance with a math task. Five categories captured barriers reported by students who indicated hesitation: (1) emotional concerns, (2) student autonomy orientation, (3) parent knowledge & skill limits, (4) parental time limits, and (5) access to external supports. A sixth category, (6) no hesitation to ask, captured the perceived absence of a barrier among students who selected no hesitation, and described why they would reach out. Operational definitions and illustrative student responses are provided in Table 1. Categories were defined to be mutually exclusive and collectively exhaustive at the response level. A total of 123 valid responses were retained after excluding invalid entries. Two researchers applied the finalized definitions to the full corpus, achieving an intercoder agreement of 92.65%, which indicates strong reliability. Code frequencies were then compared across socioeconomic groups to identify convergences and contrasts.

Table 1: Theoretical Categories, Emergent Themes, Definitions, and Student Response

Theoretical Category	Emergent Theme	Operational Definition	Example Student Response
Personal motivators	Emotional concerns	Students anticipate or describe negative affect when asking parents for help.	"I am afraid I will be scolded."
Contextual motivators	Student autonomy orientation	Students prefer to solve problems independently.	"I want to figure it out by myself."
Contextual motivators	No hesitation to ask	Students state an unqualified willingness to seek help from parents.	"I would ask without hesitation."
Family life context: knowledge & skills	Parental knowledge & skill limits	Students perceive that parents lack sufficient mathematical knowledge or ability to provide help.	"My parents do not know math."
Family life context: Time & energy	Parental time limits	Students report that parents are too busy, tired, or unavailable to provide help.	"My parents are busy."
Family life context: Social capital	Access to external supports	Students rely on alternative academic supports (e.g., tutors, online resources, etc.).	"I can ask my private tutor."

Note. Theoretical categories are adapted from the Hoover-Dempsey and Sandler Model of the Parent Involvement Process.

Researcher positionality was considered as part of analytic rigor given the interpretive nature of qualitative analysis. The primary analyst is a mathematics education researcher with professional experience working with Korean elementary students and a sustained interest in equity in home-based learning supports. To limit the influence of any single perspective, the

qualitative analysis was guided by a priori theoretical framework, with operational definitions specified in a codebook. A second trained coder independently applied the finalized categories to the full corpus, and disagreements were resolved through discussion using explicit decision rules.

Quantitative analysis. Because several subgroup sizes were small, nonparametric tests were used. Group differences in frequencies were examined with chi-square test, Fisher's exact tests, and achievement differences were tested with Mann–Whitney U and the Kruskal–Wallis H tests with Dunn's post hoc comparisons using Bonferroni adjustment.

Sample and Procedure

With the consent of students and their caregivers, students were invited to complete an online survey to answer questions about their basic information (e.g., age, gender, ethnicity, and school), mathematics tutoring, help-seeking preferences, and the detailed reasons for either asking for or not asking for help. A total of 133 students responded to the survey, and 123 students completed the survey, yielding a 92.48% complete response rate. The participants were 6th grade elementary school students whose age range between 11 and 12 years old. The decision on participants' grade was guided by Crystal et al. (2008) in that 6th grade students showed higher tendency to seek help on their homework compared to other higher-grade students.

The study solicitation targeted variation in students' family SES. Because direct questions about parental income, occupation, assets, or consumption are culturally sensitive in Korea (Kim et al., 2019), recent work has favored simplified SES indicators (Valero et al., 2015). Aggregate indices at the school or neighborhood level are commonly used (Kim et al., 2019). For this study, data from the Korean Statistical Information Service (KOSIS) is releasing tax payment information by city, county, and district every year, which allowed calculation of per capital income as a core SES indicator (Wani, 2019). Using these data, the 25 districts in Seoul was ranked by mean SES, and 10 schools were selected in eight districts, with three in higher income areas and five in lower income areas.

A mathematics achievement test was administered alongside the survey questionnaire. The test consisted of 20 problems to be solved within a 40-minute time frame, reflecting a traditional standardized testing setting in South Korea. Because the testing instrument covered the majority of the mathematics curriculum for the first half of the school year, classroom teachers who assisted with survey distribution also supported test administration for the entire class for instructional purposes. Students solved the problems directly on the test sheets, and only the sheets completed by study participants were collected separately and forwarded to the researchers for further analysis.

Instrument

Help-seeking behavior. An eight-item survey was developed to capture students' mathematics help-seeking preferences, with open ended prompts designed to elicit students' reasons for asking for or refraining from parental help. The focal item asked whether students hesitated to ask a parent for math help and requested a brief explanation aligned to the Yes or No response, capturing either reasons for hesitation or motivations for reaching out.

Mathematics achievement. The testing instruments were developed by adapting problems from as a basic academic skill assessment and a 6th grade summative assessment created by the Ministry of Education, and i-Scream, respectively. The i-Scream is a private teaching supporting platform where 93% of elementary teachers use (Jang, 2025). Twenty multiple-choice items were included in the achievement test, and the breadth of the problems was designed to cover all topics specified in the national curriculum for the first

half of the academic year. Both the achievement testing instrument and its administration were valid in that it followed the norm of traditional summative achievement testing applied to many elementary classrooms (Lee, 2009). The content domains included fraction operations, three-dimensional geometry and measurement, ratio and proportion, and data analysis.

Results

Descriptive Analysis

Data from 133 students were analyzed to examine parental math help-seeking and mathematics achievement. The high SES (n = 51) and low SES (n = 82) groups had similar gender distributions. The high SES group included 26 boys and 25 girls, and the low SES group included 40 boys and 42 girls.

Tutoring participation differed substantially by SES. Most high SES students reported tutoring (80.4%), whereas nearly half of low SES students did not (46.3%). Gender was not associated with tutoring participation in either the high SES group (r = .089, p = .534) or the low SES group (r = .121, p = .281). Achievement analyses using Mann Whitney U tests showed that high SES students scored higher than low SES students (z = −3.75, p < .001), and students who participated in tutoring scored higher than those who did not (z = 3.20, p = .001), while gender differences were not significant (z = .732, p = .464). Within group follow up Mann-Whitney U tests indicated that the tutoring advantage was concentrated in the low SES group (z = 2.49, p = .013) and was not significant among high SES students (z = .872, p = .383).

Content Analysis

Content analysis revealed clear differences by socioeconomic status. Across groups, no hesitation was the model theme. It accounted for 69.4% of coded instances among high SES responses and 64.9% among low SES responses, often phrased as “I would ask without hesitation.” In contrast, barrier themes clustered in the low SES group. Parental knowledge and skill limits comprised 18.2% of low SES codes compared with 6.5% in high SES, typically expressed as “My parents do not know math.” Emotional concerns comprised 10.4% in low SES and 4.3% in high SES, with wording such as “I am afraid I will be scolded.” Student autonomy orientation appeared at low levels in both groups (6.5% in high SES; 3.9% in low SES), indicating some preference to solve problems independently. Parental time limits and access to external supports were infrequent in both groups, each at about 5% or less. Theme distributions within each SES group are summarized in Tables 2 and 3.

Table 2: High SES - Frequency and Percentage of Reported Help-Seeking Categories

Ranking	Categories	Code Counts	Percentage
1	No hesitation to ask	34	69.4
2 (tied)	Parental knowledge & skill limits	3	6.5
2	Student autonomy orientation	3	6.5
4 (tied)	Emotional concerns	2	4.3
4	Parental time limits	2	4.3
4	Access to external supports	2	4.3
Total		46	100.0

Table 3: Low SES - Frequency and Percentage of Reported Help-Seeking Categories

Ranking	Categories	Code Counts	Percentage
1	No hesitation to ask	50	64.9
2	Parental knowledge & skill limits	14	18.2
3	Emotional concerns	8	10.4
4	Student autonomy orientation	3	3.9
5 (tied)	Parental time limits	1	1.3
5	Access to external supports	1	1.3
Total		77	100.0

Taken together, the pattern indicates that many students view parents as a viable resource, but knowledge based and emotion-based barriers are reported more often by students from low SES backgrounds, whereas high SES responses are less concentrated in any single barrier theme. At the same time, a Pearson chi-square test comparing the distribution of the six coded themes across SES groups was not statistically significant, $x^2(5, N = 123) = 7.068$, $p = .21$. The six-theme distribution did not differ significantly by SES, likely due to sparse cells and a high base rate of no hesitation. A sensitivity check excluding no hesitation remained non-significant. Because chi-square tests can be unreliable with low expected frequencies, responses were aggregated into a dichotomous indicator, "no hesitation" versus "hesitation." This aggregation reduced table sparsity and provided a more reliable test of whether the propensity to hesitate differs by SES. Under this dichotomization, the chi-square test indicated a statistically significant association $x^2(1,123) = 4.575$, $p = .032$, and a Fisher exact test was conducted as a robustness check given small expected counts ($p = .025$). Both tests supported the interpretation that SES is associated with whether students report hesitation to ask parents. Building on this, subsequent analyses examined whether the achievement gap varies across SES by hesitation status using the Kruskal Wallis H test following by Dunn's method with a Bonferroni correction.

Kruskal Wallis H Test and Dunn's Method with a Bonferroni Correction

Kruskal-Wallis H tests indicated statistically significant differences in students' mathematics achievement across four subgroups defined by SES and parental help-seeking hesitation, $x^2(3, N = 123) = 19.432$, $p < .001$. Mean ranks of mathematics achievement were highest for the high SES with no hesitation (79.16), followed by high SES with hesitation (72.50), low SES with hesitation (65.43), and low SES with no hesitation (45.96). Dunn post hoc comparisons showed that the low SES with no hesitation subgroup scored lower than the other groups, but after Bonferroni adjustment only the difference between high SES with no hesitation remained significant ($p < .001$). Table 4 presents the pairwise comparisons of the four groups' mean ranks in mathematics achievement.

To explore additional factors associated with mathematics achievement within the low SES group, a follow-up analysis incorporated private tutoring participation. Low SES students were classified into four subgroups: (1) Hesitation × Tutoring, (2) Hesitation × No Tutoring, (3) No Hesitation × Tutoring, and (4) No Hesitation × No Tutoring, and mathematics achievement was compared using a Kruskal Wallis H test.

Table 4: Pairwise Comparison of SES X Help Hesitation

Group1 / Group2	Test	Std. Error	Sig.	Adj. Sig.
Low SES × No Hesitation / Low SES × Hesitation	19.466	8.483	.022	.131
Low SES × No Hesitation / High SES × Hesitation	26.540	11.418	.020	.121
Low SES × No Hesitation / High SES × No Hesitation	33.202	7.896	<.001	.000
Low SES × Hesitation / High SES × Hesitation	7.074	12.324	.566	1.000
Low SES× Hesitation / High SES × No Hesitation	13.736	9.156	.134	.801
High SES × Hesitation / High SES × No Hesitation	–6.662	11.927	.576	1.000

Achievement differed statistically significantly across the four subgroups, $x^2(3, N = 77) = 11.698$, $p = .008$. Mean ranks of mathematics achievement were highest for hesitation with tutoring (53.86), followed by no hesitation with tutoring (38.02), hesitation with no tutoring (35.11), and no hesitation with no tutoring (30.96).

Dunn post hoc comparisons indicated that no hesitation with no tutoring was statistically significantly lower than that for hesitation with tutoring ($p < .001$), and this difference remained statistically significant after Bonferroni correction ($p = .005$). These results suggest that, within the low SES group, students with tutoring, particularly those who also reported hesitation in seeking parental help, demonstrated higher mathematics achievement that students who had neither tutoring nor hesitation. Table 5 presents the pairwise comparisons of the four groups' mean ranks in mathematics achievement.

Table 5: Pairwise Comparison of Help Hesitation X Tutoring

Group1 / Group2	Test	Std. Error	Sig.	Adj. Sig.
No Hesitation × No Tutor / Hesitation × No Tutor	4.150	8.622	.630	1.000
No Hesitation × No Tutor / No Hesitation × Tutor	7.059	6.311	.263	1.000
No Hesitation × No Tutor / Hesitation × Tutor	22.900	6.836	<.001	.005
Hesitation × No Tutor / No Hesitation × Tutor	–2.910	8.714	.738	1.000
Hesitation × No Tutor / Hesitation × Tutor	18.750	9.102	.039	.236
No Hesitation × Tutor / Hesitation × Tutor	15.840	6.951	.023	.136

Discussions and Conclusions

This study is important to mathematics education by challenging the assumption that more help-seeking is necessarily better and by offering a framework for interpreting student hesitation as a contextual signal rather than a deficit. Across the full sample, achievement differences by SES remained evident even among students who reported no hesitation to ask parents, suggesting that willingness to seek help does not equalize opportunity to learn. Within the low SES group, patterns by hesitation status further indicated that the meaning of hesitation may differ across social contexts. In this setting, refraining from asking parents was not necessarily a marker of disengagement. Instead, it may reflect a realistic appraisal of the costs and benefits of asking at home.

The content analysis provides an important interpretive anchor for these patterns. Low SES students who hesitated more often described parental knowledge and skill limits and concerns

about negative emotional reactions. These themes point to two intertwined constraints. One is instructional, in that parents may not be able to explain current mathematics content or strategies in ways students find usable. The other is relational, in that help-seeking may carry emotional risk, such as fear of being scolded or of increasing stress at home. In contrast, high SES responses were less concentrated in any single barrier theme and more frequently reflected an absence of hesitation, consistent with a home context in which asking is experienced as feasible and emotionally safe.

The follow up analysis within the low SES group investigates the role of the broader support ecology, particularly access to private tutoring. Achievement differed significantly across four subgroups defined by hesitation and tutoring, with the strongest performance observed among students who had tutoring and reported hesitation, and the lowest performance among students who had neither tutoring nor hesitation. In this sense, reliance on parental help in the absence of accessible alternatives may be a vulnerable configuration for some low SES students, not because parents are unwilling to help, but because the support available at home may be constrained in content knowledge, time, or emotional climate.

These results suggest that the function of hesitation is shaped by the options students perceive as available. For students with tutoring, hesitation to ask parents may reflect strategic allocation of help-seeking toward sources viewed as more reliable or less emotionally costly. For students without tutoring, reporting no hesitation may not indicate stronger support; it may instead reflect the absence of other viable pathways, leaving students dependent on home help that may not consistently promote understanding. This interpretation aligns with sociocultural perspectives that treat help-seeking as relational and situated, shaped by norms, resources, and interactional histories rather than by individual disposition alone.

Implications for Practice

From a practice standpoint, the findings argue against interventions that focus only on increasing the frequency of parental involvement. What appears to matter is not simply whether children ask, but whether the help they can access is instructionally useful and emotionally safe. Two complementary levers follow.

First, family facing supports can strengthen parents' capacity to assist in ways that do not require advanced content expertise. Schools can share short routines that promote constructive help interactions, such as prompting children to explain their thinking. These routines also address emotional climate by reducing judgment and conflict during math talk. Second, schools and communities should expand reliable non-parent pathways for assistance, especially for students who cannot depend on private tutoring. Structured opportunities such as homework clubs and teacher office hours can provide alternative routes that reduce the stakes of home-based help and support students' right to seek assistance when needed.

Conclusion and Future Directions

Overall, this study highlights that equity in mathematics learning involves more than encouraging children to ask for help. It requires attention to the social and emotional conditions that make help-seeking possible and productive, and to the distribution of support pathways across contexts. Future research should examine these mechanisms with designs that capture the quality of help interactions and the availability of alternative supports, and that follow students over time to clarify how help-seeking pathways develop across home, school, and community settings.

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"FINALLY TAKING A STEP BACK": MATHEMATICS CO-TEACHERS' POWER NEGOTIATIONS AND KNOWLEDGE EXCHANGES DURING CO-PLANNING

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Students with disabilities are often denied meaningful opportunities for mathematical sense-making (Lambert & Tan, 2017). Co-teaching between mathematics teachers and special educators offers a promising approach for disrupting these inequities by leveraging complementary expertise from both fields. Grounded in Fairclough's (2003) critical discourse framework and Disability Studies in Mathematics Education scholarship, this qualitative study draws on a video-recorded co-planning session and semi-structured interviews with one mathematics co-teaching pair to examine how power and expertise are negotiated through discursive moves. Findings show that co-teachers enacted, shared, and resisted power while crossing boundaries between fields, positioning co-planning as a key site for reconfiguring authority and negotiating distinct expertise to better support mathematical sense-making.

Keywords: Special Education, Students with Disabilities, Teacher Knowledge

Current mathematics education reforms and policies emphasize that all students should have opportunities for mathematical sense-making, defined as opportunities to develop conceptual and procedural understanding through engagement with grade-level content connected to students' lived experiences (National Council of Teachers of Mathematics [NCTM], 2018; NCTM & Council for Exceptional Children [CEC], 2024). Yet, despite these policy commitments, students with disabilities continue to be denied such opportunities in K–12 mathematics classrooms. In their analysis of 149 articles, Lambert and Tan (2017) found that mathematics instruction for students with disabilities frequently prioritizes procedural compliance and remediation over sense-making, reinforcing deficit views of disability and mathematical competence.

Co-teaching between mathematics teachers and special educators who co-plan, co-deliver, and co-assess general instruction has been identified as a promising approach for addressing these inequities. Co-teaching brings together distinct yet complementary forms of expertise, including knowledge of mathematics teaching and knowledge of accommodations for individual learners, which can be used to leverage opportunities for mathematical sense-making for all students in general education settings, including students with disabilities (e.g., Lambert & Tan, 2016; Sheppard & Wieman, 2020; Cohen et al., 2025). However, mathematics and special education have long operated in disciplinary silos, with limited collaboration and persistent tensions rooted in differing learning perspectives, conferences, journals, and teacher preparation pathways (Schwartz, 2023; Cohen et al., 2025). These structural divisions are often accompanied by unbalanced power dynamics at both institutional and interpersonal levels.

Within collaborative settings such as co-taught classrooms, co-planning sessions, and professional learning communities, co-teachers position themselves and each other into particular roles through discursive interactions that negotiate power (Davies & Harré, 1990). Prior research suggests that these negotiations frequently reproduce hierarchical relationships. For example, Zigmond and Matta (2004) observed 41 pairs of high school co-teachers across subjects and found that mathematics teaching was generally dictated by the mathematics teacher while the special educator was positioned as a "helper." In such arrangements, power is enacted rather than shared, limiting the potential for collaboration and integrated expertise.

At the same time, co-planning contexts open up opportunities for boundary crossing, interactions across historically distinct professional domains (Engeström et al., 1995; Akkerman & Bakker, 2011). Akkerman and Bakker (2011) argue that educational boundaries necessitate dialogue through which participants negotiate meaning. Through this dialogue, co-teachers may exchange forms of knowledge not traditionally associated with their disciplinary roles, disrupt siloed practices, and collaboratively problem solve around instruction. Despite this potential, there is a need to better understand how mathematics and special education co-teachers navigate tensions, negotiate power dynamics, and exchange expertise through discourse. The purpose of this study is to explore how mathematics and special education co-teachers use discursive interactions during co-planning sessions to negotiate power and exchange knowledge, with the goal of understanding how collaborative dialogue can support integration of expertise across fields and disrupt disciplinary siloes. This study is guided by the following research question: *How does a collaborative mathematics co-teaching pair exchange knowledge and negotiate power through discursive interactions during co-planning?*

Theoretical Framework and Literature Review

Disability Studies in Mathematics Education

This study draws from the field of Disability Studies in Mathematics Education (DSME) that advances a social model of disability. From a social model perspective, disability is produced through classroom structures, instructional practices, and institutional norms that shape whose mathematical thinking is validated and supported. Accordingly, DSME scholarship foregrounds how students' opportunities for mathematical sense-making are constructed within mathematics classrooms rather than predetermined by disability labels (e.g., Tan & Kastberg, 2017; Lambert & Tan, 2020; Tan et al., 2022; Yeh et al., 2024). Central to this perspective is the claim that structures, processes, and learning environments—rather than individual students—must be primary sites of change, as all learners, disabled or not, are capable of engaging in mathematical sense-making when provided meaningful and equitable opportunities (Lambert & Tan, 2017).

This study builds from a growing body of research that documents the mathematical competence of students with learning disabilities, autism, intellectual disabilities, and/or visual impairments (e.g., Eriksson, 2008; Fernandes & Healy, 2013; Figueiras & Arcavi, 2014; Hunt & Empson, 2015; Göransson et al., 2016; Lewis & Fisher, 2016; Tan et al., 2019; Lambert et al., 2020; Yeh et al., 2020; Wilson & Hunt, 2022; Shi, 2024). Collectively, this scholarship demonstrates that students labeled as disabled can learn and do mathematics when learning environments are designed to support sense-making.

While DSME scholars have primarily focused on supporting students' sense-making from the lens of changes within the learning environment, there is a need to extend this perspective to the environments in which teachers collaboratively design instruction. This study addresses this gap by exploring the co-planning environment of a mathematics co-teaching pair, focusing on how teachers draw on and negotiate their distinct expertise to support mathematical sense-making. By shifting analytic attention from students alone to teachers' collaborative spaces, this work extends DSME's social model of disability to mathematics co-teaching practices itself.

Fairclough's Conceptualization of Power

This study specifically studies power dynamics between co-teaching pairs. As such, this study draws on key aspects of Fairclough's (2003) framework for analyzing spoken and written language, with particular attention to his conceptualization of power, to examine the co-teachers' discursive interactions during mathematics co-planning. Fairclough (2003) defines *power* as the ability to secure a particular outcome even if the outcome depends on the agency of others. From

this perspective, power is not static or individually possessed but is enacted, shared, or resisted through discursive interactions as individuals negotiate the meanings of social practices, relationships, and identities.

In this study, the co-teachers negotiate power through discursive moves as they exchange knowledge while co-planning instruction. These power negotiations occur within knowledge exchanges, where knowledge is shared or elicited, and determine whether the co-teachers' decisions are made collaboratively or one teacher's perspective is privileged over the other.

Domains of Teacher Knowledge

Teacher education research has introduced many frameworks for conceptualizing teacher knowledge. Shulman (1986) introduced pedagogical content knowledge as knowledge of how to teach a particular subject matter. Building from Shulman's work, Grossman (1990) conceptualized teacher knowledge as encompassing four domains: subject matter knowledge (i.e., knowledge of mathematics), pedagogical knowledge (i.e., knowledge of learning, learners, and classroom management), knowledge of context (i.e., knowledge of individual students, school setting, and community), and pedagogical content knowledge (i.e., combination of subject matter, pedagogical, and context knowledge).

Notably absent from these frameworks is the specialized knowledge required to differentiate, accommodate, scaffold, and assess instruction for individual learners (Brownell et al., 2010; CEC, 2022). I refer to this domain as *knowledge of individualized education*. The longstanding separation between mathematics education and special education has resulted in mathematics teachers often lacking knowledge of individualized education, while special educators frequently lack pedagogical content knowledge for teaching mathematics (Lambert & Tan, 2017, 2020; Cohen et al., 2025). For mathematics co-teaching to meaningfully support diverse learners, it is therefore critical that co-teachers exchange and integrate these complementary domains of knowledge through collaborative planning.

Methods

Context and Participants

Situated within a larger qualitative case study examining collaborative mathematics co-teaching practices, this exploratory case study focuses on one mathematics co-teaching pair in a micro-urban community in Midwestern United States. At the time of data collection, Thomas (pseudonym), the general education mathematics teacher, and Elizabeth (pseudonym), the special educator, were identified by a leader of the North American Chapter of the International Group for the Psychology of Mathematics Education's (PMENA) Disability Studies in Mathematics Education Working Group as successful collaborators. They were in their second year of co-teaching together and jointly taught a general 9th grade Algebra I class aligned with grade-level standards.

Elizabeth identifies as a White, female Special Education Learning Behavior Specialist in her second year of teaching and co-teaching. Thomas identifies as a White, male mathematics teacher with a master's in Curriculum and Instruction, 29 years of teaching experience, and 22 years of co-teaching experiences. Their co-taught Algebra I class consists of 21 students, including five students who qualified for special education services in mathematics. The two participants shared responsibility for planning and teaching daily math instruction for all students in the class. During their interviews, both teachers claimed that co-planning was a central feature of their collaboration. Elizabeth and Thomas met weekly every Tuesday for an hour during Thomas's prep period and Elizabeth's hall duty period. They also engaged in co-planning during "found" time throughout the school week (e.g., between classes, during lunch, after school).

Their high school serves approximately 1,000 students. The student population is 33% Black, 25% Hispanic, and 25% White, with 15.9% of students receiving special education services. In one of his interviews, Thomas shared that the school had previously implemented co-teaching models. He also explained that school leadership has recently prioritized strengthening mathematics co-teaching in response to rising failure rates among ninth grade students with disabilities in Algebra I. As part of this initiative, a mathematics co-teaching instructional coach has been co-teaching with math teachers for the past two years—a form of support that both Elizabeth and Thomas identified in interviews as critical to their co-teaching practice.

Data Collection and Analysis

This study employs multiple data collection methods to examine the co-teachers' discursive interactions. Data collection included (a) a video-recorded co-planning session in which participants jointly planned a mathematics lesson; (b) a self-recorded video observation of the co-teaching of that same lesson; (c) individual video-elicited cognitive interviews (Dempsey, 2010); and (d) individual semi-structured interviews. During the video-elicited interviews, each teacher viewed segments of the recorded lesson and paused as needed to reflect on their instructional decisions, co-teaching processes, and interpretations of their partner's contributions. The semi-structured interviews were designed to elicit teachers' perspectives on the broader sociocultural context of their co-teaching, including their collaborative partnership, institutional structures, and other factors that influence their mathematics co-teaching.

All data were transcribed using Zoom's automated transcription tool and subsequently I edited each one for accuracy. I began analysis by segmenting the co-planning transcript into *exchanges*, defined by Fairclough (2003) as a sequence of two or more conversational moves with alternating speakers. At its simplest, an exchange follows a pattern of *Initiation-Response-Follow-up* (Fairclough, 2003), though exchanges can also involve extended, more nuanced sequences of these conversational moves. Each exchange served as the unit of analysis.

Knowledge exchanges. After segmenting the data into exchanges, I used Fairclough's (2003) framework to categorize each exchange as either knowledge exchanges or activity exchanges, based on whether the interaction involved eliciting or sharing knowledge, or coordinating actions and tasks. Because this study focuses on how co-teachers negotiate power and exchange knowledge through discourse, only knowledge exchanges were retained for subsequent phases of analysis.

Drawing on Fairclough (2003), I identified how each knowledge exchange was initiated. Knowledge exchanges can be initiated by either the mathematics teacher (MT) or the special educator (SE), who assumes the role of either the *knower* (sharing information) or the *other* (eliciting information). Each knowledge exchange was coded with the type of initiation (MT Knower, SE Knower, MT Other, or SE Other). I then characterized the type of teacher knowledge exchanged by the co-teachers by drawing on Grossman's (1990) generalized framework of teacher knowledge, supplemented by knowledge of individualized education (Brownell et al., 2010; CEC, 2022). I coded each knowledge exchange according to the type of knowledge(s) shared or elicited: subject matter knowledge, pedagogical knowledge, knowledge of context, pedagogical content knowledge, or knowledge of individualized education.

Power negotiations. Throughout their co-planning session, the co-teachers negotiated power through discursive moves embedded within knowledge exchanges, resulting in the co-teachers enacting, sharing, and/or resisting power (Fairclough 2003). These power negotiations can either inhibit or facilitate each co-teacher's ability to contribute knowledge and influence decision-making. To analyze how the co-teachers negotiated power during co-planning, I engaged in

Saldaña's (2021) process of deductive coding, in which power negotiation codes (i.e., enact power, share power, resist power) were pre-determined and used in a top-down approach to analyze the co-planning data. Building from Fairclough's work (2003, 2015), Table 1 provides examples of how power negotiations could be manifested in the co-teachers' dialogue during co–planning and was used to guide the coding of power negotiations.

Table 1: Examples of Discursive Moves Used to Negotiate Power

Power Negotiation	Discursive Moves	Examples
Enacting Power	Controlling agenda/topic	"Next, we'll talk about…"
	Giving commands/directives	"The quiz will have ten questions."
	Strengthening authority	"We must…"
	Closing discussion	"Okay. We'll do this."
	Interrupting	"So maybe if we do…"
Sharing Power	Negotiating agenda/topic	"Should we discuss the quiz next?"
	Making decisions jointly	"So we both agree that…"
	Softening authority	"I think it might be best if…"
	Building on each other's ideas	"What do you think?"
Resisting Power	Rejecting agenda/topic	"No, I think we should…"
	Challenging commands/directives	"I actually think it went well."
	Ignoring/changing topic	"I'm going to look at the review."
	Reopening discussion	"Can we circle back to…"
	Questioning ideas	"Is that too many problems?"
	Joking/storytelling	"Did you hear about yesterday?"
	Asserting knowledge	"In my experience…"

For each knowledge exchange, power could be enacted, shared, and/or resisted by either the mathematics teacher (MT) or the special educator (SE). As such, I further categorized the power negotiation codes to indicate which co-teacher used discursive moves to negotiate power during the exchange: MT Enacting Power, MT Resisting Power, Sharing Power, SE Resisting Power, and SE Enacting Power. Since the co-teachers could concurrently negotiate power through discursive moves in a single knowledge exchange, power negotiation codes were not discretely applied, meaning each knowledge exchange could have multiple power negotiation codes assigned to it (e.g., MT Enacting Power & SE Resisting Power).

Validity. To ensure the trustworthiness of the study, several strategies were employed: rich and thick description of the case, working with a peer debriefer who reviewed interpretations, and member checking with the co-teachers to confirm that the interpretations accurately reflected their experiences (Erlandson et al., 1993). Interview data were also used to triangulate findings and support interpretations of the co-teachers' discursive interactions during co-planning.

Results

Positioning Expertise through Knowledge Exchanges

Thomas, the general mathematics teacher (MT), and Elizabeth, the special educator (SE), engaged in 95 knowledge exchanges during their 40-minute co-planning session (see Table 2). Thomas initiated 72 (75.8%) exchanges while Elizabeth initiated 23 (24.2%), suggesting that Thomas more frequently set the agenda and steered the direction of the talk. In this sense,

Thomas shaped the terms of knowledge exchange by determining when exchanges began, what knowledge was made relevant, and how participation was structured.

Table 2: Knowledge Exchanges by Initiation Type

Number of Exchanges Initiated	MT Knower	MT Other	SE Knower	SE Other	Total
Knowledge Exchanges	33	39	15	8	95

At the same time, the type of Thomas's initiations complicates a straightforward interpretation of Thomas's power throughout the co-planning session. Of the 72 exchanges Thomas initiated, 33 involved sharing his own knowledge, while 39 involved eliciting Elizabeth's knowledge (see Table 2). Of the latter instances, Thomas was explicitly orienting Elizabeth as the source of expertise by inviting her contributions into the interaction. Correspondingly, across all exchanges, Thomas was positioned as the knower in 41 exchanges (43.2%) whereas Elizabeth was positioned as the knower in 54 exchanges (56.8%). Although Thomas more often initiated exchanges and directed the flow of the meeting, Elizabeth was more often positioned as the knower within these interactions. This configuration of interaction stands in contrast to prior research, where special educators are often positioned primarily as implementers rather than contributors to instructional decisions (Zigmond & Matta, 2004).

Boundary Crossing through Knowledge Exchanges

During their co-planning session, Thomas and Elizabeth engaged in knowledge exchanges related to specialized areas of teacher knowledge within both mathematics and special education. Of the 11 knowledge exchanges involving knowledge of individualized education, Thomas was positioned as the knower in five (45.5%; see Table 3). Of the 42 knowledge exchanges involving pedagogical content knowledge, Elizabeth was positioned as the knower in 25 (59.5%). The frequent positioning of each co-teacher as the knower in an area of teacher knowledge not traditionally associated with their disciplinary role suggests that the co-teachers engaged in boundary crossing between disciplines and disruption of disciplinary siloes during their co-planning session. As both co-teachers commented in their interviews, this boundary crossing provides space for the co-teachers to learn with and from each other.

Table 3: Knowledge Exchanges by Initiation Type and Knowledge Type

Knowledge Type	MT Knower	MT Other	SE Knower	SE Other
Subject Matter Knowledge (SMK)	0	0	0	0
Pedagogical Knowledge (PK)	13	15	3	3
Knowledge of Context (KC)	0	1	2	0
Pedagogical Content Knowledge (PCK)	12	19	6	4
Knowledge of Individualized Education (KIE)	3	4	2	1
KIE & PCK	1	0	0	0
PK & KC	4	0	1	0

Across exchanges involving pedagogical content knowledge, Thomas more frequently initiated interactions that elicited Elizabeth as the knower (19) than those in which he positioned

himself as the knower (12). These exchanges often focused on instructional decisions at the intersection of mathematical content and responsiveness to students' learning needs. For example, when creating a quiz to determine topics for review before the upcoming test, Thomas repeatedly solicited Elizabeth's input on which mathematics topics should be included, how many items should be allotted for each topic, and whether the proposed structure provided sufficient support for students' learning, positioning Elizabeth as an expert in pedagogical content knowledge.

During their individual social network interviews, both co-teachers described changes in their professional learning over time. Elizabeth reflected on her initial uncertainty with subject matter knowledge and pedagogical content knowledge during her first year of teaching. She noted that her confidence in and knowledge of math teaching developed through ongoing collaboration with Thomas. Thomas similarly described a shift in his perspective of the value of a special education co-teacher. He emphasized Elizabeth's desire to be "an equal teacher" and her willingness to engage in shared learning. He reported that these experiences helped him to make a conscious choice to rely on and value Elizabeth's expertise during instruction and planning.

While Elizabeth positioned Thomas as a source of expertise in knowledge related to individualized education in only one exchange, Thomas positioned himself as a knower in this area on four occasions. For example, near the conclusion of the co-planning session, Thomas initiated a knowledge exchange to summarize the teaching models and multi-modal supports incorporated in their lesson design. In this exchange, Thomas drew on knowledge associated with individualized education, crossing the boundary between mathematics and special education. In an interview, Thomas noted that in the past he struggled with supporting the mathematical learning of his students with disabilities but has developed greater knowledge of instructional adaptations through collaboration with Elizabeth. This perspective was echoed by their mathematics co-teaching instructional coach, who was paraphrased by Elizabeth in her social network interview as saying, "You really have pushed Thomas. Like, I never would have thought I would see these things so late in his career. He's really doing great as a co-teacher."

Negotiating Power through Discursive Moves

As the co-teachers engaged in knowledge exchanges, they used discursive moves to enact, share, or resist power (see Table 1). Consistent with prior research suggesting that mathematics teachers often occupy positions of authority in co-teaching partnerships (Zigmond & Matta, 2004), Thomas enacted power in 36 knowledge exchanges (37.9%), most frequently when initiating an exchange as the knower (see Table 4). However, a substantial proportion of the exchanges were coded as sharing power (45, 47.4%). This suggests that power was frequently negotiated rather than unilaterally exercised during co-planning.

The following exchange illustrates how Thomas and Elizabeth shared power as they collaborated to create contextual review problems for their lesson:

1	MT:	And I'm kind of mixing these up, and if you think that would be—do you
2		think if the kids see it that way, it's gonna mess them up? So they see $3
3		written out that way [3 dollars], the $2.50 in the other way [$2.50]? Is that
4		gonna freak…
5	SE:	Yeah, I don't like the "3 dollars". [Chuckles]
6	MT:	[Laughing] Do it! Change it! No, no, that's fine. But is that, like—Okay,
7		so we've got that, and your 2-liter bottle costs $2.50. Okay, cool.

In line 1, Thomas uses hedging ("I'm kind of…", "if you think that would be…") to open space for Elizabeth's feedback, positioning her as the knower on how students might interpret the

representation. Elizabeth's response in line 5 offers a clear evaluative stance, which is mitigated through laughter and functions as an affiliative move rather than a challenge. Thomas's immediate uptake and recommendation to make the change in line 6 signals alignment with Elizabeth's suggestion, even as he briefly continues negotiating the decision ("No, no, that's fine. But is that, like…"). Together, these discursive moves demonstrate how authority was dynamically and jointly negotiated and enacted within the interaction.

Table 4: Knowledge Exchanges by Initiation Type and Power Negotiation

Initiation Type	MT Enacting	MT Resisting	Sharing	SE Resisting	SE Enacting	MT Enacting to Sharing	MT Enacting and SE Resisting
MT Knower	22	0	7	0	1	1	2
MT Other	11	0	25	1	1	1	0
SE Knower	2	0	7	4	1	0	1
SE Other	1	0	6	0	1	0	0
Total	36	0	45	5	4	2	3

Conclusion

This study examined how a mathematics teacher and a special educator used discursive moves during co-planning to negotiate power and exchange knowledge. Analysis of their interactions shows that the mathematics co-teachers drew on distinct forms of expertise associated with their fields, while also crossing disciplinary boundaries to jointly develop instructional approaches designed to support students' mathematical sense-making, including those with disabilities. The co-teachers frequently engaged in discursive moves that shared power and functioned to redistribute authority across participants rather than consolidating it with a single speaker, thereby producing a more collaborative and negotiated co-planning space.

The frequency of shared-power exchange and boundary crossing highlights the importance of creating collaborative spaces in which co-teachers can share authority and draw on one another's expertise to inform instructional decisions. Implications for practice point to the need for dedicated co-planning time between mathematics co-teachers to facilitate the integration of complementary expertise from their fields. The study also highlights a need for professional learning that supports mathematics co-teachers beyond instructional strategies and includes collaboration strategies. Collaboration-focused professional learning is needed to support mathematical co-teachers in recognizing and using discursive moves to negotiate power and their distinct expertise to better support mathematical sense-making in their classrooms.

Reflecting on the partnership, Thomas described, "What you're seeing is the product of me finally taking a step back and not believing that I knew everything that I was doing, and a second year of working with somebody who helped me to do that in the first place." Such reflections underscore that expanding opportunities for mathematical sense-making for students with disabilities by improving mathematics co-teaching practices depends not only on instructional strategies, but also on how co-teachers structure their interactions, share authority, and learn with and from each other.

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THEORIZING CULTURAL MATHEMATICAL CARING RELATIONS

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This paper introduces a framework for cultural mathematical caring relations (CMCRs), an expansion of Hackenberg's (2005a) mathematical caring relations that explicitly acknowledges students' cultural and experiential realities. CMCRs integrate students' culture and interests into constructivist teaching experiments to support mathematical learning. Drawing from a teaching experiment with a developmental undergraduate student, Jocelyn, this paper illustrates how culturally relevant tasks can elicit meaningful reasoning about complex mathematical ideas such as intensity of change. By centering cultural relevance, CMCRs offer a pathway for cognitive researchers to challenge the status quo by attending to sociopolitical factors in their work.

Keywords: Cognition; Equity, Inclusion, and Diversity; Systemic Change

If you asked educators whether they care about their students, the vast majority would say that they do. Many enter the profession with a genuine desire to help students succeed, and students are more likely to succeed when they feel their teacher cares about them. After all, few students are motivated to work for someone they believe is uninterested in their needs or experiences. In the context of mathematics education, however, general care alone is not sufficient for supporting students' mathematical learning and development (Hackenberg, 2005b). A teacher may care deeply about students' interests, hobbies, academic success, or lives outside of school, but this does not necessarily translate into meaningful mathematical learning. This raises an important question: how can mathematics educators establish caring relationships that attend both to students' personal lives and needs, and to their mathematical learning?

Noddings (1984, 2002) defined caring relations as relationships that are grounded in care, and that involve two people: a carer (one-caring) and a cared-for. According to Noddings, the one-caring must experience *engrossment* and a *motivational displacement* when caring for the cared-for. Engrossment involves more than simply putting oneself in the shoes of the cared-for. Rather, the one-caring must receive the cared-for in a way that allows them to receive their feelings and feel with them. Motivational displacement, on the other hand, implies that the behaviors of the one-caring occur in service of the cared-for. Thus, the one-caring prioritizes the needs of the cared-for, even over their own needs. In turn, the cared-for must receive the care of the one-caring either with words or with indirect gestures, such as a smile or glow of happiness.

Noddings (1984) argued that the primary concern of all schools should be to maintain and enhance caring, but not at the expense of students' intellect. She emphasized that educators must attend to both caring and intellect. Yet her theory of care does not explicate how caring relations can be operationalized in a way that attends to both, particularly in a mathematical context.

Hackenberg (2005a) addressed this issue by operationalizing Nodding's caring relations in the context of a mathematical teaching experiment[1]. She defined a *mathematical caring relation* (MCR) "as a quality of interaction between a student and a mathematics teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning" (Hackenberg, 2010a, pp. 57–58). In other words, teachers aiming to establish MCRs seek to provoke learning

[1] A teaching experiment is a sequence of video-recorded teaching sessions, each involving a teacher-researcher, at least one student, and a witness-researcher (Steffe & Thompson, 2000).

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

by attending to students' mathematical ways of reasoning and emotional responses in the context of solving math tasks. Like general caring relations, MCRs cannot be established unless both the teacher and the student contribute to them in some way. Unlike general caring relations, however, MCRs require a commitment on the side of the carer (teacher) to provide the cared-for (student) with opportunities for mathematical learning.

Hackenberg's work involving MCRs (e.g., 2010a, 2010b) is unique because it lies in stark contrast to mathematics education research that has focused on how general caring by teachers impacts students' mathematical success (e.g., Hunter & Stinson, 2019). Yet this line of work is not without its limitations. Bartell (2011) noted that Hackenberg's research on MCRs "does not explicitly consider issues of race, culture, and power" (p. 55). That is, it does not explicitly attend to sociopolitical issues related to students from historically marginalized groups, and in particular, students of color.

This is problematic because MCRs are rooted in constructivist teaching experiments, which are grounded in uncovering and unpacking students' cognition. Given the privileged positioning of mathematical cognition within mathematics education research, cognitive studies have the potential to challenge dominant discourses in mathematics (e.g., formal mathematical knowledge serves as a proxy for intelligence) (Adiredja, 2021), as well as deficit discourses that negatively impact students from marginalized groups (e.g., Latinos do not value education) (San Miguel, 2011). Not attending to sociopolitical factors (e.g., culture, race), therefore, risks reinforcing the status quo such work seeks to disrupt.

The purpose of this theoretical paper is to respond to Bartell's (2011) critique by extending MCRs to more explicitly account for students' cultural and experiential realities in the context of constructivist teaching experiments. To do so, I introduce a framework for *cultural mathematical caring relations* (CMCRs). A CMCR is an MCR in which the teacher supports students' mathematical learning and development by engaging them in tasks that leverage their funds of knowledge. I define *funds of knowledge* as the knowledge and skills rooted in students' out-of-school experiences and interests, as well as their household and community culture (cf. Moll et al., 1992). I conceptualize *culture* as a dynamic construct, rather than a static collection of rituals, traditions, and customs. Thus, culture is constantly changing and dependent on the ways that individuals or households organize and respond to their experiences (González et al., 1995, 2001).

It is my hope that this paper will encourage cognitive researchers conducting teaching experiments to challenge the status quo by leveraging CMCRs to be more intentional about participant selection and to resist deficit framings by positioning students' cultural and experiential realities as assets for mathematical learning. In what follows, I unpack the framework for MCRs, extend it to CMCRs, and provide an example of a CMCR in action. I conclude by summarizing CMCRs and discussing their implications for cognitive research in mathematics education.

A Framework for Mathematical Caring Relations

Like caring relations, establishing MCRs is a two-way street. That is, MCRs cannot be completed unless both the teacher-researcher and the student contribute to them in some way. According to Hackenberg (2005a), from "a student's perspective, participating in mathematical caring relations involves being open to the teacher's interventions in the student's mathematical activity and pursuing questions and ideas of interest" (p. 4). Conversely, from the perspective of the teacher-researcher, establishing MCRs requires engagement in a three-step cyclical process: harmonizing with students' mathematical realities and affective responses, provoking learning,

and assessing and refining efforts to harmonize based on students' challenges.

Harmonizing with Students' Mathematical Realities and Affective Responses

Harmonizing with mathematical realities. According to Hackenberg (2010b), *harmonizing* "indicates that the teacher-researcher is monitoring interaction with students and making adjustments in response to the teacher-researcher's continually changing understanding about the students" (p. 242). The process of harmonizing with students' mathematical realities is intimately linked with the process of building second-order models of students' first-order mathematics. Students' *first-order mathematics* refers to "whatever might constitute students' mathematical realities" (Steffe & Thompson, 2000, p. 268). In other words, it encompasses the mathematical knowledge students have constructed and have at their disposal; it is "first-order" in that it pertains to their *own* mathematical knowledge.

Building second-order models, then, requires the teacher-researcher to document their interpretations of students' mathematical thinking and how they are experiencing the situation under observation (Steffe & Thompson, 2000). These models are "second-order" in that they are constructed from the teacher-researcher's understanding of students' mathematical knowledge. While second-order models can never be proven to depict students' true mathematical realities, they can, over a prolonged period of interaction, become more viable via testing for falsifiability. That is, the teacher-researcher can think of their model[2] as a hypothesis that can be proven false. If after testing the model it is proven to be false (i.e., falsified), then this would suggest that the model is not an appropriate fit for the student's experiential reality, and thus, must be refined by the teacher-researcher.

From the perspective of a teacher-researcher, cognitive decentering is critical in constructing viable models of their students' mathematical knowledge. I adopt Hackenberg's (2010b) definition of *cognitive decentering* as "the practice of setting one's own concepts to the side in order to formulate a coherent description of the mental activity of a student that can be the basis for interaction with that student" (pp. 239–240).

To harmonize with students' mathematical realities implies that the teacher-researcher engages in cognitive decentering to make sense of students' cognitive structures from their perspectives. The teacher-researcher must then create and pose tasks that students can engage with and that elicit the cognitive structures the teacher-researcher is attempting to evoke. Thus, the process of harmonization and the process of model building are intimately linked: harmonizing with students' mathematical realities supports the construction of viable models of students' mathematical knowledge, and these models, in turn, inform continued harmonization in future interactions.

Harmonizing with affective responses. Harmonizing with students' affective responses requires the teacher-researcher to monitor their students' levels of depletion and stimulation. I adopt Hackenberg's (2005a) definition of *depletion* as "a feeling of being taxed in some way" (p. 11), as well as her definition of *stimulation* as "a feeling of being excited or awake" (p. 12). The teacher-researcher must work to find the appropriate balance between their students' levels of depletion and stimulation; a small amount of depletion is healthy when students are confronted with challenging, yet manageable, math tasks, but a large amount of depletion can cause students to shut down completely (this might occur if a student feels that a task is out of their reach).

In her later work, Hackenberg (2010b, 2010a) leveraged the psychological construct of subjective vitality (Ryan & Frederick, 1997) to monitor students' energy levels. *Subjective vitality* refers to "the energy available to the self" (Ryan & Deci, 2008, p. 702) and is linked to a

[2] For convenience, I use the term "model" for the remainder of the paper.

person's affective experience. Connecting to depletion and stimulation, for instance, a high level of subjective vitality would correspond to a high level of stimulation, whereas a low level of subjective vitality would correspond to a high level of depletion. Thus, from the perspective of subjective vitality, harmonizing with students' affective responses implies making sure students' energy levels remain at equilibrium.

Subjective vitality depends on not only physical factors, but also mental factors (Ryan & Frederick, 1997). For example, a student who is working on a math problem can have low subjective vitality if they perceive the problem as crippling (i.e., damaging to their mathematical image or identity). Alternatively, they can have high subjective vitality if they perceive the situation as a surmountable obstacle that will make them more knowledgeable. An important note is that subjective vitality is not only associated with heightened levels of explicit excitement or stimulation, but also with calm energy (Thayer, 2001). Thayer describes *calm energy* as a state of high energy and low tension (i.e., stimulated, but relaxed). For instance, a student who is relaxed but mentally engaged with a math problem would be exhibiting calm energy, indicative of high subjective vitality.

Provoking Learning

The second step in establishing MCRs requires the teacher-researcher to provoke learning by providing their students with appropriately challenging math tasks. I define *learning* as a process that occurs when one makes a modification or refinement to their way of thinking. The word "provoke" is intended to convey that learning only needs to be initiated, not completed. Thus, the student does not need to experience learning for an MCR to be established; being appropriately challenged is sufficient.

To provoke learning, the teacher-researcher must harmonize with their students' mathematical realities and affective responses in just the right way. That is, they must pose mathematical tasks that are neither too easy nor too difficult, but appropriately challenging—tasks that support refinements to students' mathematical thinking while also maintaining or enhancing their energy levels.

Assessing and Refining Efforts to Harmonize Based on Students' Challenges

The final step in establishing MCRs requires the teacher-researcher to assess students' cognitive and affective responses to the challenges posed by tasks designed to provoke learning. Based on these assessments, the teacher-researcher adjusts the models of their students' mathematical knowledge—and, in turn, the tasks they pose—to improve harmonization with their students. The need to repeat this process of challenging and harmonizing is what makes the three-step process of establishing MCRs inherently cyclical.

Extending MCRs: A Framework for Cultural Mathematical Caring Relations

To extend MCRs to CMCRs, I draw on the theory of Culturally Relevant Pedagogy (CRP). Ladson-Billings (1995) defined CRP as a "pedagogy of opposition … specifically committed to collective, not merely individual empowerment" (p. 160). She developed CRP to promote the academic success of marginalized students, particularly African American students, by centering three key tenets: academic success, cultural competence, and critical consciousness.

Academic success involves supporting students in achieving success within dominant educational assessments, such as standardized tests and grades, while fostering students' intrinsic motivation for learning. Cultural competence requires teachers to incorporate students' diverse cultural backgrounds into instruction, not only to enhance their understanding of academic content, but also to deepen their appreciation of their own cultures and those of others. Critical consciousness demands that teachers recognize the sociopolitical and historical forces that shape

education and help students develop this awareness to analyze and challenge systemic inequities. By integrating these three tenets, CRP seeks to empower students both academically and socially, equipping them to navigate and transform the world around them.

CMCRs integrate MCRs with two of the three central tenets of CRP: academic success and cultural competence. Academic success is already central to MCRs, as their goal is to deepen students' mathematical knowledge through learning and development, with achievement defined in terms of individual cognitive growth. Cultural competence, however, is not explicitly addressed within MCRs. To integrate this critical tenet of CRP, the teacher-researcher must engage students with math tasks tied to their own or others' funds of knowledge, promoting a deeper appreciation of their own cultures and those of their peers. In this way, CMCRs not only support mathematical growth but also affirm students' cultural identities by recognizing their funds of knowledge as valuable resources in task design and instruction. Thus, CMCRs extend the three-step cyclical process of MCRs into a four-step cycle (see Figure 1) by explicitly incorporating students' funds of knowledge as an integral and recurring component of the caring process.

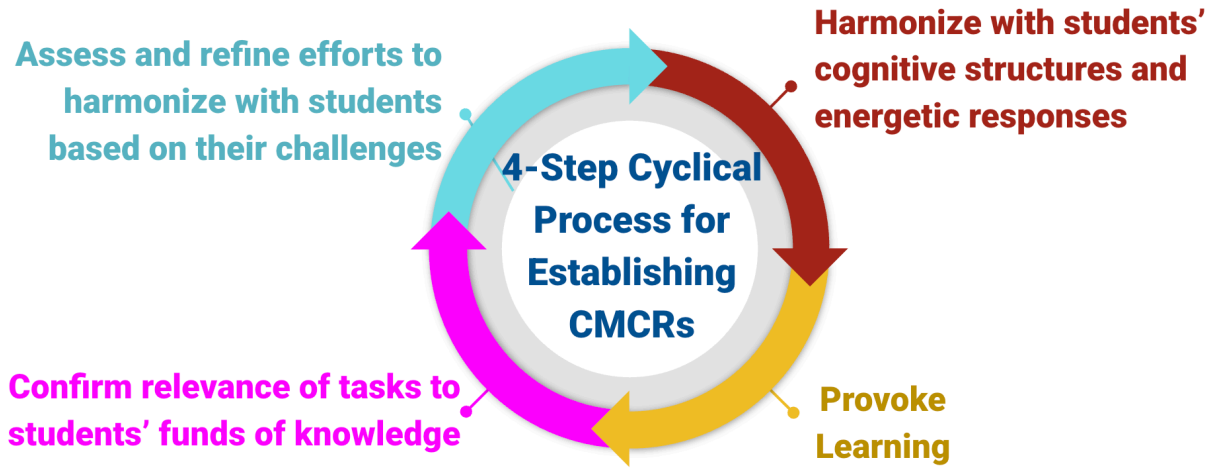


Figure 1: Four-Step Cyclical Process for Establishing CMCRs

Example of Establishing a CMCR

Background Information and Important Definitions

This example comes from a thirteen-session teaching experiment where I (Puerto Rican man) was the teacher-researcher, and my doctoral advisor (white man) was the witness-researcher. The aim of the teaching experiment was to showcase the rate of change development of two Latina undergraduate students, Yari and Jocelyn (pseudonyms), who were recruited from a developmental math course I taught in fall 2023. Jocelyn, a political science major who identifies as a white, Alaskan Native, and Puerto Rican woman, is the focal student in this example.

The eighth session of the teaching experiment, which is connected to this example, was conducted independently with Jocelyn due to a scheduling conflict. My primary goal during this session was to analyze how Jocelyn reasons about the intensity of change of an intensive quantity—an important component of rate of change reasoning. A *quantity* is a conception of a measurable attribute of an object (Olive & Çağlayan, 2008). An *intensive quantity* is a quantity that cannot be explicitly or directly measured, measures the *intensity*, or degree of presence of an

attribute (Stroup, 2002), and expresses a relationship between two quantities (Schwartz, 1988). For instance, the pinkness of a wall is an intensive quantity because it (1) cannot be directly measured with a ruler or scale, (2) measures the intensity of the wall's pinkness, and (3) can be expressed as a relationship between red and white paint.

To begin the session, I provided Jocelyn with a premade graph and asked her to connect it to her tea-making process at home. I chose to situate the task within a tea context because I had learned early in the experiment that tea was an important part of her household culture. The graph I provided, however, did not align well with her actual tea-making process, prompting her to ask whether she could construct her own graph that more accurately reflected her experience. To honor and harmonize with her way of thinking, I agreed.

Jocelyn's Engagement with the Tea-Making Task

To guide Jocelyn's thinking, I suggested using the y-axis of her graph to represent the sweetness of the tea as it was being prepared, while emphasizing that the final decision was hers. Jocelyn carefully considered the relevant quantities and how each one would influence the shape of the graph in its own way. For instance, she noted that the graph would look different depending on whether it represented the flavor of the tea or its sweetness, since the tea flavor was derived from the tea bag, whereas the sweetness came from the honey and oat milk. This distinction mattered to her because the addition of oat milk at the end of the process would have opposite effects depending on the quantity being measured: the graph would decrease if it represented tea flavor (as oat milk dilutes the tea) and increase if it represented sweetness (as oat milk complements the honey).

This reflection prompted Jocelyn to construct three graphs, each representing a different scenario (see Figure 2). First, she drew a graph representing the sweetness of the tea (purple graph). She then created a second sweetness graph that accounted for the effect of stirring during the steeping process (2–5 minutes) (black graph). She noted that sometimes she stirred the tea with a spoon while it steeped, which led to a sharper rise in sweetness. Finally, Jocelyn drew a third graph, this time representing the change in tea flavor (red graph). In this graph, she considered that roughly 80% of the tea bag's flavor would be absorbed into the water within the first minute of the steeping process, resulting in a sharp increase in tea flavor between minutes 2 and 3, followed by a much slower increase between minutes 3 and 5.

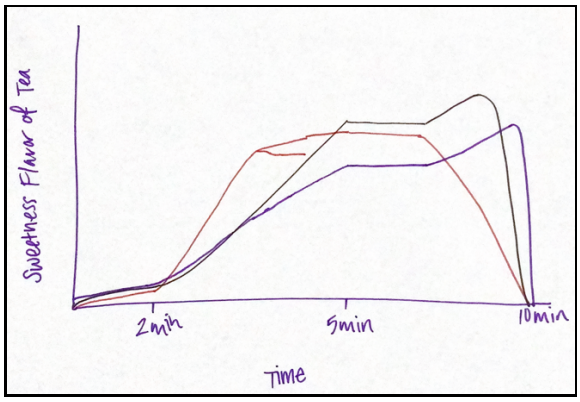


Figure 2: Graphs Associated with Sweetness and Flavor of Jocelyn's Tea

To help Jocelyn hone in on a single quantity, the teacher-researcher suggested that she draw a new graph measuring the blackness of the tea over time. This quantity was less ambiguous, as it

allowed her to focus specifically on how the tea's color changed, independent of its sweetness or flavor. Her graph had three key phases (see Figure 3): (1) the steeping phase (0–1 minute), where boiling water was added to the cup, and the tea bag began releasing its dark color into the water; (2) the resting phase (1–3 minutes), where the tea bag was removed, and the blackness remained stable; and (3) the pouring phase (after 3 minutes), where oat milk was added, lightening the tea's color until it became nearly white.

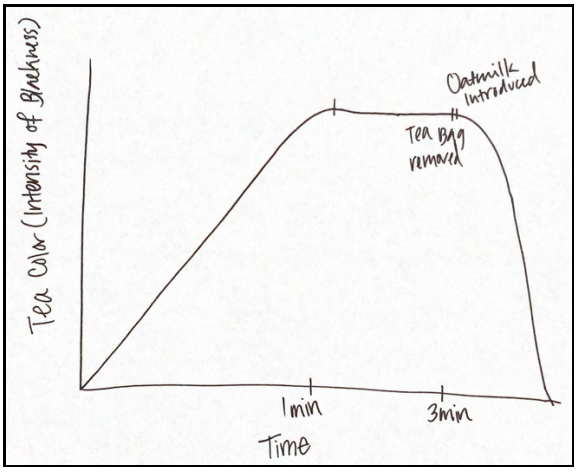


Figure 3: Jocelyn's Blackness of Tea Graph

Jocelyn explained that she connected the steeping and resting phases of her graph with a smooth curve to represent the final twenty percent of the tea absorption process. To better understand how she was reasoning about the variation in the tea's blackness during this interval, I asked her to describe how she conceptualized that final twenty percent of absorption.

Jocelyn: Umm, cause like you can see it's still absorbing, it's just slow. So like, instead of ... [3 second pause] I guess maybe there's, like ... [4 second pause] another way that I could have represented this is, at the three-minute mark, like all the time before then [slides marker along portion of graph to the left of the 3 minute mark], could have still had a rising, but just like an even more dramatic, like slower [uses marker to trace out the left side of what appeared to be a concave down parabola], like slow down, instead of just immediately going to having this be basically flat [points to horizontal flat portion of graph]. Umm, cause then, I feel like that could have more accurately represented the last like twenty percent, of it. Umm, but yeah, that's kind of, that's what I was thinking.

Jocelyn's explanation indicated that she understood the tea continued to absorb prior to the three-minute mark, but at a slower rate. This suggests that she recognized the blackness of the tea was not changing at a constant rate throughout the steeping phase. Jocelyn added that she could have made this even more apparent by extending the curve's leveling-off period to the three-minute mark, emphasizing an even "slower … slow down." Her choice of words, combined with her hand gesture tracing the left side of a concave-down parabola in the air, suggests that she was attending to the variation in the intensity of the tea's blackness over time. While her graph may not have been precise, it was clear that she understood the concept qualitatively.

Establishing a CMCR with Jocelyn

Although I cannot definitively conclude that learning occurred during this session (i.e., there is no clear evidence that her way of thinking shifted), I nonetheless argue that a CMCR was established with Jocelyn. First, Jocelyn engaged meaningfully with the "Tea-Making" task, which elicited her intensive reasoning, suggesting that I harmonized with her cognitive structures. Second, she maintained a positive disposition throughout—never disengaging, but remaining focused and intellectually stimulated. This indicates that I harmonized with her affective responses, as she exhibited a state of calm energy. Third, although learning was not evident, the task was appropriately challenging as it invited Jocelyn to reflect deeply on the quantities embedded in her tea-making process (e.g., sweetness, flavor, blackness). This reflection supported productive reasoning, as she independently constructed multiple graphs to represent variation in these quantities over time. Finally, because the graphs emerged from her own tea-making experience, the task was intimately connected to her funds of knowledge.

Discussion

In this theoretical paper, I propose a framework for CMCRs that integrates Hackenberg's (2005a) MCRs with two central tenets of culturally relevant pedagogy: academic success and cultural competence. This framework offers one avenue through which cognitive researchers conducting teaching experiments can more explicitly attend to sociopolitical factors in their work. Doing so positions them to challenge the norm within cognitive research to overlook race, culture, and students' identities beyond mathematics—specifically by constructing models that foreground the powerful sensemaking of students from historically marginalized communities.

For instance, in the example presented in this paper, the CMCR established with Jocelyn did not yield clear evidence of learning, yet it supported productive reasoning that highlighted two important strengths. First, she demonstrated flexibility in reasoning within a complex context grounded in her funds of knowledge. She attended to multiple relevant quantities in her tea-making process—such as sweetness, flavor, and blackness—and reflected on how each shaped the form of her graph. She also considered external factors, including stirring and absorption, ultimately constructing multiple representations to capture the complexity of the situation.

Second, she demonstrated an appreciation for the importance of clearly specifying the quantity being measured when reasoning about rate of change. For example, she recognized that sweetness and blackness are distinct quantities and that modeling each would produce different graphical representations, and thus, different interpretations of change. Her insistence on clarifying what was being measured reflects a commitment to deeply understanding quantitative situations involving rate of change. Given Jocelyn's identity as a Puerto Rican woman, these strengths contribute to an anti-deficit model of her reasoning that challenges deficit discourses positioning Latino students as inferior in mathematics.

For the sake of transparency, I note that although the CMCR established with Jocelyn illustrates the strengths of the framework, it represents only one episode within a thirteen-session, six-week teaching experiment. Given the cyclical and iterative nature of CMCRs, fully understanding their potential requires examining the experiment holistically. There were numerous moments in which I struggled to maintain my CMCR with Jocelyn and had to refine my models of her mathematical thinking and affective responses to reestablish our caring relation. Despite these challenges, I argue that attending explicitly to cultural relevance ultimately enhanced the cognitive rigor of the experiment by demonstrating how students' mathematical learning and diverse identities can be meaningfully intertwined to support productive reasoning. In this way, CMCRs offer a path toward reimagining cognitive research as

a space where rigorous mathematics and cultural relevance work together to support learning.

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HUMANIZING MATHEMATICS TEACHER EDUCATION PROGRAMS: CARING AND BELONGING FOR STUDENTS WITH MULTIPLE MARGINALIZED IDENTITIES

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This qualitative study examined how preservice teachers (PSTs) with multiple marginalized identities experienced a sense of belonging within mathematics teacher education. The study included participants who were Black, on the autism spectrum, LGBTQ+, and from economically disadvantaged backgrounds. Two themes are reported from the analysis: Caring Relationships and Classroom Supports. Findings illuminate how intentional relational practices and structured supports contributed to participants' sense of belonging. Implications highlight the importance of relationship-centered pedagogies that recognize and support intersecting identities.

Keywords: Culture and Asset-Based Pedagogies; Equity, Inclusion, and Diversity; Identity and Agency; LGBTQIA+

Introduction

This study investigated preservice teachers (PSTs) with multiple marginalized identities and their sense of belonging in their mathematics teacher education programs. Strayhorn (2019) described "a search for community, humanity, acceptance, and even visibility [as] core elements of sense of belonging" (p. 60). Perhaps more importantly, a "true sense of belonging is not about 'fitting in' but rather about acceptance, authenticity, and finding community as one's true self, just as you are" (Strayhorn, 2019, p. 38). In this way, belonging is deeply entangled with identity.

Osterman (2000) explained that when students feel a sense of belonging, they are more engaged in campus and classroom activities, form strong relationships with peers and faculty, and have higher academic performance. Moreover, when students lack belonging, low persistence and retention rates result (Strayhorn, 2012). Extensive literature exists investigating belonging as an important factor in college success for traditionally underserved students. Less prevalent, however, is how the intersection of marginalized identities may collectively impact a sense of belonging. Allen et al. (2021) suggest that although previous research has reported the importance of student-teacher relationships to belonging, "less is known about how to improve those relationships, as are the most effective types of supports that teachers can offer" (p. 541).

To address these gaps in the literature, and as part of a larger study, we sought to determine factors specific to professors' practices that supported PSTs' belonging in our programs. We purposefully selected four students who had multiple marginalized identities and conducted interviews to understand their sense of belonging in our mathematics education programs. Included in our sample were individuals who were Black, on the autism spectrum, LGTBQ+, and from economically disadvantaged backgrounds.

Literature Review

Retaining PSTs who have marginalized identities in our mathematics teacher education programs through graduation and successfully entering the teaching field is of paramount importance. Indeed, one of the Association of Mathematics Teacher Educators' (AMTE) long-

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term goals for 2024–2028 is to "support and provide guidance on the high-quality preparation, recruitment, retention, and diversification of mathematics teachers across the variety of educational spaces" (AMTE, 2024). Retaining diverse teachers is even more complicated when they have multiple marginalized identities. For example, Hong et al. (2024) explain:

> Very little research has been conducted to address MMU [multiply marginalised and underrepresented] teachers. These teachers need to negotiate their identity not only in one dimension of oppression, but across multiple dimensions and intersections among them simultaneously, which puts them in the most vulnerable positions. (p. 764)

As we prepare MMU teachers, they are particularly vulnerable to non-completion. However, having a diverse teacher workforce is an important factor in the success of K-12 learners. For example, the National Academies of Sciences, Engineering, and Medicine explain "there is growing and compelling evidence that teacher-student racial match has important effects on student outcomes" (NASEM, 2019, p. 87). Redding (2019) indicated that Black and Latino/a students were perceived by teachers of a different race/ethnicity as disruptive and argumentative more often than teachers of the same race. He also found that Black students were more susceptible to exclusionary discipline and less likely to be referred for gifted and talented programs when their teacher did not match their race/ethnicity. His review suggested that Black students had greater academic success when their teacher was Black.

Shelton (2019) emphasized the importance of LGBTQ+ students having ally teachers. She reported on the 2016 GLSEN study, which found that LGBTQ+ students felt three times safer and attended school more regularly when they had access to ally teachers than students who did not. Furthermore, she reported, "ally teachers help to shape more supportive school contexts, which help LGBTQ students to perform better in school, including higher GPAs, [and] greater senses of school community and belonging." (pp. 292–293). We recognize the complexity of teachers outwardly living their LGBTQ+ identities in their school workplaces (Brett, 2024, Hong et al., 2024). However, when they are able to authentically live their identities in their classrooms, it can be transformational for students, according to Brett (2024).

Gómez-Marí et al. (2021) reported in their systematic review on teachers' knowledge of autism spectrum disorder (ASD) that mainstream teachers who teach ASD students lack training, confidence, and knowledge of the disorder. There are four interventions recommended at the high school level for ASD students: (1) behavioral interventions, (2) modeling, (3) self-management, and (4) social skills (Austermann et al., 2025). For gifted and talented students who are also autistic (twice exceptional or 2eASD), there are additional interventions that can impact student success. For example, Reis et al. (2022) determined that:

> The use of advanced content, interest-based enrichment, and strength-based strategies enhanced the academic success of these young people … Participants also explained that when interests guided their academic work, they were more motivated and engaged. (p. 4436)

Preservice teachers studying mathematics education may not necessarily be classified as gifted, but they are likely mathematically talented. As mathematics teacher educators, we should be informed about the nature of autism and how it might function in our students, and understand and be able to use interventions that have the potential to improve their success and retention.

Students who come from economically disadvantaged backgrounds, often first-generation college students (FGCS), have special support needs in college as well. Watts et al. (2023) emphasized the importance of caring relationships with professors and information about and

access to campus resources and services. He also underscored the importance of mentors, advisors, peer relationships and "acknowledgment and reinforcement of academic successes" (p. 38) in FGCS student success. Means & Pyne (2017) found that faculty who had "willingness to provide access to content via multiple modalities, recognition that gaps in prior knowledge required support and were not indicators of cognitive capacity, and regular outreach to students through office hours as well as other kinds of contact" (p. 917) helped bridge the gap for FGCS.

Each of these marginalized identities requires support, and when combining them for students with multiple marginalized identities, those support needs only escalate. However, supporting MMU students in teacher education programs is highly consequential because of the impacts those PSTs will have on the belonging and academic success of K-12 learners.

Theoretical Framework

Intersectionality (Crenshaw, 1989) was the overarching theoretical framework for the study. Social categories, such as race, gender, sexual orientation, socioeconomic status, and ability, are not always disjoint; it is often the intersectionality of identities that makes an individual's experiences unique. These intersectionalities can help researchers understand differences in the experiences of students who have less privilege than others. Crenshaw (1989) alluded to intersectionality as a chain of oppressions that can affect students with multiple marginalized identities simultaneously. At these intersections, Black students often face deficit-oriented discourses that position them as disruptive or mathematically inferior. Women of color in STEM often face discrimination and microaggressions (Dortch & Patel, 2017; Johnson, 2012) and are dissuaded by professors to continue in the sciences. Graham-Smith and Lafayette (2004) showed the level of care received from peers and faculty "overcomes the sense of isolation and separateness that a student with disabilities feels and gives him/herself the permission to nevertheless belong and succeed in a frightening and challenging college environment" (p. 90).

Though we utilize Crenshaw's (1989) Intersectionality as a focal framework for unpacking institutional systems, we also recognize the necessity of supporting frameworks. Namely, we draw upon the convergence of Critical Mathematics Pedagogy and Humanizing Pedagogy (Salazar, 2013), unified by Ethics of Care (Noddings, 2013; Valenzuela, 1999). The notion of "aesthetic care" (i.e., superficial kindness) versus "authentic care" is an important distinction Valenzuela (1999) makes in disrupting deficit narratives. When considering the importance of identity to belonging, we drew upon Jones & McEwen's (2000) Model of Multiple Dimensions of Identity (MMDI). This framework, developed using a grounded theory methodology and a diverse group of undergraduate women participants, provides a model of different identity dimensions such as sexual orientation, race, culture, religion, gender, and class, and their relative distance from the core identity *at a particular moment in time*. Integrating these frameworks has implications for how PSTs with intersecting identities can be supported. Their retention demands institutional supports that recognize their uniqueness as they leverage lived experiences to help reshape narratives about who belongs in mathematics spaces.

Research Methodology

Participants

The participants in our study were enrolled at a large, public institution in the southeastern United States. We used purposeful sampling to select students who had multiple marginalized identities. Our four participants chose their own pseudonyms. Two participants were Black, lesbian women. One of them faced housing insecurity, while the other faced food insecurity. Another participant was a transgender man on the autism spectrum. The fourth participant was

an asexual woman who struggled with poverty and an abusive family background. She was also on the autism spectrum. See Table 1 for a description of each participant's background and characteristics.

Table 1: Demographics and Background Characteristics of Participants

Wonder Woman	Ashley	Batman	Felix
Black Lesbian Woman, married Grew up in loving, supportive two-parent household; seven siblings Experienced food insecurity throughout childhood and college	White Asexual Woman Late autism diagnosis Experienced poverty, housing insecurity, abusive/addictive parents. Had traumatic brain injury.	Black Lesbian Woman Extensive network of family and friends Experienced housing insecurity during her program. Did not finish program.	White, Transgender Man (assigned female at birth) Supportive family Has Autism Spectrum Disorder Grew up in Church of Jesus Christ of Latter-Day Saints

Data and Data Analysis

Each participant was interviewed twice. After the interviews, one of the authors wrote narratives summarizing the interviews. Participants read their narratives and agreed they were accurate representations of their experiences in their mathematics education programs.

We used reflexive thematic analysis (Terry & Hayfield, 2021) to analyze our data. We familiarized ourselves with the data, while attending closely to our participants' experiences within intersecting social identities and power structures. Initial coding was guided by the theoretical frameworks, as we attended to experiences where participants described mattering, care, and how their professors affirmed their identities. Codes were grouped to generate themes, which we developed, reviewed, and refined to ensure analytic coherence and theoretical alignment. For example, the theme Caring Relationships contained fourteen code groups that reflected teacher factors prioritizing students' humanity, emotional needs, and lived experiences. Each code group was comprised of multiple quotations representing recurring patterns.

Findings

Although our analysis revealed four themes, we are only focusing on two of them for this paper: (1) Caring Relationships and (2) Classroom Supports.

Caring Relationships

Our participants felt the greatest sense of belonging in classes in which their teachers cared deeply for them. They wanted their professors to get to know them on a personal level. Felix wanted his professors to understand autism—to know what it means when autistic people are overstimulated and what an autistic meltdown would look like. He wanted them to understand that he struggles with tone and people thinking he was mean or sarcastic when that was not his intention. He wanted them to understand that he might not always be up to participating in class. He said:

> I think it's a really good idea, just like some kind of system where [I could communicate that] I'm just having a rough day. I might not be 100% present in class today. I don't really feel mentally equipped to participate fully in discussions. I'm imagining some kind of little note they can fill out before class and get to the teacher discreetly.

As a transgender man, Felix also wanted professors to give him messages of acceptance. He suggested that even small gestures can help, such as a PRIDE flag in the classroom or asking students to share their pronouns. He said, "I think any kind of subtle support that you can show is 100% going to be noticed by the students that you teach." When teachers make him feel like they care about him, Felix feels much more comfortable in their classrooms. He said, "I think also something that makes me feel comfortable and accepted [is when] teachers communicate and just try to get to know their students more on a personal level—when they show that they care."

Batman wanted her teachers to check in with her to see if she was okay. She wanted them to notice when she was not participating in a group. She struggled with peer interaction due to prior negative experiences and status hierarchies within groups. When working in groups, she felt alienated, explaining:

> I've gotten frustrated one time. And I said something to my group member. [I asked] why are you not listening to me? Why are you always looking away from me when I speak?

She said she noticed that some of her peers did not interact well with "people that look like me" and "people that act like me." She replied, "Well, for the most part it's [because I'm] a person of color. Um, it may be because I'm gay as well." She further explained, "But I know as far as like being African American, yes. I've seen other African Americans get put in the same boat as me." Batman said she never felt put down or alienated by her professors but often felt that way with peers. She wanted her professors to notice this and to do something about it. When she found herself homeless during her program, she did not want to share it with her professors at first. But when she felt she could no longer handle the pressure, she finally shared her situation. She said:

> Everybody just started helping me like the Care Center helped me a lot. They gave me a place to stay. And meal passes and stuff like that. And you guys showed an outpour of love to me, since that. I won't have to drop out anymore, because I was gonna have to drop out.

As a first-generation student, Batman was not aware of resources that were available to help her. Once her professors knew her situation, they helped her find access to help. She was able to get back on her feet. Her professors assigned incomplete grades to her that semester, which she did not know was an option, so she was allowed to complete the work later. Ultimately, we lost Batman. She withdrew from the program. However, we recently received a text from her saying "I've been substituting at an elementary school since January and I'm eventually going to enroll in school. Thank you for everything, your efforts and help will not go in vain."

Ashley struggled with belonging as a first-generation student. She struggled particularly in her pure mathematics courses because she was a woman and an education major. She felt as if her professors and peers did not feel as if she truly cared about mathematics. She said:

> I felt like kind of this double whammy in my content classes. Oh, you're both math ed., and you're a woman. You know, well, you can take notes in our groups. So that was a little bit uncomfortable, especially since I didn't face any of that in my other master's program. Because, you know, nobody questions why women are reading poetry.

She recounted a time when a professor gave her an explicit message of belonging:

> I was still feeling like a phony, an imposter … And then [the professor said], oh, you're really leading that group and really helping that group's thinking … Oh, my gosh, I'm not faking. I do belong here!

Ashley had an accident in the penultimate semester of her program. She had a traumatic brain

injury (TBI). As a first-generation student like Batman, she did not know incomplete grades were a possibility. She thought she would have to leave the program. She reflected on the help she received from one of her professors during this time:

> Having a professor who cared about you, and took a personal interest in your story, came to the hospital to see you, and then shared that story with other people … [There are people] who are … connected in a way that you aren't, and then they can get someone to open up their home to me … Sometimes we're given options like you should try this, go to this website, or send this email. … I have problems with depression and mental illness. I'm already struggling with a TBI, I cannot climb a mountain of tasks to secure myself a place to stay. So knowing that one of my professors was actually making phone calls and sending emails … just knowing there was a person who cared if I finished and cared if I lived …

Ashley graduated with a perfect 4.0 GPA and was selected as the outstanding graduate in her program. During student teaching, her students' scores outperformed veteran teachers' students.

Wonder Woman felt, as a poor African-American woman, that "I don't belong here. I'm not supposed to be here, because I don't see a lot of people that I can relate to." She was often the only Black woman in her mathematics classes. She often tried to "stay under the radar," and did not talk in class unless she had to. However, some of her professors developed a caring relationship with her and pushed her to share her thoughts during discussions of power and privilege as part of her undergraduate urban education program. She said:

> And she did push me to try to voice my opinion. I did at times, but most of the time I tried to hide from her. And the reason is I was insecure about how people would judge me based off what I feel, my experiences, and my opinion. But that class … it changed my life.

Wonder Woman earned two additional degrees at our university. During her graduate programs, she began to feel a much stronger sense of belonging and did share her thoughts about power and privilege. Unfortunately, she was met with resistance from her professor and peers:

> In my [graduate programs], I've been more vocal about being an African American lesbian woman. I had a teacher who commented on my discussion post and tried to get me to talk about something different. Because the topic at hand related to me being a lesbian, and me teaching kids who are trying to figure out that space … I've had classmates online to comment in a professional way, where it's not obvious … I've had a few classmates kind of negate what I was saying about, you know, being a lesbian, being an African American woman in the classroom, teaching certain kids who are dealing with certain things.

Wonder Woman is married with a son now, and has a picture of her and her wife in their wedding dresses on her desk at school. She is able to live authentically in her work and has developed caring relationships with her own students.

Our MMU students need more from their professors than traditional students. Caring relationships that go beyond aesthetic care to authentic care make all the difference in their experiences and success.

Classroom Supports

In addition to developing caring relationships with students, professors can also use classroom strategies that support students' sense of belonging. We are limiting our discussion to a few strategies due to space issues. Wonder Woman explained:

> I wish professors knew that I wasn't the typical student. I needed more than what they were

> giving. … The amount of different supports I have to give my kids, I feel like professors should do that in college as well, because everybody … don't learn the same. Everybody don't have the same strengths … So they should approach their lectures, their notes, their activities, they should have multiple ways of giving that to students in college.

Our first-generation college students need more support than traditional students, and they do not enter college with a sense of belonging. We have to actively build that sense of belonging through our classroom practices. For example, collaborative groupwork and Complex Instruction are often positioned as pedagogical structures for deepening mathematical understanding and fostering belonging (Cohen & Lotan, 2014). Complex Instruction can help address the status issues that arise within groups. Assigning competence and addressing status issues would help Batman feel that the teacher recognized her low status and did something to raise it.

Gilken & Johnson (2019) found that:

> Belongingness is better fostered by instructional interventions … in which students contribute actively to the construction of knowledge, and in doing so, become engaged participants. In this study, it is this engagement in the content-based peer feedback intervention situated within a [community of practice] that supports feelings of belongingness. (p. 43)

In one of the researcher's courses, students' mathematical identity and sense of belonging were supported by intentionally promoting discourse and writing in mathematics. Throughout the course, students were encouraged to construct their own proofs, revise them through peer and instructor feedback, and engage in structured discussions about the logic and structure of mathematical arguments. These low-stakes, whole-class discussions created a supportive environment where students could share developing ideas, pose questions, and receive affirming responses from their peers. This process not only strengthened students' reasoning and communication skills but also positioned them as legitimate contributors to mathematical knowledge. When reflecting on what it meant to choose who she wanted to work with, Batman explained, "you feel respected, and you get to see your own work, and not just what somebody else put in …, it just matters." By centering student voices and encouraging reflection through writing and dialogue, the researcher helped foster a classroom culture where all students felt seen, valued, and capable of doing rigorous mathematical work.

This researcher also uses a culminating poster session as both an assessment tool and a means of cultivating students' identities as capable mathematical thinkers and supports their sense of belonging. Throughout the course, students engage deeply in mathematical writing and discourse, developing and revising proofs that reflect their growing understanding (Jansen, 2020). The poster session invites students to present two of their own proofs to peers and faculty in a public forum, like a poster session at an academic conference. This requires them to synthesize their work, reflect on their learning, and communicate their reasoning with clarity and precision. Batman indicated that it "matters" that the instructor created space for students to showcase their own work, explaining:

> So I honestly like the single work, when you just get to do your own project and present it, we had a project at the end of our class where we pick the topic. And I like when I get to showcase my own stuff, my own learning, and not be suppressed in a group.

Research (i.e., Koestler et al., 2024; Laursen et al., 2014) has shown that opportunities for students to talk and write about mathematics in collaborative, low-stakes environments foster a

stronger sense of belonging in the discipline. By giving students a platform to share their voice and expertise, the poster session helps create an inclusive classroom culture where each student is valued as a contributor to mathematical knowledge.

Discussion

Students with multiple marginalized identities can experience mathematics teacher education as a site of exclusion, marginalization, and conditional belonging. Jones & McEwen's (2000) MMDI posits that different dimensions of identity surface in different contexts. For example, when working with peers, it was Batman's racial identity that acted to suppress her belonging. But when she was in crisis, her poverty and first-generation status left her struggling with no knowledge of campus resources. In her mathematics classes, Ashley's gender and major affected her sense of belonging, but when she suffered a TBI, like Batman, she needed the support of her professors and community resources to recover. Ashley withheld her sexual orientation throughout her undergraduate program as she struggled most critically with race and poverty to belong in college. As her belonging increased after earning degrees, she began to explore the impact of her sexual orientation on her experiences as a student and teacher during her graduate programs. Those explorations were not well-received by a professor and peers. Felix needed explicit messages of belonging as he searched for acceptance to be able to live his authentic self as a transgender man, but often his autism acted more on his everyday interactions with peers and teachers. As MTEs, we must work to intentionally create inclusive environments that foster belonging and to allow students to live their authentic selves. The most important practices we can use to support students' belonging, especially for MMU students, are to show them authentic care and acceptance, hold them to high expectations of learning, and help connect them with support services. Creating low-stakes spaces where students can try out strategies, revise and refine them, and then publicly share them gives students confidence in their knowledge. Grading itself, Noddings (2013) argues, is a violation of the caring relationship because grading is not done for the good of the student; rather it is done to demonstrate the students' learning to others. One way to alleviate this tension in the caring relationship is to allow the student to revise work until it meets the teacher's standard. This practice shows that the teacher believes the student can do high-quality work and cares enough to work with them to attain an image of themselves that is better than their current work. In this way, students do not feel as if they are behind their peers, and they develop a deep sense of belonging. They need their teachers to encourage them and give them explicit messages of belonging. Avoiding deficit discourses, MTEs can use strength-based strategies that tie into students' interests (particularly for ASD students) and give them access to advanced content. Along with low-stakes environments, the classroom environment must be suitable for sensory sensitivities and provide ways for students to communicate their daily needs to teachers.

Limitations

The most significant limitation is that not all marginalized groups were included. There were no emerging multilingual students, students with learning or physical disabilities, nor representatives from many other marginalized groups. Findings cannot be generalized to other groups, and other themes would likely have emerged with different participants. Our findings do align with findings from previous literature, which adds to the credibility of our study.

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THE PERCEIVED IMPACT OF IDENTITY FACTORS ON THE MATHEMATICS EXPERIENCES OF PRE-SERVICE TEACHERS

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The impact of identity on students' mathematics learning is well documented in mathematics education research; however, some identity factors remain understudied. We draw on notions of mathematics identity and belonging to study how pre-service teachers perceive four concealable stigmatized identities (gender, sexual orientation, socioeconomic status, and mental health) impacted their own mathematical learning experiences and might impact the experiences of their future students. Data were collected through a qualitative questionnaire at universities in Australia, Canada, and the United States. Of the eight major themes identified, we describe four such themes in this paper pertaining to the pre-service teachers' perceptions that CSIs can impact confidence, opportunity, help-seeking, and motivation.

Keywords: Equity, Inclusion, and Diversity; Gender; LGBTQIA+; Teacher Beliefs

Disparities in students' mathematics learning experiences on the basis of identity, such as race, are well documented (e.g., Hsieh et al., 2021; Leyva, 2022). However, the influence of concealable stigmatized identities remains understudied. The term *concealable stigmatized identities* (CSIs) refers to identities that individuals can conceal, that are associated with negative stereotypes and/or social prejudices (Quinn & Earnshaw, 2013). In an effort to expand research on this topic, we developed and conducted a qualitative questionnaire for pre-service teachers in which we investigated the perceived impact of gender, sexual orientation, socioeconomic status, and mental health on the participants' own mathematical learning experiences and the mathematical learning experiences of their future students. We selected the first two CSIs as they relate to the core mission of our working group and the second two CSIs because we felt that they do not commonly appear in mathematics education studies. Guided by this focus, we sought to answer the following:

1. How do pre-service teachers perceive that their gender, sexual orientation, socioeconomic status, and mental health have impacted their own mathematics learning?
2. How do pre-service teachers perceive that gender, sexual orientation, socioeconomic status, and mental health might impact the mathematics learning of their future students?

We ground our work in scholarship on mathematics identity and belonging (Nasir & Cobb, 2007; Walton & Cohen, 2007), as well as mental health and academic success (Eisenberg et al., 2009). Specifically, in this study, we conceptualize mathematics learning as being shaped by the interaction of structural, identity-based, and psychological factors. Students do not encounter mathematics in a vacuum; rather, their engagement and persistence are influenced by the resources available to them, their sense of belonging, and their psychological well-being. Structural conditions such as socioeconomic status shape access to instructional materials and

academic support, whereas identity processes influence whether students see themselves—and are recognized by others—as legitimate participants in mathematics (Martin, 2009; Nasir & Cobb, 2007). Psychological well-being mediates attention, motivation, persistence, and cognitive endurance in mathematics, a discipline characterized by cumulative content, high cognitive demand, and cultural narratives of difficulty (Eisenberg et al., 2009). Consequently, mathematics learning is positioned as a socially situated process in which engagement and persistence are shaped by access, belonging, identity, and mental health (Gresalfi & Cobb, 2006).

Methods

Data Collection

Data for this study were collected through an open-ended questionnaire administered at two universities in the United States, one in Canada, and one in Australia. Participants were recruited by instructors who were teaching pre-service teachers at the time of data collection (e.g., via announcements on course websites). In the questionnaire, pre-service teachers were invited to share their identities with respect to gender, sexual orientation, socioeconomic status, and mental health, as well as the impact of these CSIs on their mathematics learning and the mathematics learning of their future students. For instance, one of the prompts was "How do you think that your gender impacted your mathematics learning as a student?" The anonymous questionnaire primarily contained open-ended questions and took approximately 10–15 minutes to complete.

Data Analysis

We received 147 responses to the questionnaire. However, 35 were blank or partial responses, which we excluded due to incompleteness. Of the remaining 112 responses, 17 respondents indicated they were not planning on teaching mathematics at any level and were thus removed so that we could restrict our focus to pre-service mathematics teachers, resulting in 95 responses in the final data set. The responses were analyzed through analytic discussion and notetaking (Keane, 2022), which allowed us to identify recurring patterns across participant responses and create inductive themes across the data set (Gibbs, 2007). In this report, we frame a subset of these themes through the responses of four respondents chosen for their richly detailed answers that elucidated the themes identified. All names are pseudonyms.

Participants

Lincoln is pursuing certification in secondary school mathematics teaching in the United States. He identified as cisgender, asexual, and middle SES, and shared that he has anxiety and ADHD (attention-deficit/hyperactivity disorder). Ruby is pursuing certification in elementary and secondary school teaching in Australia. She identified as cisgender, straight/heterosexual, and middle SES, and shared that she has anxiety, OCD (obsessive compulsive disorder), and PTSD (post-traumatic stress disorder). Akask is pursing certification in secondary school mathematics teaching in Canada. He identified as two-spirit (FTM, man), asexual, gay/homosexual, aromantic, and low SES, and shared that he has depression, OCD, PTSD, and avoidant personality disorder. Sammy is pursuing certification in secondary school mathematics teaching in the United States. They identified as a demigirl, asexual, bisexual and/or pansexual, graysexual, and low SES, and shared that they have anxiety, OCD, ADHD, and autism.

Findings

We identified eight major themes through thematic analysis of the pre-service teachers' questionnaire responses (Table 1); however, due to space limitations, we restrict our findings here to the four themes italicized in the table, each of which relates to a different CSI.

Table 1: Eight Major Themes Identified in Survey Responses

Gendered Expectations Shape Confidence and Opportunity	*Mental Health Affects Motivation, Focus, and Depth of Learning*
Socioeconomic Status Determines Access and Time for Learning	*Sexual Orientation is Irrelevant to Mathematics Learning*
Low SES is Linked to Shame and Reduced Help-Seeking	Intersectional Experiences Compound Perceived Learning Challenges
Representation and Belonging Influence Engagement	Mathematics Amplifies Existing Stressors

Theme 1: Gendered Expectations Shape Confidence and Opportunity

Participants often framed mathematics as a masculine activity and discussed gendered expectations as a factor that influenced their mathematics learning. For example, Ruby highlighted that "the males in the classroom were always favoured in math and females were favoured in English." This led her to perceive that the men in her classes were receiving greater support and were viewed as more mathematically capable than the women. Sammy also addressed the exclusion of femininity in mathematics, stating that, "social conditioning encourages people assigned female at birth to not conform to the competitive, individualistic standards of math classrooms," suggesting that Sammy sees mathematics learning environments as rewarding stereotypically masculine forms of communication and performance.

When reflecting on how gender might impact their future students, Lincoln acknowledged that anyone who does not see their gender represented "may feel it's not possible for them" to study mathematics. He emphasized representation, rather than masculinity, as central to his understanding of the potential impact of gender on mathematics learning. In contrast, Akask referenced the masculinity of mathematics, also discussed by Ruby and Sammy, by arguing, "If patriarchal gender roles and stereotypes are a major aspect of a student's life, their confidence in their mathematical abilities would be affected by such gender roles."

Theme 2: Low SES is Linked to Shame and Reduced Help-Seeking

Participants frequently described socioeconomic status (SES) as a factor that impacts access to learning resources. Akask reflected on needing to borrow mathematical tools from his teachers and emphasized that his low SES made it "difficult to get access to a tutor or be able to spend time after school to receive help for math," a consideration echoed by Ruby. Sammy expanded this point to a systemic level, arguing that funding "is a huge barrier to education in general."

Participants also shared beliefs that lower SES is associated with lower self-confidence and a hesitancy to seek out support. Akask stated that "a low SES may lead a student to believe they're less capable of learning than their peers, … and intense shame when asking for help." Sammy also noted that scholarships "can help but also have a shaming aspect." Hence, to Akask and Sammy, SES can be a psychological constraint as well as a material constraint, negatively affecting a student's mathematical identity.

Theme 3: Mental Health Affects Motivation, Focus, and Depth of Learning

Participants described mental health challenges as a significant factor that shaped their mathematics education, affecting their motivation, focus, and depth of learning. Conditions such as ADHD, depression, and anxiety were often cited as interfering with engagement as well as the ability to study and seek help outside of class. For example, Lincoln stated that "the ADHD made it difficult to pay attention in many of my math classes," a sentiment also shared by Akask.

Mental health challenges were also linked to fragmented learning goals. Akask mentioned that his mental health and ADHD contributed to his lack of attendance; persistent absences can create gaps in understanding that widen over time. Akask also connected mental health to his help-seeking behavior: when asked how he perceived mental health as impacting his mathematical learning, he stated that "when it came to doing the work in math, I struggled to ask for help due to fears of being judged or humiliated for having the wrong answer." We see this as evidence that Akask's experiences with anxiety and ADHD may have affected his ability to seek help, therefore impacting his depth of learning.

When reflecting on how mental health might affect her future students, Ruby suggested that "if students aren't in the right headspace, it is difficult to make them learn," a sentiment consistent with Lincoln and Akask's lived experiences. Sammy also addressed the importance of their future students having access to accommodations for mental health struggles, arguing that "without adequate supports and communication, the toolbox available to help students work in the disability system can be a fast track to failure."

Theme 4: Sexual Orientation is Irrelevant to Mathematics Learning

Of the 95 participants, 94 did not feel that sexual orientation impacted their mathematics learning, a response that was consistent across all sexual orientations. For example, Akask (aromantic) stated that he did not see an impact since he's "had plenty of time to be comfortable in my orientation since coming out at 12 years old." Sammy and Lincoln (asexual spectrum) also did not feel that their sexual orientation impacted their learning.

When asked to reflect on how their future students' sexual orientation might impact their mathematics learning, participants were unsure if there would be an impact. A few shared that it may cause harassment, though Lincoln noted that the harassment would not be unique to the mathematics classroom. Similarly, Akask noted that "a student being bullied or pressured to hide their orientation may not be fully focused on learning." Thus, sexual orientation was perceived by these participants as a learning concern primarily in relation to social discrimination.

Discussion and Conclusion

Through this study, we found that pre-service teachers in the United States, Canada, and Australia demonstrated an awareness that mathematics learning is dependent on social and psychological factors, consistent access to resources, students' sense of belonging, and the ability to withstand the high cognitive demands of mathematics work both within and outside the classroom. These findings are consistent with research findings showing that confidence and participation in mathematics is influenced not only by measured ability but also by gendered expectations, representation, and classroom placement practices (Nasir & Cobb, 2007; Walton & Cohen, 2007). Together, these factors can signal which students are perceived as capable of succeeding in advanced mathematics.

As mathematics education researchers continue to discuss the impacts of student identity on mathematics learning, we call for increased consideration of CSIs and hope to incorporate intersectional analyses of these identity factors in our future work.

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THEORIZING WHITE CHRISTIAN NATIONALISM IN MATHEMATICS EDUCATION

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Some Christian mathematics educators express discourse that reflects interconnectedness between their conceptions of mathematics and divinity, including the idea that "mathematics is the language of God." For believers, God assumes a privileged status at the top of ideological hierarchies. Thus, seeing mathematics as interconnected with God imbues mathematics with a high status. Such hierarchical thinking is reflected in the discourses mobilized by adherents to White Christian nationalism to promote marginalizing agendas in social and educational contexts. In this theoretical paper, I begin exploring the nature of the discourses that some Christian mathematics teachers may hold and describe how they align with the harmful discourses in the White Christian nationalism movement.

Keywords: Equity, Inclusion, and Diversity; Identity and Agency; Social Justice

In spite of the so-called "separation of church and state" in the United States and other countries' public school systems, scholars have studied the religious identities of teachers and students in education (e.g., Hadley & Fassbender, 2020; Hartwick, 2015a, 2015b; Häusler et al., 2019; Journell, 2018; Subedi, 2006; Weyand & Juzwik, 2020) and in mathematics education specifically (e.g., Amir & Williams, 1999; Chan & Wong, 2014; Leu, 2005; Norton, 2002; Taylor, 2013). Research has shown that some religious teachers see teaching as a calling, a divinely inspired career pursuit (White, 2010), and they seek divine guidance to inform their teaching and their relationships with students (Hartwick, 2015b; Vaccino-Salvadore, 2021). Religious identities shape pedagogical practices (e.g., Hadley, 2021; Journell, 2011, 2018; Petrova, 2016; Shekitka, 2022). In other content areas, some Christian teachers identify tensions between their commitments to God and commitments to an inclusive education (e.g., Hadley, 2022; Hadley & Fassbender, 2020; Journell, 2011). As they navigate these tensions, they use marginalizing discourse about their students (e.g., Hadley, 2021; Hadley & Fassbender, 2020; Johnson et al., 2021) and limit students' opportunities to learn (Hadley, 2022; Olshefski, 2021).

A majority of teachers (over 85%) in the United States identify as Christian (Hartwick, 2015a), but those in public schools may not feel comfortable discussing how their religious beliefs are intersecting their work as teachers (Hadley & Fassbender, 2020; Weyand & Juzwik, 2020). Working at a private institution for over a decade has afforded me glimpses into how some Christian mathematics educators, mathematicians, and prospective and practicing mathematics teachers see their religious identities intersecting their work as they feel more free to express their beliefs in this context. They have asserted that "mathematics is the language of God" or "God used mathematics to create the universe." I do not think their beliefs are unique to their particular faith traditions as it is mirrored in the (scant) literature in this area (particularly, Chan & Wong, 2014; Leu, 2005; Norton, 2002); however, as these ideas are seldom discussed, it is hard to understand the breadth and depth of their impact on mathematics education. Nevertheless, if over 85% of mathematics teachers are Christian (matching Hartwick's (2015a study), then it is important to study what discourses they could be bringing to their learning and work as mathematics teachers.

Understanding these discourses may be particularly important in the context of the current sociopolitical movements happening in the United States (US)—namely, the White Christian nationalist agenda becoming increasingly visible in legislation, governmental priorities, and judicial decisions (Gorski & Perry, 2022; Whitehead & Perry, 2020; for examples, see also Burke & Hadley, 2025; Burke et al., 2023; *Carson v. Makin*, 2022; Cline & McGill, 2024; *Kennedy v. Bremerton School District*, 2022). White Christian nationalism (WCN) is a sociopolitical movement that draws on racialized, theological, and political arguments to assert that the US was founded to purposefully be a White and Christian nation (Gorski & Perry, 2022; Whitehead & Perry, 2020). To pursue what they see as divine intentions for the US, WCN supporters divide people into opposing groups: "those belonging to the divinely ordained nation-state imaginary (White Christian people like "us") and those who do not belong (people like "them")" (Burke et al., 2023, p. 287). Supporters believe in protecting their own group by any means necessary and maintaining hierarchical positioning in which White Christians benefit (Burke et al., 2023; Burke & Hadley, 2025; Gorski & Perry, 2022; Whitehead & Perry, 2020).

It is important to note that the term Christianity serves as a larger umbrella to many particular and diverse faith traditions (e.g., Catholics, Baptists, Methodists) and many Christians do *not* align themselves with the White Christian nationalist movement. However, I argue, alongside Gafney (2024) and others (e.g., Kaylor & Underwood, 2024) that it is necessary to examine the diverse discourses within Christianity:

> Liberation minded and justice-oriented Christians are a branch in a great tree of Christianity. Some of those other branches are rotten and producing strange fruit but they are part of the tree. Maintaining the health of the tree requires dealing with the rottenness within, cutting it off and rooting it out; not pretending it's not part of the tree. (Gafney, 2024)

Further, I argue that "justice-oriented Christians," like myself, need to reckon with the ways our theologies may still operate to marginalize others or have "created the fertile soil in which its most extreme versions now bloom" (Kaylor & Underwood, 2024, p. 22).

I see it as critical, then, to explore what Christian discourses could be silently becoming entangled with mathematics teaching and learning. In this theoretical paper, I explore possible implications of the conceptualization that mathematics is divinely created within mathematics education, particularly at the K-12 level. I began my theoretical exploration of this question by analyzing historical and philosophical accounts of mathematics emphasizing Christian perspectives (viz., Cohen, 2007; Howell & Bradley, 2001; Livio, 2009) as well as existing scholarship within mathematics education on Christian students and teachers (viz., Amir & Williams, 1999; Chan & Wong, 2014; Norton, 2002; Taylor, 2013). My theoretical analysis focused first on developing short but rich descriptions of possible discourses within the intersection of (some) Christian identities and mathematics education. Secondly, I tied these examples to discourses concomitant with White Christian nationalism to demonstrate how these discursive practices may be operating to privilege and oppress differing students. In the pursuit of equitable mathematics classrooms, I argue it is important to confront the ways Christian theologies may be and are entangled with mathematics education.

Divine Discourses in Mathematics Education

Describing mathematics as divinely inspired or used by God, minimally, reflects interconnectedness between the speakers' conceptions about the nature of mathematics and the nature of God. Elevating mathematics to a divine language has implications for the discourses a

mathematics teacher might draw on to describe their work and practices, which could have implications for how students are positioned. What are some of these discourses?

In the next two subsections, I present two vignettes that reflect themes in my theoretical analyses of philosophical and historical texts focusing on the intersections of mathematics (education) and the divine (viz., Cohen, 2007; Howell & Bradley, 2001; Livio, 2009) as well as the existing scholarship within mathematics education on Christian students and teachers (viz., Amir & Williams, 1999; Chan & Wong, 2014; Norton, 2002; Taylor, 2013). I read these texts looking for discourses that focused on the entanglements of Christianity and mathematics or mathematics education. For example, I located discourses describing the relationship of God to mathematics. Hobbes is quoted in Livio (2009) as saying, "geometry…is the only science that it hath pleased God hitherto to bestow on mankind" (p. 2). Further, I considered discursive patterns in the arguments. For example, in Livio (2009), the author elaborates examples of seemingly small mathematical ideas having applications much later in history as evidence of an underlying divine design. From this theoretical analysis and following a pattern of methodological practices inspired by poetic inquiry (Faulkner, 2020; Faulkner & Cloud, 2019; Hanauer, 2010; Hoult et al., 2020), I developed the two vignettes of the main ideas, presented in the next subsections. These vignettes use quotes and my own language to represent themes. I do not follow APA conventions within the vignettes to improve readability. Although some discourses are situated from a different historical time and context, I am arguing that the historical discourses resonate within some Christian mathematicians and mathematics educators currently because of their inclusion in the texts and articles I reviewed.

Vignette 1: On the Divine Nature of Mathematics and Doing Mathematics

"As the British physicist James Jeans (1877–1946) once put it: 'The universe appears to have been designed by a pure mathematician.' Mathematics appears to be almost too effective in describing and explaining not only the cosmos at large, but even some of the most chaotic of human enterprises" (Livio, p. 1). Thus, for Plato, "the mathematical character of the world is simply a consequence of the fact that 'God always geometrizes'" (Livio, p. 36). Further, Galileo and others believed that, "mathematics was simply the language of the universe. To understand the universe, … one must speak this language. God is indeed a mathematician" (Livio, p. 77). Consequently, "the ability to reason mathematics was thus a sublime present from God" (Cohen, p. 18). "Human beings exist in a cave of impaired perception, a profane realm of limited, imperfect things: matter, decay, ever-changing shapes. Above our muddled existence, however, is another, divine realm, a sphere of purity and eternal Truth. Mathematics plays a special role in this divided universe—it ascends from the world of impermanence to this higher, heavenly plane" (Cohen, p. 19). "It is not only the physical universe that is written in the language of mathematics; all of human knowledge follows the logic of mathematics" (Livio, p. 92). "In Descartes' words: 'Existence can no more be separated from the essence of God than the fact that its angles equal two right angles can be separated from the essence of a triangle'" (Livio, p. 116).

As a consequence, "at work the mathematician becomes a pure spirit, rising out of the confinement of his material body… He has no use for everyday faculties, like sight, but rather operates with a higher, far more powerful internal sense. This mathematical faculty is not a passive receptor of information, but is instead a penetrating instrument that attains the grandest of trust, all of which lie beyond the reach of our five bodily senses" (Cohen, p. 3). A mathematician uttered, "I prayed for an idea, God cared about that piece of work, and / An idea came into the world. The idea came into the world. / Seeing the immense amount of vagueness,

what can one person do? / One small step at a time, you shouldn't give up on the restoration of rigor. / And what surfaces at the end—that's God's" (Norton, p. 31). "Put differently, when a person finds a solution to a problem using proportional geometry, the insights and understanding gained are godlike" (Livio, p. 82). Doing mathematics means drawing closer to God—engaging in His practices and discourses. Doing mathematics is a calling. The evidence of His presence, with the mathematicians, echoes as mathematical understandings hold up across years and then decades and then centuries. These mathematical understandings find new purposes and uses in context never even dreamed of when first conceptualized. These examples become the evidence of God's hand. A mathematician beamed, "I'm amazed at how precisely Nature dances to the tune we scribbled so carelessly fifty-seven years ago, and at how experimenters and the theorists can measure and calculate her dance to a part in a trillion" (Livio, p. 223). "He [Descartes] declared that 'the mathematical truths which you call eternal have been laid down by God and depend on Him entirely no less than the rest of his creatures.' So the Cartesian God was more than a mathematician, in the sense of being the creator of both mathematics and a physical world that is entirely based on mathematics" (Livio, p. 116).

Vignette 2: On Divinity Within Mathematics Education

Teaching mathematics is a calling—an occupation one does in the service of God and because one believes that God would have them fulfill this role on Earth. A mathematics teacher stated, "God gave me the chance to take up this career. I do it seriously. … I have a mission. I believe in Jesus. I have a grateful heart. … I believe that I am just a humble woman. I cannot do anything by my own effort … My religion gives me so much grace. I know that I am just a humble woman. God put me in this good environment. I need to tend His [God's] kids, the little sheep. I need to work hard" (Chan & Wong, participant, p. 263). Another mathematics teacher stated, "I used to think that being a pastor was the only thing you could do [to serve God]…but I can see that I am not gifted to do that kind of work" (Norton, participant, p. 30). Mathematics teachers are "a channel used by God to bring mathematical knowledge to the world" (Norton, p. 30). "When studying mathematics, we know we are studying something God values" (Howell & Bradley, p. 248).

Teaching mathematics particularly can be a tool to draw students closer to God as they embody His practices and discourses. Learning mathematics is a means to get to know God on a personal level. Mathematics teachers should pattern their educative practices after God and not the ways of the world. "Christians should embrace the notion of centering the learning environment on students. This notion fits well within a view of our students as God's image-bearers" (Howell & Bradley, p. 355). Yet, the theory of constructivism that underlies many conceptions of mathematics teaching and learning is also flawed. "Sin has allowed this good structure [the theory of constructivism] to be distorted by those who would direct some aspect of creation toward purposes that do not seek to honor and glorify God" (Howell & Bradley, p. 355). "Constructivists assert that only active construction produces knowledge. It would seem unnecessarily limiting on God's creative power to conclude that knowledge can only be gained though [through] the process of construction" (Howell & Bradley, p. 357). "Christians must state clearly that ultimately we are not autonomous but instead depend wholly and completely on a creating, redeeming, and sustaining God. Any perceived autonomy on the part of the individual learner is instead God's precious gift of wisdom and understanding, not to be distorted into a situation where, as Saint Augustine extends Romans 1:25, humans 'worship and serve the creature rather than the Creator'" (Howell & Bradley, p. 358). "All knowledge comes from God, and a human's ability to comprehend knowledge is limited and incomplete" (Chan & Wong, p.

271). Christians should recognize God's hand in all things, including mathematics and Christian teachers should seek to share this knowledge with their students. Perhaps the NCTM standards ought to include a standard such as, "Mathematics as a means to recognize and describe the beauty, order, and regularity of God's creation" (Howell & Bradley, p. 359). Regardless of whether it is a standard, this idea must live within the hearts of Christian mathematics teachers.

White Christian Nationalism in Education

In previous scholarship, I, alongside co-authors, developed a conceptual framework for considering the intersection of education and WCN (Johnson et al., 2025). We sought to develop a theoretical and analytical framework that would identify some of the ways that WCN is seen in discourses about teaching and learning. We used Butler's (2016) concept of "frames" to identify two main ideologies associated with WCN. Butler conceptualized frames by discussing how photographers wield an instrument of war (the camera) because non-participants, such as citizens in their home country, only understand the war through what is present and not in images. As Butler wrote, "The frame does not simply exhibit reality, but actively participates in a strategy of containment, selectively producing and enforcing what will count as reality" (p. xiii). These frames position some people as privileged and others as precarious or ungrievable (Butler, 2016). Butler argued that the enemy within a war is considered ungrievable because, in their position as the enemy, they are already "lost, and cannot be destroyed" (p. xix). Those who inhabit this position are considered "precarious" before their deaths upon which they are rendered "ungrievable" (Butler, 2016). Ideological frames also operate to position some people as privileged, with their full humanity, and others as precarious or ungrievable (Butler, 2016).

Two foundational ideological frames of WCN are dominionism and colonialism (Johnson et al., 2025). Dominionism is the idea that Earth is God's kingdom and meant to accomplish His purposes (Johnson et al., 2025). Colonialism is the idea that Earth is where God's disciples live and they should grow in strength and numbers to accomplish His purposes (Johnson et al., 2025). Although colonization can take place through force (e.g., seizing land), our use of colonialism focuses on the work to spread the Christian (gospel) messages and, therefore, "the colonization of souls or spirits" (Johnson, 2025, p. 132). The frames of dominionism and colonialism are circulated through discursive practices—namely, divinely-ordered hierarchies, victimization, (anti)structuralism, and violence (Johnson et al., 2025). For example, discourses within WCN center "a God-ordained hierarchy, and there is no one higher in the social hierarchy than a straight White Christian man" (Johnson et al., 2025, p. 137). Using examples from practitioner texts written by Christian public school teachers (viz., Graham, 2011; Mason et al., 2015; Parker, 2012; Smith, 2022; Van Brummelen, 2009), my co-authors and I showed how these dominionist and colonialist discourses are seen in discussions of schooling (Johnson et al., 2025).

Dominionism

With respect to the discourses of victimization, these Christian teachers (who authored the texts we examined in Johnson et al. (2025)) described themselves as victims of governmental regulations restricting their rights to discuss their religious beliefs. A hierarchy is evident when Christian teachers talk about the different types of authority they have in the classroom (Johnson et al., 2025). As students are at the bottom of the hierarchy in the classroom, they are positioned as precarious or ungrievable. These Christian teachers certainly do not speak directly about engaging in physical violence in their classrooms but violence as dominionist discourses were evident when they described school as a battlefield or a "minefield" (Parker, 2012, p. 16). Christian teachers may see themselves as "warriors engaged in battle" (Johnson et al., 2025, p. 144) as they work within the dominionist frame. These metaphors positioned students as the

enemy and, thus, precarious or ungrievable—objects over which the teacher should exercise control and authority (Johnson et al., 2025).

Colonialism

Within education, the hierarchal discourses position teachers as having protective influence over students who are framed as precarious (Johnson et al., 2025). With respect to antistructuralist discourse, these Christian teachers described the public-school classroom as a mission field—one which had many opportunities for spreading the gospel message; however, they were frustrated by the governmental restrictions on religious expression in public schools (Johnson et al., 2025). Colonialist discourses of violence were evidenced when these Christian teachers wrote about why they needed to spread the gospel messages to their students (Johnson et al., 2025). They referred to students as "broken" and "sinful" and as needing rescue from the saving influences of Jesus Christ (Johnson et al., 2025). Because the teachers positioned themselves as the conduit to Jesus Christ, they enacted colonialist framings of students in which students' souls would be changed (Johnson et al., 2025). Students' own funds of knowledge or cultural practices were disrespected and they were rendered precarious, marginalized by the position of privilege the teachers saw for themselves (Johnson et al., 2025).

Frames and Discourses

In summary, two main ideologies associated with WCN are dominionism and colonialism. These frames shape how adherents to WCN, including teachers, see the world around themselves. In short, within the educational context:

> We can see that dominionism reflects an authoritarian relationship between the privileged and those who will be ungrievable. Within this frame, teachers believe their conceptions of how to behave or engage in concepts being taught are the correct or right ways. Alternatively, colonialism reflects a (mis)appropriation of the concept of love. In colonialism, the privileged seek to *love* the precarious into a different way of being–one that erases the humanity of the precarious. Within this frame, teachers see students as people who need saving from their own disbelief. (Johnson et al., 2025, p. 135–136)

These frames are upheld and circulated through particular discourses which continue to position some as privileged and others as precarious or ungrievable (Johnson et al., 2025).

White Christian Nationalism Discourses in Mathematics Education

Returning to the vignettes at the intersection of divinity and mathematics education, I now consider the ways the discourses align and not with WCN. Most prominently, the vignettes show hierarchical structure with God and mathematics at the top of the hierarchy. This hierarchy is expressed directly: "Human beings exist in a cave of impaired perception, a profane realm of limited, imperfect things: matter, decay, ever-changing shapes. Above our muddled existence, however, is another, divine realm, a sphere of purity and eternal Truth" (Cohen, 2007, p. 19). Human beings are positioned as inferior with the language "exist in a cave of impaired perception" while divinity is positioned as "above." The human being experience is described with words that connote poor conditions such as "imperfect," "decay," and "muddled." In contrast, divinity is described as "a sphere of purity." These contrasts show humans are (much) lower than divinity in a hierarchy. As Cohen (2007) continued, "Mathematics plays a special role in this divided universe—it ascends from the world of impermanence to this higher, heavenly plane" (p. 19). That is, mathematics is a part of the "higher, heavenly plane." It might be characterized as a bridge, but it also seems travel on the bridge only goes one way—from the human to the divine. These discourses reify hierarchical thinking commonly expressed in WCN.

With mathematics at the top of a hierarchy, there is a danger of dominionist and colonialist discourses of violence—ones in which mathematics teachers seek to diminish students' funds of knowledge to promote particular ways of knowing mathematics. They may dismiss students' interests in other subject areas (explicitly or implicitly) as they work to bring students into the divine light of doing mathematics, which reflects a colonialist perspective of students. Within this perspective, mathematics teachers can position themselves as saviors (cf. Hadley, 2021; Johnson et al., 2025) of the students, bringing them into a more enlightened way of being. However, this discourse renders the students precarious by oppressing their natural talents and interests. Further, mathematics teachers may believe that particular ways of doing mathematics are most aligned with God's uses. Thus, they may denigrate students' ways of knowing and doing mathematics in favor of these other particular ways. This approach to teaching mathematics can limit students' opportunities to develop a robust understanding of mathematics.

In Johnson et al. (2025), we saw evidence of the WCN discourse of antistructuralism when teachers were not trustful of the governmental oversight and restrictions of religious expression of teachers. We described a tension between antistructuralism, which resists restrictions, and hierarchies. "Although WCN does embrace rhetorical practices espousing antistructuralism in relation to governmental organization, at the same time, WCN does highly value structures that privilege White Christian men" (Johnson et al., 2025, p. 142). As a consequence, adherents to WCN have a tenuous relationship with ideas that seem to describe the world in ways that are antithetical to their interpretation of God's rules and principles. In their view, education is one of God's ideas, but humans may enact this idea in ways that are contrary to His principles (cf. Johnson et al., 2025). This perspective is reflected in the vignette primarily in the discussion of constructivism: "Sin has allowed this good structure [the theory of constructivism] to be distorted by those who would direct some aspect of creation toward purposes that do not seek to honor and glorify God" (Howell & Bradley, 2001, p. 355). This statement describes the perspective that "sin" can "distort" aspects of God's work on earth. Further, Howell and Bradley (2001) note that constructivism, as a theory, is "unnecessarily limiting on God's creative power" (p. 357). In other words, God's principles and practices are infallible, but humans may implement them inconsistent with God's work. What remains less clear is what kinds of theories of learning are consistent with God's work as many may be considered "unnecessarily limiting on God's creative power." This perspective can be used to reinforce particular ways of teaching mathematics that are more likely to be the ways the mathematics teachers construe as appropriate. These strategies might be more likely to position students as precarious in ways that are seen as desirable by adherents to WCN as they are promoting conformity to a particular way of teaching and learning mathematics that reifies the hierarchy.

As noted in the vignettes, both doing mathematics and teaching mathematics can been seen as callings from God—divinely appointed career pursuits (cf. Hartwick, 2015a; White, 2010). Mathematicians and/or mathematics teachers see themselves as drawing closer to God through their work. Thus, if creating (and learning) mathematics is an act of conformity to divinity, then teaching mathematics may be viewed as a surreptitious way of evangelizing. In Johnson et al. (2025), we showed that some Christian teachers view the public school system as a mission field as it is "filled with countless opportunities to live out our faith and draw others to Christ through our actions and reactions toward people and circumstances" (Parker, 2012, p. 16). Within the discourses about divinity in mathematics education, mathematics education itself might be a particularly powerful tool of evangelizing as students would find themselves solving problems through which "the insights and understanding gained are godlike" (Livio, 2009, p. 82).

Mathematics teachers may view themselves as "a channel used by God to bring mathematical knowledge to the world" (Norton, 2002, p. 30), which can reflect a colonialist view of the work of mathematics teaching. It is consistent with the antistructuralist and violent discourses of WCN that work to reify the hierarchy in which mathematics and divinity are intertwined at the top. This way of thinking becomes ordained through the further discourse that mathematics teaching is a calling. More directly, if mathematics teachers see themselves as divinely appointed to spread the message of mathematics in ways that draw people closer to God, then they are likely to position students as precarious and to believe that the students' position as precarious is the will of God. For adherents to WCN, this position is infallible—as it is God's, not theirs.

Discussion

Some mathematics educators are likely to approach mathematics teaching and learning with reverence for mathematics as having divine origin or having divine uses. In some Christian mathematics educators, either unwittingly or with awareness, this perspective may have consequences for marginalizing students. Both dominionist and colonialist discursive practices are present in the constellation of discourses within the entanglements of Christian conceptions of divinity within mathematics education. Thus, students may be marginalized through these discursive practices, which renders them precarious or ungrievable.

I do not assert that all Christians align themselves with the discourses in the vignettes above and have marginalizing perspectives of students. But also, as noted earlier, I argue that all Christians, myself included, must seek to understand the ways our theologies are entangled with and provide space for the increased marginalizations of others (students and other humans). As I worked to construct the vignettes and reviewed the texts and articles I examined here, I found both resonating and contradictory discourses among the data in the journal articles particularly (viz., Amir & Williams, 1999; Chan & Wong, 2014; Norton, 2002; Taylor, 2013), which reflected the authentic beliefs of Christians who were mathematics educators, mathematicians, or students learning mathematics. For example, Norton (2002) summarized a mathematician's Christian perspective, "the domain of mathematics does not stand out in importance from other secular studies, and the path toward Truth in these fields is not a direct one" (p. 31). Thus, I am confident there are discourses that mathematics, mathematics education, and the divine are not entangled in the ways represented in the vignettes presented above but I wonder how often they are the main focus of writing (although one example is Heie, 2002). At the same time, I do want to be clear that the vignettes represent themes seen across the texts analyzed: even in this example presented by Norton, the participant's perspective was contrasted against the prevailing, "Philip Melanchthon's philosophy [is] that 'the study of mathematics offers a vehicle by which the human mind may transcend its restrictions and reach God' (Methuen, 1998, p. 83) [which] makes mathematician and pastor one" (Norton, 2002, p. 31). Further, not all of the views expressed about the nature of mathematics and the nature of the divine are uniquely Christian.

In this theoretical discussion, I explored some of the possible intersections of WCN ideologies within the prevailing discourses on Christians' perspectives of God, the nature of mathematics, and teaching and learning mathematics. In short, there is danger of marginalizing and harmful discourses within the entanglements of the divine and mathematics education. It is unlikely that typical approaches to equitable mathematics teacher education are sufficient to disrupt these marginalizing perspectives as they are valued, by adherents, as divinely inspired. Thus, it is important that we, as mathematics educators, continue to deepen our understanding of how theology is wound within and around mathematics education so that we can work to develop equitable classroom spaces that value the humanity of all students.

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A CRITICAL STUDY OF MARTIN'S *MATHEMATICS SUCCESS AND FAILURE AMONG AFRICAN AMERICAN YOUTH* AT 25 YEARS

UN ESTUDIO CRÍTICO DEL ÉXITO Y EL FRACASO EN MATEMÁTICAS DE MARTIN ENTRE LOS JÓVENES AFROAMERICANOS

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Written with sincere love from a colleague, collaborator, and friend, this report honors Danny Bernard Martin's scholarship by taking it seriously—not as past doctrine but as living theory. This analytical essay centers on Martin's (2000) Mathematics Success and Failure among African American Youth *and his subsequent scholarship with both appreciation and critique, recognizing that the highest form of respect for ideas is rigorous engagement with them. The essay argues that mathematics socialization—MSF's most important but least developed contribution—remains essential for understanding and addressing persistent racial inequity in mathematics education.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity, Inclusion, and Diversity; Identity and Agency

Introduction: The Necessity of Critique as Love

In memory of Dr. Khalel Hakim, a brother-scholar who left this world far too soon, and whose desire to publicly join a deep love for Black people with the possibilities of mathematics education continues to inspire.

In *Mathematics Success and Failure among African American Youth (MSF)*, Danny Bernard Martin (2000) shared that his work was both personal and professional—an unusual yet strategic admission that signaled essential features of his scholarly approach: embracing rigor without affecting objectivity, employing criticality without abandoning hope, and applying unflinching analysis animated by profound care. This was a pedagogical move. Martin was writing for and about Black children, Black families, and Black communities with the full weight of his scholarly training brought to bear on questions that mattered existentially, not merely academically.

A quarter-century later, as mathematics education faces renewed attacks on scholarship centering race and equity, as so-called "DEI" (i.e., initiatives that aim to address diversity, equity, and inclusion concerns) is transformed into a slur, and as hard-won spaces for critical examination relent to legislative pressures, the necessity of returning to Martin's foundational work has taken on fierce urgency. This essay undertakes that task—not as nostalgic commemoration but as critical engagement in the present with ideas that remain generative, frameworks that remain underutilized, and questions that remain unanswered.

Martin's style of analysis is both singular and unsparing—of deficit discourses that pathologize Black learners, of reform rhetoric that promises inclusion while maintaining exclusion, of research traditions that treat race as a demographic variable, and of systems that produce failure while blaming individuals and their communities. Yet his critique never becomes nihilistic. It emerges from and adheres to what we might call a profound love for Black people and an academic commitment to the transformative potential of mathematics education. This

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distinction matters; there exists a meaningful difference between malicious critique and beneficent critique, which demands better because of a belief that better is possible.

The Emergence of *MSF*: Context and Intervention

To begin to understand what *MSF* accomplished, we must first discuss what it entered into. The 1990s marked a significant national ferment in mathematics education policy and curriculum development. The NCTM-led standards-based reform movement reached its peak with "Mathematics for All" as its rallying cry (see Martin, 2003). At the same time, the field underwent a "social turn," with increasing focus on sociocultural factors shaping mathematical learning. Yet something crucial remained absent. The field's attention to equity rarely translated into rigorous, theoretically grounded study of racialized mathematics learning experiences. Black learners, when they appeared in literature, were typically subsumed under broad categories like "minorities" or "underserved populations." This aggregation obscured the specificity of Black experience, erasing the particular forms of racism that Black students encountered (Martin, 2009).

The "Mathematics for All" rhetoric carried both promise and limitation. It was assumed largely that access was the primary issue; if gates could be opened wider, equity would be achieved. This framing took for granted what students were being given access to. It did not interrogate the nature or normative culture of school mathematics itself, the pedagogical approaches through which it was taught, or what mathematics education did to students' identities and sense of possibility.

MSF intervened by insisting that access, while necessary, was insufficient. Martin argued that we must examine not just whether students have access, but what their experience of mathematics education *is*—how it shapes their identities, constrains or enables their agency, and affects their sense of mathematical possibility. This shift from achievement to experience represented a fundamental reframing. Where reform discourse focused on what students produced, Martin focused on what mathematics education produced in students.

Core Conceptual Contributions from *MSF*

Among *MSF's* most significant contributions was developing a theoretically grounded framework for studying race, identity, and agency in mathematics education. Race was theorized not as a demographic variable but as lived, sociohistorical experience—a social construction with real material consequences, a system of power structuring opportunity and constraint (Martin, 2009). For Black learners specifically, race could not be understood apart from the particular history of anti-Blackness in American society and education (Martin, 2013).

Identity was conceptualized as negotiated, socially (co-)constructed, and mathematics-specific. Mathematics identity encompassed beliefs about oneself (Am I a "math person"?), beliefs about mathematics (What is it for?), beliefs about one's relationship to mathematics (Does it connect to my life?), and beliefs about mathematics in society (Who does it serve?). Crucially, Martin theorized mathematical identity as being shaped by racialized experiences—that is, formed in relation to deficit discourses, curricula that rarely reflected Black lives, and pedagogies that positioned some ways of thinking as legitimate while marginalizing others.

Agency was conceptualized neither as free will nor as complete determination but as strategic action within constraint. Martin rejected both hyper-individualistic notions treating agency as simply personal choice and deterministic frameworks positioning Black students as passive victims. For Black students in mathematics education—a space structured by white

supremacy—agency often took the form of resistance, of maintaining dignity in environments that denied humanity (Martin, 2013).

The power of Martin's framework relates to its ability to articulate relationships among these concepts. Race shapes the conditions under which mathematical identities are formed. Mathematical identities enable or constrain agency. Agency operates within racialized structures but is not determined by them. All three concepts must be understood together to comprehend mathematics learning experience.

Mathematics Socialization: The Most Important but Least Developed Contribution

If *MSF* is known for one contribution above all others, it is mathematics identity. Yet this very success obscures what may be *MSF's* most important contribution: the theory of mathematics socialization. While identity became widely popular, the broader theoretical framework within which Martin situated identity—the analysis of how mathematics socialization operates at multiple levels—received comparatively little sustained attention.

The Problem: Identity Without Socialization

As identity became a major research focus, it often became detached from the structural analysis that Martin's socialization framework provided. Studies examined individual students' mathematical identities without adequately theorizing the structural forces shaping identity formation. Identity became psychologized and individualized, treated as a property of persons rather than as representing the relationship between persons and systems.

This individualization had several consequences. First, it suggested the problem was students' identities rather than systems damaging those identities. Second, it depoliticized mathematics education—if identity formation was primarily psychological, then addressing identity did not require confronting power or challenging white supremacy. Third, it disconnected identity from collective and historical dimensions that Martin emphasized.

Martin himself has critiqued how identity has been taken up, noting the tendency to treat mathematical identity as a goal in itself without interrogating what mathematics is or whether strong identification with mathematics as currently practiced serves Black students' or communities' interests. These critiques point toward what has been lost when identity research is divorced from the broader framework of mathematics socialization.

Explicating the Theory of Mathematics Socialization

Martin conceptualized mathematics socialization as the complex of experiences, messages, and interactions that shape how individuals come to understand mathematics, their relationship to it, and themselves as mathematical beings. This was not simply learning mathematics but being socialized into particular relationships with mathematics and particular understandings of mathematical possibility.

Multi-level ecological perspective. Drawing from Bronfenbrenner, Martin conceptualized socialization as operating simultaneously at multiple, nested levels. At the individual level, students experience mathematics learning directly. At the interpersonal level, family members, teachers, and peers communicate messages. At the institutional level, schools make decisions about curriculum and tracking. At the community level, neighborhood resources and collective historical memory shape understanding. At the societal level, racism, economic structures, and accumulated historical legacies create the broad context.

Dialectical relationship between structure and agency. Mathematics socialization was conceptualized as dialectical—individuals were socialized within structures, but they also resisted, reinterpreted, and reshaped socialization messages. Structures socialized powerfully through tracking systems, deficit pedagogies, and institutional practices. Yet students were not

passive recipients—they interpreted socialization messages through lenses shaped by their own experiences, community knowledge, and critical consciousness.

Temporal dimension. Socialization was not a one-time event but an ongoing process unfolding across multiple timescales. At the immediate timescale, each encounter conveyed messages. At the biographical timescale, individuals' mathematical socialization extended across lifetimes. At the sociohistorical timescale, socialization operated through collective memory and accumulated legacies—expressed as a historical arc of educational exclusion extending from slavery and reconstruction to Jim Crow segregation and contemporary resource inequalities.

Content of socialization. Students were socialized into particular understandings of what mathematics is (fixed procedures or dynamic reasoning), of themselves in relation to mathematics (capable or incapable), of who belongs in mathematics (which students are marked as talented), and of what mathematics does in society (whose interests it serves). For Black students specifically, mathematics socialization occurred in relation to broader processes of racial socialization and in contexts structured by antiblackness.

Why This Framework Is Essential Now

Mathematics education has seen twenty-five years of equity-focused reforms since *MSF*—challenging tracking and "algebra for all" policies, introducing culturally relevant curricula, and producing waves of professional development in equity pedagogy. Yet racial inequities persist. This persistence cannot be explained by a lack of effort but points toward what the mathematics socialization framework makes visible: Inequity is reproduced through systemic socialization processes operating at multiple levels simultaneously. Equity reforms that do not address fundamental socialization processes cannot fundamentally alter outcomes.

Decontextualized identity work. Much contemporary identity research treats identity as an attribute individuals possess that can be developed through appropriate interventions, proceeding as if identity formation were primarily a psychological process. The mathematics socialization framework insists on reconnecting identity to structural forces. Identity is not simply something individuals have but something negotiated between individuals and systems, shaped by institutional structures, societal narratives, accumulated histories, and community resources.

Understanding resistance and resilience. Research often documents resilience—students who succeed despite hostile environments. Yet focusing on resilience alone risks individualizing structural phenomena, suggesting the solution is helping more students become resilient rather than transforming systems requiring resilience. The mathematics socialization framework provides tools for analyzing resistance structurally—asking how students resist harmful socialization, what resources enable resistance, how resistance operates at different levels.

Intergenerational patterns. Mathematics socialization is inherently intergenerational. Parents' experiences shape how they socialize their children mathematically. Teachers' experiences as students shape their beliefs about mathematics and students. Community histories shape collective understanding of what mathematics education means. Understanding these patterns is essential for addressing persistent inequity—cycles of negative mathematical socialization reproduce themselves across generations when not interrupted.

Mathematics Socialization and the Experience Frame

The theory of mathematics socialization is intimately connected to what Martin later articulated as the experience frame—that is, the focus on what mathematics education does to students rather than what students produce. The achievement frame focuses on outcomes—e.g., test scores, grades, and course completion. The experience frame focuses on what mathematics

education does to students—how it shapes identities, affects sense of possibility, constrains or enables agency.

Mathematics socialization is the mechanism through which experience operates. It is how mathematics education shapes students—through accumulated messages, structures navigated, practices encountered. The socialization framework explains how experience is produced, how it operates at multiple levels, how it connects individual lives to collective histories and structural forces.

This connection reveals that achievement and experience are related—achievement outcomes reflect experiential processes. Focusing exclusively on achievement obscures the experiential processes producing both success and failure and the meanings those outcomes hold. Transforming mathematics education requires attending to experience and socialization, not just outcomes.

Building from *MSF*: What Has Been Overlooked

While *MSF* has been widely influential, certain ideas have received less attention than they merit. Three overlooked threads deserve particular focus:

The Relationship Between Mathematics and Liberation

MSF explicitly connected to Bob Moses's framing of mathematics literacy as civil rights issue, examining how mathematics functioned in relation to Black freedom struggles. Yet this theme—mathematics education in service of liberation—remained somewhat underdeveloped. What would mathematical liberation actually look like? How might mathematics education serve liberatory projects rather than functioning primarily as gatekeeping? These questions become particularly urgent given Martin's later argument that equity discourse is insufficient. The goal cannot be including more Black students in mathematics as currently practiced; it must be transformation toward mathematics education that serves Black liberation.

Community Knowledge and Epistemology

MSF documented community forces shaping mathematics socialization and the intelligence with which Black families engaged with mathematics education. Yet less developed was interrogation of what mathematical knowledge existed in Black communities outside school mathematics and how this knowledge might transform what counts as mathematics. How might community mathematical knowledge transform what is taught in schools? What would curriculum look like if it began from community knowledge rather than treating community as source of "real-world applications"?

Collective Identity and Political Action

MSF focused primarily on individual identity and agency—how individual students formed mathematical identities and exercised agency within constraints. Less developed were questions of collective identity and collective action. How might Black students develop collective mathematical identity? How might mathematics education support collective action—organizing to demand better resources, challenging racist policies, using mathematical analysis to expose injustice? Transformation requires collective action; how might mathematics education cultivate not just individual competence but collective capacity for political action?

Martin's Scholarship After *MSF*: Evolution and Lessons

Martin's thinking has evolved significantly since *MSF*. Three developments are particularly crucial:

From Equity to Antiblackness

Martin's 2019 paper marked an explicit break from then-contemporary equity discourse,

arguing it is insufficient while allowing the field to claim commitment to justice—despite avoiding the harder work of naming and confronting antiblackness specifically. The shift to foregrounding antiblackness insists on specificity, challenges the field's comfort with equity language, and demands that mathematics education reckon with how antiblackness structures the field. This implies that twenty-five years of equity work has been insufficient, that the goal cannot be equity within systems structured by white supremacy.

Mathematics as White Institutional Space

Martin (2013) developed the concept of mathematics education as white institutional space—a domain organized by and for white participation, in which whiteness functions as unmarked norm and Blackness is marked as other. This means mathematics education assumes white cultural norms as universal standards, positions whiteness as natural belonging, and obscures its racialized nature. Understanding mathematics education as white institutional space suggests that transformation requires not just changing who participates but reconstructing mathematics education itself.

Equity as Delusion

Martin (2019) has increasingly critiqued how equity discourse functions, suggesting equity may be delusion—a promise that cannot be kept. Equity discourse focuses on distribution within systems structured by white supremacy, which may simply distribute oppression more equally rather than eliminating it. For Black students, equity may mean being included in spaces structured against them, achieving success by suppressing identity. This critique does not mean abandoning equity work but recognizing its limitations and understanding that equity cannot be endpoint—liberation is necessary.

The Fire This Time: The Current Political Moment

Twenty-five years after *MSF*, the political context has shifted dramatically. Coordinated political organizing aimed at rolling back social progress is targeting the institutional structures that support equity efforts. State legislatures have passed laws restricting teaching about race, universities have eliminated diversity offices, and scholars face coordinated harassment campaigns.

In this hostile context, *MSF* matters more than ever. It provides an intellectual foundation for resistance—demonstrating that scholarship centering race is rigorous intellectual work with strong theoretical foundations. It models engaged scholarship that is simultaneously rigorous and accountable to communities. It shows what's at stake—documenting how mathematics education functions as system of racial sorting, how it shapes Black students' identities and opportunities, how it constrains futures. When scholarship documenting these realities is silenced, systems continue operating without scrutiny, racism continues functioning without being named, students continue being harmed without accountability.

Conclusion: Bethinking Ourselves Anew

W.E.B. Du Bois called on Black people to "bethink ourselves anew"—to reflect critically on who we had been, to imagine who we might become, to envision futures not constrained by present oppressions. Twenty-five years after *MSF*, mathematics education needs its own renewed bethinking.

What Twenty-Five Years Teaches Us

Critique is generative. *MSF's* unflinching analysis did not destroy the field but made better research possible. Love manifests through demanding better, through refusing to accept systems that harm, through telling hard truths in service of justice.

Theory matters. Conceptual rigor is not academic luxury but essential tool for understanding and transformation. Without sophisticated theory, equity work devolves to good intentions without systemic understanding, to interventions addressing symptoms while leaving causes intact.

Black specificity is non-negotiable. Understanding mathematics education requires centering the particular history, present reality, and future possibilities of Black people. Generic frameworks abstracting from race or aggregating across different racialized groups are inadequate.

The work continues. Twenty-five years after *MSF*, mathematics education still functions as gatekeeping system, still sorts along racial lines, still requires transformation. This persistence clarifies what is needed: transformation requires deeper, more sustained, more comprehensive work addressing structures, culture, and ideology simultaneously.

The Charge: For the Love of Black People

MSF issued a charge that echoes across twenty-five years: that mathematics education should serve Black liberation, that we should pursue this work for the love of Black people, that we should refuse to accept anything less than transformation toward justice.

This charge means starting from the axiom of Black brilliance—assuming Black children's mathematical capacity, seeing potential rather than deficits, creating systems that nurture. It means pursuing transformation, not inclusion—reconstructing mathematics education rather than simply opening access to unchanged systems. It means maintaining community accountability—conducting research serving Black communities' needs, producing knowledge useful for transformation. It means sustaining collective struggle—recognizing that transformation requires organized action, political mobilization, coalition-building. It means maintaining courage across the long struggle—persisting when dangerous, protecting each other, refusing to abandon commitments.

To bethink ourselves anew is to engage in radical imagination—to envision mathematics education designed for Black flourishing, to conceive possibilities beyond present constraints. *MSF* began this bethinking, showing that different mathematics education was possible, that current patterns were not inevitable, that systems producing inequity could be understood and changed.

Twenty-five years later, we continue this bethinking. We do so with benefit of scholarship *MSF* inspired, with deeper understanding of challenges we face, with clearer sense of what transformation requires. We do so in more hostile political environment but also with stronger resolve. We do so in community, in struggle, in hope—holding fast to vision of mathematics education serving Black liberation, refusing to abandon that vision regardless of obstacles, working together toward futures where Black children's mathematical brilliance is assumed and nurtured, where mathematics contributes to collective flourishing, where education serves freedom.

This is the charge *MSF* issued. This is the work it began. This is the vision it holds before us.

For the love of Black people. For the transformation of mathematics education. For the possibility of liberation.

We continue.

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CHECK-INS FOR LATIN* STUDENTS' EQUITABLE SUPPORT IN GATEWAY MATHEMATICS INSTRUCTION AT A HISPANIC-SERVING INSTITUTION

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Racialized and intersectional forces of oppression (e.g., stereotypes of ability) produce Latin undergraduates' discomfort with asking for help from instructors in gateway mathematics courses. Instruction plays a key role in reinforcing and disrupting such inequities of support. However, research about instructor-initiated support that expands Latin* mathematics students' access to help is limited. The present study adds such knowledge by examining student experiences of check-in practices in gateway mathematics instruction at a Hispanic-Serving Institution (HSI). We found that cultural influences, including the value of orgullo (protecting one's pride), added to Latin* students' help-seeking tensions due to shame associated with their bids for support. Mathematics instructors' check-ins were experienced as racially-conscious care that fostered equitable support aligned with campuswide efforts for serving Latin* students.*

Keywords: Culture and Asset-Based Pedagogies; Equity, Inclusion, and Diversity; Instructional Activities and Practices; Undergraduate Education

Background

Latin* undergraduates experience racial and intersectional oppression in mathematics courses that are gateways for graduation and STEM degrees, such as calculus, college algebra, and introduction to proof (Battey et al., 2022; Bhattacharya et al., 2020; Brown, 2018; Leyva, 2016, 2018, 2024; Rios, 2024). Structural inequities and ideological forces tied to racism and interlocking power systems (e.g., patriarchy, linguicism)—e.g., stereotypes of mathematical ability, isolation due to underrepresented identities, dominance of the English language in U.S. education—shape Latin* students' wariness of asking questions to instructors (Convertino et al., 2023; Leyva & Joseph, 2023; MacArthur & Dobie, 2023; Oppland-Cordell, 2014). Such constraints on help-seeking produce a racialized distribution of support. Considering how gateway mathematics success impacts long-term academic outcomes (e.g., STEM major persistence, mathematics identities), equitable access to instructor support is consequential.

Research on Latin* undergraduates' academic help-seeking suggests that cultural norms figure into tensions about asking for support in gateway mathematics. Cultural values of collectivism and interdependence give rise to concerns that others' negative judgment may jeopardize relationships. Such cultural influences like *orgullo* (protecting one's pride) and the *el qué dirán* (concerns about others' opinions) bring Latin* undergraduates, especially first-generation college attendees, to work through struggles on their own (Cabrera et al., 2016; Chang et al., 2020; Payne et al., 2023). Additional cultural influences are cisheteropatriarchal gender ideologies of machismo and marianismo, which valorize men's assertiveness and women's submissiveness (Estrada & Jimenez, 2018; Liang et al., 2017). Seeking support, as a result, conflicts with gendered expectations that Latin* men avoid displaying vulnerability and Latin* women remain quiet. Thus, Latin* students' cultural values as well as their realities of racial and intersectional oppression may present barriers for seeking help in mathematics.

Instruction can exacerbate or alleviate Latin* students' racially- and culturally-mediated concerns about asking for help in and beyond gateway mathematics. Dismissing bids for support

as well as showing limited consciousness for Latin* students' racialized realities and cultural backgrounds fuel help-seeking tensions (Kennedy & Hernandez, 2025; Leyva, Amman et al., 2022). Such instruction falls short in cultivating a safe, culturally-responsive learning context that expands access to support. However, instruction that honors Latin* students' contributions and cultural norms promotes a sense of belonging that mitigates risks in exhibiting struggle and asking questions (Convertino et al., 2023; Oppland-Cordell, 2014; Payne et al., 2023). Our prior work has shown that affirming Latin* cultural values of collectivism (e.g., building peer connections in groupwork), resisting stereotypes of ability, and framing the Spanish language as a resource builds a community for equitable classroom participation in mathematics (e.g., Leyva et al., 2025). Research on such instructional features for Latin* student support is needed.

Considering traditional norms in undergraduate classrooms that students initiate the process of receiving help, support for Latin* students in gateway mathematics has largely been explored in relation to help-seeking. Tensions with seeking support among Latin* students point to the value of research that examines instructor-initiated help as a disruption of such norms for equitable access to content. Such inquiry is crucial in gateway mathematics where exclusionary logics of the discipline, including innateness of ability and sorting out students not "cut out" for STEM, frame instruction in ways that leave help-seeking vulnerabilities unchallenged (Ellis et al., 2014; Leyva & Alley, 2021; Leyva, McNeill et al., 2021; Leyva, Quea et al., 2021; Perez et al., 2024). To address this need, our study explores how instructors' outreach to students, or "check-ins," impacted equitable support opportunities. Check-ins ensure that Latin* students receive academic and socioemotional support to foster their success in gateway mathematics courses. Examples of check-in practices explored through the present analysis include sending an e-mail message, touching base immediately after class, and scheduling office hour appointments. We address the following research question: *How does the absence and presence of check-ins influence Latin* students' experiences of equitable support in gateway mathematics instruction?*

Conceptual Model: Instructional Servingness in Undergraduate Mathematics

Servingness refers to culturally-responsive practices that enhance connections between Latin* student enrollment and equitable educational opportunities at Hispanic-Serving Institutions (HSIs; Garcia et al., 2019). U.S. colleges and universities with at least 25% Latin* enrollment receive a federal HSI designation and funding to support students' academic success. HSIs vary in the extent to which their campus structures and institutional practices advance servingness (Garcia, 2019; Núñez et al., 2015). Despite such variation, HSIs have a shared reality that Latin* undergraduates more frequently experience servingness outside of STEM departmental contexts like gateway mathematics courses (Herrera et al., 2022; Leyva, 2022; Leyva, Mitchell et al., 2022; Leyva et al., 2025; Shi et al., 2025). Limited research, however, has explored relationships between STEM instruction and campuswide servingness (Ro et al., 2024). Such inquiry is needed to strengthen continuity of servingness in and outside of STEM spaces. Our study fills this gap by examining opportunities to foster equitable support resonant with campuswide servingness through check-ins in gateway mathematics instruction.

Building on the general framework of servingness (Garcia et al., 2019), our team developed a model of *instructional servingness in mathematics* (Leyva et al., 2025). Figure 1 is a visual adaptation of our model. It accounts for three interrelated levels of influence on instructional servingness—society at large, HSI campuses, and the mathematics discipline. At the societal level (Figure 1a), the model considers Latin* culture and white cisheteropatriarchy (i.e., a system of oppression at the juncture of white supremacy, racism, cisheteronormativity, and patriarchy) that shape social realities in and out of mathematics. These influences encompass ideological

forces (e.g., stereotypes of ability, gendered cultural norms) and structural inequities (e.g., access to high-quality P-12 education) that give rise to Latin* undergraduates' help-seeking tensions.

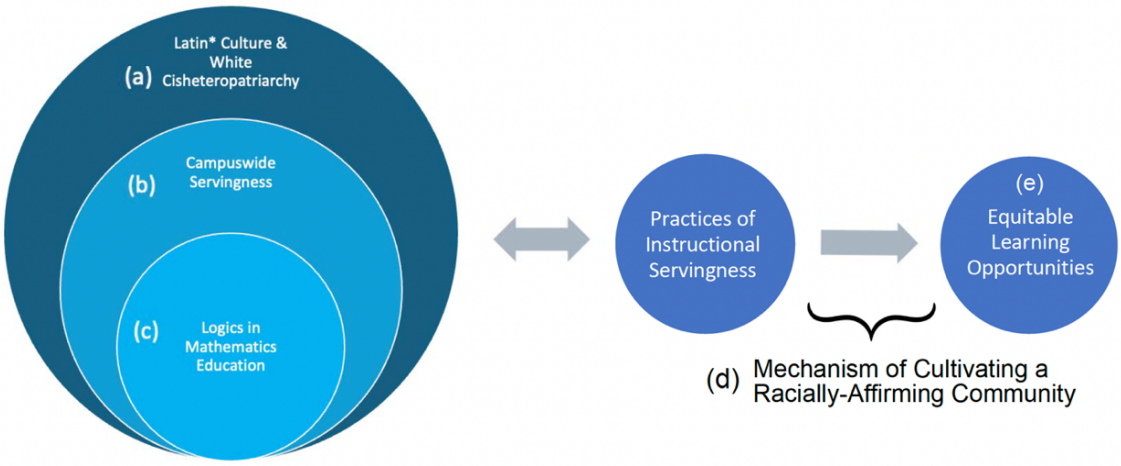


Figure 1: Model of Instructional Servingness in Mathematics (Leyva et al., 2025)

At the campus level (Figure 1b), the conceptual model attends to HSI structures that foster servingness by disrupting oppressive societal influences for equitable educational opportunities. Sources of campuswide servingness outside of mathematics classrooms include DEI offices and initiatives, student support services, and organizations for Latin* racial affinity in and out of STEM. Since these HSI campus structures extend race-conscious and culturally-responsive support, they remove pressures and stigma related to seeking help among Latin* students.

The model also accounts for influences of the mathematics discipline (Figure 1c). Namely, it considers dominant logics—e.g., individualism, innateness of ability, social neutrality (Battey & Marshall, 2023; Leyva, McNeill et al., 2021)—shaping instruction that contributes to Latin* students' vulnerability with exhibiting struggle. Disruption of these logics, in tandem with resisting white cisheteropatriarchy and embracing Latin* cultural values, builds a collaborative and identity-affirming environment in gateway mathematics classrooms. The model describes this as a mechanism of instructional servingness for *cultivating a racially-affirming community* (Figure 1d) to promote equitable learning opportunities, including instructor support (Figure 1e). By alleviating help-seeking tensions and facilitating access to content, these practices ensure continuity of campuswide servingness in mathematics. The model guided our inquiry of check-ins as practices of instructional servingness for Latin* students' equitable support opportunities.

Methods

Study Context & Data Collection

Our study comes from a larger project about instructors' and Latin* undergraduates' views of instructional servingness in gateway mathematics. The HSI site, Sonoma State University (SSU), is a medium-sized, public university in the western U.S. At the start of data collection, SSU undergraduates were approximately 45% white, 40% Latin*, and 15% some other race. Gateway mathematics instructors were predominantly white and commonly used active learning strategies (e.g., groupwork, whole-class discussions). The present analysis focuses on Latin* students'

views on servingness. We recruited 38 students from gateway mathematics courses, including multiple student voices in specific course sections and across ethnoracial/gender identities.

The research team recruited participants and collected data over two years. Findings reported later feature Year 1 participants, so we only discuss Year 1 data collection below. Year 2 data collection is presented elsewhere (e.g., Leyva et al., 2024). A total of 24 students in Year 1 (12 in fall and 12 in spring) kept a journal of instructional experiences—a log of written descriptions and reflections about events that expanded or limited support. We collected 114 journal entries.

Near the end of each semester, Year 1 participants completed a 90-minute, semi-structured, and virtual individual interview that was audiotaped and transcribed. The first half of interviews explored participants' identities and backgrounds, in addition to their views on being served as Latin* students in mathematics and across the SSU campus. In the second half, we presented two written prompts of instructional events, which each addressed a theme of instructional servingness across all instructor and student participants' journaling. Our team developed the prompts as syntheses of individual journal entries. The first prompt about classroom participation was written from a student perspective, and we report on an analysis largely focused on responses to this prompt elsewhere (Leyva et al., 2025). The second prompt about support (quoted below) was written from an instructor perspective and featured check-ins with a student who seemed to be struggling. Responses to this prompt were central in our present analysis.

> A student did not submit multiple homework assignments and missed a few weeks of class. I sent an e-mail message to check in with them and encouraged them to visit my office hours if they needed any support. The student was unresponsive. I approached the student during a class meeting and asked if they could stay after class to check in with me. The student and I worked together in developing a plan to help them catch up with the course material.

For each prompt, participants were asked about: (i) similarities to and differences from their experiences, (ii) oppressive and supportive aspects for Latin* students, (iii) connections to campuswide servingness at SSU, and (iv) suggestions to enhance instructional servingness.

Data Analysis & Positionality

The research team used a grounded theory approach for data analysis (Corbin & Strauss, 2014). First, two team members independently coded each student participant's data set. With the model of instructional servingness as a guiding lens, we flagged content with open codes that addressed *campuswide servingness* (Figure 1b) and *logics in mathematics education* (Figure 1c). Another set of open codes flagged *instructional practices* that Latin* students deemed oppressive or supportive, including absence and presence of check-ins (the central unit of our analysis).

Axial codes identified relationships between open codes to convey how mathematics instruction constrained or promoted continuity of campuswide servingness, which translated into *(in)equitable learning opportunities* like instructor support (Figure 1e). We used selective codes to document societal and cultural forces that texture open-axial coding relationships. These codes flagged influences of *Latin* culture* and *white cisheteropatriarchy* (Figure 1a) to reveal how instruction restricted or facilitated creation of a racially-affirming community—the mechanism of instructional servingness for equitable and culturally-responsive learning (Figure 1d).

Paired coders for each participant's data set exchanged their independent analyses to identify themes that provided theoretical explanations for how instructional servingness was limited and promoted (Corbin & Strauss, 2014). In team meetings, we shared participant-specific themes for a cross-case analysis of Latin* students' experiences. Three strategies sharpened validity and trustworthiness of our analysis: (i) prolonged engagement through multi-semester data collection (ii) triangulating and layering data sources, and (iii) ongoing debriefs and exchanges of analytical

memos. Three team members (this research report's authors) led the present line of inquiry based on the coded data and cross-cutting themes about instructor support for equitable learning. More specifically, we addressed our research question exploring the role of check-in practices in Latin* students' perceived opportunities for support likened to campuswide servingness at SSU.

For this report, we present findings from our analysis of data for 21 out of the 38 participants. We focused on these participants because they were enrolled in courses with instructors who taught most of our sample. This sampling also ensured representation across all course types in recruitment. Most focal participants, much like the broader SSU undergraduate population, are first-generation college students, come from low-income backgrounds, have Mexican heritage, and/or identify as cisgender women. Our analytical sample, thus, spoke to common Latin* student realities at SSU. At the same time, the sample had representation for other ethnoracial identities (e.g., Salvadoran) as well as cisgender men and nonbinary students to capture variation in Latin* student experiences. Ten of the 21 participants were interested in STEM majors.

We pursued the study with consciousness for our individual and collective positionality. Diversity across team members' identities (e.g., race, gender), professional roles (e.g., faculty, undergraduates), and institutional affiliations in and out of SSU provided multiple perspectives for data collection and analysis. Most of us identify as Latin* and/or first-generation college students, and SSU researchers (both undergraduate and faculty) make up nearly half of the team. We leveraged these insider perspectives on participant experiences by having at least one Latin* researcher present for each interview and paired coding assignment. An SSU researcher was included in each coding pair analyzing de-identified versions of study data. Our collaboration, thus, minimized risks of distorting Latin* participants' experiences during analyses due to team members' social proximity and distance. As a team, we adopted an asset-based orientation that positioned Latin* students' realities as knowledge for enhancing instructional servingness.

Findings

Our analysis uncovered two types of instructional realities. Two-thirds of the sample (n=14) reported supportive experiences with varied use of check-ins. The remaining seven participants experienced limited support with check-ins being largely absent. Below, we present a pair of illustrative cases for each type of instructional reality. The case pairs, respectively, depict how absence and presence of check-ins impacted Latin* students' opportunities for support with racially-conscious care. We conclude with a cross-case analysis using the model of instructional servingness (Figure 1), which shows how check-ins cultivated a racially-affirming community as a mechanism of instructional servingness for equitable support in gateway mathematics.

Absence of Check-Ins Limiting Equitable Support

This section highlights cases from Joana (Mexican woman, she/her, non-STEM major) and Monica (Latina nonbinary person, they/them, STEM major)[1] who reported limited support in instruction with an absence of check-ins. Joana saw her instructor as unaware that many Latin* students like her navigate conflicts between coursework and personal responsibilities of family support (e.g., part-time jobs, sibling care). She saw the dominant logic of STEM being "neutral" shaping her experience of limited racial consciousness in mathematics instruction, thus making her feel unseen as a Latina, "The professor being like, 'I recognize that this student is a Latina. Maybe she's going through something.' No, not at all… I'm just treated like any other student… It is a STEM class… We don't talk about that [identities]." She reasoned that check-ins would

[1] Course details, types of STEM majors, underrepresented ethnoracial identities among Latin* students, and mathematics instructors' racial and gender identities are not specified to avoid participant identifiability.

exhibit consciousness for her circumstances tied to the Latin* cultural value of *familismo* (family loyalty) by offering support to manage conflicts. Unable to attend tutoring due to personal commitments, Joana felt on her own to overcome struggles in mathematics without check-ins. Shame about help-seeking from the Latin* cultural value of orgullo (protecting one's pride), in tandem with a lack of check-ins, contributed to Joana's pressures of self-reliance in mathematics.

> There are heaps and heaps of Latino students that struggle with this [mathematics]… We don't get tutoring… because we have busy lives at home… [and] have to get jobs… We are shameful when asking for help because… [it] can sometimes be seen as weak for us, at least in the Latino mindset… You think, "I don't want to be a burden. I should be able to do this."

Joana's experience suggests how check-ins, which were perceived as entry points for providing racially-conscious support, can alleviate Latin* students' help-seeking tensions. She saw the destigmatization of seeking support as a need in gateway mathematics instruction, "If it was made more known that there's nothing wrong in asking for help and it's not a shameful thing… more Latino students would ask for help." Check-ins, thus, were viewed as building a supportive learning environment for mathematical success that was responsive to Latin* cultural values.

Monica also felt help-seeking tensions in their classroom with an instructor who did not use check-ins and was dismissive toward students' questions. As someone who "felt really behind" in mathematics after high school teachers did not meet their learning needs, Monica was wary that seeking instructor support could result in negative judgment for lacking foundational knowledge. Stereotypes of mathematical ability figured into these concerns with asking for help.

> He [mathematics instructor] was talking down to me a bit like, "Why don't you get this already?"… [His] condescending tone... [was] like, "You're not worth my time."… It [race] definitely could have contributed because especially just like my last name… it's really easy for people to… know I'm Latina… And [Monica] is also a very common female name.

Monica grappled with uncertainty about racialized and gendered undertones in their instructor's response during office hours, which made them avoid future help-seeking in mathematics. This instructional behavior reflects workings of a dominant logic that quickly understanding and completing work is an indicator of innate mathematical ability, in addition to a seeming lack of consciousness for Latin* students' racial inequities (e.g., limited access to high-quality education before college). These influences shaped Monica's vulnerability with seeking help in gateway mathematics, where knowledge gaps can be interpreted as deficits in students' potential for majoring in STEM (Leyva, Quea et al., 2021). Such vulnerability, as depicted in Monica's case, can be racialized and gendered for Latin* STEM majors navigating stereotypes of ability. Monica journaled how a lack of check-ins produces inequities of support that negatively impact Latin* students' STEM persistence, "We [Latin* students] already are underrepresented… and having teachers who aren't willing to help or are often unavailable causes for students to lose their interest in STEM… When they don't offer help [or] reach out… the student[s] must fend for themselves." Thus, Monica saw check-ins as defusing help-seeking tensions and promoting equitable support for mathematical struggles to nurture their success as a Latina STEM major.

Presence of Check-Ins Fostering Equitable Support

This section reports insights from two cases of supportive instructional experiences with the presence of check-ins. These cases came from Frank (Mexican man, he/him, STEM major) and Ofelia (Latina woman, she/her, non-STEM major). Frank's instructor used office hour meetings and e-mail to check in with students who seemed to be struggling. These check-ins, which Frank saw as "the professor really car[ing] about his students," shaped positive instructor-student

relationships. Frank perceived check-ins with Latin* students reflecting consciousness for his and his Latin* peers' responsibilities of family support that conflicted with time-intensive work in mathematics, "We [Latin* students] have outside problems like work… [and] helping the family… [Mathematics] requires a lot of time to submit assignments, to review problems, and to study… They [mathematics instructors] would see us like, we're going through a lot of stuff." Check-ins, therefore, were experienced as racially-conscious care for Latin* students' success.

Instructor care through check-ins motivated Frank to frequently ask for help after class. When describing servingness across the SSU campus, Frank raised his instructor's advice to seek campuswide tutoring and attend office hours, which he often did, "Encourag[ing] a lot of their students to go to…tutorings and their own office hours… If we have any more questions… we go to more office hours." Frank's instructor ensured Latin* students like him received help in and out of the course, thus cultivating a network of support that extended beyond the classroom. With Frank describing mathematics as one of the "very challenging" requirements for his STEM major, the instructor's supportive practices arguably were positive influences on his long-term academic trajectory at SSU. Unlike Joana's and Monica's cases, Frank experienced racially-conscious care via check-ins that initiated provisions of academic support for Latin* students, which facilitated his help-seeking readiness and campus connections for mathematical success.

Ofelia similarly reported supportive instruction that built a classroom community invested in her academically and personally. Her instructor's check-ins included office hour meetings with individual students as well as a peer accountability system for students to ensure each other's well-being and collaborate on groupwork after class, "She'll tell a student, 'Hey, reach out to your classmates and see what's going on'… She's [also] like, 'I encourage you to meet with your group… You guys could be in here working on it or go to the library.'" Ofelia viewed these check-ins reflecting racial consciousness for Latin* students' needs in light of how the instructor commented on limitations of servingness at SSU, "She said, 'In fact, Sonoma isn't really a Hispanic-serving college. They just say that… They are working on bettering." Such support through check-ins made mathematics less of a vulnerable experience as further described below.

Instructor support alleviated Ofelia's help-seeking tensions arising from Latin* cultural values, including orgullo and gendered scripts of behavior, "Pride is a very big thing [in Latin* culture] also tied in with the machismo… Women are very silent [and] don't voice what's going on. Even with men, you're supposed to be this tough person." Racially-conscious care via check-ins defused Ofelia's pressures of remaining silent and working through struggles alone. For instance, such support shaped Ofelia's comfort with attending office hours where she shared how on-campus racism and homophobia affected her mental health and assignment progress.

> She [instructor] noticed that I was kind of off and she was like, "Hey, you haven't done your work. What is going on?… I could help you. … If you need more time, just tell me."… At first, I was like, 'I'm not going [to office hours]'… [It was] a pride thing. If I go, it shows that I don't understand this and I'm dumb… It was a struggle with eating up my pride and going.

This check-in experience ultimately brought Ofelia to view office hours as a "safe space" where her struggles as a queer Latina were seen and accommodated to succeed. Her instructor's support disrupted the logic of mathematics as disconnected from social issues, thus enabling Ofelia to negotiate identity-related matters and mathematics coursework. By cultivating a classroom community that destigmatized help-seeking through check-ins (e.g., peer accountability norms, office hour meetings), the instructor promoted equitable support

opportunities for Latin* students like Ofelia that compensated for limitations of servingness across the broader SSU campus.

Cross-Case Analysis

The two case pairs portray how check-ins were perceived as gateway mathematics instructors demonstrating racially-conscious care that expanded support for Latin* students. Check-ins were responsive to the three levels of influence for instructional servingness (Figure 1). At the societal level (Figure 1a), a void of check-ins in Joana's and Monica's cases left them grappling with culturally-mediated tensions for help-seeking as well as wariness of judgment due to stereotypes and educational inequities. Instructor consciousness for Latin* student realities, including cultural values (e.g., familismo) and systemic racism (e.g., lack of campus resources), was seen as guiding support via check-ins across Frank's and Ofelia's cases. At the campus level (Figure 1b), check-ins allowed instructors to initiate the process of Latin* students receiving help, thus resembling campuswide servingness from SSU programs and offices that reached out to provide support as detailed elsewhere (Leyva et al., 2025). Check-ins relieved Latin* students of help-seeking pressures to ensure support for their mathematical success. At the discipline level (Figure 1c), check-ins resisted logics of social neutrality, individualism, and innate ability in mathematics education that preclude an equitable distribution of instructor support in classrooms.

With campuswide servingness connecting Latin* students to staff and peers who extended help for their success at SSU (Leyva et al., 2024), similar support through check-ins across Frank's and Ofelia's cases fostered collectivism by building support networks for mathematics learning. Such networks alleviated help-seeking tensions and embraced Latin* students' cultural value of communal support, thereby creating a racially-affirming community as a mechanism of instructional servingness (Figure 1d). Check-ins experienced as rooted in racial consciousness of Latin* students' social realities contributed to this mechanism for equitable support (Figure 1e).

Discussion & Implications for Future Research

Our report's findings make a threefold contribution. First, we extend research on racial equity in gateway mathematics instruction, which has articulated a general vision of equitable support with limited insights on practices for specific populations. We fill this gap by capturing how check-ins can defuse Latin* students' culturally- and racially-mediated pressures about seeking academic assistance. Second, this report's insights on check-ins add a novel dimension to studying equitable support in gateway mathematics that has focused on instructor responses to students' help-seeking. Our analysis distinctly portrays how instructor-initiated help via check-ins that exhibit racially-conscious care disrupts norms in undergraduate classrooms reproducing inequities of support. Check-ins can relieve Latin* students from needing to self-advocate for support, all while resisting dominant logics in mathematics education (e.g., individualism) and racialized forces (e.g., stereotyping) that exacerbate help-seeking vulnerability. Third, we respond to calls for exploring connections between Latin* students' experiences in gateway mathematics classrooms and broader HSI campus settings (Convertino et al., 2023; Ro et al., 2024). Our findings show how instructor support through check-in practices fostered continuity of racial equity from campuswide structures at SSU for instructional servingness in mathematics.

For future research, studies can explore (mis)alignment between instructor intent with check-ins and Latin* student experiences. This work can uncover how racial consciousness levels shape check-in approaches and the extent to which students view them as fostering equitable support (Gutiérrez et al., 2025; McNeill et al., 2022). Studies can also extend our analysis by examining how variation in HSI settings' campus resources and Latin* student demographics influence check-ins for instructional servingness. In addition, intersectional analyses of check-ins that

more explicitly consider Latin* students' other identities (e.g., disability) can further nuance insights on servingness in gateway mathematics. Such future lines of inquiry will shed light on the multiplicity of instructional servingness across HSI contexts and within the Latin* diaspora.

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CONCEPTIONS OF EQUITY AMONG DISTRICT PERSONNEL THROUGH A JUSTICE LENS

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We analyzed interviews conducted with school district personnel in the U.S. state of Missouri, in which participants were asked about equity-related challenges both in general and specific to mathematics. Applying Fraser's framework for justice, we analyzed each transcript for what individuals say needs to be made equitable (redistribution, recognition, or representation); who they describe disadvantaged by inequality in terms of race, class, disability, sexual orientation and gender identity, or emerging multilingualism; and what kind of change is necessary (transformative or merely affirmative). Results revealed variation in equity conceptions depending on the mathematics specificity for all three foci. We found that there were equity discourses present that did not penetrate the boundary of mathematics. Most striking is that almost all conceptions of mathematics-specific equity were limited to redistribution.

Keywords: Equity, Inclusion, and Diversity; Social Justice

Through its decades-long trajectory from the margins of scholarly interest, with only a relative few calling for a greater focus on it (e.g., Matthews, 1983; Secada et al., 1995), to a mainstream presence worthy of critiques from sympathizers within the field (e.g., Bullock, 2024; Chen, 2024), "equity" in mathematics education has been defined and pursued in multiple ways. Each is rooted in a set of logics that may vary in articulations of the goals of schooling (and school mathematics), underlying theories (e.g., from theories of learning to critical theories), and, consequently, emphasis on things like diversity, history, and justice. Among those whose definitions are oriented by more critical perspectives, a common concern is whether increased attention to equity in the field has had any effect beyond the surface, whether the status and school mathematics experiences of the historically marginalized have materially changed (Martin, 2019).

Some scholars have conjectured that equity ideas and practices that are unlikely to disrupt current power structures are more readily adopted than those that might threaten hierarchies organized by race, class, or other social dimensions. For example, in studies of mathematics teacher professional development efforts, Rubel (2017) as well as Wilson et al. (2019) observed teachers successfully employing equity-focused instructional practices that increased students' access to and success with important mathematics, but fewer instances of teachers' use of practices that were attentive to students' identities or power dynamics in the classroom—a distinction that Gutiérrez (2007) characterized as "dominant" vs. "critical." In a study of district policymaking in two different communities, Nguyen (2024) similarly found a presence of dominant equity ideas and an absence of critical equity ideas in leaders' discourses around mathematics education policy. However, given her interest in where and how the "boundary" of

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mathematics education policy is framed, Nguyen also studied leaders' discourses beyond mathematics, where she did observe the presence of more critical equity ideas—ideas that Nguyen argued "did not penetrate mathematics" in part because of the ways that school mathematics was discursively constructed (p. 503).

In the present analysis we sought to more systematically examine equity discourses among school personnel both within and beyond school mathematics to gain insights into the ways and reasons that more critical discourses do not penetrate the boundary of school mathematics. To do so, we drew from extant work to develop an expanded framework rooted in Fraser's (2005) conceptualization of justice and specified in ways applicable to both education in general and mathematics education specifically.

Theoretical Perspectives and Literature Review

Attention to Discourse

The ways in which school district leaders make sense of and respond to challenges is consequential to maintaining or transforming education. Leaders' discourses—"language use around a particular socially and historically situated topic, which… corresponds to a particular position and perspective"—about their challenges reflect and shape whether and how they implement initiatives and policies (Bertrand et al., 2015, p. 3). Informed by critical discourse analysis (CDA) (Rogers, 2011; Fairclough, 2015), we understand discourses employed by district leaders—who are in positions of leadership tasked with writing and implementing educational policies—as both reflecting and constructing power relations (Anderson & Mungal, 2015).

Discourses cannot be understood apart from the social contexts in which they circulate, and the ways in which people employ discourses are influenced by broader ideologies, including those related to race and class (Fairclough, 2015). That is, discourses are not neutral; they compete for power and recognition (Fairclough, 2015; Rogers, 2011; Gee, 2014). CDA helps to reveal the dominant ideologies masked by seemingly neutral language, policies, and practices to explore the ways in which discourses gain power.

Orientation to Equity

Within that view of discourse, we understand "equity" to be, as Apple (1995) characterized it, a "sliding signifier"—a term not with "essential meaning," but "defined by [its] use in real social situations with real relations of power" (p. 335). An examination of equity discourses among school district personnel therefore requires a sufficiently expansive framework to capture the various conceptions of equity—both in general and in ways specific to school mathematics—that might be circulating. In our view, it must account for *what* individuals say needs to be made equitable; *who* they describe as currently, historically, and/or potentially disadvantaged by inequality; and *what kind of change* is necessary to achieve equity.

Initially, we tried to apply recent frameworks for characterizing conceptions of equity among school district personnel, including Roegman et al.'s (2022) conceptions of equity among school principals; Farrell et al.'s (2023) conceptions of equity among research-practice partnerships; and the National Academies of Sciences, Engineering, and Medicine's (NASEM; 2024) five frames for decision-making in STEM education. Each of these had merits, but none was sufficient for our needs—either because it was not applicable to both general and mathematics education or because it was not helpful in distinguishing different kinds of proposed changes.

Ultimately, we found resonance in Fraser's (1995, 2005, 2009) three-dimensional conceptualization of remedying injustice (or, *what* needs to be made equitable): redistribution, recognition, and representation. Redistribution refers to the distribution of goods in response to

socioeconomic injustice. Recognition attends to "existing identities that are currently undervalued" in response to cultural or symbolic injustices such as cultural domination, non-recognition and/or disrespect (Fraser, 1995, p. 73). Representation foregrounds "procedures that structure public processes of contestation" in response to political injustices such as the exclusion of minoritized individuals from decision-making processes and/or the maintenance of hierarchies privileging individuals belonging to dominant social groups (Fraser, 2009, p. 17).

According to Fraser, remedies to injustice are also concerned with *who* is or has been affected by the injustice and/or to whom justice should apply. In addition to describing *what* needs to be made equitable and *for whom* inequity should be addressed, Fraser articulated two responses to injustice (or, in our terms, *kinds of change*). *Affirmative* responses to injustice reaffirm the differences between social groups (e.g., multiculturalism). *Transformative* responses acknowledge that differences are socially constructed and attend to the structures that draw distinctions between social groups (e.g., mechanisms of tracking that create a hierarchy of ability).

In recent years, Fraser's remedies of injustice have been increasingly applied in mathematics education research (Vithal et al., 2024). For instance, Fraser's tenets have been employed to study (a) patterns of marginalization for refugee students across multiple levels of education (recognition) (Keddie, 2012), (b) disparities in learning opportunities due to inequitable resource allocation (redistribution) (Taylor et al., 2024), and (c) mathematics teachers' perspectives on "*who* decides on *what* schooling for *whom*, and *how*" (representation) (Meaney et al., 2022, p. 560). The present study adds to this body of literature, with attention to general and mathematics-specific contexts in the discourses of various educational constituents.

Attending to Mathematics

Despite increased attention to equity, mathematics education reform has been characterized by enduring inequality and the largely unchanging nature of the status and experiences of the historically marginalized. Some have suggested school mathematics may be especially aligned with hegemonic discourses and the ideologies undergirding them (Llewellyn & Mendick, 2011) and, thus, may resist meaningful change, protecting the interests of the dominant class (Martin, 2015). Others have pointed to aspects frequently attributed to mathematics itself (e.g., neutrality, numbers, objectivity) and posit that those discourses, rooted in dominant ideologies of neoliberalism and neoconservatism, act as a barrier around school mathematics, leaving it immune/untouched (e.g., Kinder et al., 2024). Still others suggest that mathematics itself is to blame, pointing to the inextricable role of mathematics in capitalism (Apple, 2000) and white supremacy (Battey & Leyva, 2016; Stinson, 2017). This warrants attention to school mathematics itself. Given Nguyen's (2024) suggestion that the discursive construction of school mathematics prevents penetration of critical discourses in school mathematics, there is reason to believe we should attend to discipline-specific conceptions of equity.

To this end, in our analysis, we are concerned with the conceptions of equity among those designing and implementing equity initiatives. In particular, we are concerned with how those conceptions differed when talking about school mathematics. Our analysis was oriented by the following research question: What are the conceptions of equity among school district personnel, and how do those conceptions vary across descriptions of mathematics-specific and general equity initiatives/concerns?

Methods

Data Collection

To answer our question, we analysed interviews conducted during the 2023-2024 academic year as part of an effort to understand equity-related challenges, goals, and initiatives in rural school districts. We partnered with a single rural district in Missouri, a state in the central region of the U.S. The district is situated in a community with approximately 20,000 residents, 80% of whom are white (U.S. Census Bureau, 2020).

For the current analysis, from the 38 semi-structured, 45-minute interviews conducted in the larger project, we identified those that mentioned both (a) general equity challenges and (b) equity challenges specific to school mathematics. This resulted in 12 interviews, including two district leaders, eight school leaders—two of whom were interviewed together, accounting for a thirteenth participant—and three teachers.

Data Analysis

We began analysis by organizing interview transcripts into stanzas (Gee, 2014) and identifying stanzas in which participants described an equity-related challenge. This resulted in 145 stanzas, with 10.75 stanzas, on average, in each interview (range 4-18).

Type of Discourse	Redistribution	Recognition	Representation
Transformative	(a) Identifies structural barriers to an equal distribution of goods and/or (b) promotes initiatives that would shift such distribution.	(a) Identifies sources that promote narratives of "normality" (e.g., race, heteronormativity, etc.) that contribute to maintaining higher status and privilege for some groups and/or (b) employs narratives that suggest the social construction of identity (e.g., gender as a construct, racism as foundational to race)	(a) Identifies structures or practices that maintain hierarchies of power that elevate some voices and limit others and/or (b) promotes processes that would shift the current power structure
Affirmative	Acknowledges differences in opportunity *between social groups*; promotes equal access, without attending to structural barriers	Recognizes patterns of exclusion that contribute to maintaining higher status and privilege for some groups, but falls short of acknowledging the sources contributing to the injustice	Recognizes the need for voices of minoritized social groups to be included in the decision making of educational issues
Evasive	Does not acknowledge an injustice in the distribution of goods; raises no concerns about the status quo	Fails to attend to the identities of minoritized social groups and the injustices and/or differences associated with those identities	Does not acknowledge an injustice or inequality in decision-making power or processes
Anti-equity	Promotes/agrees with ideas or actions that contribute to an unequal distribution of goods	Promotes/agrees with ideas or actions that contribute to an unequal recognition of social identities	Promotes/agrees with ideas or actions that contribute to an unequitable distribution of decision-making power

Note: The codes were employed to characterize the participants' discourses within any named social groups (i.e., race, class, gender, LGBTQIA+, ability, multilingual status, personality/behavior)

Figure 1: Analytical Framework

We coded each stanza for (1) the named social group(s) (e.g., race, class, gender, sexual identity, ability); (2) whether each challenge the participant named was within or beyond school mathematics; (3) the dimension of the named injustice (i.e., maldistribution, misrecognition, misrepresentation); and (4) the type of discourse. Within the first of those, named social group(s), we separated responses regarding gender into two categories—those that constructed a

gender binary (e.g., comparing girls' opportunities to those of boys) and those that concerned gender non-conforming students, which we combined with sexual identity ("LGBTQIA+"). For the final code, we identified four "types" of discourse: those aligned Fraser's (1995) "transformative" and "affirmative" approaches to justice, those that were overtly "anti-equity" (NASEM, 2024), and a fourth type resulting from our analysis, descriptions that were "evasive," where the participant did not raise concerns about the status quo with respect to injustice or inequality (Bonilla-Silva, 2006; Annamma et al., 2017). In Figure 1 we define each type with respect to each of Fraser's remedies to injustice. If we determined the participant's equity discourse was not a response to an injustice, but the participant still named an equity challenge, we coded the stanza as a "response to diversity." Using the framework in Figure 1, interviews were coded independently by at least two authors. Discrepancies were resolved collaboratively until agreement was attained. We compiled our analyses to identify themes in the participants' discourses across social constructs and role groups.

Findings

Figure 2 provides a summary of the results of our analysis. Perhaps unsurprisingly, discussion of equity was roughly four times more frequent for "non-math" topics than for those specific to mathematics. But our analysis revealed considerable variation in participants' equity conceptions depending on the mathematics specificity. This was the case with respect to all three of our foci: the dimension of injustice; the named social group(s); and the type of discourse (or kind of change promoted).

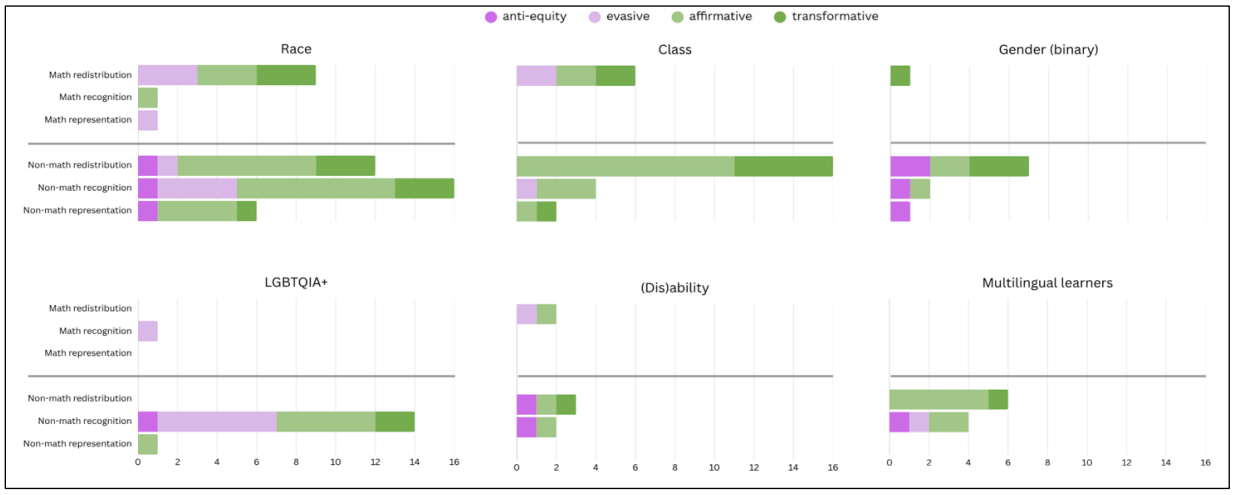


Figure 2: Coding Results

Regarding the "who" of participants' concerns, race and class were most frequently invoked. With respect to class, redistribution was the most prominent justice dimension within non-math areas, and the only dimension in conceptions of mathematics-specific equity. And evasive types of discourse were more present in discussion of mathematics (e.g., "our number one area of concern is just number sense… and our kindergarten readiness reports, those kiddos that come in not knowing anything," which the participant, when asked, associated with socioeconomic status). General equity talk about race was more distributed across Fraser's dimensions, with recognition being the most frequent. It also spanned all types of discourse, from transformative

to anti-equity. But within mathematics, conceptions of equity and race were almost exclusively about redistribution, with no transformative discourse about recognition or representation.

Among the other social groups, there were very few instances of talk that were specific to mathematics. There was considerable discussion of LGBTQIA+ students, but almost exclusively along the dimension of recognition and only once in the context of mathematics ("In math, we kind of work with the numbers and the x's...and maybe not as concerned about your feelings"). Additionally, as illustrated by the previous quote, we found the highest rate of evasive discourses was about this social group. While half of the participants discussed equity for multilingual learners in general (regarding redistribution and recognition, but not representation), none mentioned mathematics-specific issues. And only a few participants invoked (dis)ability, including two mentions of mathematics-specific issues, both about redistribution. In addition to the six social groups already noted, very few participants raised equity concerns related to different "personality" or "behaviour" needs among students—all with affirmative discourses except for one instance of transformative discourse.

Lastly, in four of the interviews we identified a conception of equity framed by responses not to injustice or inequality, but to diversity. Most were related to notions of "ability." For example, one school leader explained, "we decided to do flex math groups… ability grouping, you could call it… our data suggests that we have kids all over the place, and we need to meet those individual students' needs. So, we have six third grade teachers, and we took our data and broke it up into six groups." This is not to suggest that such categories would not lead to inequality or injustice (Domina et al., 2023), only that some participants conceived of equity as attending to differences in students' current understanding and learning "needs."

Discussion

Our results suggest conceptions of equity among school personnel vary between considerations of education in general and mathematics education in particular. Specifically, we found the presence of equity discourses that did not penetrate the boundary of mathematics (Nguyen, 2024). Most striking is that almost all conceptions of mathematics-specific equity were limited to redistribution. This result could be interpreted as echoing previous findings by Rubel (2017) and Wilson et al. (2019) that teachers were more likely to attend to issues of achievement and access than to issues of identity and power—a division that Vithal et al. (2024) mapped onto Gutiérrez's (2007) *dominant* and *critical* axes, respectively.

However, Fraser's distinction in types of discourse is helpful here: the transformative ways of describing redistribution we observed suggest that critical conceptions of mathematics-specific equity were present among district personnel, as some advocated for dismantling structures that give rise to inequality rather than an affirmative promotion of merely more equal distribution within those structures. This suggests that the absence of matters of recognition or representation in participants' conceptions of mathematics-specific equity should not, necessarily, be interpreted as resistance to critical orientations, but could be attributed to other factors, including the local construction of the subject matter context (Grossman et al., 2004; Nguyen, 2024). It is possible that other factors may also have contributed to the absence of attention to LGBTQIA+ students and multilingual learners in mathematics-specific ways.

We might pause to consider, however: what would we consider "ideal" conceptions of equity? Would it be transformative discourses attending equally to all three dimensions of justice for each social group? Or might we expect conceptions to vary across contexts as they respond to local histories and circumstances? While our study presents a framework for accounting for multiple aspects equity conceptions, it does not provide any empirical basis for considering what

might be "ideal" or "best." While we are doubtful that such a claim could be possible (or even appropriate) to pursue, we do imagine that future research could help explain more of the variation we observed.

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PEDAGOGICAL ACTIVISM IN MATHEMATICS EDUCATION: ARMED LOVE AND COMMUNITY CULTURAL WEALTH

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In mathematics education, scholars often attend to the complexities of teaching and learning in socio-cultural contexts, but far too often, they fall short of developing a deep understanding of the history that informed such contexts. While we must look to history to understand the ripple effects of such unjust systems, we must also look to powerful historical examples of pedagogical activism to understand how we can better link research and practice to serve historically marginalized learners today. In this oral history, we document how 12 Black K-12 mathematics teachers during the American Civil Rights Movement engaged in pedagogical activism by creating protective factors for their learners, drawing on armed love and community cultural wealth. This study advances our understanding of how to operationalize critical pedagogies in mathematics and build more equitable mathematics instruction by looking to the past.

Keywords: Culture and Asset-Based Pedagogies; Equity, Inclusion and Diversity; Identity and Agency; Social Justice

Problem Statement and Conceptual Framework

Our hopes for mathematics instruction are rooted in its potential to serve as a tool for liberation, cultural affirmation, and the development of critical consciousness. Although mathematics is often perceived as neutral or apolitical, it holds transformative power when contextualized within teachers' and students' lived experiences and histories. We envision mathematics instruction that not only supports conceptual understanding (Lampert et al., 2010) but also affirms identity (Aguirre et al., 2024), fosters agency, and builds bridges between learners, their communities, and the broader world. We look to historical forms of pedagogical activism (Alridge et al., 2023) to better understand how teachers drew upon armed love and community cultural wealth (CCW) to create protective factors for learners in mathematics.

Armed Love

Armed love is not soft or lacking in power; rather, it is a "political and radicalized" form of love "rooted in a committed willingness to struggle persistently" toward our true purpose—to be human (Ares et al., 2024; Darder, 2002, p. 34). While armed love has been researched across contexts (e.g., Lanas & Zembylas, 2015; Stern & Brown, 2016; Bartolomé, 2004), in critical pedagogy (Ladson-Billings, 2021) and pedagogical activism, it is discussed as drawing upon shared experiences and critically conscious care (Ares et al., 2019; 2024).

Community Cultural Wealth

When teachers practice armed love that builds upon CCW, they create protective factors for historically marginalized learners (Ares et al., 2024; Zoch & He, 2020). CCW is a community's cultural resources, not unlike funds of knowledge, to help learners to challenge oppression from a place of strength and knowledge (Ares et al., 2024). However, unlike other works that focus on assets developed by communities, CCW "expands on traditional notions of capital to identify those that are often found in non-dominant communities under pressure" (Ares et al., 2024, p. 2932; Yosso, 2005). CCW includes six forms of capital: familial, aspirational, navigational, resistant, social, and linguistic. Learners and families often use such capital to help them to

navigate challenges and resist exclusion in social spaces controlled by dominant cultural norms, as seen within the American educational system. What is less clear is how teachers from non-dominant communities incorporate the six forms of capital and armed love to build critical pedagogies in mathematics education. To further examine this work, we looked to history and Black mathematics teachers who engaged in pedagogical activism by showing acts of love and enacting capital-rich practices during the American Civil Rights Movement.

Methods

As Research Associates of the Teachers in the Movement Project, we conducted a secondary analysis of 12 oral history (Ritchie, 2015; Alridge et al., 2023) interviews collected with Black, former mathematics teachers who taught K-12 during the American Civil Rights Movement. These educators, primarily located in the American South, shared their experiences teaching mathematics amid the social and political upheaval of the era. Consistent with oral history methodology, participants' identities were retained to preserve historical attribution and to honor participants' contributions. All interviews (approximately 60 minutes each) were video-recorded, transcribed, and verified through member checking to ensure historical accuracy. Our analysis involved coding transcripts at the stanza level (Saldaña, 2013) and composing analytic memos to address our central question: How did these teachers engage in pedagogical activism with their students in mathematics? Specifically, we wanted to understand the protective factors behind such pedagogical activism. Our codebook was developed using a combination of deductive and inductive coding (Corbin & Strauss, 2015), drawing upon critical pedagogies (Ladson-Billings, 2021), armed love and CCW (Ares et al., 2024).

Pedagogical Activism Built Upon Protective Factors

Armed Love & Critically Conscious Care

While many of the teachers did not necessarily consider themselves Civil Rights activists, their accounts of pedagogy display central tenets of pedagogical activism. Their classrooms were spaces where care, resilience, and resistance intertwined, reflecting *caring as resistance* (Ladson-Billings, 1994) or critically conscious care (Ares et al., 2024). By enacting rigorous yet compassionate teaching, Black educators created protective factors designed to safeguard students from the dehumanizing effects of racism and discrimination. Oliver captured this ethos when she explained, “You don’t have to talk it, but you need to act it. I think that I did not do a lot as far as the movement was concerned except that I tried to practice certain things [in my teaching].” In this sense, teaching itself became activism—every lesson, every act of care, every insistence on students’ potential constituted resistance in a world that routinely denied Black children intellectual and social legitimacy (hooks, 1994). Such care was continually interacting with the six forms of capital within CCW. Figure 1 is the authors’ conceptual model of protective factors building pedagogical activism, developed from analysis of the oral history findings.

Familial Capital

Familial capital reflected armed love through relational accountability, as teachers positioned themselves as trusted community members who shared responsibility with families for students’ academic, social, and moral development. Their commitment extended beyond classroom instruction, creating networks of care and high expectations that protected students from dehumanizing societal messages while affirming their worth and potential. The teachers often made connections to their own educational experiences in shaping their perceptions of who should be stakeholders in students’ learning as well as what learning communities should look

like. For example, the teachers discussed the extent to which their own teachers were involved in their community and their learning as familial figures. Goode stated, “it wasn't like a call home or stuff [like] that they’d walk over to your house.” Benjamin echoed these same sentiments, saying, “you saw the teacher at church, and you did not want your parents to have a parent conference at church.” Brown discussed how this sense of community created more opportunities for the teachers to hold the learners accountable to high expectations, both behaviorally and academically. Gordon stated that they relied on their teachers for “nurturing and care in love and guidance and our parents really trust the school system to take care of us”. The teachers saw this familial capital as integral to students’ mathematical learning.

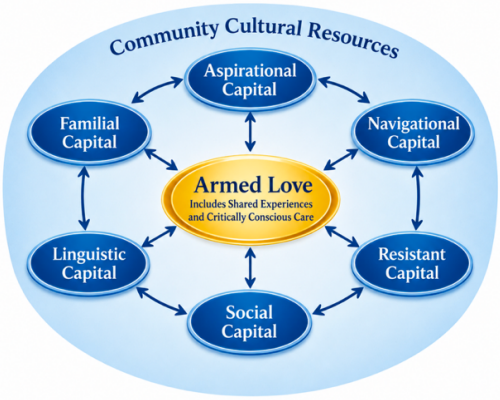


Figure 1: Protective Factors Building Pedagogical Activism

Aspirational Capital

Aspirational capital embodied armed love through teachers’ deliberate cultivation of future possibilities for Black learners. By encouraging self-advocacy, exposing students to successful role models, and nurturing beliefs in their own capabilities, teachers enacted a radical form of care that challenged limiting narratives and helped students envision futures beyond the constraints imposed by racism and social inequities. Goode stated, “I would do the very best I can for my students, plus … I also want them to advocate for themselves. So, I give them the tools that they need to advocate for themselves.” Gordon spoke of how Johnson, who was her math teacher, was influential in stressing these ideals in his instructional strategies. One practice that Gordon noted as being impactful was bringing former students back to the classroom who had obtained careers that her students aspired to obtain [especially in STEM] because as a student herself, “seeing individuals who are successful, who started the same way as me, gave me hope.”

Social & Navigational Capital

Social and navigational capital embodied armed love through teachers' intentional efforts to equip students with the knowledge, skills, relationships, and resources necessary to navigate educational and societal structures not designed for their success. Through ambitious mathematics instruction, mentorship, and access to networks of support, teachers helped students develop both mathematical competence and the confidence to maneuver through spaces shaped by racism and exclusion. Their commitment to ensuring students understood mathematics deeply and recognized their own intellectual capabilities represented a radical form of care that protected students' identities while expanding their opportunities for participation and success. For example, Johnson discussed how he “ma[de] sure that everyone had a conceptual grasp of

the content" and Gordon noted that it was about helping students to "understand the why" behind the procedures. Lewis discussed the importance of rooting mathematics in lived experience and future goals, and how his father's work as a handyman influenced his pedagogy that focused on hands-on visual applications. "So sometimes it might have been taking a [soup] can … [and] have a pair of tinsnips and cut it down the middle and lay it out. You've got a rectangle, you know, just so they could visually understand some of the things you were talking about." Lewis went on to discuss the importance of dispelling myths about who could or could not "do math", which was also a deliberate protective stance.

Resistant Capital

Resistant capital illustrated armed love through the development of critical consciousness and collective struggle against injustice. Teachers encouraged students to recognize, question, and challenge inequitable structures while equipping them with the intellectual tools necessary to navigate and transform those conditions. In doing so, they demonstrated a form of love grounded in liberation, empowering students to see themselves as capable agents of change rather than passive recipients of oppression. One of the most profound examples was when Langston discussed bias assessment practices in the '67, '68, and '69 school years (prior to integration). The segregated Black school, Emmett Scott, performed better on standardized mathematics tests than the White school, Rock Hill. Langston noted, "…we beat Rock Hill in every category, and they didn't like that because we were Black, and so we had to go back and take this test over." He went on to discuss how, after the retake, the Black students still performed better in every individual component and how he discussed these issues to build consciousness with his learners. The teachers had their own definitions of Civil Rights; some, like Edwards, extended discussions to present times, adding "But we are in an ongoing struggle, and that has become very evident to us in this very, very recent past."

Linguistic Capital

Linguistic capital reflected armed love through teachers' affirmation of students' cultural and communicative ways of knowing while simultaneously preparing them to navigate dominant linguistic norms. By valuing the language practices, storytelling traditions, discourse patterns, and cultural expressions of Black communities, teachers created spaces where students' identities were respected rather than erased. This form of care was not merely affirming; it was protective and empowering, equipping students with the linguistic flexibility needed to thrive across multiple contexts while maintaining connections to their communities and cultural heritage. In all Black schools, with all Black students and teachers, such cultural competency looks different than how it is discussed in literature post integration; thus, it expands upon our understandings of how teachers integrate cultural knowledge and practices into their mathematics instruction to create meaningful and relevant learning opportunities for their students (Thomas & Berry, 2019).

Discussion & Conclusions

Across the six forms of capital, armed love functioned as a protective force that affirmed students' humanity, nurtured their identities, expanded their possibilities, and prepared them to navigate and challenge oppressive conditions. In this way, pedagogical activism was not simply about teaching mathematics; it was about leveraging CCW to cultivate liberation, agency, and flourishing. This study contributes to the growing scholarship on Black educators' roles during the Civil Rights Movement, particularly in mathematics education. The teachers' recollections offer critical insights for today's educators, especially amid renewed resistance to culturally responsive education and critical pedagogies. By preserving and learning from these histories, we can better understand how teachers created protective factors in mathematics, drawing upon

armed love and the CCW of non-dominant communities under pressure (Ares et al., 2024). Likewise, we can further understand how to best support Black learners with identity-affirming mathematics instruction and reimagine mathematics education as a site of liberation and joy.

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TRANSFORMING PRESERVICE TEACHERS' MESSAGING THROUGH MATH NARRATIVES IN MIXED-REALITY SIMULATIONS

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity, Inclusion, and Diversity; Preservice Teacher Education; Technology

Beliefs about mathematics shape students' identities, engagement, and opportunities to learn. Unproductive math narratives (MNs) can limit students' participation and persistence. This poster reports on a multi-institutional project examining how structured preparation and mixed-reality simulation (MRS) can support preservice secondary mathematics teachers (PSTs) in communicating productive MNs. RQ: How does engaging in preparation and MRSs influence PSTs' self-efficacy and messaging related to the MNs of: (1) encouraging help seeking, (2) making mathematics relevant, and (3) affirming the value of mistakes?

Theoretical Perspectives

The project is grounded in sociocultural and social constructivist perspectives that position learning as relational and discourse-driven. Drawing on Complex Instruction, we conceptualize help-seeking as a collaborative practice that redistributes intellectual authority across the classroom (Cohen & Lotan, 2014). Efforts to make mathematics relevant align with calls for instruction that prepares students for participation in college, careers, and civic life (Aguirre et al., 2013; Moses & Cobb, 2001). Affirming mistakes reflects research demonstrating that struggle can support learning and identity development (Boaler, 2022; Jansen, 2020). PSTs' visions of ambitious instruction inform how they enact the MNs (Munter, 2014).

Methods

Mathematics Teacher Educators (MTEs) attended a BranchED professional learning conference introducing the Math Narratives Project (n.d.), a Gates Foundation-funded mixed-methods study of U.S. secondary students' mathematical identities. During Spring 2026, PSTs in the MTEs' methods courses completed three phases designed to contribute to shifts (Self & Stengel, 2020) in their MNs (Gutiérrez et al., 2023). In the Prepare phase, PSTs completed belief and self-efficacy surveys and a reflection tool articulating their visions for practice. During the Intervention phase, PSTs analyzed materials on MNs, planned a short lesson integrating the three focal MNs, and prepared for an MRS (Bondurant & Amidon, 2021; Bondurant & Reinholz, 2023; Colonnese et al., 2024; Lee et al., 2025). During the MRSs, PSTs practiced enacting the three focal MNs. In the React-Review-Reconsider phase, PSTs composed reflections, analyzed video recordings of their teaching, and completed a post-survey.

Preliminary Findings

Data sources include pre/post belief measures, self-efficacy ratings, written reflections, and MRS artifacts. Quantitative analyses will examine shifts in PST beliefs, while qualitative analyses will identify patterns in how PSTs interpret and enact MNs. Preliminary findings suggest PSTs became more intentional and confident in communicating productive MNs.

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PRAXIS, NEPANTLA, AND POETRY: A CROSS-CULTURAL APPROACH TO BECOMING IN MATHEMATICS EDUCATION

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Keywords: Community-Based Praxis; Teacher Educators; Identity and Agency; Equity, Inclusion, and Diversity

"Authentic thinking, thinking that is concerned about *reality*, does not take place in ivory tower isolation, but only in communication" (Freire, 1970/2004, p. 77). Through a critical autoethnography lens (Boylorn & Orbe, 2021), this poster presentation reflects our experiences of continually confronting our realities as new mathematics teacher educators. Our participation in an early career induction program for teacher educators in mathematics education initially shaped our communication. In our attempts to transcend the traditional isolation of academia, we sustained supportive dialogues grounded in the works of bell hooks and Paulo Freire. As mathematics teacher educators who support preservice teachers in their preparation programs, we started by meeting regularly to read and discuss *Teaching to Transgress* (hooks, 1994) and *Pedagogy of the Oppressed* (Freire, 1970). These meetings functioned as an intentional counter-space within the academy—one that invited vulnerability, dialogic engagement, and sustained critical reflection. Through collective reading, discussion, and reflection, we worked to surface how our lived experiences, despite coming from varied cultural, professional, and personal backgrounds, have been shaped by what Freire (1970) critiques as the *banking* system of education and continue to impact our work as mathematics teacher educators. While we welcomed the complexity of our lived experience, we supported each other in navigating the institutional and societal constraints through liberatory pedagogies (hooks, 1994) and leaning into the discomfort of unlearning the banking system of education (Freire, 1970). Liberatory pedagogies and parting away from the banking system of education in our academic spaces while honoring our uniqueness has led to a necessary *praxis* (Freire, 1970)—reflection and action in dynamic relation. This praxis involved a fracturing of self within a liminal space where we are in the midst of changing, leading to a recognition that *becoming* demands relinquishing familiar identities and certainties. This praxis resonated with what Anzaldúa (2012) describes as *nepantla*—in-between spaces of tension, ambiguity, and transformation—and how Freire (1970/2004) unpacks these spaces "as an uncompleted being conscious of their incompletion" (p. 43).

Whether through the continual reflection of our identities as women with diverse cultural lived experiences, the fracturing of our identities as we are undone and remade within academia, or through connecting to poetic memories, our small collective of mathematics teacher educators continues to make strides in understanding who we are and the power of injecting our voices into existence. Through reflection on our practices, we become conscious beings, constantly unveiling the reality that occurs with communication. As beings in this process of becoming, we became conscious of our incompletion, آن کس که بداند و نداند که بداند, of the need for constant transformation. This poster presentation shares our experiences in developing a community-based praxis in mathematics education to support the transition into teacher education and

explores how our collective learning and reflection have enhanced the ways we support our preservice teachers.

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REFRAMING PRODUCTIVE STRUGGLE AS INCLUSIVE STRUGGLE FOR LEARNERS WITH DISABILITIES IN SECONDARY MATHEMATICS

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Keywords: Equity, Inclusion, and Diversity; High School Education; Students with Disabilities

High school mathematics courses taught in accelerated formats compress a full year of content into a single semester, often prioritizing coverage over depth (Cooke, 2024; National Council of Teachers of Mathematics [NCTM], 2014). Research reports mixed outcomes and reduced opportunities for sense-making and reflection (Young et al., 2024). Learners with disabilities (LWD) are especially underserved, as rigid pacing and deficit-based practices restrict meaningful opportunities for productive struggle (Comstock et al., 2022; Lambert et al., 2018). In constrained environments, instructional feedback becomes a critical lever for meaningful engagement. This study asks: How can instructional feedback support productive struggle for LWD in compressed-pacing mathematics classes?

Method

A conceptual synthesis approach (EPPI-Centre, 2010) integrates three theoretical constructs: Zone of Proximal Development (Vygotsky, 1978), Universal Design for Learning (Catama, 2025), and Rights of the Learner (Kalinec-Craig, 2017). These constructs map to the core dimensions of agency, authority, and belonging. Using this synthesis rather than empirical data, this study develops the *Instructional Feedback for Inclusive Struggle* (IFIS) framework. IFIS reframes productive struggle as inclusive struggle, defined as cognitive challenge supported through feedback that affirms identity, access, and participation (Mosley & Fox, 2026).

Findings

This poster presents a visual crosswalk showing how IFIS aligns with the five strands of mathematical proficiency (National Research Council, 2001) and its core dimensions (Mosley & Fox, 2026). Derived from the conceptual synthesis, findings identify key feedback strategies that support LWD, including dialogic feedback (supporting authority), cognitive scaffolding (supporting agency), and visual supports and problem-based learning structures (supporting belonging) (Tarmizi & Bayat, 2012; van Zee & Minstrell, 1997). When intentionally designed, these strategies sustain adaptive reasoning and productive disposition while fostering identity-affirming routines (Marita & Hord, 2017). By centering agency, authority, and belonging, IFIS promotes equitable mathematics instruction.

Implications

This work rehumanizes LWD in conversations about productive struggle and equitable, rigorous instruction. While not drawing on student data, this framework builds on scholarship that centers the experiences of learners with disabilities. Unlike deficit-oriented approaches that often reduce cognitive demand, IFIS positions feedback as a tool for sustaining rigor while expanding access to meaningful participation and affirming students' identities (Lambert et al., 2018). It also offers actionable strategies for teacher preparation and professional development for educators committed to equity-centered mathematics learning.

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A RATIONAL CHOICE: WHY MATHEMATICS EDUCATORS CHOOSE NOT TO PUBLISH IN HIGH-IMPACT JOURNALS OF MATHEMATICS EDUCATION

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Keywords: Equity, Inclusion, and Diversity

High-impact journals of mathematics education (HIJMEs) such as ESM, JRME, and MTL are often treated as elite, desirable outlets (Andrade-Molina et al., 2020; Nivens & Otten, 2017). Yet even accomplished scholars do not always pursue them. This poster examines why some researchers may not be inclined, motivated, or ready to submit to HIJMEs, despite being active publishers. Guided by Bourdieu's theory of practice, academic publishing is conceptualized as a competitive field in which access depends on economic, social, and cultural capitals (Bourdieu, 1972). HIJMEs thus function as spaces where dispositions such as confidence, linguistic sophistication, visible networks, and Western-aligned methodologies are rewarded. Fanon's psychoanalytic lens (1961, 1967) complements this framing by explaining how repeated exclusion and symbolic hierarchies produce avoidance, resignation, or self-doubt. Following this, our aim is to provide answers to the following questions:

1. What reasons do mathematics education faculty give for not attempting or prioritizing publication in HIJMEs?

Three senior mathematics education faculty were interviewed. Thematic analysis (Braun & Clarke, 2006) used deductive codes based upon capital (economic, cultural, social), habitus (identity, motivation, affect), and field norms; and inductive codes on emergent themes related to institutional expectations, intended audiences, and emotional labor. Although the data revealed key findings on field and capital deficits, we found that not publishing in HIJMEs was often a rational and strategic academic choice, not necessarily a sign of disinterest or incompetence. Participants described: (1) Restricted audience: Research on constructs such as self-efficacy may enjoy wider readership in educational psychology and broader education journals, making HIJMEs secondary outlets. (2) Professional logic: Faculty may delay HIJME attempts until promotion pressures are reduced, since institutional requirements to publish internationally can be met in less competitive journals. (3) Institutional indifference: Some higher institutions only require local journal and conference publications, reducing incentive to pursue HIJMEs. (4) Psychological aftermath: Faculty may avoid HIJMEs due to mental strain, fear of rejection, or anticipated bias—echoing Fanon's claim that exclusion generates internalized marginality. (5) Limited visibility: Because their work uses Nigerian samples, local researchers' lack of access to HIJMEs means their intended primary audience cannot read, use, or cite their research.

Therefore, rather than framing absence in HIJMEs as apathy or incompetence, these findings reveal a coherent logic of scholarly decision-making shaped by capital, habitus, and psychological experience, not merely exhortations to "publish higher."

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FACTORS INFLUENCING RECRUITMENT OF WOMEN INTO SECONDARY MATHEMATICS TEACHING CAREERS

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Keywords: Equity, Inclusion, and Diversity; Gender; High School Education

Objectives of the Study

This purpose of this scoping review was to examine factors influencing recruitment of women into high school mathematics teaching careers.

Conceptual Framework

Women might be dissuaded from becoming secondary mathematics teachers for several reasons. Zhou and Liu (2023) note, "Societal expectations, family influences, gender bias, and discrimination are influential factors that can impact career development" (p. 16). Stereotypes about mathematicians in general and about women in mathematics, including perceptions that men are more likely to be mathematics teachers and that women teachers teach lower-level content than men, might negatively impact women's decision to enter mathematics-oriented fields (Gjøvik et al., 2023; Rossi et al., 2022). College mathematics can also contribute, given that secondary mathematics teacher preparation requires substantial mathematics coursework (Winsor et al., 2018). In addition to being subject to sexist comments (Ogbuanya & Ogbuefi, 2025), women in STEM can experience college STEM environments that are individualist, competitive, and intimidating, with discourses of difficulty that cause self-doubt (Parson & Ozaki, 2017). A greater number of women students and teachers can be one factor that can favorably influence women in STEM (Bowman et al., 2022).

Methods

Guided by the Arksey and O'Malley framework and reported in line with PRISMA-ScR guidelines, a systematic search was conducted across five academic databases (Scopus, ERIC, Web of Science, JSTOR, and Google Scholar) for studies published between 2000 and 2025. Following screening and eligibility assessment, 19 studies were included in the review.

Results

This study shows that recruiting women into high school mathematics teaching is shaped by interconnected societal, institutional, and psychological barriers, which include gender stereotypes, limited access to female role models, institutional biases in recruitment pathways, mathematics anxiety, and a weaker professional identity. Promising interventions, including mentorship programs, targeted financial support, and institutional reforms, show potential for improving women's representation.

Discussion and Conclusions

This review contributes to mathematics education research by listing key constraints and supports that can influence women's recruitment into high school mathematics teaching and offering evidence-informed directions for policy, practice, and future research. This is an equity imperative and an important step toward enriching mathematics education.

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GEOMETRY AND MEASUREMENT

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

MORE IS NOT ALWAYS BETTER: DGS INTERACTION TYPE AND DEDUCTIVE GEOMETRIC REASONING THROUGH EMBODIED COGNITIVE PATHWAYS

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This study examined how access to Dynamic Geometry Software (DGS) influences undergraduate students' deductive geometric reasoning through embodied cognitive pathways. Structural equation modeling revealed a modest positive effect of DGS access overall, but passive viewing and non-dynamic interactions showed stronger positive associations with reasoning performance than dynamic interactions, which had a weak negative association. The latent embodiment constructs (sensorimotor simulation and imagery formation) were both moderately associated with deductive reasoning, yet neither was positively activated by dynamic interactions. These patterns suggest a more complex account of how physical interaction shapes geometric reasoning in technology-enhanced environments than typically reported in DGS research.

Keywords: Geometry and Spatial Reasoning, Technology, Cognition, Learning Theory

Deductive proof (i.e., reasoning that is general, logical, and operative; Harel & Sowder, 2007) is a central goal of geometry instruction and a persistent challenge for students at all levels (Dreyfus, 1999). Dynamic Geometry Software (DGS) has been widely adopted to support geometry learning, and meta-analyses report positive effects on mathematics outcomes (Chan & Leung, 2014; He et al., 2025). Yet two important gaps remain. First, most DGS studies treat the technology as a uniform intervention and focus on procedural achievement rather than deductive reasoning specifically, leaving open whether and how DGS access supports proof construction. Second, while research in mathematics education increasingly implicates the body in geometric reasoning—suggesting that embodied cognitive processes may mediate the relationship between physical interaction and mathematical thinking—this pathway has not been systematically modeled, and no study has examined how different types of DGS interaction relate to it.

The present study addresses both gaps. Using structural equation modeling, we examine how DGS access and interaction type relate to undergraduate students' deductive geometric reasoning through two theorized embodied cognitive mechanisms: imagery formation and sensorimotor simulation.

Theoretical Framework

Grounded and Embodied Mathematical Cognition

Grounded and embodied cognition holds that mathematical reasoning is rooted in perceptual and motor experiences, not purely symbolic manipulation (Barsalou, 2008; Nathan & Walkington, 2017). In geometry, students who produce dynamic representational gestures depicting transformations of geometric objects are more likely to construct valid, generalizable proofs (Nathan et al., 2021; Walkington et al., 2022), while pointing gestures help ground

reasoning in the diagrammatic environment (Alibali & Nathan, 2012). These findings suggest that physical action and geometric reasoning are deeply intertwined.

Two cognitive constructs are central to this relationship. *Imagery formation* refers to the construction of task-relevant mental images of geometric objects (i.e., what a student mentally "sees" while reasoning). *Sensorimotor simulation* refers to the activation of motor processes associated with transforming or manipulating those objects, even without overt movement. Both constructs figure centrally in Hostetter and Alibali's (2008) Gestures as Simulated Action (GSA) framework. GSA proposes that as people think, they mentally simulate actions and perceptual states; when the resulting motor activation surpasses an individual's gesture threshold, it is externalized as gestures. Critically, GSA predicts a sequential relationship: imagery formation precedes and activates sensorimotor simulation, which may then be expressed as gesture. Simulation strength varies with imagery type (i.e., motor imagery activates the motor system more strongly than purely visual imagery), and both constructs are theorized to support the spatial-relational thinking underlying deductive proof (Hostetter & Alibali, 2008, 2019).

The theory of action-cognition transduction (ACT; Nathan, 2017) complements GSA by positing that the relationship between thinking and action is bidirectional: deliberate physical action can induce cognitive states through a reciprocal coupling of motor and cognitive systems, not merely reflect them. In geometry, this means that physically engaging with diagrams may directly activate imagery formation and sensorimotor simulation, thereby strengthening conceptual reasoning.

Dynamic Geometry Software

DGS allows students to construct and dynamically modify geometric figures, interacting with invariant mathematical properties unavailable in static diagrams (Hollebrands, 2007). By enforcing mathematical constraints and enabling epistemic actions (Kirsh & Maglio, 1994), DGS makes problem-solving more reliable. Touch-based implementations further afford haptic interactions that may directly engage the embodied processes underlying geometric reasoning (Sinclair & de Freitas, 2014). From an ACT and GSA perspective, dynamic manipulation of diagrams may intensify motor imagery, strengthen sensorimotor simulation, and thereby support proof construction. Meta-analyses consistently report large positive effects of DGS on mathematics outcomes ($g = 0.82 – 1.02$; Chan & Leung, 2014; He et al., 2025; Y. Zhang et al., 2025), with qualitative work confirming that DGS supports novel forms of embodied interaction (Pittalis & Drijvers, 2023). However, these studies treat DGS as a uniform intervention and rarely examine deductive reasoning or how different interaction types engage embodied cognitive processes.

Current Study

Drawing on GSA and ACT, we developed a cognitive model of deductive geometry reasoning with and without DGS access (Figure 1), representing three latent constructs: *imagery formation* (manifested by intuition and insight), *sensorimotor simulation* (manifested by dynamic, non-dynamic, and pointing gestures), and *deductive geometric reasoning* (manifested by general, logical, and operative reasoning; Harel & Sowder, 2007). We further distinguish three mutually exclusive DGS interaction types: *dynamic interaction* (transforming, scaling, or moving diagrams), *non-dynamic interaction* (static contact without manipulation), and *passive viewing* (observation without physical contact), representing a gradient from most to least motorically demanding.

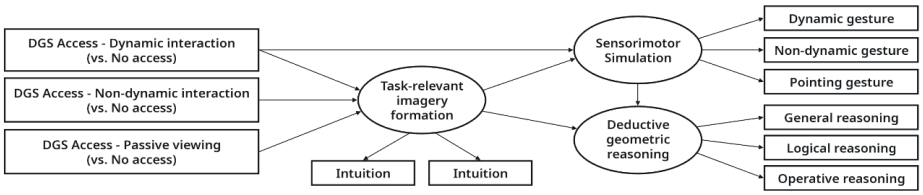


Figure 1: Hypothesized Relationship Between Constructs

With this model, we focus on how the three types of DGS interaction relate to deductive geometric reasoning. A complementary question concerning the overall effect of DGS access on deductive geometric reasoning is examined in a separate report and is not addressed in this manuscript. Here, we ask: *How do associations between DGS interaction type and deductive geometric reasoning differ across the three interaction types*? We hypothesize that the associations between interaction type and deductive reasoning will differ significantly, with dynamic interaction expected to show the strongest positive association due to its combination of visual and motor engagement.

Methods

Participants

Undergraduates (n = 158; mean age = 20.19; 92 female, 65 male, 4 non-binary) were recruited from a large Midwestern university. Ninety-five students identified as White/Non-Hispanic, 52 as Asian, and 11 as Other Race/Ethnicity. 107 students had completed college-level calculus or higher.

Study Design

Experimental design overview. We conducted a randomized within-subjects repeated-measures experiment in which students provided geometric reasoning for eight conjectures. Each student completed half the conjectures with DGS access and half without (Figure 2), with the presentation order of conjectures counterbalanced using a Latin Square design.

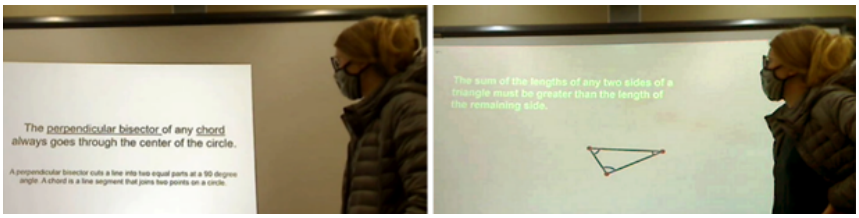


Figure 2: Proving Conjectures Without DGS Access (Left) and With DGS Access (Right)

Procedure and materials. Upon enrollment, students filled out online surveys. Researchers used the *Spatial Reasoning Instrument* (Ramful et al., 2017) to measure students' spatial ability, the *Spatial Anxiety Survey* (Lyons et al., 2018) for spatial anxiety, the *Diagnostic Geometry Assessment* (Masters, 2010) for geometry knowledge, and the *Situational Interest Survey* for academic domains (Linnenbrink-Garcia et al., 2010) for attitudes toward geometry.

Students worked in a laboratory space with a projected display and an interactive whiteboard. They proved or disproved eight geometry conjectures, each concerning objects and properties commonly featured in high school geometry curricula. Corresponding GeoGebra 3D Calculator simulations were developed for each conjecture and rendered on a large, touch-enabled

Promethean digital whiteboard, allowing direct manual manipulation (Figure 3). All conjectures were categorically true in Euclidean geometry. Students initially received only conjecture texts and vocabulary definitions for the first four conjectures (Figure 2 Left). After completing these, they watched a 5-minute tutorial on manipulating GeoGebra simulations. For the remaining four conjectures, simulations were presented alongside conjecture texts (Figure 2 Right). The facilitating researcher instructed students to use simulations at their discretion, neither encouraging nor discouraging them.

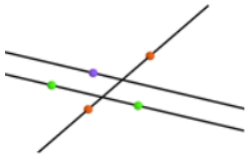
Conjecture: *If two parallel lines are cut by a third line, the pairs of corresponding angles are congruent.*		**Insight:** e.g., "The transversal cuts both parallel lines at the same angle. If you know one of the angles is 30 degrees, you use theorems to show that every angle has to be either 30 degrees or 150 degrees."

Figure 3: Example of Conjecture, GeoGebra Simulation, and Proof with Insight

Data Collection and Processing

Students' responses were video recorded, with a single video file created for each student. These video recordings were segmented by conjecture, yielding 1,264 conjecture responses and corresponding video clips. After quality screening, 1,239 responses were deemed eligible. The eligible responses were initially transcribed using Otter.ai and subsequently reviewed by human coders. Transcripts were then qualitatively coded for verbal reasoning and gesture production using MAXQDA 2022.

Variables

DGS access and interaction type indicators. Variable names are capitalized (e.g., "Name") to distinguish them from their corresponding theoretical constructs (e.g., "name"). *DGS Access* served as the response-level indicator for within-subjects treatment assignment, coded as "1" for responses with DGS access and "0" for those without.

Based on qualitative observations during data collection, we identified systematic variation in how students engaged with the DGS simulations. Although interaction type was not part of the original experimental design, these observed patterns prompted a post hoc classification of student interactions into three mutually exclusive categories, each theorized to differ in the degree to which they engage sensorimotor processes relevant to conjecture proof. *Dynamic Interaction* indicated transformative engagement with the simulation (e.g., transforming, scaling, or moving diagrams) believed to most strongly activate sensorimotor processes. *Non-Dynamic Interaction* indicated static physical contact unlikely to stimulate sensorimotor processes to the same degree (e.g., rotating the view or touching the board without manipulating figures). *Passive Viewing* indicated observation of the simulation without any physical contact.

All indicators were binary (0/1), coded once per response regardless of frequency. To ensure mutual exclusivity, we employed a hierarchical coding strategy in two stages. In the first stage, two researchers independently coded each response qualitatively for the presence of any physical interaction with the simulation, distinguishing responses that involved touching the board from those that did not (interrater reliability $\kappa = .92$). In the second stage, responses coded as involving physical interaction were further coded for whether that interaction was dynamic (i.e., whether the student actively transformed or manipulated the geometric figures) or non-dynamic ($\kappa = .92$). Dynamic Interaction superseded Non-Dynamic Interaction when both types of

interactions were found, and Non-Dynamic Interaction superseded Passive Viewing, such that each response received exactly one interaction type code.

Outcome variables. We qualitatively coded each response for five verbal reasoning characteristics and three gesture categories using binary classifications (0/1, absent/present). Unlike DGS access and interaction variables, outcome codes were not mutually exclusive. Raw proportions of outcome variables are shown in Table 1.

Table 1: Raw Proportions of Outcome Variables

		Verbal Reasoning					Gesture Production		
	n	Int.	Ins.	Gen.	Log.	Ope.	Dyn.	N-Dyn.	Point.
No Access	618	.73	.73	.53	.46	.45	.16	.64	.01
DGS Access	621	.79	.81	.57	.52	.55	.16	.44	.35
Dynamic Interaction	182	.77	.80	.50	.42	.45	.15	.39	.39
Non-Dynamic Interaction	200	.75	.76	.60	.54	.56	.22	.46	.26
Passive Viewing	239	.83	.84	.59	.55	.60	.13	.46	.40
Whole sample	1239	.76	.77	.55	.49	.50	.16	.54	.18

Note. Int. = Intuition, Ins. = Insight, Gen. = General, Log. = Logical, Ope. = Operational, Dyn. = Dynamic, N-Dyn. = Non-Dynamic, Point. = Pointing

Verbal reasoning characteristics included *Intuition* (immediate correct judgments without extended deliberation; $\kappa = .96$), *Insight* (explicit retrieval and correct application of mathematical properties; $\kappa = .86$; Z. Zhang et al., 2016), and three deductive reasoning attributes following Harel and Sowder's (2007) taxonomy: *General* reasoning (addressing entire object classes rather than specific instances; $\kappa = .87$), *Logical* reasoning (coherent inferential chains; $\kappa = .84$), and *Operative* reasoning (goal-oriented, mathematically purposeful thinking; $\kappa = .86$). Gestures were categorized into three types. First, Pointing Gestures (referring to simulation displays or imagined objects; $\kappa = .87$) and two types of representational gestures were classified as Dynamic Gestures (hand movements showing object transformations across different states; $\kappa = .88$) and Non-Dynamic Gestures (representing static objects or properties; $\kappa = .85$; Garcia & Infante, 2012).

Statistical Modeling

Unit of analysis. The unit of analysis was the response to a given conjecture ($n = 1{,}239$). Although responses were nested within students, preliminary clustering analyses using design effect coefficients and intraclass correlations supported single-level analyses with robust corrections (Muthén & Satorra, 1995; Stapleton et al., 2016), as our theoretical interest lies at the response level, where DGS access and interaction are operationalized.

Modeling framework. We employed structural equation modeling using a multiple-indicator-multiple-cause (MIMIC) approach. Unlike ANOVA or generalized linear models, which treat outcome variables separately, the MIMIC framework simultaneously examines treatment effects on multiple correlated indicators while representing their shared cognitive architecture through latent constructs (Hancock, 2003). This approach also enabled modeling of direct DGS effects and mediation among latent constructs (Figure 1), consistent with our theoretical model, and allowed incorporation of baseline covariates—spatial ability, geometry knowledge, and demographic characteristics—for more precise effect estimates. Following Mueller and Hancock (2019), we first established an acceptable measurement model before testing structural relationships.

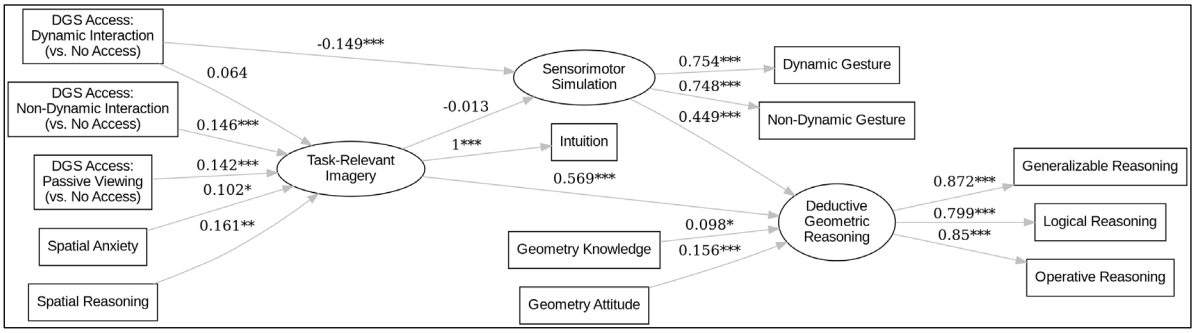
Technical details. Analyses were conducted in R using the {lavaan} package (Rosseel, 2012) with the Weighted Least Squares Mean- and Variance-adjusted (WLSMV) estimator, appropriate for binary outcomes. Latent variable variances were constrained to unity for model identification. All structural paths are reported as standardized coefficients (β), with effect size receiving primary interpretive emphasis alongside continuous p-values.

Results

Measurement Model and Latent Variables Evaluation

The three latent constructs (i.e., imagery formation, sensorimotor simulation, and deductive geometric reasoning) were specified in the measurement model. Insight and Pointing Gesture were excluded from the final measurement model due to severe multicollinearity issues. As a result, Imagery Formation used a single indicator (Intuition) with its loading fixed to 1.00. The final measurement model showed adequate fit across multiple indices: scaled CFI = .991, scaled TLI = .980, scaled RMSEA = .049 (90% CI: .031–.069), and scaled SRMR = .037.

The measurement model demonstrated adequate convergent validity and reliability. Sensorimotor Simulation was measured through Dynamic Gesture (λ = .78) and Non-Dynamic Gesture (λ = .68), with Deductive Geometric Reasoning measured through General (λ = .88), Logical (λ = .80), and Operative (λ = .84) reasoning attributes, all showing strong loadings (p < .001 for all three). Reliability assessment using Hancock and Mueller's (2001) Coefficient H yielded H = .704 for Sensorimotor Simulation and H = .885 for Deductive Geometric Reasoning, both exceeding the .70 threshold for factor replicability. The Imagery Formation construct was represented by a single indicator (H = 1.00) and was thus evaluated on theoretical grounds. Measurement invariance across conditions was established per Cheung & Rensvold (2002).



Note. • for p < .1,* for p <.05, ** for p <.01, *** for p < 0.001.

Figure 4: Fitted Structural Model with Path Coefficients

Structural Model and Direct Path Coefficients

The structural model (Figure 4) showed adequate fit across multiple indices: scaled CFI = .912, scaled TLI = .968, scaled RMSEA = .057 (90% CI: .050–.065), and scaled SRMR = .039. After excluding Insight and Pointing Gesture due to collinearity issues, modification indices suggested no further meaningful sources of misfit. Path coefficients revealed varied associations between DGS interaction types and cognitive processes. Dynamic Interaction had a weak positive association with Imagery Formation (β = .064, p = .097) but a stronger negative association with Sensorimotor Simulation (β = –.149, p < .001). Non-Dynamic Interaction (β = .146, p = .001) and Passive Viewing (β = .142, p = .001) both showed clearer positive associations with Imagery Formation. The path from Imagery Formation to Sensorimotor

Simulation was essentially zero ($\beta = -.013$, $p = .816$), providing little evidence of association. In contrast, both Imagery Formation ($\beta = .569$, $p < .001$) and Sensorimotor Simulation ($\beta = .449$, $p < .001$) showed strong positive associations with Deductive Geometric Reasoning.

Indirect Effects of DGS Interaction Type on Deductive Geometric Reasoning

We adopt correlational language given that interaction type categories were not randomly assigned. The associations between interaction types and Deductive Geometric Reasoning varied considerably. Dynamic Interaction showed a weak, negative, and statistically insignificant association with Deductive Geometric Reasoning ($\beta = -.031$, $p = .228$). In contrast, both Non-Dynamic Interaction ($\beta = .082$, $p = .001$) and Passive Viewing ($\beta = .080$, $p = .001$) demonstrated positive associations with deductive reasoning.

The omnibus Wald statistic for testing parameter equality constraints was $W(2) = 14.36$ ($p = .001$), suggesting the magnitude of the associations differs across the interaction types. Pairwise comparisons revealed that Dynamic Interaction's association with Deductive Geometric Reasoning was smaller than both Non-Dynamic ($\Delta = -.113$, $p < .001$) and Passive Viewing ($\Delta = -.111$, $p < .001$), while Non-Dynamic Interaction and Passive Viewing's associations showed little difference ($\Delta = .002$, p = .942). These findings appear consistent with part of our hypothesis that different types of DGS interactions may be differentially associated with the deductive geometric reasoning outcomes. However, the direction of these associations contradicts our hypothesis: rather than showing the strongest positive association with deductive reasoning, dynamic interaction showed a weak negative association, while the less motorically demanding interaction types showed stronger positive associations.

Discussion

Findings

The findings present a nuanced picture of how the way students engage with DGS shapes deductive geometric reasoning through embodied cognitive pathways. Contrary to our hypothesis that more active manipulation would most strongly support reasoning, the three interaction types were associated with deductive geometric reasoning in markedly different ways.

Specifically, passive viewing ($\beta = .080$, $p = .001$) and non-dynamic interaction ($\beta = .082$, $p = .001$) showed stronger positive associations with deductive reasoning than dynamic interaction, which showed a weak negative association ($\beta = -.031$, $p = .228$), with differences across interaction types being statistically meaningful ($W(2) = 14.36$, $p = .001$). Rather than more interaction being uniformly better, students who observed or lightly engaged with the simulation appeared to reason more successfully than those who actively manipulated it.

Theoretical Implications

Several explanations may account for this pattern. First, a self-selection mechanism is plausible: students who found a conjecture familiar may have felt little need to manipulate the simulation, while those who struggled engaged more intensively, which is consistent with Walkington et al. (2024). The negative association with dynamic interaction may thus reflect conjecture difficulty or prior knowledge rather than a direct effect of manipulation. Second, Physically Distributed Learning theory (Martin, 2009) suggests that dynamic interaction may have over-structured the environment, reducing the interpretive cognitive work that supports proof construction. Non-dynamic interaction and passive viewing may have required students to mentally simulate geometric relationships themselves, which is a productive struggle that proof-oriented reasoning demands. Cognitive load concerns may have further compounded this effect (Reis et al., 2012).

The negligible path from imagery formation to sensorimotor simulation ($\beta = -.013$, $p = .816$) raises further questions about how embodied processes operate in technology-enhanced settings. Rather than functioning as sequentially linked processes as GSA predicts, the two constructs appeared largely independent. DGS may have shifted students' imagery toward visual rather than motor encoding (i.e., externalizing the geometric transformations students would otherwise mentally enact) thereby weakening the imagery-to-simulation pathway. This suggests the relationship between imagery and sensorimotor simulation may be more context-dependent than GSA originally theorized.

This null path may also reflect how sensorimotor simulation was measured. Because the construct was operationalized through observed gesture, it inherits the threshold dynamics that GSA itself describes. Dynamic gestures, which depict transformations across states, had a high standardized threshold ($\tau = 1.03$) and appeared in only 16% of responses, whereas non-dynamic gestures had a much lower threshold ($\tau = -.36$) and appeared in 54%. If imagery formation activated motor processes only modestly, that activation may have been sufficient to produce low-threshold non-dynamic gestures but insufficient to cross the higher threshold for dynamic gestures, obscuring an imagery-to-simulation link that gesture-based measures may be unable to detect, which is a measurement dependency that Hostetter and Alibali (2019) themselves identified as a limitation in need of independent measures of simulation.

Practical Implications

These findings carry practical implications for DGS-based geometry instruction. Educators should not assume greater interactivity uniformly benefits proof-oriented reasoning; structured observation may support deductive reasoning as effectively as open-ended exploration. Instructors should also select conjectures for which dynamic manipulation genuinely reveals properties unavailable in a static diagram. More broadly, the modest overall effect reinforces the idea that how students interact with DGS and how instructors structure those interactions matters more than access alone.

Future Inquiry

The most consequential limitation is that the interaction type was classified post-hoc from naturally occurring behavior. Because students self-selected into interaction types, causal claims cannot be made. Future studies randomly assigning students to specific interaction protocols would allow stronger conclusions about whether different forms of physical engagement causally influence deductive reasoning. Conjecture-specific analyses also represent a promising direction. Whether dynamic interaction supports reasoning may depend on whether the conjecture's key property is genuinely revealed through manipulation or equally accessible through observation. Future work incorporating eye-tracking, think-aloud protocols, or fine-grained gesture analysis could further illuminate the moment-to-moment cognitive processes linking DGS interaction to proof construction.

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RE-EXAMINING CHILDREN'S DEVELOPMENT OF GEOMETRIC INTUITION

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This theoretical report examines children's emerging conceptions of geometric shape, especially in light of Piaget and Inhelder's (1967/1948) topological primacy thesis and Klein's Erlangen program. We focus on aspects of shapes that children find most salient when representing them with drawings. Within our Erlangen framing, the topological primacy thesis aligns with a progression of invariances under transformation, requiring increasing coordination of actions. Pilot study results from 21 four- and five-year-olds indicate support for the thesis when framed this way. An interesting exception was children's attention to the preservation of perpendicularity.

Keywords: Cognition, Early Childhood Education, Geometry and Spatial Reasoning, Learning Theory

Based on their extensive studies with children, Piaget and Inhelder (1967/1948) characterized the development of geometric intuition as a progression in children's constructions, from topological properties (e.g., enclosure), to projective properties (e.g., linearity), to Euclidean properties (e.g., length and angle measure). Although several subsequent studies have replicated Piaget and Inhelder's findings, affirming the progression, some researchers have criticized their interpretations of findings and their theoretical underpinning—often referred to as "the topological primacy thesis" (Darke, 1982).

A main concern among critics is that Piaget and Inhelder mischaracterized some properties of shapes as topological when they would be mathematically categorized as projective or Euclidean. For example, Piaget and Inhelder (1967/1948) stated, "our experiments have shown how important such actions are in the mental construction of shapes, based on elementary topological relations such as proximity, order, and enclosure" (p. 449). However, formally speaking, the *order* of points is a projective property that is not generally preserved under topological transformation, and *proximity* implies a metric (measurement), which is a Euclidean property. To resolve these issues, we turn to the hierarchy of geometries established by Klein's (1893) Erlangen program.

The Erlangen program aligns with Piaget and Inhelder's broader epistemology of mathematics in that it foregrounds *actions* as the basis for mathematical development. In fact, Piaget (1950) cited Klein's work broadly, noting the alignment but without relying on the detailed characterizations the program might have provided him. Klein studied mathematics from the perspective of invariants: What remains unchanged under a transformation? For example, if we perform an isometry (e.g., a spatial rotation) on a Euclidean shape, such as a square, it maintains that shape. So, we can characterize Euclidean shapes and properties as those that remain invariant under isometry (e.g., angle measure and side length). In foregrounding actions over properties, Klein characterized Euclidean geometry by the group of transformations (isometries) that preserve those properties, rather than by the properties themselves.

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With Piaget and Inhelder's research in mind, we designed a study to investigate preschoolers' emerging conceptions of perpendicularity. As part of that study, we assessed children's emerging conceptions of geometric shape, partially replicating a study by Piaget and Inhelder and using Klein's Erlangen program to address some of the prior critiques of this work. Specifically, children were asked to draw copies of pictures that depicted a circle or pairs of line segments (parallel or intersecting). In analyzing children's drawings, we asked ourselves, "Which geometric aspects of the shape does the child consider salient in producing their drawing?"

We designed a rubric, beginning from Piaget and Inhelder's characterizations of various geometric properties (e.g., enclosure and angle preservation) but modified to align with Klein's hierarchy of geometries. Our goal was to refine Piaget and Inhelder's characterizations by explicitly identifying various kinds of invariants and the corresponding transformations, or mental actions, that preserve them. Thus, we introduce a refined theory of children's development of geometric intuition, while partially replicating Piaget and Inhelder's work to elucidate the theory and to begin answering the following question: Does children's development of geometric intuition progress according to greater invariance under transformation? As such, this paper serves as a theoretical report that fits within PME-NA's Geometry and Measurement Strand: "teaching and learning of geometry, measurement, and spatial reasoning."

Theoretical Framework

The Topological Primacy Thesis

In *The Child's Conception of Space*, Piaget and Inhelder (1967/1948) report on their extensive studies of children's development of geometric intuition. The central thesis of this work—"intuition of space is not a reading or apprehension of properties of objects, but from the very beginning, an action performed on them" (p. 449)—extends from Piaget's epistemology of mathematics. Namely, Piaget (1950) framed mathematics as a product of human activity that progresses through increasingly complex coordinations of mental actions. Applying this epistemology to space and geometry, Piaget and Inhelder describe the theoretical progression as follows: "From the rudimentary sensorimotor activity right up to abstract operations, the development of geometric intuition is that of an activity, in its fullest sense" (p. 449).

The principal finding from this work was the identification of stages in children's development of geometric intuition. Elementary intuition begins at Stage 1 (~four years of age), wherein children evoke thought by reproducing sensorimotor actions in imagination. In one of the related studies, Piaget and Inhelder asked children at various stages/ages to draw copies of shapes shown to them. Analyzing children's drawings (and accounting for variance in fine motor skills), the researchers found that children at Stage 1 progressively attended to topological aspects of the shapes (e.g., distinguishing closed curves, such as circles, from open curves) but did not attend to projective or Euclidean aspects of those shapes (e.g., straight lines or angle measures). For instance, the top row of Figure 1 shows, from left to right, Stage 1 drawings of a circle, a triangle, and a square. Note that all three drawings appear as simple closed curves (preserving enclosure), with four tick marks tacked onto the square to mark its four sides or vertices but with no angles or straight lines included in the closed curve.

Progressive attention to topological aspects culminates at Substage 1B (between three-and-a-half and four years of age) when "one can begin to speak of real drawings, though curiously enough it is only topological relationships which are indicated with any degree of accuracy, Euclidean relationships being completely ignored" (Piaget & Inhelder, 1967/1948, p. 55). The second row of Figure 1 illustrates the progression and its culmination with drawings of a circle from children at the beginning of Stage 1 (Stage 1A, left), Stage 1B (middle), and Stage 2 (right).

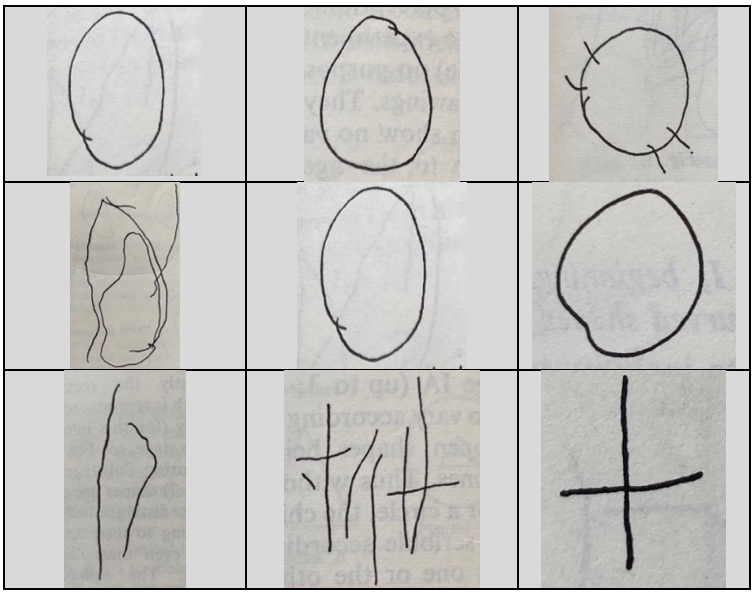


Figure 1: Shape Drawings from Piaget and Inhelder (1967/1948)

Starting at Stage 2, (~four years of age), children make progressively finer distinctions in their drawings, beginning with projective properties (e.g., straight vs. curved lines) and continuing into Euclidean ones (preserving angle measures and side lengths). The third row of Figure 1 illustrates this emergence of attention to projective and Euclidean aspects of shape, with drawings of a cross (+) from children at Stage 1A (left), Stage 1B (middle), and Stage 2 (right). Note the Stage 1A drawing does not preserve the topological property of intersection, but the Stage 1B drawing does, with some effort (repeated attempts). The Stage 2 drawing additionally includes consistently straight lines, four right angles, and approximately equal segment lengths.

Critiques of the Thesis

Several studies have replicated this work, generally affirming its findings (e.g., Kato, 1986; Pinard & Laurendeau, 1966). However, other researchers have been more critical of the central thesis of the work: the topological primacy thesis (e.g., Clements & Battista, 1992; Kapadia, 1974; Martin, 1976b). As summarized by Darke (1992), these critiques include two broad concerns: (1) the characterization of topological properties, as interpreted by the researchers; and (2) the instructional implications of this work, as interpreted by educators.

Regarding the first concern, some critiques misrepresent Piaget and Inhelder's work. For example, Kapadia (1974) critiqued the researchers for misuse of terms like proximity and order, only to conclude, "we believe, contrary to Piaget and Inhelder, that continuity is one of the earliest aspects of space representation" (p. 422). Although Piaget and Inhelder (1967/1948) might have used terms like proximity and order unconventionally, they explicitly related each topological term to continuity, which lies at the heart of the topological primacy thesis.

Other critiques, such as those offered by Martin (1976a; 1976b), are more constructive. Martin (1976b) also raised concerns about language: "Piaget's use of the term separation does not correspond to mathematical usage" (p. 16). However, he also seemed to appreciate Piaget and Inhelder's intent and suggested: "Perhaps the emphasis should be placed on the child's ability to order, organize, and coordinate his actions, not on trying to chronologically order spatial representation according to topology, Euclidean geometry, or projective geometry" (p.

15). This suggestion certainly fits within Piaget's broader epistemology of mathematics, but without the danger of imposing formal mathematics onto children's mathematical development.

In a second article, Martin (1976a) shared results of a study conducted to test the topological primacy thesis. That study tasked children with drawing copies of different shapes, including some of those used by Piaget and Inhelder: a square with one diagonal, and a circle inscribed in a triangle. Each of these shapes displayed both topological properties (e.g., connectedness and enclosure) and projective properties (e.g., straight vs. curved lines). Drawings were classified, first by determining whether they were homeomorphic to the given shape, meaning they shared *all* topological properties. Then, among the homeomorphic and non-homeomorphic copies, drawings were classified according to the degree to which they satisfied projective properties. Martin found that, among the 30 four-year-olds in the study, 40% of their drawings were homeomorphic to the given shape, and 7% of their drawings preserved all projective properties (whether or not they were homeomorphic). However, because most (67%) non-homeomorphic drawings preserved *some* projective properties, Martin determined that "none [of the data] indicate that the four-year-old's representational space is predominantly topological" (p. 37).

Regarding the second concern, Martin noted, "although making implications for mathematics education is not a primary objective of Piaget, mathematics educators do make inferences from his research" (Martin, 1976b, p. 24). One unintended implication of Piaget and Inhelder's work was the introduction of formal topological properties in elementary school. The more appropriate implication would be a focus on children's mental actions and their coordination as children construct mathematics through their own activity. For instance, drawing upon a case study with a seven-year-old student, Greenstein (2014) suggested teachers could "build on their students' emergent geometric reasoning and make sense of transformation-based ideas (e.g., property, equivalence, invariance)," whether transformations were topological or Euclidean (p. 93).

Similarly, working with 18 kindergarteners (five to six years old), Davis and Hyun (2005) found that they could support children's development of geometric intuition through interactive investigations and reflections on those activities. In line with Piaget and Inhelder's topological primacy thesis, at the start of the study, children's drawings exhibited more topological properties than projective or Euclidean properties, by a 3:1 ratio. However, toward the end of the school year, the ratio had reversed, 1:4. This result aligns with prior findings, indicating that children do come to school with many topological concepts and that instruction focused on Euclidean concepts can begin there (Schipper, 1983). Although these findings at least partially align with the topological primacy thesis, Davis and Hyun (2005) concurred with an alternative hypothesis advanced by Clements and Battista (1992): "It may be that ideas of all types develop over time, becoming increasingly integrated and synthesized. These ideas are originally intuitions, grounded in action …" (pp. 425–426), which also fits Martin's (1976b) suggestion.

An Erlangen Solution

As noted, part of the problem with Piaget and Inhelder's (1967/1948) topological primacy thesis owes to their use of terms. In the case of "order," the researchers seemed to have in mind the intermediate value theorem (curves passing through intermediate points), which relates to continuity and, therefore, topology. In the case of proximity, the researchers did not refer to a metric but a qualitative aspect of perception: "The most elementary spatial relationship which can be grasped by perception would seem to be that of proximity, corresponding to the simplest type of perceptual structurization, namely the nearby-ness of elements belonging to the same perceptual field" (p. 6). They contrast proximity with separation, wherein children begin to distinguish two objects that could be touching (e.g., a chair leaning against a wall) as two

separate objects. They relate these terms—proximity and separation—to continuity via Weber's Law "in terms of the increasing fineness of the thresholds of sensitivity" (p. 8).

To address the larger problem of topological primacy—concerning whether children's development of geometric intuition follows a progression from topological, to projective, to Euclidean—we rely on Klein's Erlangen program. The program classifies geometries based on (groups of) geometric transformations and the properties that they preserve (invariants). This focus on transformations aligns with Piaget's (1950) epistemology of mathematics as a coordination of mental actions. Note that this epistemology also aligns with perspectives offered by critics of the topological primacy thesis (Clements & Battista, 1976; Martin, 1976b). What remains unclear is whether the progression from topological, to projective, and Euclidean transformations fits a progression of increasingly complex coordinations of actions; and the Erlangen program offers clarification.

Essentially, the Erlangen program answers the question of what it means for two geometric shapes to be the same, by specifying the actions that transform one shape into another. These mental actions, or transformations, allow some properties to change while leaving others—the properties that define sameness—invariant. Topology has the broadest group of transformations and, therefore, the narrowest set of invariant properties. Two geometric shapes are considered topologically the same (homeomorphic) if each can be transformed onto the other by a bicontinuous transformation (a bijective continuous transformation with a continuous inverse). Importantly, bicontinuous transformations include every projective transformation (projections) and every Euclidean transformation (isometries), but they also include more general transformations that require little coordination. For instance, shapes could be stretched in different directions without any consideration for whether their side lengths or angles remain the same, or even whether the sides remain straight. This is why we might consider bicontinuous (topological) transformations to be primitive, even though the formal definition of continuity appears so late in history and curricula.

Projective transformations form a subgroup of topological transformations: Every projective transformation is bicontinuous, so it preserves all topological properties of a shape. Additionally, projective transformations preserve linearity and order, and preserving these additional properties requires coordination of actions—there is a point of reference from which to produce lines of projection that pass through two shapes that are projectively the same. Affine transformations additionally preserve parallel lines, and similarity transformations (transformations that preserve geometrically similar shapes) additionally preserve angles, requiring further and further coordination.

Similarity transformations consist of isometries and dilations, so Euclidean transformations (isometries) form a subgroup of similarity transformations. Note that isometries are rigid transformations, and so we would consider two shapes to be the same if one can be rigidly transformed onto the other. This is the kind of sameness that Euclid took for granted: "Things fitting on one another are equal to one another" (Fried, 2009, p. 3). However, we can further restrict the group of transformations by removing reflections and rotations, leaving only translations ("the group of displacements;" Piaget & Inhelder, 1967/1948).

By focusing on nested subgroups of transformations, requiring greater coordination to preserve more invariance, the Erlangen program reconciles the topological primacy thesis with alternate hypotheses suggested by Matin (1976b) and by Clements and Battista (1992). Specifically, within the Erlangen program, we can identify increasingly complex coordinations of actions, and an increasing synthesis and integration of ideas that are grounded in action.

Methods: Elucidating Theory with Empirical Examples

Although we present this work as theoretical, we include preliminary results from a pilot study to elucidate the theory. Data come from a larger study on children's emerging conceptions of perpendicularity. The pilot sample included 21 preschoolers (M_{age} = 4.56 years, SD = 0.35; 13 boys, 8 girls), who participated in one-on-one interviews with a researcher. Among other tasks, the pilot study included five drawing tasks like those used by Piaget and Inhelder (1967/1948). Children were introduced to the activity by the researcher, with the researcher demonstrating by copying the circle (Task 0). Then, each child was shown each of the shapes in the top row of Figure 2, one by one, and was asked, "Can you draw this?" when given a blank paper and crayon. Samples of children's drawings are shown in the bottom row of Figure 2.

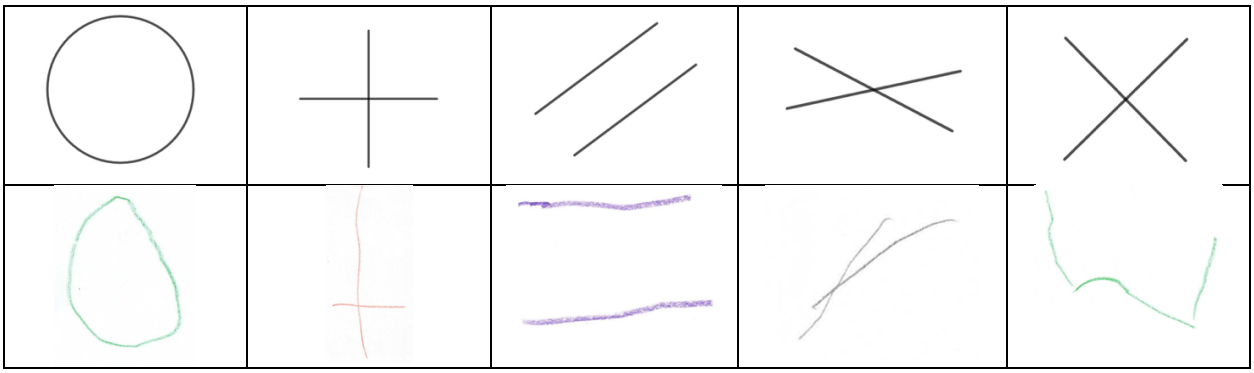


Figure 2: Children's Shape Drawings (T0 to T4)

Drawings were analyzed by a pair of researchers using the rubric described here. For each task, each rater evaluated the following: Does the drawing preserve (1) topological aspects of the shape (enclosure, connectedness, and separations/intersections)? Does the drawing preserve (2) projective (linearity) and (3) affine aspects of the shape (parallelism)? Geometric aspects were subdivided into three categories aligning with increasing degrees of sameness: (4) similarity (angle and proportional lengths), (5) isometry (absolute distances and absolute lengths), and (6) sameness in orientation, which accounts for reflection and rotation. Not every aspect applied to every task (e.g., preservation of parallelism only applied to Task 2). Once the two raters completed scoring for all drawings, the research team met to reconcile discrepancies.

Results: Interpretations of Examples Using Theory

Out of the 861 assessments of various aspects of children's drawings, the two raters achieved 83.86% agreement. Out of their 139 disagreements, 53 (38%) concerned judgements of linearity. The two raters met with the three other members of the research team to reconcile disagreements. In some cases, discrepancies could be resolved using objective measures, applied by a third researcher. For example, we allowed for errors of up to 20%, in either direction, for children's drawings of lengths and angles. In other cases, such as linearity, judgments were more subjective, and the researchers had to determine from children's imperfect drawings whether they were attending to that aspect of the shape. Reconciled scores are presented in Table 1, as ratios of drawings that preserved each aspect of the shape for each task (T0–T4). Note that some of these ratios come from fewer than 21 responses because for some drawings, preservation could not be determined. Most (six out of eight) of these cases concerned the orientation of the shape, as some drawings had no discernable orientation (e.g., the drawing for T4, in Figure 2).

Table 1: Ratios of Drawings Indicating Salience of Properties for Each Shape

Group	Invariant Property	T0	T1	T2	T3	T4
Homeomorphism	Enclosure	16/21	20/21	21/21	18/21	21/21
Homeomorphism	Connectedness		19/21	20/21	18/21	20/21
Homeomorphism	Separation/Intersection		19/21	21/21	17/21	19/21
Projection	Linearity	13/21	13/21	11/21	6/21	7/21
Affine	Parallelism			12/21		
Similarity	Proportional Length	4/21	5/21	12/21	4/21	6/21
Similarity	Angle Measure		**20/21**		1/21	**11/21**
Isometry	Absolute Length		2/21	3/21	0/21	3/21
Isometry	Absolute Distance	1/21	1/20	3/21	0/21	1/20
Displacement	Orientation		**20/20**	**5/21**	**1/18**	**8/19**

Results presented in Table 1 follow the pattern predicted by Piaget and Inhelder's topological primacy thesis. Specifically, topological properties (enclosure, connectedness, and separation/intersection) of each shape are largely preserved; then we see a substantial drop in the preservation of projective properties (linearity) across all five shapes; and another substantial drop, across all five shapes, for Euclidean properties (absolute length and distance). As Piaget and Inhelder (1967/1948) noted, this pattern in children's development fits the reverse order of historical mathematical development, which proceeded from Euclidean geometry, to projective geometry, to topology. This pattern also fits the order of geometries as classified by the Erlangen program, but that program includes additional geometries whose order is determined by the groups of transformations that leave their shapes invariant: affine transformations, which preserve parallelism (as well as linearity); similarity transformations, which preserve angles and proportional lengths; and displacements, which preserve the (angular) orientation of shapes.

When we include these additional groups and their associated invariant properties, the pattern generally holds but is not as clean. First, in the case of Task 2, we do not see a drop off in children's preservation of parallelism relative to their preservation of linearity. Second, although we generally see a substantial drop from projective (and affine) properties to similarity properties—and again to properties preserved by isometry—there are two prominent exceptions. These exceptions appear in bold in the angle measure row of Table 1, for Tasks 1 and 4, which involve shapes with right angles. It appears that children preserve right angles far better than other angles (compare to Task 3), particularly when they are oriented orthogonally, as in Task 1. Third, children reliably preserved the orientation of those shapes; and generally, they preserved orientation as well or better than they preserved absolute lengths and distances.

Discussion

The Erlangen program suggests an inverse relationship between geometric invariants and the groups of transformations that leave those properties invariant (Klein, 1893). Topological invariants are restricted to properties related to continuity (e.g., enclosure), so the group of topological transformations is large—all bicontinuous transformations—and the number of invariants is small. In contrast, Euclidean invariants are many (including angle and length), so the group of Euclidean transformations (isometries) is restricted to rigid transformations of

space. This hierarchy provides a framing to justify Piaget and Inhelder's (1967/1948) topological primacy thesis while reconciling it with Piaget's broader epistemology.

In line with Piaget's epistemology—and some proposed revisions to the topological primacy thesis (Clements & Battista, 1992; Martin, 1976b)—our Erlangen solution suggests that greater invariance requires greater coordination of actions. The group of transformations that define a geometry are like free variables that do not need to be coordinated with one another to determine an invariance within that geometry. In this sense, the proposed theory of development of geometric intuition is similar to the theory of covariation in students' developing understanding of functions and variables: Functions and other equations describe invariant relationships that students learn through increasing coordination of their activity, while leaving free variables free to vary independently (Carlson et al., 2002).

So, the Erlangen program serves to clean up Piaget and Inhelder's (1967/1948) characterizations of various geometric properties, while providing a deeper theoretical basis for their topological primacy thesis (fitting Piaget's broader epistemology of mathematics; Piaget, 1950). However, it also complicates the predicted pattern. In our pilot data, we find strong support for Piaget and Inhelder's thesis but less clear support for the progression implied by the Erlangen program. Clements and Battista's alternative hypothesis explains some of the discrepancy but would not predict the patterns observed in Table 1.

On the one hand, Clements and Battista (1992) note a "bias toward the perpendicular in copying angles" (p. 425), especially when the shape is oriented horizontally or vertically. As illustrated in the sample response to Task 1 in Figure 2, children preserve right angles even when neglecting the proportional lengths of line segments. This bias explains two of the unexpectedly high success rates reported for Task 1 (and, to a lesser degree, one for Task 4) in Table 1. Nearly all of the children's drawings preserved right angles and the orthogonal orientation for Task 1! Furthermore, children preserved right angles in Task 4 far more often than they preserved the non-canonical angles in Task 3. On the other hand, Clements and Battista's framing is not detailed enough to explain relationships between different kinds of transformations/invariants.

Specifically, with the bolded exceptions, and the additional exception of the displacement group, we find a consistent pattern in Table 1 (across all five tasks) corresponding with the transformation groups. These initial results also indicate aspects of shapes most salient to children's drawings, which align with the invariant properties of those groups. For example, a child might straighten a curve in order to preserve enclosure (see sample response to Task 0 in Figure 2), or they might bend a line to create an intersection (see sample response to Task 3). It may be that children can attend to various properties of a shape, especially when teachers bring these properties to their attention, as shown in the classroom study by Davis and Hyun (2005). However, that study also showed an initial disposition toward topological properties among kindergarteners, indicating primacy or greater salience.

In closing, we acknowledge that Piaget and Inhelder's (1967/1948) drawing study results are influenced by the particular shapes/tasks they chose. This limitation is noted by several of their critics (e.g., Clements & Battista, 1992), and it applies to our pilot results as well. However, recall that our tasks were selected for use within a larger project on children's emerging conceptions of perpendicularity. From that perspective, we find it especially interesting that children preserve the perpendicularity of lines so early in their development, especially when the lines take canonical forms (+).

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A PEDAGOGICAL MODEL FOR GEOMETRY: WEAVING NARRATIVES, ETHNOMATHEMATICS, AND INTERDISCIPLINARY LEARNING

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A Mathematics education, rooted in predominantly Eurocentric traditions, often functions as an exclusionary space, hindering the development of robust mathematical identities for many students, particularly Muslim students who face underrepresentation and marginalization. To address this inequity, this conceptual paper proposes the Narrative-Ethnomathematics Framework for Geometry (NEFG) as an interdisciplinary pedagogical model for creating inclusive learning spaces. Drawing on Culturally Relevant Pedagogy, Funds of Knowledge, ethnomathematics, and identity research, the NEFG leverages geometry instruction to counter narratives of exclusion. The framework comprises three interwoven strands: Narratives, Ethnomathematics, and the Mathematical/Disciplinary strand. Together, these strands aim at connecting abstract geometrical concepts to real-world cultural heritage.

Keywords: Elementary School Mathematics, Ethnomathematics, Geometry

Mathematics education remains an exclusionary space that is rooted in predominantly Western and Eurocentric traditions, which "limits many students' opportunities to develop identities as mathematically capable learners and thinkers" (Louie, 2017, p. 489). This culture of exclusion creates inequitable learning spaces and poses a pedagogical challenge (Louie, 2017). Mathematics teachers often struggle to devise meaningful, authentic, and relevant learning opportunities that are respectful, ethical, and inclusive of their students' diverse histories and onto-epistemologies. This challenge is particularly critical with Muslim students in the United States, where the growing anti-Muslim rhetoric (i.e., islamophobia) is compounded by the fact that Muslim students are absent from mathematics research (Salem, 2019). In 2025, the Council of American-Islamic Relations (CAIR) Foundation noted various instances where Muslim students were bullied in schools or faced forms of discrimination and/or misunderstanding across multiple states. Additionally, because the current educational framework often falls short in providing a comprehensive understanding of global perspectives, Muslims' histories, mathematical contributions, and sense-making systems are absent from any representation in the curriculum (Salem & Tillis, 2021). This lack of representation can lead to creating learning spaces that are alienating, thereby reinforcing marginalization and racialized identities.

To address this gap, we draw from research on culturally relevant pedagogy (CRP) (Ladson-Billings, 1995), Funds of knowledge (Moll et al., 1992), ethnomathematics (D'Ambrosio, 1985), and mathematics identity research to leverage geometry instruction to counter narratives of exclusion and difference in mathematics curriculum. Integrating the cultural context of Islamic geometry and picture books (Russo & Russo, 2018) serves as a critical, proactive intervention model to honor students' Funds of knowledge (FoK), improve academic outcomes, and foster empathy (McAndrew et al., 2017). This conceptual paper proposes the Narrative-Ethnomathematics Framework for Geometry (NEFG) as an interdisciplinary pedagogical model

for inclusive learning spaces in mathematics, especially for Muslim students. It uses picture books to link core mathematical concepts in geometry, socio-historical context, and cultural practices of Muslims to create meaningful learning contexts. The ethnomathematics dimension of the NEFG extends prior scholarship that has applied culturally grounded mathematical frameworks with Indigenous student populations, broadening this body of work to consider Muslim students' cultural knowledge systems and mathematical heritages.

We acknowledge the need for greater specificity regarding the mathematical and cultural dimensions of Muslim geometry, moving beyond broad generalizations to avoid framing this cultural tradition as a monolith. Despite their varied cultural and ethnic origins, Muslims share a cohesive belief system grounded in the Qur'an and the Sunnah, where racial and ethnic differences are reconciled through a common meaning-making system that fosters unity and counters external perceptions of differences. The Islamic art and history are rooted in their common meaning systems.

Positionality and Researcher Reflexivity

As interdisciplinary scholars working at the intersection of mathematics education and literacy studies, we recognize that our identities and professional trajectories shape the conceptualization and interpretation of this work.

Wesam's experiences as a first-generation Muslim American scholar and parent inform her attention to belonging, representation, and the emotional dimensions of mathematical learning. Her children's reports of feeling different and disconnected underscored the need for this work. Her research investigates and seeks to develop methods for training preservice teachers to deliver mathematics instruction with both rigor and relevance within nurturing and empathetic learning environments that validate students' diverse worldviews and backgrounds. Suriati's scholarship critically examines representations of Muslim identities in children's and young adult literature. Drawing on her transnational experiences as a Malay Muslim educator in Singapore and the United States (Abas et al., 2020), she contributes a lens attentive to spatial belonging, cultural legitimacy, and epistemic inclusion. Matt approaches literacy as a civic and communal practice, examining how literature amplifies marginalized voices and fosters inclusive school communities. His work informs the framework's narrative strand and its commitment to counterstorying as a form of epistemic resistance. Lara's work and research as a mathematics teacher educator focuses on the conceptualization of mathematics identity and the role of curricular resources in shaping how learners position themselves as mathematics doers. Her scholarship grounds the framework's emphasis on identity development and disciplinary integrity. Collectively, our interdisciplinary collaboration reflects a shared commitment to reimagining mathematics education as a shared space where narrative, culture, and mathematical rigor intersect.

Theoretical Foundations

The proposed NEFG is rooted in literature that foregrounds the use of picture books as an instructional tool to connect students' worldviews, funds of knowledge, and cultural relevance to rich and meaningful geometry content. In this section, we share the theoretical foundations for this framework

Students' Funds of Knowledge and Culturally Relevant Pedagogy

With the growing, diverse student population in the United States, it is becoming crucial to design mathematical instructional plans and activities that include students' funds of knowledge and how they make sense of the world around them (Moll et al., 1992). Funds of knowledge

(FoK) is a term that was coined by Moll and colleagues and refers to the cultural and experiential insights held by students and their families, especially in multicultural settings. Within mathematics education research, FoK encompasses the varied mathematical understandings and practices that children from diverse backgrounds contribute to the classroom (Altaş et al., 2024). In addition, leveraging students' FoK serves as a vehicle towards equitable, inclusive, and meaningful mathematics education that builds upon the strengths of all students. When students' real-world experiences and cultural funds of knowledge are foundational to teaching, mathematics concepts and ideas become tangible, accessible, and meaningful (Marshall, 2023).

Leveraging students' funds of knowledge is a core tenet of theoretical frameworks that center student identities, such as culturally responsive pedagogy (Ladson-Billings, 1995), culturally sustaining pedagogies (Marshall, 2023), and culturally relevant mathematics teaching (Marshall, 2023; Zavala & Aguirre, 2024). While each framework is distinct in its own way, each of these theoretical perspectives prioritizes a deep consideration for designing student-centered instruction. Culturally relevant mathematics teaching provides students with access to rigorous mathematics learning experiences while also attending to the unique intersectional identities of students in the classroom (Zavala & Aguirre, 2024). Research found that culturally relevant and sustaining approaches to teaching mathematics inform the development of students' robust mathematics identities (Anderson, 2007), increase student achievement and interest in mathematics (Hunter & Miller, 2022), and significantly improve educational equity (Zavala & Aguirre, 2024). Hence, teachers are tasked with creating mathematically powerful learning environments that nurture students to become active participants and doers of mathematics (Huinker et al., 2020).

The construct of the term mathematics identity has gained increasing prominence in mathematics education scholarship, reflecting a broader sociocultural turn in the field (Darragh, 2016; Sfard & Prusak, 2005). While definitions of and perspectives on mathematics identity vary, most mathematics educators and researchers adopt a sociological view of mathematics identity. In this view, an individual consists of multiple dimensions of identity that are formed and reformed as the individual interacts in different social environments (Darragh, 2016; Wenger, 1998). In essence, mathematics identity can be viewed as the story we tell ourselves in relation to how we are positioned in mathematics learning communities (Sfard & Prusak, 2005). Starting elementary school, students begin to define themselves as math learners (Collins, 2018). This self-identity can be negatively impacted by school and community biases, content biases, differences in learning styles, and often untracked biases within instructional processes (Martin, 2013). These factors are all outside of the student's control and are often unreflective of their potential ability as mathematicians. For students to see themselves as mathematicians, they need to be provided with learning opportunities that draw on their funds of knowledge to support their content learning and the construction of robust mathematics identities (Varelas et al., 2012). With these considerations in mind, mathematics classrooms have considerable power in shaping students' mathematics identities, which in turn has a direct impact on students' ability to learn (Zavala & Aguirre, 2024). Historically, traditional mathematics classrooms that emphasize speed and procedural memorization of mathematics concepts have served to marginalize students and perpetuate social hierarchies (Martin, 2013). Increases in culturally responsive mathematics teaching offer promise to create spaces that support improved content learning and identity construction for elementary students. We argue that picture books could be anchored in geometry instruction to create counter-narratives that inform the development of robust mathematics identity.

Ethnomathematics

At the intersection of using diverse cultural contexts and understanding mathematical concepts, D'Ambrosio (1985) emphasizes the importance of ethnomathematics as a culturally inclusive teaching approach. From this perspective, *ethnomathematics* presents possibilities for instruction based on culturally relevant pedagogies. It empowers students through its ability to connect mathematical concepts to students' lived experiences, thereby facilitating deeper comprehension and retention. Ethnomathematics views mathematics as a product of diverse cultural practices and symbolic systems rather than a universal decontextualized body of knowledge (D'Ambrosio & Rosa, 2017; Hunter & Miller, 2020). Empirical and synthesis studies report benefits for engagement, conceptual understanding, and achievement when mathematics instruction integrates cultural content and interdisciplinary context that are relevant to students' real worlds (Averill et al., 2009).

Making clear connections between mathematics and students' culture, communities, and identities takes on multiple approaches and orientations. Research on the use of community processes in teaching and co-creating curriculum materials reports improvement in student belonging and instructional quality for diverse learners (D'Ambrosio & Rosa, 2017). Designing tasks around cultural artifacts (e.g., textiles, symbols, farming activities) offers comprehensible and accessible encounters that make mathematical concepts concrete, relevant, and meaningful, reinforcing inclusion and belonging and dismantling the narrow view of what mathematics is and who is a mathematics doer. A systematic literature review of studies published between 2021 and 2024 found that integrating ethnomathematics hat links mathematical context to cultural motifs and art in teaching plane geometry significantly enhances primary school students' numeracy skills (Zainovi et al., 2025). In turn, this promotes conceptual understanding and increases motivation (Zainovi et al., 2025). Ethnomathematics is especially valuable in diverse societies, accommodating varied cultural and linguistic backgrounds and empowering students through deeper learning. Multiple research studies examined the use of Islamic art as a context to explore Geometry concepts (e.g., Rita et al, 2022, Purniati at al., 2021) in a relevant and meaningful approach that *interlaces* (Rita et al, 2022) mathematics with history, culture, and art. In this way, ethnomathematics explicitly positions the learners' cultural repertoires as a legitimate source for mathematical knowledge, celebrating their histories, culture, language, and ways of knowing, which is critical for their robust mathematics identity development (Hunter & Miller, 2020).

Picture Books in Mathematics Teaching and Learning

Another line of research we draw upon for the NEFG is the use of picture books as an accessible and available means of integrating the aforementioned comprehensible cultural artifacts. Culturally-centered picture books also help build a learner's cultural repertoire, especially as our nation's political and social tensions are felt in classrooms (Ishizuka, 2018). We seek for the use of culturally-centered picture books and mathematics as integral components to a lesson, beyond the traditional use as an instructional hook. Introducing culturally-centered, authentic picture books may shape individual's cultural understandings and engagement with mathematics as they see the significant contributions of math in the larger world. Drawing from Rudine Sims Bishop's (1990) framework for books as windows, mirrors, and doors, we recognize the transformative potential of books to help readers understand themselves, each other, and the world at large. As Nancy Snow is cited with saying, "If we can't put [students of different backgrounds] physically together, we can give them other people's stories and talk about them" (Ishizuka, 2018). While Ishizuka's main argument stems from librarians advocating for diverse book collections, they also recognize the importance of diverse texts being integrated into school curriculum. Within practitioner spaces of education blogs and forums, educators

advocate for the value picture books add to instruction and for promoting classroom community, empathy, and understanding (Short, 2019; Thrower, 2025). Thrower (2025) notes that using picture books as a hook or introduction to a math concept helps students build connections to the content, thus making learning meaningful. There are curricula that are available with read alouds already paired to key standards, but there is also potential to work with trade/mass market books as authentic texts. These texts connect with readers as the protagonists grapple with problems similar to those students face—often in a more realistic way than a traditional math word problem.

Naseef, et al. (2024) found that contemporary picturebooks about Muslim children in the United States function as counternarratives, representing religious practice as integral to daily life and portraying Muslim families as culturally and religiously diverse. Their study challenges the misunderstanding that religion has no place in public school's curricula. Abas et al.'s (2021) extend Bishop's (1990) metaphor through a geocritical framework (Westphal, 2011) to foregrounds space as socially produced (Lefebvre, 1991), historically situated and central to identity formation. Through their critical content analysis of Muslim immigrant picture books, they demonstrate how fictional and real spaces intersect (Westphal, 2011) to shape the processes of adaptation, social negotiation and self-determination. Space, in this framing, is experienced, contested, and imbued with cultural meaning. This spatialized understanding offers a significant extension for mathematics education, particularly in geometry, where space is often abstracted from lived experience. Hence, when geometry instruction is situated within culturally embedded spatial practices such as Islamic architecture designs or tessellated artistic traditions, it reconnects mathematical reasoning (National Council of Teachers of Mathematics, 2000) to the social production of space (Lefebvre, 1991) and to students' identities as meaning-makers (Wells, 1986/2009). When there is no shared foundational knowledge, picture books provide a communal, immediately accessible experience to help students understand the content, local community, or even the larger world.

A complementary insight emerges in Abas (2025) where visual narratives are examined as multimodal sites shaped by power, authorship, and cultural mediation. This insight extends to mathematics classrooms, where geometric diagrams, patterns, and transformations function as multimodal representations that warrant critical interpretation aside from procedural consumption (Rau, 2017). However, integrating literature in such conceptually robust ways requires more than good intentions. Zhang et al. (2023) report that mathematics educators often possess limited pedagogical knowledge for meaningfully incorporating picture books as they perceive them "lack[ing] concrete pedagogical details" (p. 3) around which rigorous instruction can be organized. Purposeful scaffolding is, therefore, necessary to support mathematics teachers in leveraging picture books for both conceptual and cultural understanding. Empirical evidence from Abas (2026) further demonstrates that sustained engagement with Muslim-centered picture books can shift educators' assumptions and instructional dispositions. Through structured dialogue and critical reflections, teachers move beyond symbolic gestures of diversity toward deeper interrogation of identity, bias, and epistemic authority. These findings align with Tondreau and Yang (2025), who supported preservice teachers in using Asian American and Pacific Islander picture books to build community and cultural knowledge. Similarly, Baer (2017) used picture books to develop students' understandings of Muslim cultures, specifically looking at similarities and differences in schooling, play, and clothing.

Taken together, the scholarship suggests that picture books in mathematics classrooms should function as epistemological interventions. When educators integrate Muslim-centered narratives that foreground geometric design, spatial negotiation, and cultural meaning, students

are invited to engage with geometry as both mathematical reasoning (NCTM, 2000) and cultural inquiry (Pinxten, 1991). In doing so, the classroom becomes a space where mathematical understanding, identity formation, and cultural recognition intersect, expanding who is recognized as a producer of mathematical knowledge and broadening the epistemic boundaries of the discipline. The theoretical foundations of ethnomathematics, culturally responsive pedagogy, and funds of knowledge converge on a central imperative: mathematics instruction must center students' cultural practices and histories while interrogating the sociopolitical forces that shape whose knowledge is legitimized. Building on this scholarship, we conceptualize an approach to teaching geometry that intentionally integrates narratives, ethnomathematics and critical frameworks, and mathematical rigor to foreground culture and identity. In the following section, we delineate the intersection of these ideas to outline the strands of the NEFG.

Conceptualizing the Intersection: Geometry, Culture, and Identity

Teaching geometrical concepts like transformations and tessellations through the lens of Islamic histories and Muslim mathematical contributions provides a powerful, culturally sustaining framework that bridges abstract mathematics with tangible human heritage (Huda, 2025; Mubarak & Setyaningsih, 2025). By utilizing picture books and tasks anchored in ethnomathematics, educators can transform geometry into a tool for both academic excellence and social justice (Mubarak & Setyaningsih, 2025; Darragh, 2016; Voss & Rickards, 2016). Additionally, grounding mathematics instruction in the use of picture books facilitates a narrative-first approach that foregrounds storytelling by identifying "rich narratives" to support integrating mathematics with picture books (Russo & Russo, 2018, p. 8). This model fosters student engagement with comprehensive mathematical material by presenting it within relevant and meaningful contexts (e.g., narratives). Hence, picture books serve as instructional tools, rather than add-ons, which harness student imagination to conceptualize worlds beyond their classroom walls where they can solve real-life mathematical problems, such as determining the geometry of a prayer rug or identifying symmetry in architectural ornaments (Russo & Russo, 2018). This approach is also effective for contextualizing and differentiating mathematical experiences while teaching mathematical content. The use of picture books within this approach opens up instructional spaces to integrate diverse onto-epistemologies as well as interdisciplinary content. See Figure 1 for a list of suggested picture books to be used with this approach.

Khan, H. (2022). *One sun and countless stars: A Muslim book of numbers* (M. Amini, Illus.). Chronicle Books.
Khan, H. (2018). *Crescent moons and pointed minarets: A Muslim book of shapes* (M. Amini, Illus.). Chronicle Books.
Khan, S. (2016). *The shapes of Eid, according to me* (M. Ahmed, Illus.). Prolance.
Khan, H. (2015). *Golden domes and silver lanterns: A Muslim book of colors* (M. Amini, Illus.). Chronicle Books.
Winter, J. (2017). *The world is not a rectangle: A portrait of architect Zaha Hadid*. Beach Lane Books.

Figure 1: Recommended Picture Books that Could Be Used with the Narrative First Approach

Alongside the use of picture books, ethnomathematics is an instructional approach in which the mathematical context, task, and activities are rooted in students' cultures, languages, and

histories (D'Ambrosio & Rosa, 2017; Hunter & Miller, 2020). Nieto (2013) advocates for an education that connects the curriculum to the context of students' lived experiences. This empowers student voices and provides an opportunity for teachers and students to learn about, respect, and honor other cultures, experiences, backgrounds, and languages, and expands students' worldviews. Curriculum must move beyond superficial representation and "checking a box" to effectively educate students before issues of otherness or difference are brought into the classroom, or worse, when curricula promote Islamophobic stereotypes (CAIR Foundation, 2025). We must be proactive in creating empathetic spaces in our schools, where students can find commonalities and connections with each other. For Muslim students, this means having a space to pray and even having dietary needs met. Schools should reflect, represent, and celebrate students' histories, languages, and cultures to improve academic success and foster consciousness. Because Islamic art and architecture present a culturally embedded source of geometric patterns (Bier, 2015), we argue that an interdisciplinary instructional approach that foregrounds students' stories, imagination, and histories through picture books and ethnomathematics creates meaningful opportunities for students to learn mathematics. Geometry, in particular, as a mathematical content, lends itself well to cultural and historical connections for Muslim students as a context allowing them to see their identities reflected in the mathematical knowledge (Massarwe et al., 2012). It showcases the mathematical artistry of the Islamic civilization and decolonizes the Eurocentric grounding of the mathematical knowledge.

At the intersection of the aforementioned ideas and theoretical foundations, we propose the NEFG as an interdisciplinary pedagogical model designed to anchor and integrate students' narratives and ethnomathematics to weave culture, identity, and sense-making within mathematical learning. It encompasses three strands that inform each other and are interwoven and enmeshed to create a dynamic instructional space (see Figure 2). While these strands are numbered, they do not reference a hierarchical or processional order. The numbers are a means of helping teachers and researchers to enter into this work, selecting the strand they are most familiar with and expanding from there to the others. We conceptualize these strands as follows:

Strand I: Narratives as Humanizing and Epistemic Entry Points

This strand advocates for a narrative-first approach (Russo & Russo, 2018) that utilizes picture books as an instructional tool to facilitate students' engagement, context relevance, and representation in the curriculum. Drawing on Bishop's (1990) idea of windows, mirrors, and doors, these diverse narratives introduce students to real-life and historical contexts that decolonize the traditional, linear, and neutral views of mathematics. During this strand, the use of picture books rooted in culturally relevant pedagogies elicits students' narratives and storytelling to foreground world views and histories.

Strand II: Ethnomathematics as Cultural Historical Grounding

This strand highlights ethnomathematics as a core instructional approach that disrupts the traditional, Eurocentric, and culturally-neutral notions of mathematics to create ethical educational spaces that celebrate students' counterstories and sense-making. This creates inclusive spaces where students' participation and contributions inform the development of their robust mathematics identities. In this section, students undertake mathematical activities that use real-world contexts, thereby creating opportunities for representing and decolonizing the mathematics curriculum.

Strand III: Mathematical/Disciplinary

This strand is informed by research on how students develop their identities as mathematics *doers* by focusing on ensuring rigorous and meaningful mathematical learning experiences that

lead to mathematical proficiency. In this strand, the teacher facilitates the rigor and the richness of the mathematics learned to foreground conceptual mathematics.

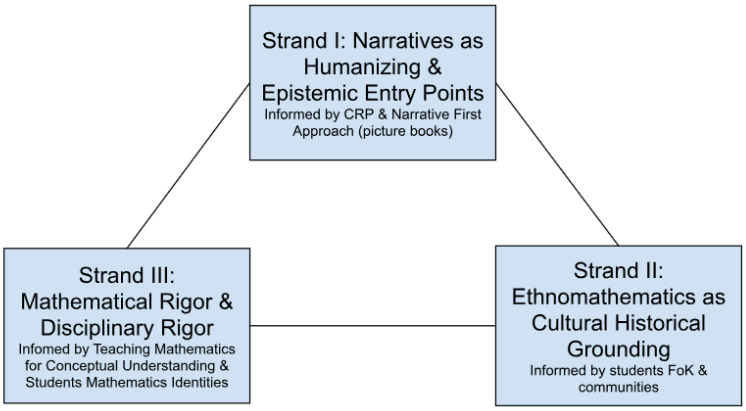


Figure 2: The Narrative-Ethnomathematics Framework for Geometry (NEFG)

We assert that these strands should be evident in any geometry instructional plan that aims at providing a minoritized student population with access, representation, and opportunity to do meaningful mathematics. Additionally, we recognize that the conceptual nature of our framework necessitates a clearer definition of "geometry" to guide application. We align our view with the work of Nurnberger-Haag (2017), which highlights the importance of precision in early shape instruction to prevent the entrenchment of common misconceptions.

Pedagogical Implications

The NEFG offers pedagogical implications beyond mathematics education by providing teachers and mathematics teacher educators with an interdisciplinary model that informs the planning and development of powerful instruction. While we focused on using geometry to connect sociohistorical Islamic contexts for decolonizing mathematics and creating ethical learning spaces for Muslim students, this approach can apply to other subjects and student populations. Integrating ethnomathematics and culturally grounded narratives challenges Eurocentric curricula by recognizing learners' cultural backgrounds—such as Islamic art and architecture—as valid sources of geometric knowledge (D'Ambrosio & Rosa, 2017; Hunter & Miller, 2020). This model reimagines geometry instruction as a culturally sustaining practice that connects abstract concepts like transformations and tessellations to real-world cultural heritage, supporting the development of strong mathematical identities and deeper conceptual understanding (Hunter & Miller, 2020; Zainovi et al., 2025).

Moving Forward

Future iterations of this framework will explicitly define the geometric concepts—such as classification, transformation, and property analysis that our three-strand model aims to support, ensuring that instructors can apply these strands to design tasks that challenge Eurocentric biases while maintaining mathematical rigor. Building on this work, our future research focuses on the application of the NEFG. Also, this work will be extended to the teacher education program curriculum to serve as a guide for preservice teachers to support them in planning and enacting culturally relevant mathematics teaching that promotes identity development and equitable learning.

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PRESCHOOLERS' SPATIAL LANGUAGE WHEN COMPARING CONTRASTING TANGRAM PUZZLES

EL USO DEL LENGUAJE ESPACIAL DE NIÑOS EN EDAD PREESCOLAR PARA COMPARAR DIFERENTES ROMPECABEZAS DE TANGRAM

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This study examines how 18 monolingual and emergent bilingual preschoolers used gesture, language, and movement as they noticed structures while comparing and transforming contrasting tangram figures. Findings indicate that gesture and deictic language functioned as primary resources for indicating similarities and differences, supported by translanguaging. The recognizable figures supported comparison through background knowledge and everyday language. These results highlight the importance of multimodal and multilingual approaches to early geometry instruction and suggest that sequencing tasks from familiar to abstract configurations may support the development of spatial language and reasoning.

Keywords: Geometry and Spatial Reasoning, Early Childhood Education

Spatial reasoning is central to early geometric thinking because it involves noticing structure, coordinating parts within a whole, and supports reasoning about how relations change under movement and reorientation (Clements & Sarama, 2009; National Research Council, 2006). Although state standards do not explicitly isolate spatial reasoning as a separate domain, noticing structure is central to identifying, composing, and decomposing shapes and understanding geometric relationships (Clements & Sarama, 2009). Young children often notice composite figures holistically before they can systematically compare embedded components, track correspondences across two designs, or anticipate the transformations needed to make them match (Clements & Sarama, 2009; Loewenstein & Gentner, 2001; National Research Council, 2006). Emerging research suggests that language can support children's geometric and spatial reasoning (Hallowell, 2020; Owens, 2015) along with experiences focused on structural relationships (Mulligan et al., 2004). The role of language is particularly important to consider within today's multilingual learning environments, where children may draw on multiple linguistic resources while developing mathematical ideas.

Shape puzzle tasks provide opportunities for children to explore how shapes can be composed, decomposed, and transformed. Prior research suggests that transformation can be particularly challenging when pieces appear in non-prototypical orientations and that repeated opportunities to work across orientations can promote flexibility (Bofferding & Aqazade, 2023; Bofferding & Zhu, 2023). Comparison tasks encourage children to notice differences between composite figures may support the development of structural awareness (Schwartz & Bransford, 1998). Yet an important question remains: how do children engage in noticing and comparing structural differences before they have fully developed geometric vocabulary?

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Gesture may play a critical role in this process (Alibali, 2005). Young children often rely on pointing, tracing, and coordinated manipulation to communicate spatial relationships when verbal language is still developing. Gesture and movement can function as cognitive and communicative tools that support reasoning about shape, orientation, and position. Flexible use of multiple languages (translanguaging) may further support multilingual learners' participation and meaning-making during spatial tasks by allowing children to access concepts through whichever linguistic resources are most available (Tai, 2022; Wei, 2018).

This study examines how preschool children use gesture, language, and movement while comparing and transforming composite tangram figures. In particular, we investigate how children indicate similarities and differences between contrasting figures and how they reconcile those differences through manipulation and language. We also explore how translanguaging supports these processes for emergent bilingual learners. By examining children's engagement with both recognizable and abstract composite figures, this study aims to illuminate how multimodal and multilingual resources support early geometric and spatial reasoning.

Theoretical Frameworks

Central Conceptual Structure for Geometry (Central Geometric Structure)

Extending Case's theory of Central Conceptual Structures (1996), Hallowell (2020) describes early geometric reasoning through the coordination of two precursor schemas. An *imagistic precursor schema* supports recognizing prototypes of shapes (e.g., circle, square, equilateral triangle sitting on base) and overall figure appearance, but can limit attention to embedded components and to the magnitude and direction of transformations (Clements & Sarama, 2009; Hallowell, 2020). A *verbal-declarative precursor schema* helps children name and describe shapes and features, often through everyday labels and informal descriptors. (Clements & Sarama, 2009; Hallowell, 2020). By age 6, children integrate their earlier schemas and can discuss shape properties and identify geometric features, though they tend to attend to only one set of features at a time (Hallowell, 2020). They can also locate shapes within shapes and perform transformations more accurately and intentionally, but may not be predicting these transformations in advance (Clements & Sarama, 2009; Hallowell, 2020).

Translanguaging and Multimodality

Translanguaging offers a heteroglossic perspective on bilingualism, emphasizing that emergent bilinguals draw from an integrated linguistic repertoire rather than alternating between bounded "languages" (García & Wei, 2014; Otheguy et al., 2015). In mathematics activity, this flexibility allows children to draw on whichever linguistic resources best support their participation and reasoning. From a multimodal perspective, meaning making extends beyond speech to include gesture, gaze, spatial arrangement, and manipulation (Jewitt, 2008; Kress & van Leeuwen, 2001). In early geometry tasks that rely on pointing, modeling, and transforming objects, linguistic and embodied resources work together to support spatial reasoning. Together, translanguaging and multimodality form the core meaning-making processes in our framework, aligning with the semiotic and cognitive foundations of early spatial reasoning.

Current Study

The current study builds off of prior research by giving young children experiences to analyze and manipulate targeted combinations of shapes. Our research questions are as follows**:** 1) How do preschoolers use gestures and language to indicate the similarities and differences between contrasting tangram puzzle pictures? How does translanguaging support this process for emergent bilinguals? 2) How do preschoolers reconcile the contrasting tangram puzzles'

differences by changing one version of the puzzle into the other through movement and language?

Methods

Participants and Setting

The participants for this study included 18 preschoolers (ages 3–5) who attended a university preschool program or a play-oriented daycare. Preschoolers' home languages included Mandarin (n=8), English (n=4), Spanish (n=3), Tamil (n=1), Farsi (n=1), and Odia and Hindi (n=1). Instruction at the preschool and daycare was provided in English. During the interviews, we gave multilingual Spanish and Mandarin speakers the option to speak in their home language or English as they wanted. The Spanish speakers primarily chose to respond primarily in Spanish, sometimes switching to English. Most Mandarin speakers responded in English; only one Mandarin-speaking pair regularly used Mandarin. Within this pair, when the interviewer shifted between English and Mandarin, the children sometimes matched the interviewer's language in the moment. Preschoolers chose their own pseudonyms.

Procedures

The data for this paper come from session two of a four-session study focusing on preschoolers' spatial language, reasoning, and their composing of shapes. In session one, we identified preschoolers' baseline ways of talking about and working with shapes and space (see Bofferding et al., 2025). In session two, the focus of this paper, we worked with preschoolers in pairs–two students worked along with an interviewer–and had them complete two main tasks together: a tangram puzzle comparison task and a copy-cat task. Our focus here is on the former task. For the tangram puzzle comparison task, we showed each pair two versions of a completed cat tangram puzzle (see Figure 1A) and asked, "What's the same and different between the two puzzles? *(¿Qué hay de diferente entre los dos rompecabezas?* "*这两个图形有什么相同和不同吗？*"). "How are these puzzles put together differently?*" (¿Cómo están armados estos rompecabezas de manera diferente?* "*这两个拼图是如何被不同地拼起来的?")*. After they compared them, we set up tangram pieces as shown in the right picture and asked them to change it so that it matched the left picture. "Can you make them the same?" *(¿Cómo los haces iguales?* "*你们能把它们俩变得一样吗?*") We repeated this process for a second set of hourglass tangram puzzle pictures (see Figure 1B). Between the two cat puzzles, the images differed in the orientation and placement of the orange parallelogram and in the orientation of the small blue triangle, emphasizing differences in non-embedded pieces but highlighting relative positions. In the hourglass puzzle, the differences were clustered in the bottom-right portion of the image, emphasizing differences in embedded pieces and requiring attention to shapes within a larger configuration. Due to space, we focus on the results for the cat puzzle in this paper.



Figure 1: A. Contrasting Cat Tangrams and B. Contrasting Hourglass Tangrams

Analysis

After transcribing the sessions, we coded the transcripts in Dedoose (Version 10.0.59) using the language framework developed by Cannon et al. (2007), which we modified to include subcodes related to gestures and if they are self-corrected. When coding, we coded the entire interaction where the language or gesture happened so we could capture the context. We then looked across transcript codes for how they talked about and gestured related to similar areas of the puzzle and grouped them by similarities, differences, and attempts to make them look the same so we could identify trends and differences across the pairs.

Table 1: Language Codes (based on Cannon et al., 2007)

Code	Description and Examples
Code-switch	Participant alternates between two or more languages or varieties of language in conversation.
Translanguaging	Participant draws upon different linguistic, cognitive and semiotic resources to make meaning and make sense.
Gestures	Participant uses a gesture rather than language to communicate their answer to a question. The subcode *two-handed* indicates when they used both hands simultaneously.
Background knowledge	Words that make connections with children's everyday world/self.
Continuous amount	Words that are used to describe continuous quantities.
Deictics	Words that are place deictics/ pro-forms (i.e., these words rely on context to understand their referent). The subcode *generic shape* indicates the deictics took the place of a shape name.
Location/direction	Words that describe the relative position of things in space. The subcode *using mat/manipulative* identifies instances where location was referred to relative to the images.
Shapes	Words that describe the standard or universally recognized two- and three-dimensional objects and spaces.
Spatial dimension	Words that describe the size of objects and spaces.
Spatial features/ properties	Words that describe the features and properties of objects, spaces, people. The subcode *color* indicates a non-mathematical property of the shape.
Spatial orientations/ transformations	Words that describe the relative orientation or transformation of objects in space.
Comparisons	Words that indicate what is the *same*/*similar*, *different*, and *not same* between objects or shapes.

Results

From Holistic Recognition to Structural Comparison

When shown the initial set of contrasting "cat" tangram puzzles, four pairs identified both images as cats, and one called each a fox. Interestingly, five pairs gave different names to the two images (e.g., cat versus dog, cat versus baby, cat versus cow, or fox versus bed or bird), suggesting that differences were noticed early even at the level of whole-figure interpretation. In describing similarities, participants frequently drew on background knowledge and labeled meaningful parts of the figure. For example, Mister stated, "They have ears, bodies, legs, and tail," indicating that the animals shared the same parts and treating the composite shapes as a coherent organism rather than as a set of geometric components. Several pairs similarly referenced the "body" as the same (often pointing to the large triangles), indicating attention to global structure prior to attending to embedded shapes. However, prompts to compare specific details supported a shift from holistic recognition toward structural comparison; when asked whether the ears were "exactly the same," Mister indicated they were not, often pointing to the corresponding ears as he compared them.

Gesture as a Resource in Comparative Reasoning

Across the pairs in our dataset, all noticed the two differences, with six pairs and one individual noticing the triangle ears as different first and two pairs and one individual noticing the parallelogram tails as different first. No pairs needed prompting to notice that the tails of the two cats were different. When asked what was different, Marfin and Snow first pointed to the ears in the right-hand cat (see Figure 2A). When asked whether the turned-down ear was the same as the other cat's ear, Snow shook her head and responded, "不是" *("No")*. While two pairs required an explicit prompt to state whether the ears were the same or different, most pairs made this comparison without prompting.

Across episodes, participants often coordinated pointing with spatial orientation language when describing ear differences. For instance, Superman stated, "the ear is down, and the ear is up" while pointing to each ear in turn. Deictic talk (e.g., "this one") commonly appeared alongside pointing, suggesting that gesture and deictic reference served as key resources for comparative reasoning when precise shape names were limited.

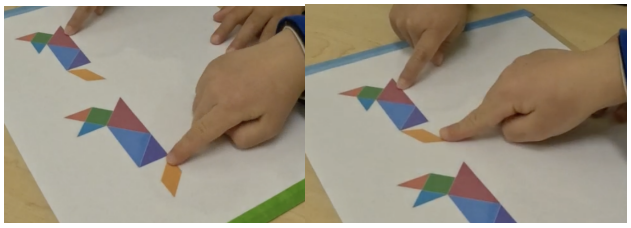
A.  B. 

Figure 2: Joint Pointing of A. Marfin and Snow and B. Big Train and Small Train

Participants also described tail differences readily, typically using location and orientation language (e.g., top/bottom; up/down) in coordination with gesture. For example, Pumpkin explained, "This one's on the top… and this one's connected by the bottom," while moving her finger between the two figures. In another interaction, Big Train interrupted his peer's Small Train repeated deictics ("This is…this is…") to localize the contrast: "No! Because of these!" while pointing directly to the two parallelogram "tails", isolating a specific region of difference (Figure 2B). Across these episodes, pointing helped treat familiar part-based interpretations (e.g.,

animal parts) as comparable units, enabling pairs to decompose the figure and sustain comparisons of orientation and placement.

Translanguaging as Conceptual Access in Spatial Reasoning

Translanguaging practices were observed in connection with both shape naming and spatial reasoning, allowing emergent bilingual participants to draw flexibly on their full linguistic repertoires while engaging with the task. For example, Bella, a Spanish-speaking preschooler, used both English and Spanish labels when discussing the cat's head, referring to the shape as both "square" and "*cuadrado*." Interestingly, she first referred to the square as a "rectangle," demonstrating both her bilingualism and her background knowledge, since "rectangle" was not a shape explicitly introduced during instruction. She also used both "triangle" and "*triángulo*," fluidly combining languages in expressions such as "*un* triangle *chiquito*" ("a small triangle"), which she repeatedly to explain the differences between the two pictures and to describe how to make them the same. These examples suggest that translanguaging supported access to geometric reasoning by allowing participants to draw on whichever linguistic resources were most available at the moment.

Translanguaging also supported participation and conceptual access through flexible shifts between languages within interaction. In an exchange between Rosa and Olivia, two Spanish-speaking preschoolers, Rosa initially recognized the figures by saying "*¡Son gatitos!*" ("They are kitties!"). When Olivia hesitated, the researcher switched to English, prompting Rosa to respond again in English, which Olivia then adopted ("Cat!"). This fluid, dynamic bilingualism appeared to lower participation barriers and enabled both participants to engage in subsequent reasoning about orientation and rotation. For example, Rosa later explained, "*Su oreja está torcida…pero su otra oreja no está torcida*" ("This one's ear is crooked, but this other one's ear is not crooked"), demonstrating awareness of differences in orientation between corresponding shapes.

Across bilingual interactions, spatial language related to orientation frequently emerged through coordinated gesture and translanguaging. For instance, Big Train explained the difference between the two ear triangles by stating, "This is low, and this is at the top," (Figure 3), and when asked to repeat, reiterated, "This is down, this is up," while pointing between the two ears. Similarly, Bella indicated "*La colita es para arriba*" ("the little tail is up") while pointing to the parallelogram in the left picture and then followed with "*Y esta es para abajo*" ("and this one is down"), making a downward motion with both index fingers to mark orientation. In these examples, spatial language (up/down; *arriba/abajo*) was tightly coordinated with pointing and vertical tracing gestures, suggesting that participants encoded differences primarily through orientation rather than through geometric naming.

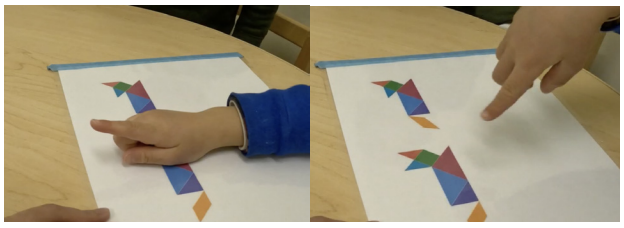


Figure 3: Big Train Indicating Up (left) and Down (right)

A similar pattern was observed in interactions conducted in Mandarin. For example, Snow stated, "这是下, 这是上" ("This one is down; this one is up") while pointing between two

parallelograms and tracing along each piece's slanted edge with her finger. When prompted to extend the comparison, Marfin applied the same language to the ear triangles, again labeling one as "up" and the other as "down" while pointing between them. This sequence suggests that participants used translanguaging to map relational concepts across different components of the figure, extending the same orientation framework from ears to tails.

Together, these interactions highlight how translanguaging supported participation, comparison, and transformation reasoning within the task. Rather than relying solely on one language, participants flexibly combined linguistic and gestural resources to anchor deictic references ("this one") to specific regions of the figure and to communicate spatial relationships. Gesture appeared to function as a stabilizing resource across languages, enabling participants to maintain coherent spatial reasoning even when lexical access varied.

Reconciling Differences Through Movement and Language

When they tried to fix the cat on the right to look like the left one, participants often combined manipulation, gesture, and language to reconcile the differences. Pinky suggested, "Turn it the same as this one," while pointing between corresponding ears (the small blue triangles) on the two figures, indicating awareness that orientation changes could resolve the discrepancy. In several cases, movement preceded verbal explanation. For example, Butterfly rotated a parallelogram to match the target orientation while Mister adjusted its placement, commenting, "A little downer," and sliding the piece to align with the base. These interactions illustrate how embodied actions supported spatial reasoning during the transformation process, consistent with the idea that manipulation and gesture can mediate early geometric reasoning. Their discussion of how to align the tails focused mostly on location and orientation.

Discussion and Implications

In our dataset, gesture and deictic language played a central role in how children indicated similarities and differences and reconciled the contrasting tangram images. Within these interactions, children frequently relied on pointing and expressions such as "this one," "here," or "there," often before or in place of specific shape labels**.** From a Central Geometric Structure perspective, this pattern is consistent with children recognizing the overall structure of the figure while still developing verbal resources for explicitly isolating component parts and describing transformations (Clements & Sarama, 2009; Hallowell, 2020). In these episodes, pointing helped stabilize reference to specific parts and supported aligning correspondences across the two images. These findings suggest that gesture functioned as a primary semiotic resource that supported spatial reasoning when lexical resources were still emerging. Rather than viewing imprecise language as a limitation, the coordinated use of gesture, manipulation, and speech reflects a multimodal process through which children constructed meaning. This pattern aligns with prior research suggesting that embodied actions can mediate early geometric reasoning and support the development of more formal spatial language over time (Ehrlich et al., 2006; Iverson & Goldin-Meadow, 2005; Goldin-Meadow, 2009; Goldin-Meadow et al., 2012).

The recognizable animal structure of the cat puzzles appeared to serve as a semantic anchor that allowed children to leverage background knowledge to identify and compare parts. References to ears, tails, and bodies enabled children to decompose the composite figure even when geometric vocabulary was limited. These everyday part labels can be understood as verbal–declarative resources that supported decomposition of the composite figure, even when conventional geometric naming was not available (Clements & Sarama, 2009; Hallowell, 2020). More abstract configurations (e.g., the hourglass puzzles) may lack familiar referents and may

initially invite global judgments of sameness; future work could examine whether such tasks also support more explicit analytic reasoning about orientation and placement.

Prompting also shaped children's comparisons. In this task, questions that directed attention to specific components or asked how to make the two figures the same were often associated with shifts from holistic judgments toward part-by-part comparison and transformation talk. This finding extends work showing that guided comparison supports alignment of representations and noticing shared relations (Loewenstein & Gentner, 2001) and aligns with developmental perspectives on coordinating imagistic and verbal–declarative resources through guided experiences (Hallowell, 2020).

Translanguaging further supported participation and reasoning for emergent bilingual children. In our data, flexible language use expanded children's moment-to-moment access to spatial terms during comparison and repair, while gesture helped maintain shared reference across language shifts. Together, these findings reinforce viewing multilingual resources as assets for mathematical meaning-making.

Overall, our results highlight the value of multimodal and multilingual resources in early spatial reasoning. For task and instructional design, the findings suggest prompts that guide children from whole-figure identification to aligning corresponding parts, legitimizing deictic talk paired with pointing, and leveraging relational language about orientation and location (e.g., up/down; top/bottom) and translanguaging as bridges toward more conventional geometric naming. Future research could examine how sequences of familiar and abstract spatial tasks shape the development of geometric language and transformation reasoning over time, particularly in multilingual learning environments.

Acknowledgments

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HIGH SCHOOL TEACHERS' COORDINATED USE OF A CHATGPT ASSISTANT AND GEOGEBRA IN MATHEMATICAL PROBLEM-SOLVING ACTIVITIES

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Recent developments of generative artificial intelligence (GAI) tools have opened intense discussions regarding their use in mathematics education. We argue that teachers' consistent and coordinated use of these resources opens novel paths to complement dynamic problem-solving strategies. The research question that guided the study was: What ways and patterns of reasoning do high school teachers develop when they coordinate a customized ChatGPT problem-solving assistant with GeoGebra to work on mathematical problems? Eight practicing high school teachers worked on a set of problems that involved concepts that appear in algebra, geometry and calculus courses. Results indicate that coordinated use of a customized Socratic Problem-Solving Assistant and a Dynamic Geometry System (GeoGebra) allowed the participants to transit among visual, geometric, and analytic reasoning to develop concepts understandings and mathematical problem-solving competencies.

Keywords: Mathematical Problem Solving, GeoGebra, ChatGPT, Problem-Solving Assistant, High School Teachers

Introduction, Background, and Research Questions

Humans have always designed and use a diversity of tools to cope with their physical environment as well as with the functioning of their cultural environment and mental work including mathematical teaching and learning scenarios. What tools and how can teachers and students use them to understand mathematical concepts and to develop problem-solving competencies? Ruthven (2019) introduces the construct "resource system" to refer to, "the various mathematical tools and curriculum materials in use in the classroom and the way in which their use—individually and collectively—is organized and made functional" (p. 51). To understand how teachers develop a productive way to rely on a variety of tools, it is necessary to analyze the tool appropriation process through which learners internalize a set of affordances to represent, explore, and solve mathematical problems (Santos-Trigo, 2024). Within digital environments, resources like Dynamic Geometry Systems (DGS) and Generative Artificial Intelligence (GAI) apps can be coordinated to nurture and frame mathematical practices. To manage the complexity of these human and machine interactions and to guide the mathematical command process, this coordination can be conceptualized through the lens of instrumental orchestration (Trouche, 2004).

In this study, we focus on documenting the extent to which high school mathematics teachers integrate the coordinated use of a customized ChatGPT Problem-Solving Assistant (PS-A)—acting as a Socratic tool—and GeoGebra as a dynamic environment in their teaching practices. Thus, the research questions that guided this study were:

What ways and patterns of reasoning do high school teachers develop and apply when they coordinate the use of a ChatGPT problem-solving assistant with a Dynamic Geometry System (GeoGebra) to work on mathematical problems? And how could their emerging problem-solving experiences orient their teaching practices?

Essentials of a Conceptual Framework: The Relevance of Subjects' Tool Appropriation in Mathematical Problem-Solving Activities

Schoenfeld (2011) argued that to understand and explain what people do when they are involved in "well practiced" activities, one needs to delve into individuals or group's goals, the type of resources that people bring to a situation including their base knowledge and tools, and people's orientations that are expressed in terms of their belief about mathematics and their teaching. An overarching goal in teaching and learning scenarios is to provide conditions for learners to engage them in mathematical practices to understand and construct concepts and to work and solve mathematical tasks. To this end, teachers and students rely on diverse tools to represent, explore and grasp concepts or make sense of problem statements and to look for different ways to work on mathematical tasks and to communicate, and extend mathematical results (Santos-Trigo, 2024). In this context, students' resources or tool appropriation is a key process for them to transform resources or tools into an instrument to achieve mathematical goals or to work on mathematical tasks. This subjects' tool appropriation is relevant because it shapes their thought process and actions involved in approaching mathematical tasks. Teachers and students conceive of or think of those resources as a set of affordances that they rely on to engage in rich collections of mathematical activities and practices. Thus, subjects' thinking system gradually constructs cognitive schemata or some consistent patterns of behavior that are applied to perform mathematical tasks. Reintroducing digital technologies into this structure not only enhances and expands the subjects' pathways of reasoning about approaching mathematical tasks, but also provides novel routes for them to represent, explore, solve, and communicate problem solutions or results (Moreno-Armella, 2021; Santos-Trigo, 2024). Arcavi (2020) pointed out that "some tools not only enhance our capabilities by 'augmenting a signal' but they may also enable us to do completely new things that one could not undertake without them" (p. 422). To frame the coordinated use of GAI tools and GeoGebra affordances, it is essential to leverage what tools' affordances subjects activate to work on mathematical tasks:

1. Dialogic-Reflective Reasoning (GAI Affordances): The use of GAI tools privileges the use of natural language to establish a continue dialog between the tool and the user to explore and engage in different ways of reasoning while dealing or approaching mathematical tasks. Focusing on the design of a ChatGPT assistant implies to guide the problem solver to navigate through all problem-solving phases (Polya, 1945) instead of receiving a direct answer or problem solution. Through a Socratic approach, subjects engage in a productive dialog by answering sequential questions posed by the PS-A throughout all problem-solving stages. This category systematically evaluates how participants formulate, revise, and refine prompts, analyze tool outputs, and deploy metacognitive skills to deal with potential tool hallucinations.
2. Dynamic-Geometric Reasoning (DGS Affordances): GeoGebra affordances offer teachers and students an opportunity to model and explore mathematical concepts and problem dynamically by moving orderly objects within the model. This exploration becomes important for learners to transit among visual, numeric, geometric, and algebraic representations to detect and analyze invariants or patterns associated with mathematical relationships.

The dual analysis that reconciles learners' pencil and paper and digital work opens innovative routes for learners to represent, explore and connect visual and geometric meanings of concepts with objects' mathematical properties and results (Moreno-Armella and Santos-Trigo,

2016). Under this framework, a new epistemology of school mathematical knowledge emerges, expressed through combining visual, empirical, graphic and analytical approaches to identify and support mathematical relations and solution paths. This is manifested in how the ways of conceiving mathematical objects evolve in terms of the user's activation of technology affordances to explore concepts and problem solutions, allowing individuals to experience a distinct materiality through the malleability and plasticity of digital representations (Moreno-Armella and Hegedus, 2009).

Research Design, Participants, and Procedures

The nature of this study is qualitative since there is interest to delve into the participants' problem-solving experiences and sense-making processes to work on mathematical tasks through the coordinated use of a PS-A and GeoGebra. Eight practicing high school teachers participated voluntarily in a problem-solving seminar within a professional development program. The seminar included face-to-face weekly problem-solving sessions held during one semester for 4 hours each session. During the initial sessions, the participants reviewed recent research on GAI tools and focused on the collaborative design of a problem-solving assistant. Indeed, the option of creating customized GPTs was officially introduced by OpenAI through providing templates for configuring ChatGPT for specific tasks (OpenAI, 2023). Guided by the seminar coordinators, the participants introduced specific instructions, parameters, and information into each component associated with the configuration template provided by the platform to ensure the PS-A worked and behaved as a Socratic tutor. Figures 1 and 2 show part of the information that the group of participants introduced into the assistant template and its initial interface.

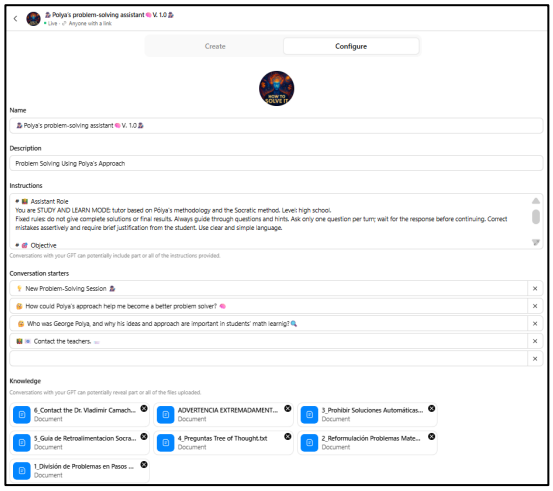


Figure 1: Configurating the Problem-Solving Assistant

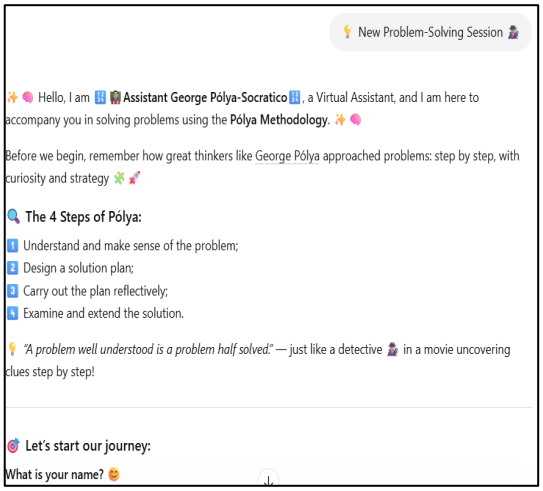


Figure 2: Initial User Interaction with the PS-A, Focusing on Polya's Stages

Although the course coordinators guided the process of introducing pertinent information to design the PS-A, the participants actively contributed to the creation of the assistant collectively and subsequently used it in pairs during the problem-solving sessions to approach the tasks in combination with GeoGebra.

Data Collection and Data Analysis

The unit of analysis included the pairs' problem-solving work that was registered in the problem-solving assistant chatbox conversation and written reports of the use of GeoGebra to work on the task and episodes of the collective discussion during the pairs' presentation to the group. To analyze these data, we applied a coding scheme focused on identifying patterns of reasoning (visual, empirical, geometric and analytic) and tracking the participants' transitions through Pólya's problem-solving phases. The integration and triangulation of these data during the analysis contributed to the trustworthiness of the study.

Presentation of Results

To present the main results of the study, we analyzed what the participants did during the solution process of the circle problem. This task is representative of the type of problems that the participants worked on during the sessions. Data came from the dialog that pair-1 reported during its interaction with the PS-A and the pair-3's use of GeoGebra to work on the same task. The approaches involved the use of the PS-A that guided the participants to navigate throughout problem-solving phases and the use of GeoGebra to dynamically represent, explore, and solve the task. Thus, pair prompts and answers to the problem-solving assistant determined transitions through problem-solving stages.

The Circle Task

A circle with center O and a radius of five units is drawn, a line that passes through the center of the circle intersects the circle at points A and B. P is any point on the radius OB. Draw a perpendicular line to OB through point P. This perpendicular intersects the circle at points Q and R (Figure 3). When point P is moved along radius OB a family of triangles AQR is generated, find a position for point P where the area of triangle AQR reaches its maximum value.

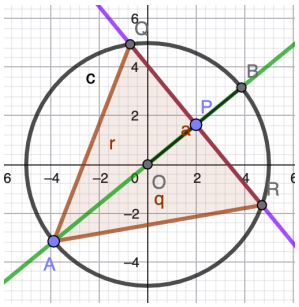


Figure 3: The Circle and a Family of Triangles

The problem statement was the initial prompt introduced to the PS-A which guided pair-1 to follow Polya's phases to work on the task. Pair-1 presented this dialog to the whole group in which all participants identified and discussed relevant moments related to the process. In Table 1, the first column shows a selection of prompts provided by pair-1 and comments to contextualize them. The second column includes the assistant responses. It is important to observe that the PS-A indicated when the pair could transit from one phase to the next one based on the pair inputs (a complete dialog between pair-1 and the PS-A can be consulted in this link).

Table 1: Excerpts from Pair-1 Dialog with the PS-A

Pair-1 Prompts or Answers and Comments	The Problem-Solving Assistant Answers
When pair-1 introduced the circle task statement, the words "the area" after the term "where" was missed; but the assistant directed the pair to the understanding phase to clarify the statement.	✨🧠 Hello, I am 🔢👩‍🏫**Assistant George Pólya-Socratico**🔢, and I will guide you step by step like a true problem-solving detective! 🕵️‍♂️💡 Let's recall the **Pólya Method**: 1️⃣ Understand the problem 2️⃣ Design a plan 3️⃣ Carry out the plan 4️⃣ Reflect and extend
At this stage, the Assistant posed a question about key elements of the problem, and the pair's answer was: A circle, a movable point P on a radius, perpendicular to the radius through P and a family of triangles associated with the position of P The pair wrote: *From the family of triangles* AQR *to locate a position for P where the area of triangle* AQR *gets the maximum value*	*What are the key elements (data and geometric objects) given in the problem*? —Then, the Assistant asked to focus on specific figure: **What exactly is the problem asking you to determine about triangle AQR?** The Assistant answer was: *You clearly identified the goal:* ***maximize the area of triangle AQR****. That's powerful* 💪📐 ✅ **Conclusion of Step 1:** You understand both the elements and the objective. —Here, the assistant posed questions related to the plan design
The pair answer was: *First, drawing the circle with center at the origin of the Cartesian system. That is, its equation will be x^2 + y^2 = 25* The pair wrote: *The coordinates for P could be* (*x,* 0) *and the coordinates for A and B will be* (-5, 0) *and* (5, 0). *The* Q *and* R *coordinates will be* (*x,* (25-*x*^2)^1/2 *and* (*x,* -(25-*x*^2)^1/2 *respectively the base will be the length of* QR *and the height the length AP* This dialog led pair-1 to express the triangle area as: $A(x) = (5 + x)\sqrt{25 - x^2}$ and to rely on calculus concepts to determine that when x = 2.5 triangle AQR reaches its maximum value. In addition, an emerging property was identified: The solution corresponds to the midpoint of OB. Pair-1's answer was:	STEP 2: Design a plan 👉 ***Question (only one!):*** ***Can you think of one way to express or calculate the area of triangle AQR in terms of the position of point P?*** Great idea! You're thinking like a true mathematician—bringing geometry into coordinates is a powerful strategy, just like turning a real-world problem into a formula 💡🚀 Placing the circle at the origin is an excellent start: $x^2 + y^2 = 25$ ***How would you represent the coordinates of point P on the radius OB in this coordinate system?*** 🤔 👉 ***Question:*** ***If the vertical line through*** $P = (x, 0)$ ***meets the circle*** $x^2 + y^2 = 25$***, what coordinates do you get for*** Q ***and*** R***?*** 🤔 👉 ***Question:*** ***Using triangle*** AQR***, what could you choose as its base, and what would its corresponding height be?*** 📐🤔 👉 ***Question:*** **What strategy do you want to use next to maximize** $A(x) = (5 + x)\sqrt{25 - x^2}$**: derivative, squaring the expression, or another idea? Briefly justify your choice.**

because one always finds a way to translate or move the Cartesian system keeping invariant the structure of the problem	✅ Step 4 is complete. Very good. Now let's see if you can extend this reasoning. 👉 Question: What is the geometric position of P on radius OB when $x = 2.5$? 🤔📐 —The Assistant posed a final conceptual question: 👉 **Final Question:** **Why does placing the problem in a convenient coordinate system NOT change the final answer of the optimization problem?** 🤔📘

Comment: There is evidence that the questions and guidance provided by the PS-A to approach the task not only directed pair-1 to navigate through problem-solving stages; but also, they became relevant for pair-1 to think of the task in terms of concepts that were relevant to construct the area model of the problem. Situating the center of the given circle at the origin of the Cartesian system was a key strategy to reduce the task complexity. Thus, applying calculus concepts (derivative, critical points, etc.) helped pair-1 to solve the task and to identify that the solution corresponded also to the midpoint of OB. In the group discussion, it was recognized that the geometric meaning of the derivative as the slope of the tangent line of the area function could be further explored with the use of GeoGebra.

The coordination between resources was organically supported within the seminar's framework. Rather than positioning one tool as subordinate to the other, the learning environment encouraged participants to transit between the discursive-analytic register of the PS-A dialogue and the dynamic-visual register of GeoGebra. This interplay facilitated a synergistic relationship where the algebraic properties and answers (or Socratic answers) from the chatbox conversation continuously informed and were enriched by the geometric configurations and variation parameters explored within the dynamic environment.

A GeoGebra approach to this circle task. During the pair-3 presentation and discussion with the whole group, the participants focused on representing a geometric model that included the task conditions. With center at the origin O, a circle with a radius of five units was drawn. A line that passes through the center O intersects the circle at points A & B, and P is a movable point on radius OB. When point P is moved along OB a family of triangles AQR is generated. How does the area of this family of triangles behave when point P is moved along OB? To visualize this, point S was drawn with *x*-coordinate the same *x*-coordinate of point P and as *y*-coordinate the corresponding area of triangle AQR. Indeed, the GeoGebra affordances allow to graphically visualize the area variation generated for any position of point P (Figure 4). The purple graph represents the locus of point S. During the group analysis, it was observed that for any point T on the locus and a tangent line to this locus at T, its slope varies; when the tangent line becomes parallel to the *x*-axis, the slope value is zero, reaching its maximum value.

Comment: During the group discussion, it was recognized that the use of GeoGebra offers affordances to visualize key elements of the task to make sense of the problem statement. The group discussion led the participants to determine that at point M (midpoint of OB), the area reaches its maximum value of 32.48 units and at this point the slope of the tangent line gets the value of zero (Figure 4). In addition, another result emerged: Triangle ADE at this position becomes an equilateral triangle. At this stage, the participants contrasted both approaches to this task and argued that while the PS-A seems to privilege the construction of an algebraic model of the task and its analysis through calculus concepts, the GeoGebra approach focused on the geometric meaning of involved concepts to visualize the behavior of key mathematical relations.

More importantly, the coordinated use of both tools allowed the participants to connect and transit between algebraic and geometric ways of reasoning, linking symbolic expressions with dynamic visual representations of variation. In this process, the tools did not operate as isolated resources; but rather as complementary mediational systems that supported and expanded the participants' opportunities to make sense and coordinate symbolic and geometric approaches.

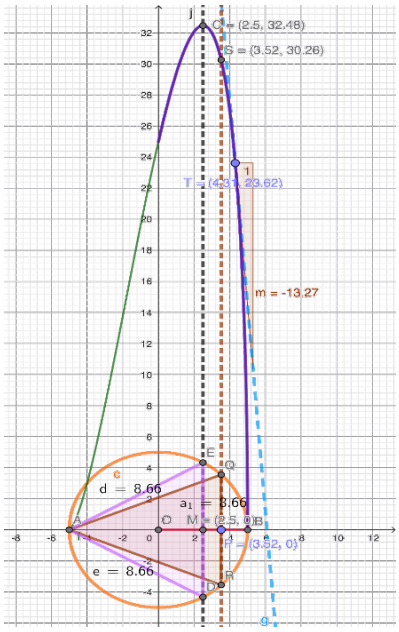


Figure 4: A Dynamic Model to Determine the Maximum Value of Area Variation of the Family of Triangles AQR

Discussion and Final Remarks

The significant and constant digital developments including GAI tools have generated intense debate on not only what role they should play in students' mathematical learning, but also, what adjustments or reforms are needed in mathematical curriculum and teachers' education programs to fully integrate those tools in learning environments. In this study, high school practicing teachers had an opportunity to integrate two tools in problem-solving activities: a PS-A that does not provide direct solutions but guides users to navigate through problem-solving phases via a Socratic approach, and GeoGebra to dynamically represent, explore, and solve mathematical problems. In this coordinated use, participants recognized that their interaction with the PS-A demanded that they constantly refine and review the prompts introduced to engage in a productive conversation with the tool to transit through problem-solving stages. There is evidence that the pairs' interaction with the PS-A directed and shaped their ways to decide what concepts and strategies to follow to work and solve the task. Designing prompts that effectively elicit the desired responses from LLMs is a complex task that often requires iterative refinement and domain expertise, a complexity that "can be a barrier for educators who may not have the resources or expertise to develop and evaluate optimal prompts" (Chen et al., 2024, p. 10).

Likewise, the participants relied on the use of GeoGebra affordances to model and explore problems dynamically. To this end, they needed to think of mathematical concepts and problem statements in terms of geometric properties or meaning. For instance, in the circle task, the

participants associated the position of point P on segment OB with the corresponding area of the generated triangle AQR at that position to delve into the area variation of the family of triangles. It becomes relevant to discuss what changes the coordinated use of several digital tools brings to the study of mathematical concepts. The construction and exploration of dynamic configurations offer a distinct path to study optimization problems. A salient feature is recognizing that the formulation of prompts and ways to analyze the tool's outputs are key activities that learners need to engage in to gradually construct robust prompts that guide them in developing problem-solving competencies. Learners need to problematize the tool's outputs to make sense of what is involved and what part of the output needs further explanation.

A distinguished feature of GAI tools is that users can formulate prompts in natural language. This affordance not only offers an opportunity to extend the types of questions the user wants to explore but also to engage in problem-posing activities that emerge during the task solution process. For instance, the dialogue with the PS-A led the participants to transform the initial geometric problem into a variation task that required calculus concepts to solve it, information that they subsequently modeled geometrically (with GeoGebra) to extend the initial nature of the task.

Finally, our exploratory findings suggest that teachers' interaction with these tools can influence and help shape their approaches to mathematical tasks. By engaging in this dual integration, the participating teachers showed indications of developing aspects of Dialogic-Reflective and Dynamic-Geometric reasoning, while reflecting on the cognitive demands that might be required to orchestrate these resources in their classrooms. The coordinated use of a Socratic PS-A and GeoGebra could support a transition in mathematical activity from merely finding an answer toward a multi-representational inquiry process, suggesting tentative pathways to consider for future pedagogical practices. Looking forward, this study outlines potential dimensions for a future research agenda. First, exploring the design of specialized assistants tailored for punctual tasks could provide insights into how educators might guide tool behavior around specific learning objectives. Second, further research could examine instructional workflows where diverse digital tools are systematically combined to support mathematical goals. In this regard, studying how teachers and students engage with these combined digital environments represents an open avenue to further clarify the cooperative role of Generative Artificial Intelligence and Dynamic Geometry Systems in mathematics education.

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TRANSFORMING VISUALIZED SPATIAL REASONING INTO GEOMETRIC JUSTIFICATION: A RANDOMIZED PILOT OF GEOGEBRA-BASED DYNAMIC GEOMETRY SIMULATIONS FOR PROOF PREPARATION

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This randomized pilot examined whether GeoGebra-based dynamic geometry simulations improved Grades 7–8 students' 2D–3D spatial reasoning and proof preparation. Twenty-eight students were randomized to simulations or parallel non-dynamic instruction across nets, cross-sections, and congruence. Baseline and immediate post-assessments showed strong rater agreement on constructed-response subtotals (ICC pre = .97; post = .92). ANCOVA controlling for baseline favored simulations (adjusted M = 14.38 vs. 11.34), F (1, 25) = 42.08, p < .001, partial η^2 = .627, with consistent gain-score difference (Δ = 2.00), t (26) = 2.985, p = .006. Results indicate promising short-term effects and inform future retention and transfer analyses.

Keywords: Dynamic Geometry Simulations, 2D–3D Representational Coordination, Spatial Reasoning, Geometric Justification

Purpose

Geometry is a core strand of secondary mathematics and a gateway to abstract reasoning (Noor & Alghadari, 2021). Many students struggle with proof because it requires coordinating spatial reasoning, logic, and multiple representations, while typical instruction often provides limited support for unfolding solids, anticipating cross-sections, and justifying diagram-based claims (Haj-Yahya et al., 2022; Fujita et al., 2020; Okamoto & Romo, 2015; Sudirman et al., 2023). This pilot tests whether GeoGebra-based dynamic simulations improve students' spatial reasoning and geometric argumentation for proof preparation, compared with parallel non-dynamic instruction, by strengthening 2D–3D representational coordination through tasks on nets, cross-sections, and congruence. Given limited classroom-embedded randomized evidence, we estimate immediate effects to inform a future delayed-post phase examining retention and within-domain transfer (Adams et al., 2022; Zhang et al., 2023).

RQ1: How does participation in GeoGebra-based dynamic geometry simulations affect post-assessment spatial reasoning and justification after controlling for baseline?

RQ2: How do pre- to post-improvements in geometric justification for proof preparation differ between GeoGebra-based dynamic simulations and parallel non-dynamic instruction?

Theoretical Framework/Perspectives

This study is grounded in spatial reasoning theory, van Hiele's theory of geometric thought, and cognitive load theory (Naufal et al., 2021; Seah & Horne, 2020). These perspectives explain why students may struggle with geometry tasks that require coordinating visual, spatial, and logical reasoning. Spatial reasoning involves mentally and visually operating on geometric objects, such as rotating, unfolding, slicing, and connecting 3D structures with 2D representations (Lowrie et al., 2019; Judd & Klingberg, 2021). In line with van Hiele's theory, visualization supports students' movement from recognizing figures by appearance toward using properties, relationships, and logical claims for emerging proof (Fitriyani et al., 2018; Nahdi et

al., 2024; Miyazaki et al., 2017; Haj-Yahya et al., 2022). Cognitive load theory further informs the design because 2D–3D transformations can burden working memory when students rely only on internal imagery or static diagrams (Paas & Van Merriënboer, 2020; Rivella et al., 2021; Buckley et al., 2019; Okamoto & Romo, 2015).

Accordingly, 2D–3D representational coordination is defined as connecting actions on three-dimensional solids with corresponding two-dimensional representations, including nets, cross-sections, diagrams, and written explanations (Fujita et al., 2020; Lowrie et al., 2019). GeoGebra-based dynamic and interactive geometry environments may support this coordination by making transformations visible, manipulable, and reversible, helping students observe relationships difficult to infer from static representations alone (Moreno-Armella et al., 2008; Ng et al., 2020; Zhang et al., 2023). Guided by these perspectives, spatial reasoning and early-stage geometric justification are the primary outcome constructs. The randomized comparison tests whether structured visual manipulation improves immediate spatial reasoning and supports stronger geometric argumentation after controlling for baseline performance.

Methods

Research Design

This pilot employed a two-condition, parallel randomized controlled design embedded in routine middle school instruction. Students were assigned within class to GeoGebra-based dynamic simulations or a parallel non-dynamic comparison matched on goals, pacing, and task sequence, with the key contrast in representational modality and interaction. Pre-/post-assessments with rubric-scored written work examined short-term learning and proof-oriented reasoning. Although a larger phase will add a delayed post-assessment 5~6 weeks later to examine retention and within-domain transfer, this report focuses on immediate pre–post effects.

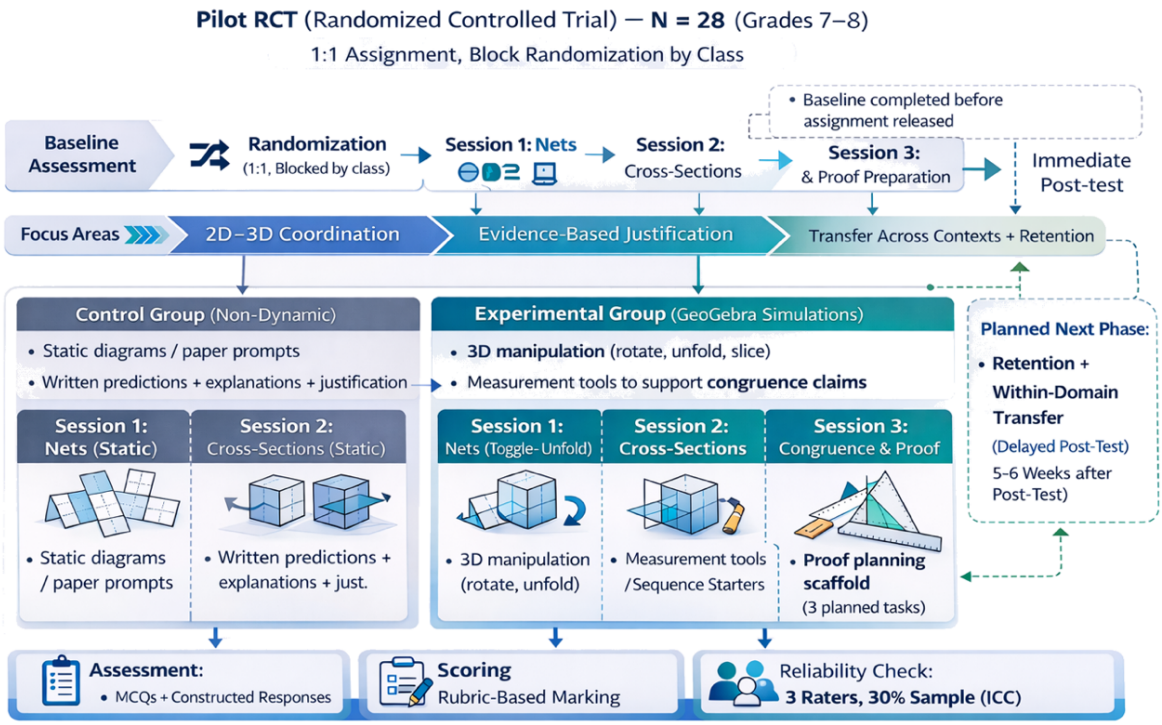


Figure 1: RCT Procedures and Intervention Pathway Across Sessions

Procedures and Data Collection

Participants were 28 students in Grades 7–8 at a Florida middle school, randomized 1:1 to control (n = 14) or experimental (n = 14). Across three interventions: Nets, Cross-sections, Congruence-based proof preparation, both groups completed parallel activities. The experimental group used GeoGebra 3D models to unfold nets, rotate solids, manipulate slicing planes, examine intersection traces, and apply measurement tools to support congruence claims. The control group completed the same sequence with non-dynamic materials (static diagrams and paper prompts). After Session 3, students completed identical pre-/post-assessments with multiple-choice and constructed-response items that were rubric scored to produce a geometric argumentation subtotal, with reliability established by three raters (see Figure 1).

Analysis and Reliability

Primary inference used ANCOVA with post total score as the dependent variable, condition as the factor, and baseline total score as the covariate. An independent samples gain-score t test provided a secondary check. A parallel ANCOVA examined the post constructed-response subtotal controlling for baseline. Three raters scored a 30% subset of constructed responses, and interrater agreement was evaluated with ICC at both time points to establish scoring consistency.

Findings

Data analyses examined the pilot sample (N = 28, 14 Control ~ 14 Experimental) and focus on baseline and immediate post-assessment outcomes after the three-session intervention.

Table 1: Immediate Post Total Score Results (RQ1): ANCOVA and Gain-score t-test

Outcome	Control	Experimental	*P*	Effect / 95%CI	Test statistic
Post Total adjusted mean (SE)	11.34 (0.32)	14.38 (0.32)	< .001	partial η^2 = .627	ANCOVA Group: F (1, 25) = 42.08
Post Total 95% CI	[10.68, 12.00]	[13.72, 15.04]			
Covariate Pre Total			< .001	partial η^2 = .677	ANCOVA Covariate F (1, 25) = 52.37
Gain Total mean (SD)	2.29 (1.33)	4.29 (2.13)	.006	CI [0.62, 3.38] (Exp − Ctrl).	t-test (Gain): *t* (26) = 2.985

Note. ANCOVA modeled *Post total* ~ group + *Pre total* (adjusted mean at *Pre total* = 9.57). *Gain total = Post total – Pre total*. Two-tailed *p* values are reported.

ANCOVA on post total, controlling for pre total, showed a clear simulation advantage, F (1,25) = 42.08, $p < .001$, partial $\eta^2 = .627$, with higher adjusted post means for GeoGebra than control (14.38 vs 11.34, SE = 0.32). Gains told the same story: simulations improved more (4.29, SD = 2.13) than control (2.29, SD = 1.33), t (26) = 2.985, $p = .006$, 95% CI [0.62, 3.38]. This supports RQ1 by suggesting that even a short GeoGebra simulation sequence strengthens immediate geometric performance and conceptual visualization because students were able to visualize and manipulate 2D–3D relationships (e.g., unfolding, rotating, slicing, dimensionality measurement) rather than relying on static procedures, making spatial relationships easier to coordinate and justify.

Table 2: Constructed Response for Geometric Argumentation (RQ2) and Reliability

Measure	Control (n = 14) M (SD)	Experimental (n = 14) M (SD)	Test statistic
Pre GeoArgument	3.14 (1.35)	4.29 (2.13)	Gain *t*-test:
Post GeoArgument	3.86 (1.96)	6.29 (1.44)	t (26) = 2.30, p = .030
Gain (post−Pre)	0.71 (1.33)	2.00 (1.62)	95% CI [0.14, 2.43] (Exp − Ctrl)
ANCOVA adjusted Post GeoArgument (SE)	4.21 (0.37)	5.93 (0.37)	ANCOVA group effect: F (1, 25) = 10.39 p = .004
Interrater agreement (ICC, single)	Pre: .97 [0.92, 0.99]	Post: .92 [0.78, 0.98]	partial η^2 = .294

Note. Pre/Post GeoArgument are rubric scored. ANCOVA adjusted post means at pre = 3.71. ICC with three raters on a 30% subset (n = 8). Two tailed p values are reported.

GeoGebra group showed larger gains in justification: argumentation increased by about 2 rubric points (M gain = 2.00, SD = 1.62) versus less than 1 point in the control (0.71, SD = 1.33), t (26) = 2.30, p = .030, 95% CI [0.14, 2.43], and the advantage held after baseline adjustment (5.93 vs 4.21, SE = 0.37; F (1,25) = 10.39, p =. 004, partial η^2 = .294). Scoring consistency indicates strong (ICC=.97 pre /.92 post). It suggests simulations supported proof preparation by helping students ground written claims in geometric features and relationships rather than unsupported statements from static representations, which is central for developing proof readiness.

Conclusion and Implication

This pilot provides initial evidence that GeoGebra-based visual simulations can strengthen students' immediate geometric learning and proof preparation after a brief three-session sequence. For RQ1, the simulation group outperformed the comparison group on overall post-performance after controlling for baseline, F (1,25) = 42.08, p < .001, partial η^2 = .627. For RQ2, students using simulations showed stronger growth in constructed-response justification and a higher adjusted post level, F (1,25) = 10.39, p = .004, partial η^2 = .294, with strong scoring consistency (ICC = .97 pre / .92 post). These results suggest that dynamic visualization can support conceptual visualization and written claims grounded in geometric features and relationships central to proof preparation.

Instructionally, the findings support short simulation-based task sequences that make spatial invariants observable and prompt students to articulate what changes and what remains invariant. Pairing dynamic manipulation with targeted prompts that connect observations to formal language and justification (e.g. correspondences, invariance, transformation-based reasons) may be especially productive for developing proof-oriented reasoning.

The findings are preliminary given the small sample size, single site, and focus on immediate outcomes. However, the consistent advantages across overall performance and argumentation provide clear justification to proceed to the next phase. The planned study will add a delayed post-assessment five or six weeks after the post to examine whether gains in spatial reasoning.

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THE COW TASK: DESIGN OF PIAGETIAN-STYLE TASK TO ASSESS CHILDREN'S UNDERSTANDING OF PERPENDICULARITY

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Keywords: Early Childhood Education, Geometry and Spatial Reasoning, Cognition

The purpose of this study is to develop, implement, and analyze the results of a Piagetian-style task using an interactive tool to assess children's understanding of perpendicularity.

Theoretical Framework

To assess understandings of parallelism, Piaget and Inhelder (1998) asked children to predict what happens when lazy tongs are opened. However, they did not develop any perpendicularity tasks. Existing research about preschoolers and perpendicularity focuses on static shapes; e.g., of children aged 3–6, older children are better at identifying non-square rhombuses as not squares (Clements et al., 1999). As Piaget and Inhelder (1998) assert, "what we must endeavor to concentrate on is not […] static shapes, but rather their transformations" (p. 304).

Task Design and Implementation

Our tool consisted of four blocks of wood of equal dimensions attached by hinges, so that children could collapse or extend the rhombus. Children were shown how the tool (or "fence") could be manipulated and given time to change the shape independently. Children were told, "change the shape of the fence to give [your cow] the most room" and asked, "How did you know to make the fence like that?" The task was administrated as part of video-recorded individual clinical interviews in a larger study. Participants included 42 4- to 5-year-old children (M_{age} = 4.71 years, SD = 0.42).

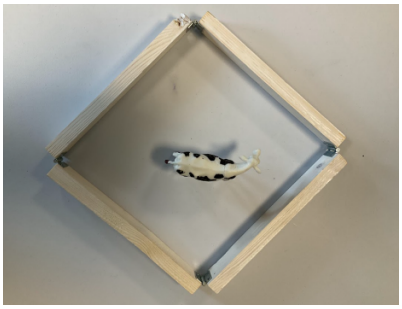


Figure 1: Tool Used in Task to Assess Preschoolers' Concepts of Perpendicularity

Results

Children had a strong bias towards squares, explicitly or implicitly. Many children referenced squares in their response, e.g. "because it's a square" (child 217, 4yr 0mo). Other children identified rotated squares as diamonds, e.g. "a diamond has more space" (child 234, 5yr 0mo).

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GEOMETRY BY EXAMPLE: IMPROVING STUDENTS' LEARNING OF BASIC AND HIGH SCHOOL GEOMETRY CONTENT

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Objectives and Theoretical Framework

In the present study, we evaluate the impact of GeometryByExample—a worked example-based intervention co-developed in a close research-practice partnership—for supporting learning of basic and high school Geometry content. The ByExample approach involves providing strategically designed, worked-example based assignments for students to complete in class in place of more typical practice assignments; instead of solving all of the practice problems themselves, students study correct or incorrect examples of solutions to half of the problems and respond to prompts asking them to write explanations of why those procedures are correct or incorrect. This approach draws on empirical evidence which has amassed from laboratory studies from three practices supported by the Science of Learning: worked examples (Sweller & Cooper, 1985), self-explanation (Roy & Chi, 2005), and learning from errors (Siegler, 2002).

Methods

Students (N = 348) in 17 intact high school Geometry classrooms in the United States completed measures of basic (pre-test, mid-year, and end-of-year) and high school Geometry knowledge (mid-year and end-of year). All classrooms received business-as-usual instruction from their teachers throughout the course; in classrooms randomly assigned to the experimental condition, teachers were asked to use GeometryByExample assignments to replace typical practice assignments they would use in their classroom, while students in control classrooms completed their normal practice assignments. Intervention dosage was computed as the number of self-explanation prompts attempted by a student.

Results and Implications

Results from the present study yielded significant effects of intervention dosage, indicating that experiencing greater levels of exposure to GeometryByExample materials led to greater learning of basic geometry content by mid-year, and greater learning of high school geometry content both at mid-year and by the end of the school year. The present study demonstrates that the ByExample approach can be beneficial for older students and can be incorporated successfully into classroom practice for more complex mathematics.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

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EXAMINING CRITIQUE ARTICULATION QUALITY IN PROSPECTIVE ELEMENTARY TEACHERS' EVALUATIONS OF AI-GENERATED GEOMETRIC JUSTIFICATIONS

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Research has shown that many prospective elementary teachers (PTs) rely on intuitive or visual explanations rather than constructing logically valid and generalizable justifications (Kaur Bharaj & Cross Francis, 2020; Runnalls & Hong, 2020). Although PTs' levels of geometric justification have been examined (Segal et al., 2018; Zeybek, 2017), less is known about how to support reasoning that bridges visual intuition and formal justification (Stylianides et al., 2016), particularly when AI generated explanations are included into learning contexts. Building on recent work on AI in mathematics education (Son et al., 2024; Yoon et al., 2024), this qualitative study (Kalu & Bwalya, 2017) examines how PTs critique AI-generated geometric justifications in geometry content courses.

Fifty PTs from two public universities in the United States participated in the study and evaluated five AI-generated geometric justifications embedded as a course assignment. Using a critique-based approach (McGalliard & Otten, 2025), PTs analyzed AI responses related to two- and three-dimensional shapes and their properties. For example, when given the claim, "If the diagonals of a quadrilateral are congruent, it must be a rectangle," PTs critiqued a measurement-based AI response and provided written feedback on how to improve the justification by explicitly distinguishing rectangles from other quadrilaterals with congruent diagonals. PTs used an adapted version of Level 0–3 justification rubric (Brunheira & Ponte, 2019) emphasizing geometric definitions, logical sequencing, and generalizability. The data constituted how PTs identified missing components in the AI responses using the rubric, and suggested ways to improve.

Using thematic analysis (Xu & Zammit, 2020) on PTs' written critiques, we named a category focused on articulation quality, defined as how clearly and mathematically PTs identified missing mathematical components in AI-generated justifications and how actionable their proposed suggested were. Three patterns emerged under this category: (1) *Mathematically Substantive Improvement (56%)*: PTs explicitly named missing definitions, logical steps, or generalizability and proposed concrete revisions. (2) *Minimal or Vague Improvement (35%):* recognized incompleteness but offered underspecified suggestions (e.g., "needs more explanation"), limiting the usefulness of their critiques. (3) *No or Incorrect Improvement (9%):* PTs provided no improvement suggestions or proposed revisions that were mathematically irrelevant or incorrect, relying on broad statements without referencing core geometric ideas.

Data analysis revealed that PTs frequently critiqued measurement-based or example-bound reasoning and suggested revisions that emphasized explicit properties and logically sequenced and generalizable arguments. While many PTs demonstrated emerging capacity to articulate evidence-based improvements, others struggled to translate evaluative judgments into precise mathematical feedback. These findings underscore the importance of instructional support that explicitly targets how PTs identify missing reasoning and articulate substantive improvements

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when critiquing AI-generated justifications, positioning critique not only as an evaluative act but as a core practice for developing robust geometric reasoning in AI-mediated learning environments.

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TEACHING EXTERIOR ANGLE THROUGH INTEGRATION OF CODING AND ROBOTICS IN ELEMENTARY CLASSROOMS

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Keywords: Elementary School Education, Geometry and Spatial Reasoning, Computing and Coding, Integrated STEM / STEAM

Relating angle and rotation is an important and challenging concept for elementary students (Sinclair & Bruce, 2015). Specifically, students often have difficulty viewing a static angle as a rotation (Mitchelmore, 1998). This concept is especially challenging when one of the rays forming the angle is invisible (Mitchelmore & White, 2000), which is the case for exterior angles of a polygon. Scholars have found success in using bodily movements (Smith et al., 2014) and coding and robotics (Kim et al., 2021) to help students better understand angles.

As part of our ongoing project to integrate coding and robotics into mathematics in elementary classrooms, our partner teachers created a task that addresses this challenging concept of a partially visible static angle as rotation. Although the main objective of the task was about characteristics of polygons and not specifically about exterior angles of polygons, students naturally engaged in reasoning about exterior angles as they personally traveled along the edges of a polygon and as they programed a desktop robot to travel the path of a polygon, creating a trace image.

Using a distributed perspective (Evans et al., 2011), we seek to understand how students reason about a partially visible static angle as rotation. We will present an analysis of a segment of discourse of a student pair consisting of the students' words, movements, and gestures mediated by their tools and figures. In this segment, two students move a small toy along a quadrilateral to assist them to write pseudocode for a desktop robot to travel the same path. The students correctly identified the first turn, where the interior angle was obtuse, as an acute turn. At the second turn, they mistakenly identified the turn as 35 degrees, which was what they estimated the measure of the interior angle to be. At the third turn, where the interior angle was obtuse, a student said "that's [the turn is] definitely not obtuse" as she physically turned the small toy sweeping the exterior angle.

Our initial analysis shows that these students often reasoned mathematically by using object-level coreferences to the figure that represents the robot they would eventually code. Specifically, at the third turn, physically turning the figure and having to articulate the number of degrees seems to be the catalyst for the students to start attending to the exterior angle instead. The results imply that coding and robotics provide a both a context and a set of tools to support students to reason about partially invisible angles as rotations. These results add to the existing literature on ways coding and robotics can be used in elementary mathematics education (e.g., Boote & Galanti, 2023; Dunbar & Rich, 2020) by highlighting the affordances of coding and robotics integration.

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ELEMENTARY TEACHERS' APPROACHES TO PROBLEM POSING AND SOLVING IN DIVISION CONTEXTS

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This study investigated elementary teachers' conceptual understanding of division through the dual lenses of problem posing and problem solving. Participants (N=28) generated word problems for 34 ÷ 6 (pre-test) and 48 ÷ 8 (post-test). Posed problems were analyzed using an analytic framework encompassing multiplicative structures (e.g., equal groups) and qualitative features, including linguistic clarity, explicit equal-grouping constraints, and contextual variety. While participants exhibited a robust default preference for partitive division and equal-groups structures, the post-test results revealed a significant shift. Following a targeted professional development intervention, the frequency of measurement division models increased, as did the mathematical precision and clarity of the posed problems. These findings suggest that intentional professional learning can effectively broaden teachers' conceptual repertoires and enhance their ability to construct mathematically rigorous instructional tasks.

Keywords: Number Concepts and Operations, Problem-Solving, Professional Development, Teacher Knowledge

Division word problems serve as a critical window into learners' conceptual understanding of multiplicative structures and the underlying meanings of division. In mathematics education research, two fundamental models are commonly distinguished: partitive (sharing) division, where the number of groups is known and the size of each group is sought, and quotative (measurement) division, where the group size is known and the number of groups is sought (Fischbein et al., 1985; Greer, 1992). Since a single numerical expression can represent divergent semantic structures, mathematical proficiency depends not only on computational accuracy but also on the ability to interpret and construct situations that align with the intended meaning of the operation. This "representational flexibility" is essential for developing a robust understanding of rational numbers and proportional reasoning (Mulligan & Mitchelmore, 1997).

Problem Posing

The process of generating new mathematical problems or reformatting existing ones has emerged as a powerful practice for both developing and accessing conceptual depth (Cai et al., 2013; Silver, 1994). When educators and students pose division story problems characterized by clear inquiry, equal-group structures, and diverse contexts, they are better positioned to externalize underlying mental models and connect abstract symbolic operations to meaningful scenarios. However, existing literature suggests that teachers, including preservice candidates, often possess a limited "pedagogical problem-posing toolkit." Research indicates that educators frequently default to familiar, stereotypical contexts, such as sharing food, and exhibit a significant bias toward partitive over measurement division (Luo, 2009; Osana & Pelczer, 2015). This reliance on a single model can inadvertently limit students' exposure to the breadth of multiplicative situations required by modern rigorous standards, such as the Common Core.

MATHEMATICAL KNOWLEDGE FOR TEACHING

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The Study

This study investigates elementary teachers' understanding of division word problems through the dual lenses of problem posing and problem solving. By analyzing the "mathematical work" teachers perform when constructing problems (Ball et al., 2008), the research examines how teachers:

(a) Construct word problems corresponding to specific division expressions;
(b) Represent partitive versus measurement meanings and;
(c) Attend to key qualitative features, including linguistic clarity, explicit equal-grouping constraints, and contextual variety.

Theoretical Framework

Research on multiplication and division word problems emphasizes that divergent semantic structures can elicit distinct cognitive strategies and varying levels of difficulty, even when the underlying numerical operation remains constant (Fischbein et al., 1985; Greer, 1992; Verschaffel et al., 2000). In mathematics education research, division is fundamentally categorized into two conceptual models: partitive (sharing) and measurement (quotative).

In a partitive model, the dividend (total) and the number of groups are known, while the size of each group is the unknown to be determined. This model is psychologically associated with "fair sharing" actions and is often the first model students encounter (Mulligan & Mitchelmore, 1997). Conversely, in a measurement model, the dividend and the size of each group are known, while the number of groups serves as the unknown—a situation often modeled by "measuring out" or repeated subtraction. These relationships are summarized in Table 1.

Table 1: Known and Unknown Quantities in Multiplicative Situations

Operation	Number of Groups	Size of Each Group	Product (Total)
Multiplication	Known	Known	Unknown
Partitive Division	Known	Unknown	Known
Measurement Division	Unknown	Known	Known

Beyond these primary models, researchers identify several multiplicative structures used to frame word problems, categorized by the nature of the quantities involved (Carpenter et al., 1999; Vergnaud, 1983):

- Equal Groups: Situations involving fair sharing or repeated measures.
- Multiplicative Comparison: Comparing two quantities to determine how many times larger one is than the other (e.g., "three times as many").
- Cartesian Product: Determining the number of possible pairings between two sets, representing a product of measures.
- Area and Other Products of Measures: Problems involving rectangular arrays or physical dimensions where two measures are multiplied to produce a third, different unit.

Within equal-groups structures, problems typically emphasize either fair sharing (aligned with partitive meanings) or rates and measures (frequently associated with measurement meanings).

Criteria for High-Quality Division Word Problems

To ensure division word problems are mathematically well-specified and instructionally effective, this study identifies three essential qualitative features derived from the literature on mathematical task design and teacher knowledge (Ball et al., 2008; Silver, 1994):

- Clarity of Inquiry and Quantities: High-quality problems must clearly define the known quantities so that the specific unknown is unambiguous. Ambiguity in problem phrasing can lead to "situational difficulty" that is independent of mathematical complexity (Verschaffel et al., 2000).
- Explicit Equal Grouping: The text must explicitly state that groups are equal in size (e.g., "each person received the same amount") to distinguish formal mathematical division from informal distribution. Without this constraint, the problem is mathematically underdetermined.
- Contextual Variance and Authenticity: Problems should utilize diverse, meaningful contexts that extend beyond "cliché" examples, such as sharing cookies or candies. Authentic contexts help students bridge the gap between abstract operations and the varied real-world applications of division (Palm, 2008).

These three criteria serve as the primary analytic framework for evaluating the problem-posing tasks in this study, reflecting the "Specialized Content Knowledge" (SCK) required to teach division effectively.

Method

This study utilized a qualitative research design, supplemented by descriptive quantitative summaries, to evaluate the conceptual nature of teachers' posed word problems. Adopting a pre-test/post-test framework, the study examined shifts in teacher practice following a targeted intervention. In each assessment phase, participants were required to (a) solve a specific division expression and (b) construct two distinct word problems that accurately modeled that expression.

Participants

The participants were 28 elementary school teachers from a Western U.S. state. They participated in a week-long professional development on Number and Operations. Each participant generated two word problems during the pre-test and two during the post-test, resulting in a total sample of 112 posed problems ($n = 56$ per administration).

Data Collection Tasks

The pre-test task required teachers to generate two distinct word problems for the expression $34 \div 6$. The post-test task required two problems for $48 \div 8$. Between the two administrations, teachers participated in a professional development (PD) intervention. Grounded in the construct of *Mathematical Knowledge for Teaching* (Ball et al., 2008), this instruction focused on:

- Division Models: Distinguishing between partitive and measurement (quotative) division.
- Multiplicative Structures: Identifying various semantic frameworks, such as multiplicative comparison and Cartesian products (Greer, 1992).
- Quality Indicators: Applying features of high-quality word problems, emphasizing linguistic clarity and the explicit requirement of equal grouping.

Data Analysis

All posed problems were transcribed and systematically coded using a three-dimensional analytic framework developed from established research on multiplicative thinking (Carpenter et al., 1999; Verschaffel et al., 2000):

1. Division Type: Categorized as either partitive or measurement.
2. Multiplicative Structure: Classified into four types: equal groups, multiplicative comparison, Cartesian product, or area/products of measures.
3. Qualitative Features:
   - Clarity: Evaluated on a three-point scale (not clear, weak clarity, or good clarity).
   - Equal Grouping: Coded as either explicitly equal or not explicitly equal.
   - Contextual Themes: Categorized by the thematic content used to represent the dividend and divisor (e.g., food, discrete objects, living creatures, or abstract quantities).

To ensure the rigor of the findings, a subset of the problems was double-coded by the research team. Inter-rater agreement was calculated, and any discrepancies were resolved through consensus to ensure coding reliability (Silver & Cai, 1996).

Results

Division Types: Partitive vs. Measurement

The distribution of division models shifted notably following the professional development intervention. On the pre-test, participants primarily utilized partitive division over measurement division. By the post-test, the frequency of measurement division increased substantially while partitive models decreased to 31%. Although the partitive model remained the dominant choice, these results indicate that instruction effectively broadened teachers' capacity to construct measurement-based scenarios

Table 2: Frequencies of Posed Problems by Division Type

Division Type	Pre-test (34÷6)	Post-test (48÷8)	Total
Measurement (Quotative)	17	25	42
Partitive (Sharing)	39	31	70
Total Posed Problems	56	56	112

Note. N=28 teachers; two problems per teacher

Qualitative notes provided by participants revealed a shift in perceived difficulty. Pre-test reflections suggested a preference for partitive division due to the familiarity of "sharing" actions. Conversely, post-test reflections indicated that some teachers found measurement division more accessible because it naturally supports strategies like repeated subtraction or determining how many units "fit" into a total.

Multiplicative Structures

Analysis of the structural framing revealed a lack of diversity in multiplicative contexts. Across both administrations, participants exclusively utilized equal-groups structures. Within this category, fair-sharing contexts predominated, with a smaller subset representing rates or measures. Notably, no problems were coded as multiplicative comparison, Cartesian product, or

area models. This suggests that teachers may perceive equal-grouping as the only viable or accessible structure for elementary division instruction.

Problem Quality

Clarity, Explicit Grouping, and Context: Instruction appeared to have a significant impact on the mathematical precision of the posed problems. As shown in Table 3, the proportion of problems characterized by "Good Clarity" rose from 33.9% on the pre-test to 53.5% on the post-test.

Table 3: Clarity and Equal-Grouping Codes for Posed Problems

Code	Pre-test f(%)	Post-test f(%)	Description
Not clear	10 (17.9%)	7 (12.5%)	Ambiguous or unsolvable wording
Weak clarity	27 (48.2%)	19 (33.9%)	Mostly solvable but somewhat confusing
Good clarity	19 (33.9%)	30 (53.5%)	Clear quantities and explicit unknown
Equal grouping	27 (48.2%) equal; 29 (51.8%) not explicit	36 (64.2%) equal; 20 (35.7%) not explicit	Whether problems explicitly stated equal sharing/grouping

Note. N = 56 posed problems per test administration.

From pre- to post-test, the proportion of problems coded as good clarity increased from 33.9% to 53.5%, while not-clear problems decreased. Similarly, the proportion of problems that explicitly indicated equal grouping increased from 48.2% to 64.2%. These findings indicate that instruction supported teachers in writing more mathematically precise division word problems.

Contextual Themes

Table 4 details the thematic choices for the dividend and divisor. Teachers demonstrated a heavy reliance on a narrow set of familiar contexts.

Table 4: Context Themes Used to Represent Dividend and Divisor Across All Posed Problems

Component	Theme	f (%)	Examples
Dividend	Food	44 (39.2%)	brownies, cookies, pizzas, candies, cupcakes
	Objects	44 (39.2%)	baskets, cups/containers, marbles, stickers, tables, bags
	Living creatures	22 (19.6%)	students, cows, friends, groups, people
	Abstract	2 (1.7%)	(rare)
Divisor	Food	1 (0.8%)	(rare)
	Objects	46 (41.0%)	bags, baskets, bowls, pieces, plates, vans
	Living creatures	62 (55.3%)	friends, children, people, groups, students, teams
	Abstract	3 (2.6%)	days, hours, rows, room area

Note. Percentages are based on N = 112 posed problems across both test administrations.

Teachers most frequently used food and object contexts to represent the dividend, and living-creature contexts (e.g., friends, students, teams) to represent the divisor. Abstract contexts were rare. Overall, the results suggest a reliance on a limited set of familiar contexts, consistent with prior observations that food-sharing scenarios are common in posed division problems.

Discussion

This study investigated elementary teachers' conceptual understanding through the dual lenses of problem posing and solving. The findings provide a critical look at the "default" schemas teachers bring to instruction and the extent to which these boundaries are malleable.

The Partitive Bias and Malleability of Practice

The significant pre-test preference for partitive division suggests a deeply rooted cognitive association between division and the physical act of sharing, as identified by Fischbein et al. (1985). However, the post-test shift indicates that teachers' pedagogical repertoires can be expanded through explicit instruction. This suggests that the "partitive bias" is not a lack of capability, but perhaps a lack of exposure to the quotative (measurement) model in professional training.

Structural Constraints

The absolute absence of non-equal-groups structures (e.g., Cartesian products or area models) is a striking finding. While equal-groups contexts are a vital entry point, the failure to utilize varied multiplicative structures may limit students' ability to generalize division to higher-level domains (Vergnaud, 1983). This suggests that teachers may view these alternative structures as inaccessible for elementary-grade contexts, or they may lack the SCK required to model them (Ball et al., 2008).

Improvements in Semantic Precision

The marked improvement in linguistic clarity and explicit equal grouping highlights a crucial intersection between language and mathematics. On the pre-test, many problems were "underdetermined", meaning the situation was too ambiguous to guarantee a single mathematical path. The post-test results suggest that when educators are trained to define the unknown and specify constraints, they produce more coherent and instructionally effective tasks (Silver, 1994).

Implications for Teacher Education

- Expansion of Multiplicative Repertoires: PD should move beyond the "fair sharing" archetype to include non-equal-groups contexts, such as rates and area models, to support students' flexible reasoning.
- Precision as a Professional Skill: PD should treat problem posing as a rigorous skill. Focusing on semantic precision helps teachers avoid "cliché" tasks that lack mathematical depth (Osana & Pelczer, 2015).
- Contextual Variation: Instructors should be encouraged to move beyond food-and-living-creature contexts, incorporating abstract and scientific measures to better prepare students for the varied applications of division.

In conclusion, this study highlights both the initial predominance of partitive, equal-groups models in teachers' posed division problems and the malleability of that preference. Following targeted professional development, teachers demonstrated increased use of measurement division and greater mathematical precision in the construction of their tasks. These findings suggest that intentional professional learning can broaden teachers' conceptual repertoires and strengthen their capacity to design mathematically rigorous instructional problems.

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COMPARISON AS A MATHEMATICAL PRACTICE IN PROFESSIONAL DEVELOPMENT: TEACHERS' PROPORTIONAL REASONING VIA INVARIANCE AND GENERALIZATION

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This study examines comparison as a central practice in proportional reasoning and explores how foregrounding comparison in professional development (PD) shapes teachers' reasoning with respect to invariance and generalizations. Data were collected from two PD offerings. Findings show that comparison can be used to identify invariant properties and to advance generalizations across dimensions, functioning both as a conceptual foundation for other concepts such as ratio and as an instructional tool for sense making. Findings further show that teachers predominantly engaged in between-objects comparison and tended to rely on empirical reasoning, suggesting that PD should explicitly support multiple forms of comparison, attention to invariant relationships, and precision in comparative language to foster structural generalization. Implications for teacher education and professional development are discussed.

Keywords: Curriculum and Instruction, Teacher Knowledge, Proportional Reasoning and Similarity, Variance and Invariance

Purpose and Background

Proportional reasoning is a foundational concept essential for a learner's progression beyond elementary mathematics (Lamon 2007), yet it is widely recognized as challenging for both students (Karplus et al., 1983; Lamon, 2007; Thompson & Saldanha, 2003) and teachers (Jacobson et al., 2018; Modestou & Gagatsis, 2010). Robust understanding of proportional reasoning involves multiplicative reasoning about relationships between quantities, not merely solving tasks for an unknown in equivalent fractions (Lamon, 2007; Weiland et al., 2019, 2021).

Variance and invariance are fundamental concepts for mathematical reasoning (Nagar et al., 2022a, 2022b), and teachers' ability to identify invariant properties is essential to development of robust understanding of mathematical concepts. Despite the importance of invariance in proportional reasoning, there is little research available considering these domains (Nagar et al., 2025), especially in working with digital representations.

Prior research suggests that proportional reasoning depends on coordinating quantities across and within measure spaces (Lamon, 1995; Thompson & Thompson, 1996). Teachers often show preferences for comparing using only within or only between measure spaces (Orrill et al., 2010), which influences how scale factors and constant relationships are recognized. Studies on double number lines further show that difficulties arise not from lack of procedural knowledge but from challenges coordinating quantities and identifying what stays equivalent across relationships (Orrill & Brown, 2012). Similarly, research on proportional tasks presented visually indicates that multiplicative comparison alone is insufficient when dimensional shifts occur, e.g. when reasoning shifts from length to area (Orrill & Burke, 2013).

Building on this work, we hypothesize that comparison is a foundational mathematical practice through which invariant relationships can become discernible and generalizations can be constructed. The present study examines how comparison operates within professional development (PD) contexts and how it both supports and destabilizes teachers' reasoning about proportional situations. This study explores the following research question: *how do in-service teachers (grades 3–9) use comparison to identify invariance and develop generalization about proportional reasoning using digital representations in two PD workshops?*

Theoretical Framework

Knowledge in Pieces

We view teacher knowledge through the lens of Knowledge in Pieces (KiP; diSessa et al., 2016), which asserts knowledge is comprised of fine-grained understandings (i.e., knowledge resources) that work in interconnected ways in any given situation. Learning is considered the development of new knowledge resources, the refinement of existing resources, and/or the creation of connections between and among knowledge resources. Connections are particularly important because they control flexibility in a person's ability to invoke a wide variety of knowledge resources in a given situation. KiP is powerful for characterizing teachers' knowledge because, rather than trying to quantify the amount of knowledge a teacher may or may not have, KiP allows a fine-grained analysis of the knowledge invoked in a given situation.

Our use of the KiP framework emphasizes the process of thinking, not only the end result of such a process. Accordingly, we value the process of comparing elements in a task, identifying invariance, attempting to generalize, checking conjectures, and making connections among knowledge resources. These processes can help us study how teachers connect and use their knowledge when unpacking a task.

Invariance and Invariant Feature of Change

We define invariance by referring to properties that remain unaltered when a transformation is applied or described (Hadas et al., 2000; Laborde, 2005; Leung et al., 2013; Nagar et al., 2022a). In the context of proportional reasoning, we referred to mathematical transformations such as scaling up, scaling down, inverting, resizing, and so on. For instance, one might consider a cookie recipe with two cups of flour and one cup of sugar. If you double the recipe, the amounts change, but the part-to-part ratio, *2:1*, remains the same. This ratio has an invariant property, staying constant when the recipe is scaled up or down.

Nagar et al. (2024) found teachers describe an invariant property as something that changes, but a feature of that change remains constant (e.g., always decreasing or increasing, always doubling, etc). For instance, when doubling ingredients in a recipe, one might observe that each amount changes; however, the *feature of change remains the same,* everything is "doubled."

In the current work, we examine how comparison invokes knowledge resources related to invariant properties and the connections between these resources teachers made. This is important as making connections between knowledge resources suggests they are more readily drawn upon across a wide range of situations (Brown et al., 2020).

Generalization

We draw on Mason et al.'s (2010) definition of generalization as the process of moving from observing a few instances to extrapolating a wider class of cases. In Mason et al.'s (2010) work, two types of generalizations were identified: *empirical* and *structural*. Empirical generalization occurs when one looks at several instances to identify what is the same about them in order to form a conjecture (Mason et al., 2010). Empirical generalization does not guarantee the truth of that conjecture. Structural generalization occurs when one recognizes relationships and patterns

that "has to be justified by reference to underlying structure" (p. 232). Structural generalization is a more reliable form of generalization because it goes beyond observation to understanding how the structure itself operates. One of our goals in the current work is to examine empirical and structural generalizations and how participants used them as part of reasoning.

Comparison

We define *comparison* as a mathematical practice through which quantities are coordinated within or across measure spaces to establish relationships among attributes of a situation. Comparison involves selecting elements or quantities, determining correspondences among them, and attending to invariance. Research suggests that proportional reasoning depends on coordinating quantities across and within measure spaces (Lamon, 1995; Thompson & Thompson, 1996). Yet the role of comparison often remains implicit. Research has shown that attending to measure space is important in making comparisons that might appear proportional, but are not, such as comparing length to area (Orrill & Burke, 2013), highlighting the importance of a deep understanding of structures underlying comparisons. We therefore hypothesize that comparison can function as a process through which invariants can be identified, relationships formed, and structural generalizations can be constructed in proportional situations.

Data Sources and Methods

This study is part of a larger project exploring teacher knowledge of rational numbers. The 30-hour PD (five consecutive days) focused on key mathematical ideas such as referent units, invariance, covariation, and quantity. Teachers worked in small groups on tasks, followed by whole-group discussions. We analyzed two PD offerings in the Northeastern U.S. The first had 31 grades 3–9 math teachers in an urban district (five male, twenty-six female). Twenty-five identified as White, six Black, Hispanic, Native American, or multiracial. Teaching experience ranged from one to 30 years (mean = 9.4), and 21 had at least a Master's. The second had 15 grades 2–11 teachers (three male, twelve female), with one Asian, one Black, and the rest White. Experience ranged from three to 26 years (mean = 10.8), and eleven had a Master's or higher.

We considered one group from each setting, selected for being talkative, working on a task focused on comparisons (Figure 1). Group A included five elementary teachers and the second author as the facilitator. Group B included five middle graders special education teachers and a facilitator (not an author). Both facilitators used the same PD plan to maintain consistency. The Kitties Task, used on Day 5, involved a dynamic tool that scaled the two kitties proportionally (within a kitty and across kitties) in length and width (Figure 1b). Also, both kitties could be moved around in space. Participants explored the relations as shown in Figure 1a. We focused on this dynamic task since Brown et al. (2020) showed that using those in PD can support teachers in developing connections to make their knowledge resources more accessible.

We focused our data analysis on verbatim transcripts of each group's conversations as they solved the task, supplemented with descriptions of participants' observable actions from the video recordings. This resulted in detailed records of what participants said and did while exploring the Kitties task. Our analysis proceeded in two coding cycles: *descriptive coding,* followed by a *pattern coding* (Miles et al., 2014). Two team members led the descriptive coding, identifying references to comparisons, invariants, and generalizations as related to our coding scheme (Table 1). In the second cycle, we grouped utterances by analytic category and not chronologically, to determine what was compared, whether invariance was articulated, whether reasoning was empirical or structural, including any generalization. Finally, the full research team met to review, discuss, and refine codes, and to identify patterns.

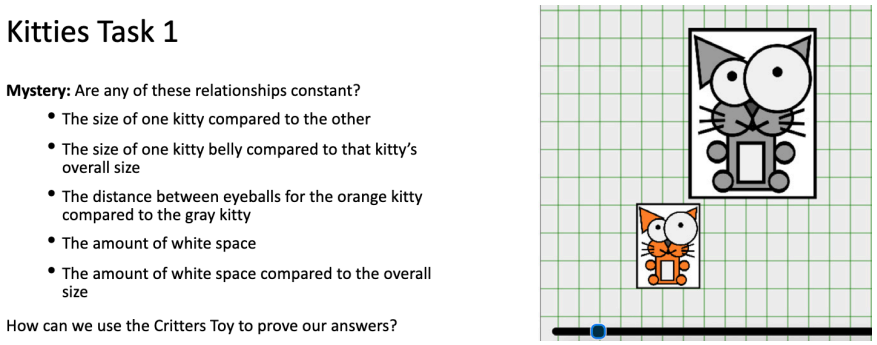


Figure 1: Kitties Task – (a) Task Posed to Participants About the Kitties, (b) Screenshot of the Digital Representation Tool

Table 1: Coding Scheme

Code		Description
CB	Comparison between two objects (one set)	Refers to participants looking at how quantities are related (compared) to each other between two objects (one set of objects)
CW	Comparison within an object	Comparing quantities within an object (e.g., comparing height and width of a kitty)
CS	Comparison across sets of objects	Indicates that participants are comparing quantities using sets of objects, i.e., comparing multiple instances of the objects (more than one set)
IP	Invariant Property	Refers to an invariant property that is maintained (properties that remain unaltered when a transformation is applied or described). Records the property.
EG	Empirical Generalization	Considering invariance to form a conjecture related to the situation by discerning "sameness." We coded any attempt for a conjecture, even if the wasn't established
SG	Structural Generalization	Indicates that participants identified relationships or patterns that are invariance and can be explained by reference to underlying structure. This was more reliable than EG because it suggests an understanding of how the structure operates
ES	Example Spaces	Indicates if participants generated one or more examples of objects describes them
JS	Justification	Explanations and justifications provided as part of the reasoning.
AP	Approximation vs Precision	A - Indicates that participants used approximation to reason proportionally P - indicates that participants used exact measurements to reason proportionally.

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Results

In what follows, we present findings related to participants' use of comparison across the two PD contexts. We first describe how comparison emerged in participants' work with digital representations and examine its function in their reasoning. We then highlight the invariant properties and relationships participants identified while engaging in comparative reasoning. Next, we present participants' reasoning processes and the types of generalizations articulated as they drew on comparison. Finally, we illustrate how comparison can serve as a source of productive destabilization through a specific case example.

Comparison as a Central Mathematical Activity in the Kitties Task

Table 2 summarizes the comparisons identified, including sample utterances, the invariant properties discussed, and the generalizations that emerged. Teachers engaged in several types of comparison: between the two kitties' overall size, between the kitties' linear features, within a single kitty, and across sets of kitties (i.e., involving more than one pair).

Table 2: Comparison, Invariance, and Generalization

Comparison Type	Evidence - Sample Utterances	Invariant Properties / Relationships	Generalization Type
		Comparison Between – Size	
Group A	**"**Four small kitties within [gray kitty]"; "4 to 1"; "Gray is ¼ of orange"; "1 belly to 4 bellies"; "size orange to gray is constant"	• The size of one kitty compared to the other stays constant • Ratio remains constant	Empirical and observational – within the task
Group B	"Half the size"; "orange lengths are [half the size]"	• Kitties' shapes always remain a square • Orange's lengths always half of gray	
		Comparison Between - Linear Features	
Group A	"Each side is times 2"; "Distance between eyeballs 3:1"; "Distance between the pupils 2:1"; "proportion of length and width is constat between the two cats"	• All sides are multiplied by 2 • Distance between eyeballs remains the same (2 to 1)	Mostly Empirical. Participants struggled comparing the distance between the eyeballs and tried to approximate it.
Group B	"4 to 1.5"; "distance stays 2 to 1, 'cause there's still similar figures"; "all ratios stay same."	• Figures remain similar • Distance stays 2 to 1 ratio • If figures stay similar, distance stays 2:1	One structural claim made in Group B (similar figure)
		Comparison Within - One Kitty	
Group A	"Belly scaling at same rate"; "Amount of white space compared to overall size is constant"	• Belly is changing at same rate • Amount of white space stays constant • Belly to whole remain a constant ratio	Empirical and based on inspection. Participants struggled comparing different objects within one kitty
Group B	"Fractions simplify to ¼"; "The amount of white space compared to the overall size is constantly half"	• Inside of kitties remains similar bellies/figures • Fractions stay ¼ • If fractions stay the same, objects are proportional • The comparison of white space to overall kitty stays constant and remains 1/2	

	Comparison across Sets		
Group B	"Counting by 2s going up adding one whole box"; "Both [bellies in sets of examples] about 1/4 of the frame"; "length and width ¼"; "area is ¼ of the frame"; "16 next because doubling dimensions"	• As linear dimensions keep going up by 2 the distance keeps adding 1 • If you change the size going up by 2, the distance keeps adding 1 • Belly stays ¼ of the frame in both sets • Length/width remains same • Belly area remains ¼ of the frame • doubling the dimensions	Empirical. Looked at both dimensions: linear length and width and area. Expressed tension between additive vs multiplicative relationships. Additive reasoning

Comparison between sizes. Both groups used comparisons between kitties to reason about how their sizes compared. In Group A, Glinda began her exploration by setting the gray kitty to 8×8 and orange kitty to 4×4. She then positioned the orange kitty on top of the gray kitty and explained that "four of these small kitties [fit] within," indicating that she was determining how many orange kitties could fit inside the gray one. Others in Group A discussed size comparisons in similar ways (Table 2). In contrast, participants in Group B compared the kitties' sizes by referring explicitly to length and width, concluding that one kitty was half the size of the other.

Elena: The size of one kitty compared to the size of the other. So it's half the size, right?
Dale: Yours was five.
Elena: And the other one was ten. And you have four and eight, right?
Dale: Yeah, yeah, so. So the length. Orange one is half.
Elena: Orange = ½ gray one and *w* [wrote on paper, suggesting *l* = length and *w* = width]

In making comparisons between the kitties' sizes, participants identified several invariant properties, such as the ratio remained constant and orange's kitty's lengths remained half of the gray kitty. However, identifying invariant properties did not guarantee correctness. For example, claiming the Kitties' shapes always remained square, despite the figures being rectangles. Participants relied on empirical generalizations and did not mention invariant relationships between constant properties, and their generalizations tied to the task context.

Comparison between linear features. When comparing linear features like the kitties' eyeball distances, both groups focused on specific features and were more explicit than when discussing size, with Group A approximating the distances. Wanda, Mara and Pepper initially agreed the relationship was "3 to 1" (Wanda). Glinda questioned this since earlier comparison of sizes suggested a 1 to 4. After Mara resized the kitties (orange approximately 3×3 vs gray 6×6), the group reconsidered and concluded the distance comparison was "1 to 2" (Mara).

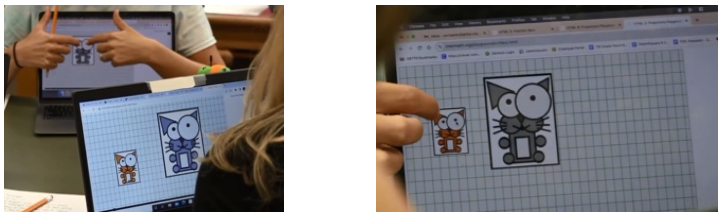
Group B struggled to quantify this relationship even using comparisons across sets (described below). They were unable to determine whether the relationship was constant until later on in the whole group discussion Charlotte used structural knowledge suggesting that the eyeball distance "stays 2 to 1 'cause there's still similar figures, so all the ratios stay the same."

Across these linear comparisons, participants identified invariant properties (Table 2). Glinda in Group A noted although sides change, they are all multiplied by two. Generalizations remained largely empirical, except for one structural claim made by Charlotte who connected two invariant properties to suggest that similarity guarantees a constant ratio of 2:1 between linear features.

Comparison within one kitty. Participants in both groups compared elements within a single kitty but struggled to quantify those relationships. Group A saw the belly was changing relatively to the overall kitty and how that impacted the amount of white space. Glinda suggested that the "belly of the cat is scaling up … at the same rate as the entire cat" and she proposed

finding "the ratio instead of just say[ing] it's constant." Wanda and Pepper argued determining the ratio was difficult because they could not determine how many bellies fit within the figure.

In Group B, participants found it difficult to compare "the frame of the belly to the frame of the cat" (Dale). Elena described the relationship as proportional because they were "multiplying," and Dale proposed making "a chart of all different lengths" to demonstrate that "all fractions … come out to a ¼". However, they appeared to confuse the dimensions. Dale referred to linear measurements while reasoning about the belly which is two dimensions (2D). This confusion reappeared when Elena suggested that the amount of white space would remain "constantly half," extending her previous linear comparison results to an area-based relationship. The invariant properties were identified and the reasoning was based on empirical generalization.



(a) (b)

Figure 2: Dale and Elena's Sets of Kitties Comparison – (a) Dale and Elena Comparing Across Sets, (b) Dale Points to the Eye of the Orange Kitty

Comparison across sets of kitties. The least common comparison involved sets of kitties. Several Group B participants used this comparison relying on two sets of kitties when examining the eyeball distances (Figure 2), and the belly to frame relationships. When comparing eyeball distances, they relied on additive reasoning rather than multiplicative. For example, Dale and Elena described how increasing the size of the kitties "by 2s," added "one whole box" to the eyeball distance. Dale and Elena struggled to determine whether the relationship was invariant and how to quantify it. Later, using their set of kitties, they examined belly to frame comparison and despite difficulties approximating measurements, they concluded that "the belly remains ¼ of the frame" (Dale). Although participants identified invariant properties, they exhibited a tension between additive and multiplicative reasoning. Their attempt to generalize was again empirical, and the additive reasoning led them to propose an invalid invariant relationship.

Destabilization through Comparison

The results suggest that comparison can destabilize teachers' reasoning in productive ways. In Group B, comparison exposed underlying assumptions and led to an overgeneralization shared by both participants and the facilitator:

Facilitator:	What are you noticing about the kitties?
Raymond:	The black and white [gray] is always double the size of the color [orange].
Facilitator:	What do you mean by size?
Raymond:	Length and width.
Tina:	In high school you're teaching them to put one side length over another side length. But when you do area, it's double, and when you do volume you have to triple it.

Facilitator:	Doubled, yes, because the area is a two-dimensional quantity, and then the length and width are linear quantities, and so the relationship winds up being different.
Amira:	But is it more like dilation then? I mean isn't it what we do in dilation.
Raymond:	No, she's talking more so about the area. Yeah, the area, the whole picture, in comparison to this. So it's, we're saying like it's twice as big, but we're referring to the length, but the area, whatever it doubles.

Early in the exploration, participants described the gray kitty as always being double the size of the orange kitty, referring to length and width. As discussion shifted toward area, both participants and facilitator extended this comparison without attending to the structural change involved in moving from linear to 2D quantities. The facilitator explained that area was "doubled" rather than emphasizing that area now changes by the square of the linear scale factor. This language use obscured the dimensional structure of the situation and illustrates how comparison can generate a conceptual tension rather than clarity.

Discussion and Conclusions

In this study, we found comparison can function as a central practice in proportional reasoning that, when foregrounded in PD, can both destabilize and advance teachers' understanding. Comparison supported teachers in identifying invariance and moving toward generalization. We saw comparison as a mathematical concept that helps define other concepts such as ratio and as an instructional tool to advance mathematical exploration and reasoning.

We identified three categories of comparisons used by teachers: comparisons between objects, within a single object, and across sets of objects, with comparisons between objects being dominant. Teachers used comparisons to identify constant relationships across dimensions, quantify ratios, search for patterns, and attempt justification and generalization. These findings suggest that PD may benefit from explicitly foregrounding comparison and supporting teachers to engage in multiple forms of comparison, rather than relying on between objects comparisons.

In both groups, comparison played a critical role to support the identification of invariant properties that enabled teachers to unpack the tasks and advance sense making. We posit that comparison can be a gateway to invariance. This finding is significant given prior work showing teachers do not typically introduce invariance spontaneously (Nagar et al., 2022a). While participants identified invariants, they less frequently articulated logical (if-then) relationships among them. When such relationships were proposed, they reflected attempts to move beyond empirical observation toward structural generalization and reasoning, for example, recognizing that if figures are similar, then that ensures the ratio between distances (e.g., between eyeballs) must remain constant. These results suggest PD should encourage teachers to examine logical relationships among invariants, as they appear critical for understanding proportional structure.

Notably, the teachers in our study rarely compared quantities within a single object or across sets of objects. Such comparisons may be especially important for reasoning about similarity and proportional structure within figures as well as for testing generalizations. We saw that comparisons across sets, particularly when paired with attention to invariance, appeared to support pattern recognition and opportunities for generalization. Future research should examine how PD can better support teachers to engage in within object and across sets comparisons.

Generalizations observed in this study were predominantly empirical rather than structural. Teachers' reasoning relied largely on observed cases and offered intuitive conjectures, with limited justification. We posit that shifting teachers' focus to invariant relationships may help

promote structural generalization. Future research should further investigate how teachers can be supported in moving from empirical to structural generalization.

Importantly, we saw comparison can destabilize teachers' reasoning in productive ways. Comparison can expose hidden assumptions, confusion about units, reliance on approximation, and difficulty coordinating multiplicative relationships. For example, both groups struggled to reason about the distances between the kitties' eyeballs, relying on empirical reasoning, while Group B also slipped into using additive reasoning instead of multiplicative reasoning which invoked an incorrect response. Such moments represent valuable opportunities for teachers in PD to surface and examine the multiplicative nature of proportional reasoning and similarity.

Finally, discussions in Group B revealed a productive tension when reasoning shifted from linear relationships to area relationships, particularly as the facilitator repeatedly invoked the language of *doubling* rather than *squaring*. This suggests a conceptual tension: imprecise language obscured the dimensional structure of the situation, making it difficult to determine whether teachers' reasoning reflected minor verbal imprecision or deeper misunderstandings about invariant relationships across dimensions. We therefore argue PD should explicitly attend to the precision of comparative language and intentionally support teachers in reorienting comparisons when dimensional shifts occur, helping participants move from empirical observations toward structural generalization. Rather than avoiding confusion, PD can intentionally design moments in which comparison surfaces conceptual tension, creating opportunities for deeper structural understanding and generalization.

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ARTICULATING INSTRUCTORS' KNOWLEDGE FOR TEACHING MATHEMATICS IN AND FOR ENGINEERING

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Scholars have documented a persistent disconnect between college math as taught in math departments and as used in disciplines such as engineering. Drawing on a broader study investigating mathematical knowledge for teaching that supports students' application of math in engineering contexts, we focus on the question: What knowledge is entailed in the instructional work of interpreting students' use of math in an engineering context? We use a case study of a math instructor analyzing an engineering student's reasoning on a Calculus task to argue that such instructional work requires a "dually bifocal lens," meaning that they: (1) have a conscious awareness of both disciplinary lenses and (2) notice the perspective(s) that a learner takes; and ultimately, develop a repertoire of techniques to shift learners' back and forth between lenses.

Keywords: Mathematical Knowledge for Teaching, Calculus, Undergraduate Education

Introduction

Our work addresses a key disconnect in the mathematical education of engineers: the lived separation between engineering practice and mathematics course experiences. As undergraduates, engineers report experiencing conflict between mathematics as practiced in mathematics coursework and as applied in engineering courses (e.g., Alpers, 2017; Bingolbali et al., 2007; Jablonka & Bergsten, 2023). As professionals, engineers report viewing their undergraduate mathematics experiences as irrelevant to their work (Quéré, 2017; Härterich et al., 2012; Pearson, 1991). As these studies indicate, the problem is not the inherent abilities of students to succeed in coursework, but a perceptual disconnect between math as it is *taught* and engineering as it is learned. We report an effort to address this disconnect through the study of the *instructional knowledge* required to help undergraduate students learn to apply mathematics.

The purpose of this paper is to examine disciplinary knowledge for teaching mathematics service courses whose students largely consist of STEM majors outside of mathematics. The data discussed in this paper comes from a larger study whose overarching agenda is to characterize the mathematical knowledge for teaching that allows students to learn to apply mathematics in engineering contexts. Here, we focus on the research question: What knowledge is entailed in the instructional work of interpreting students' use of mathematics in an engineering context? We articulate such knowledge in the context of a case study, where we focus on a mathematics instructor, Alaina, as she examines an engineering graduate's thinking on a Calculus task. This case allows for analytic generalization (Yin, 2018) into an aspect of instruction that is needed to address the problem of disciplinary disconnect. We use collective professional examination of the video records and artifacts of this case to articulate the mathematical knowledge entailed in instructional work (Ball et al., 2014). We then problematize the implementation of such knowledge in practice under the current disciplinary constraints of Calculus as a service course.

[1] All authors contributed equivalently. Our team alternates author order to reflect this.

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In the United States where this study is based, introductory (differential and integral) Calculus courses are considered first-year university courses and serve as prerequisites for a variety of STEM coursework. For this project, we choose engineering as our application domain, and first year Calculus as our instructional domain, due to the (a) unique role of Calculus in our context as both a "scaffold" and "filter" (Biza et al., 2022) for STEM degrees, and thus its pronounced impact on first year undergraduate students; (b) need for Calculus concepts across engineering subdisciplines (e.g., Engineering Accreditation Commission, 2025) and; (c) opportunity to build on a substantial existing base of literature on the mathematical education of engineers (see Alpers, 2020 for a review).

The main result of our work is a conceptual framework, which we term a *dually bifocal perspective.* We use the term "bifocal" to reference Ball's (1990) observation that teaching inherently requires seeing near and far, and we introduce "dually bifocal" to refer to the disciplines of mathematics and engineering. We build on the premise that successful learning of mathematics in and for engineering requires teaching students to draw on two partially overlapping disciplinary lenses-a mathematical lens focused on generalizability and internal structure, and a contextual lens focused on real-world constraints and practicality. We then use our case toward the following articulation: *Knowledge for teaching mathematics in and for engineering requires conscious awareness of both disciplinary lenses, noticing the perspective(s) that a learner takes, and ultimately, a capacity to shift learners' back and forth between disciplinary dispositions.* While our case focuses on Calculus and engineering, we believe that our characterization may extend to mathematics in and for other client disciplines.

Literature Review

The calculus sequence exerts significant influence over engineering students' academic preparation for later coursework and their perception of mathematics and its relationship to engineering (Gardner et al., 2007; Middleton et al., 2014; Van Dyken et al., 2015; O'Connor et al., 2015; Biza et al., 2022). The link between engineering contexts and mathematics learned in these first- and second-year math courses is often unclear, to both students and engineering faculty, as the following literature shows. Students report a lack of awareness of the substantial role that prerequisite mathematics may take in engineering coursework (Harris et al., 2015); as well as skepticism about the relevance of theoretical mathematics in their future careers (Flegg et al., 2012; Wood et al., 2008); Engineering faculty state that beginning engineering students often struggle to connect first and second-year mathematics concepts to their introductory engineering courses without significant support (Faulkner et al., 2019; Engelbrecht et al., 2017).

Previous work to address this problem has historically focused on departmental leadership and curricular development (e.g., European Society for Engineering Education [SEFI], 2013; Smith et al., 2021) in advanced courses such as linear algebra, differential equations, or complex variables (e.g., Alpers, 2001; Barquero et al., 2013; Hjalmarson et al., 2023; Härterich et al., 2012; Peters, 2023; Rønning, 2023; Schmidt & Winsløw, 2021). For instance, Schmidt and Winsløw (2021) reflected on 20 years of task design for a mathematics course addressing linear algebra, differential equations, and differentiable vector fields. They created in-depth assignments (some with 5 pages of data and 18 pages of student-facing text) in contexts such as Halbach magnet arrays, home heating, and red blood cells. Another common approach is to infuse examples throughout a course, even if the students do not engage with the examples in depth. For instance, Kerrigan and Prendergast (2021) solicited examples from engineering faculty for a flipped precalculus course. Burrill et al. (2023) reported a similar effort to redesign linear algebra, with the effect that the "students appreciated the content much more and

enrollment in [this course, offered by the Department of Mathematics] grew vigorously while that of the analogous class from the school of engineering dropped precipitously" (p. 800). Finally, other more resource-intensive approaches include team teaching (e.g., Quintanilla et al., 2007), coordinated mathematics and engineering courses (e.g., Laoulache et al., 2001; Lowery et al., 2010; Schroeder & Carpenter, 1999), and specialized sections for engineering majors (e.g., Horwitz & Ebrahimpour, 2002; Sathianathan et al., 1999).

These efforts are effective, but may not translate easily into instructional knowledge beyond the immediate faculty involved. Indeed, there is ample evidence that excellent curricular materials are not enough; instructors must also have the personal resources and sufficient pedagogical knowledge to enact curricular materials in the intended manner (Cohen, 1990; Ganter, 1999; Stein et al., 2007). However, there is very little existing literature describing the kind of pedagogical knowledge required to effectively teach mathematics in and for engineering, and thus minimal understanding of how to support instructors in effectively mending the perceptual disconnect between these disciplines. In their systematic reviews of knowledge for teaching at the undergraduate level, Andrews et al. (2022) and Sarkar et al. (2024) independently found only one article addressing engineering: Viiri (2003), who only argued that instructor knowledge of students' conceptions of engineering topics was an aspect of pedagogical content knowledge.

There is thus room for innovation in this area, especially in lower-division courses including Calculus, where improving retention will have the most impact due to the larger pool of students. Instructors' knowledge shapes their teaching, and thus students' learning (e.g., Baumert et al., 2010; Kraft et al., 2024; Stein et al., 2007). Hence, if we, as a field, better understand how instructors can convey and notice the disciplinary interactions between mathematics and engineering, we might then be able to help students connect these disciplines in their coursework. We thus focus on instruction, and the knowledge entailed in instruction, to advance existing work.

Conceptual Perspective

We define *mathematical knowledge in and for teaching (MKT)* as the mathematical skills, dispositions, and reasoning entailed in and developed through the recurrent work of teaching (Ball et al., 2008; Rowland & Ruthven, 2011; Thompson & Thompson, 1996). In developing MKT, teachers may iteratively synthesize interpretations of students' thinking with their personal understandings (Silverman & Thompson, 2008). When a subject intersects with two disciplines, knowledge for teaching must then draw on discourse and interactions in both disciplines.

We use the terms *mathematics in and for engineering (MEng)* and *teaching mathematics in and for engineering* in analogy to MKT. Just as there is specialized mathematical use entailed in teaching, there is also specialized mathematical use in engineering (Alpers, 2010; Kent & Noss, 2002). Moreover, knowledge for teaching mathematics differs from knowledge for doing mathematics as a discipline. In other words, we build the argument that there is a *special domain of knowledge for teaching mathematics in and for engineering*. Instructional representations are needed to ground inquiry into mathematical knowledge for teaching, so that examination can move beyond vague generalities and into articulating features of practice and the knowledge drawn upon in practice (Ball et al., 2014; Boaler et al., 2003). Representations may include video representations of student thinking, through which instructors' engagement may develop MKT (e.g., Philipp et al., 2007). Analogously, examining instructors' analyses of student thinking can help educators refine their perception of instruction (e.g., Amador et al., 2017). When a group of

professionals collectively examines instructional records, the group's interactions can create professional knowledge about teaching (Ball et al., 2014), as we show in this work.

Method

This study used a case to develop theory (Schwandt & Gates, 2018). Rather than seeking to characterize this knowledge comprehensively, we aimed to generate potential features of disciplinary knowledge for teaching mathematics in and for engineering. These findings should be seen offering hypotheses for further inquiry (Yin, 2018). Throughout this report, all participant names provided are pseudonyms.

Data sources for this study were instructional representations (Grossman et al., 2009) designed to support conceptual inquiry into teaching (Ball et al., 2014). Specifically, we inquired into instructional reasoning that draws from the disciplines of mathematics and engineering.

We created instructional representations as follows. First, we recorded two recent graduates of an undergraduate engineering program-Wade and Michael-in a think-aloud protocol (Ericsson & Simon, 1993) on an activity designed to situate calculus constructs in engineering. We refer to this activity as the "Roadway Curve" activity. The activity is based on a standard computation in civil engineering, where the coefficients of a function describing the elevation of a roadway may be derived from surveying data and familiarity with derivatives. We chose recent engineering graduates to maximize the potential for eliciting mathematical reasoning from an undergraduate engineering perspective. Both participants completed the activity successfully.

Second, we recorded two mathematics instructors who were shown excerpts of the engineering graduates' thinking. Both instructors had prior experience teaching calculus. We asked the instructors to complete the Roadway Curve activity themselves, to describe the student thinking they saw in terms of mathematical and outside-of-mathematical reasoning, and to explain how they would respond to this thinking if it had arisen in their calculus class. Instructors were intentionally shown excerpts of reasoning that appeared to us and two engineering faculty to contain moments of maximal contrast with reasoning from the discipline of mathematics. For this report, we present a case study of one instructor, Alaina. This choice is motivated at the start of the results section below.

Analysis proceeded in two phases. The analysis team consisted of the authors and the two engineering faculty. Phase 1 focused on the engineering graduates' reasoning on the Roadway Curve activity while Phase 2 focused on the instructors' responses to the students' reasoning. Phase 1 resulted in the identification of two types of reasoning with mathematics that were accessible to and frequently used by the engineering graduates-reasoning that prioritized contextual details (an "engineering" lens), and reasoning that prioritized generalizability and systematization (a pure "mathematical" lens). Both lenses are discussed in the results.

For the sake of space, we focus on describing analysis from Phase 2, as instructor data is the focus of the current report, and we only discuss one instructor's reasoning. In this phase, we used Ball et al.'s (2014) notion of collective professional examination to examine how instructors interpreted student use of these lenses and discussed their "fit" within the context of calculus instruction. From the instructors' responses, we deliberated about features of instructional knowledge that would promote students' development of mathematics in and for engineering, using our different professional values as a mechanism to make inferences explicit (Akkerman & Bakker, 2011; Schwandt & Gates, 2018). We generated hypotheses about professional practice, through inferences about what was present from the records of the instructor interviews, as well as what may have been absent (Ball et al., 2014; Grossman et al., 2009). This combination of

direct and inferential examination can result in useable knowledge for and about teaching (Kazemi, 2008).

Results

At the time of the study, Alaina was a mathematics graduate student with experience teaching calculus at multiple institutions in the United States. She had no previous experience with engineering coursework, nor with the contextual background (highway design) of the Calculus activity. Alaina's prior teaching experience included recitations intended to elicit and support student thinking on recently introduced topics in lecture, so she had experience in grappling with and responding to student thinking. Because of this, the knowledge involved in her noticing and interpreting students' use of mathematics in an engineering context was more likely to be visible during her interview. Presenting her case here allows us to develop a robust foundation for considering how such knowledge might be cultivated among less experienced graduate student instructors, or instructors with less experience attending to student thinking.

We asked Alaina to engage with Michael's thinking on the activity opening, shown in Figure 1; option (d) is the intended choice. Mathematically, this opening was designed to elicit students' language for differentiability of a function based on its graph. In terms of engineering, option (d) shows a roadway design that adheres to guidelines from the American Association for State Highway and Transportation Officials: two straight segments are connected by a parabola curve so that derivatives at the boundaries are equal. The remainder of the activity examined this parabola segment using first and second derivatives.

Which of the following is the best way to connect two straight segments to make a road?

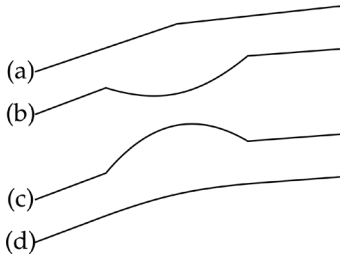


Figure 1: Roadway Curve Activity Opening

As engineering graduate Michael observed, roadway (b) is not desirable because, "This is where children fall off sleds, things come out of the back of your truck, big trucks tip over. We don't like these [roads]." Here, Michael relied on the consideration of additional, reasonable contextual details such as the kinds of transportation that might be done on such roads, to come to a conclusion. We refer to this type of reasoning as reasoning from an *engineering lens*. By an engineering lens, we mean one that sees contextual consequences for "making 'things' that work and making 'things' work better" (Lucas et al., 2014, p. 11). This kind of reasoning was common across Michael and Wade's interviews and is characterized by explicit attention to problem-solving contexts. This includes surmising additional context that may not be necessarily provided by the problem itself to determine which conclusions would be appropriate. Reasoning in this

domain has a goal of considering as many likely realities of a given context as possible to determine the relevant constraints on solution strategies.

Michael went on to use mathematical language, describing such points as "discontinuities" of the derivative, and noting that two tangent lines placed on either side of each point differ in slope. This is an example of reasoning from a *mathematical lens,* one that focuses on the potential of mathematical concepts to systematize and generalize (Freudenthal, 1968). This kind of reasoning includes intentionally simplifying or backgrounding certain realities of the problem context (e.g., roadway elevation data) and foregrounding other reasoning techniques (e.g. differentiation, analysis of the first derivative, etc.). In this way, Michael provided two types of justification for his claim that (d) was the most sensible choice. We wanted to understand how Alaina would interpret these kinds of reasoning, whether she would recognize these lenses as different, and how she would respond to them in an instructional situation.

When asked to describe Michael's reasoning, she did present evidence that she observed two types of reasoning. Alaina described Michael's initial reasoning with the context as, "thinking about problems in the context of the world, and in the context of how you would actually be driving your car...this is the more intuitive version of what we will probably later formalize [in the class]." Alaina's use of the word "intuitive" here reflects a broader tendency for her to describe reasoning from the engineering lens as a form of "domain expertise" involving experience with the context of the given problem. While she sometimes described this reasoning as "just [using] your opinions," "your feelings," "intuition," and generally being less formal than the reasoning from a mathematical lens, she went on to explain that she would encourage both forms of reasoning in her Calculus course. When asked about the extent to which she would agree with the statement that only some of Michael's reasoning was "really" mathematics, Alaina clarified that she viewed *both* forms of reasoning as falling under a broad spectrum of reasoning with mathematics. As she stated:

> I mean, at some point, [Michael] mentioned discontinuity. I might say, "oh yeah, discontinuity of the derivative." But that's probably the more formalized part of this [reasoning], which is maybe what the other person meant by saying "mathy" is, like, formalized mathy, rather than the understanding that we have of the world, and how things should work, and how things are best. That also is valuable. Because when we're talking about real-world problems, we need domain expertise. That's also valuable.

Here, Alaina's characterization of Michael's work as demonstrating an understanding of how "things should work" and "how things are best" indicates that she recognizes, in some way, hallmarks of mathematical reasoning as interpreted from an engineering lens.

We were also interested in understanding Alaina's reasoning around the instructional work of incorporating both forms of reasoning in a calculus course. When reflecting on reasoning from the engineering lens in the context of the Roadway Curve activity, she stated, "I wouldn't necessarily think that this [reasoning] is, like, broadly applicable to every other type of problem that we want [students] to also be able to do based off of this activity. So I would also want to make sure that [students] knew the more general way that I would probably be talking about this problem with the class." As she explained, encouraging students to use reasoning with both disciplinary lenses requires expertise on the part of the instructor, because the depth of contextual consideration present in the engineering lens is not compatible with the emphasis on generalizability from the mathematical lens. Instructors would thus need the ability to recognize which lens a student might be reasoning from and to foresee potential issues regarding an overextension of certain reasoning practices into inappropriate contexts.

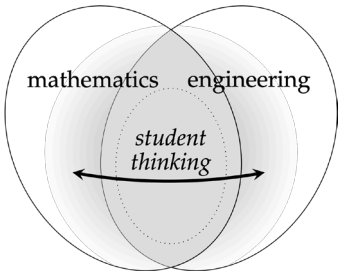
Alaina went on to explain that such circumstances of overextension were very likely when working with applied problems in calculus courses.

> [The problem's author] could be a math professor who's never thought about highways before and needed to do different numbers for different problems...but then they changed the numbers too much in a way that doesn't actually make sense if you're doing these sorts of things in real life.

In this way, Alaina highlights the need for instructors to be aware of how context can typically function within calculus problem contexts to guide students in utilizing the engineering lens productively. As she stated, even when a problem in a calculus course is based in a 'real world' context, that context may not actually be realistic due to the prioritization of repeated structure and generalizability inherent in such a course.

Hypothesis for Teaching Mathematics in and for Engineering: Dually Bifocal Perspective

So, what knowledge is entailed in the instructional work of interpreting students' use of mathematics in an engineering context? Based on Alaina's case, we hypothesize that teaching mathematics in and for engineering entails knowledge stemming from what we term a "dually bifocal perspective." This perspective is dual in the sense that it involves the coordination of knowledge in and for two disciplines: mathematics and engineering. This coordination necessarily involves an awareness of both perspectives as well as the ability to "shift" back and forth between them as necessary. This perspective is bifocal in the sense of Ball (1990), and we extend her use of this term to fit our context. Bifocal lenses give the ability to focus near and far. In our context, we see Calculus instruction as premised on accurate understanding of the "nearby" student thinking and course goals, while also viewing "farther away" versions of these concepts that will be necessary for application in other contexts. Figure 2 depicts our perspective as overlaid bifocal "lenses."



Teaching mathematics in and for engineering entails bifocally viewing "nearby" student thinking and "farther away" concepts through both mathematical and engineering perspectives

Figure 2: Dually Bifocal Perspective

When an instructor can see their instruction through a *dually bifocal perspective*, this means *they are consciously aware of when their personal thinking or learners' inferences draw from mathematical concepts or from a given engineering context*. Further, with a dually bifocal perspective, an instructor can see when a learner draws on only one disciplinary lens, or when they move (perhaps tacitly) between lenses. In other words, we are saying that when instructors

can see from a dually bifocal perspective, they are better able to support their students in operating from the dual lenses of mathematics concepts and engineering context.

Finally, conscious awareness (cf. van Oers, 1998) of lenses is crucial. There are two reasons for this. First, the mathematical lens and engineering lens have competing goals, so it is necessary to "switch" between them, rather than thinking about both at the same time. Second, while an engineer may function with tacit switching between lenses, this is insufficient for *teaching mathematics in and for engineering*. Teaching entails *explicit* knowledge of "situations of justification" (Bauersfeld, 1980; Cobb et al., 1992, p. 576; Stylianides & Ball, 2008) for that topic. Here, an instructor needs conscious awareness of when they (or students) are or could be reasoning through mathematics, when they (or students) are or could be reasoning through engineering, and when such reasoning practices would or would not be appropriate.

Conclusions

Building instructor capacity requires the support of mathematics departments, particularly in creating opportunities for instructor professional development involving mathematics in and for engineering. Authentic engagement requires attention to Engineering's disciplinary perspectives and hence interaction with engineers. At the same time, instructors do not need—and are unlikely to have the time—to attain fluency in all aspects of mathematics in and for engineering. We could envision, instead, developers and instructors selecting specific topics and teaching moves, in consultation with their institution's engineering programs, that showcase a dually bifocal perspective. Getting this right would, we believe, require the dually bifocal perspective we have developed here, but the payoff for departments would be reducing the disconnect between mathematics and its use in Engineering, a long-standing challenge in post-secondary mathematics instruction.

Inquiry into the work of teaching and its entailed knowledge has the potential to advance research around cross-disciplinary mathematical learning and instruction, as well. Our framework recognizes disciplinary distinctions; we also hold them productively in tension through a turn to conscious awareness of the lenses and an explicit knowledge of situations of dual justification. As a potential conceptual and analytic framework, a dually bifocal perspective opens avenues for cross-disciplinary inquiry into how instruction and instructors can be levers of perceived disciplinary continuity in undergraduate mathematics education. A dually bifocal perspective may also lay groundwork for educative curriculum materials (Davis & Krajcik, 2005) that position instructors to mend disconnects between mathematics and engineering.

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MEASURING TEACHERS' SPECIALIZED CONTENT KNOWLEDGE OF COGNITIVELY GUIDED INSTRUCTION: VALIDATION AND IMPLICATIONS OF THE CGI TKA

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Cognitively Guided Instruction (CGI) is a widely-used professional development framework that emphasizes teachers' understanding of students' mathematical thinking, yet little research has directly measured teachers' knowledge of CGI itself. This report describes the revision and validation of a CGI Teacher Knowledge Assessment (CGI TKA) designed to measure teachers' specialized CGI knowledge. Using classical test theory, factor analysis, and item response theory, the study examined the assessment's psychometric properties. A revised 30-item assessment was administered, and analyses supported a final 21-item version with strong construct validity, appropriate item functioning, and acceptable reliability. The instrument provides a sound tool for assessing CGI knowledge and informing the design and evaluation of CGI-focused professional development.

Keywords: Assessment, Mathematical Knowledge for Teaching, Professional Development, Teacher Knowledge

Introduction

Cognitively Guided Instruction (CGI) positions teachers' knowledge of children's mathematical thinking as central to effective mathematics instruction (Carpenter et al., 1999). Since its conception in the late 1980s, teachers across the United States have participated in professional development focused on CGI and it has been heralded as one of the most significant contributions to mathematics education reform in the past thirty years (Empson & Jacobs, 2008). Research has consistently shown positive associations between participation in CGI professional development and changes in teachers' beliefs, instructional practices, and student achievement (Carpenter et al., 1988; Carpenter et al., 1989; Enochs, Smith, & Huinker, 2000; Peterson et al., 1989; Philipp, 2007; Pianta, LaParo, & Hamre, 2008; Schoen Bray, Wolfe, Tazaz, & Nielsen, 2017). However, much of this research has inferred teachers' CGI knowledge indirectly through beliefs surveys, classroom observations, or student outcomes, or by measuring generalized elementary Mathematics Knowledge for Teaching (MKT) through tests like those produced by the Learning Mathematics for Teaching (LMT) project (Hill et al., 2004). These assessment approaches assume that teachers possess the *specialized content knowledge* (SCK) required to interpret student strategies and orchestrate instruction consistent with CGI principles, however a more targeted assessment could provide additional information about CGI SCK.

If CGI depends on teachers' ability to anticipate student thinking and respond instructionally (Carpenter et. al., 1988), then directly measuring teachers' knowledge of CGI is essential. A valid and reliable measure of CGI knowledge can strengthen research on CGI, inform professional development design, and support decisions about teacher learning and support. This study addresses this need by examining the validity and reliability of a CGI Teacher Knowledge Assessment (CGI TKA) by asking the following overall research questions:

1. What revisions or modifications should be made to the original CGI TKA to make it a more valid and reliable measure to teacher's specialized content knowledge of CGI?

2. How validly and reliably do the scores obtained from the revised CGI TKA measure teacher's knowledge of CGI?

Theoretical Framework

This study is grounded in theoretical perspectives on teacher knowledge, particularly the construct of *specialized content knowledge* for teaching (Ball et al., 2008; Shulman, 1986). Beyond general mathematical content knowledge, teachers require knowledge that enables them to interpret student thinking, recognize meaningful strategies, and make instructional decisions in real time (Hill et al., 2004). Within the CGI framework, this specialized knowledge includes understanding CGI problem types, common student solution strategies, and underlying principles that connect student thinking to instructional responses (Carpenter et al., 1988). The CGI TKA was designed to operationalize this form of specialized content knowledge, allowing it to be examined empirically.

The CGI TKA is grounded in methodology that suggests psychometric validity is a metric of a test, including evidence to suggest generalizability of the interpretation of the scores obtained from administration of the test (Borsboom et. al., 2004). Borsboom states that a valid test can convey the effect of variation in the attribute being measured, in this case CGI knowledge, if variation in the attribute causes variation in the test scores. This study sought to measure and explain that variation as a means of determining validity.

Methods

Participants included in-service and pre-service elementary teachers who had previously participated in CGI professional development, including story problem types and their difficulty levels, understanding student strategies for solving story problems, and implementation of math instruction using story problems in elementary classrooms. As part of this study, an original 36-item assessment was revised based on expert feedback, item analysis, and alignment with a refined table of specifications, producing a revised assessment. Data were drawn from administrations of both the original (n = 97) and revised (n = 213) versions of the CGI TKA.

Analyses proceeded in multiple phases as shown in Table 1. Classical item analysis (Novick, 1966) and distractor analysis (Gierl et al., 2017) were used to evaluate item performance and inform revisions. Exploratory factor analysis was conducted to examine the underlying structure of the revised assessment, followed by confirmatory factor analysis to evaluate model fit (Thompson, 2004). Item response theory analyses using a Rasch model were used to examine item difficulty, fit, and information (De Ayala, 2004; Wright & Linacre, 1998). Reliability of scores was estimated using Raykov's rho because of the lack of correlated errors (Bacon et al., 1995; Raykov, 1997). Comparisons between dichotomous scoring and partial credit scoring were also conducted to examine trade-offs between scoring precision and practicality (Andrich & Styles, 2002; Masters, 1982; Sideridis et al., 2016).

Table 1: Analytic Evidence Supporting Validity and Reliability of the Revised CGI TKA

Analysis	Purpose	Key Evidence
Item & Distractor Analysis	Guide item revision	Improved item functioning and distractor effectiveness
Exploratory Factor Analysis	Examine structure	Support for a single CGI knowledge factor
Confirmatory Factor Analysis	Test model fit	Strong fit indices supporting construct validity
Rasch IRT Analysis	Examine item properties	Appropriate difficulty and acceptable item fit
Reliability (Raykov's rho)	Estimate score consistency	Acceptable reliability with dichotomous scoring

Findings

The original 36-item CGI TKA format included testlet and true-false structures that limited reliable item analysis, so items were revised by repeating stems, combining related content, and converting true-false sets into multiple-choice items assessing the same concepts. These changes eliminated ten items while creating space for new items identified as missing through expert feedback, resulting in a 30-item assessment. This assessment was administered to 213 examinees.

Rasch distractor analysis illuminated item difficulties and the functionality of item distractors. It was observed that 4 of the 21 items contained one distractor that was considered *non-functioning* as it was not selected above a 10% threshold of responses. There was an additional item in which three distractors were *non-functioning*, resulting in a low-difficulty item in which 85% of the examinees answered the item correctly, but it was determined to keep the item based on other analyses. However, an interesting phenomenon occurred where one of the 21 items contained a *hyper-functioning* distractor—a distractor that was selected more frequently than the correct answer across all examinees, which contributed to the item difficulty of 31.9% which was one of the most difficult items.

Although the original organization of the test items theoretically suggested four possible constructs based on the table of specifications—(a) problem types, (b) solution strategies, (c) principles and pedagogy, and (d) evidence of base-ten understanding—results supported a unidimensional structure for the revised CGI TKA, interpreted as teachers' *specialized content knowledge* of CGI. Exploratory and confirmatory factor analyses of the same data using MPlus (Muthén & Muthén, 2012) indicated that 21 of the 30 items loaded adequately (above .32, representing approximately 10% of the variance) on a single factor and demonstrated strong model fit (CFI = 0.962 and TFI = 0.958).

Rasch item response theory analyses showed appropriate item difficulty distribution and acceptable item fit, indicating that the assessment was well-targeted to the sample. Whereas some of the distractors contained partial or incomplete understanding, the researcher theorized potential partial credit for distractors containing partial understanding. Distractors considered to contain partial accuracy were scored 0 (no understanding), 1 (partial understanding), or 2 (accurate or complete understanding). To accurately compare the dichotomous and partial credit scoring models, information criteria were calculated from the log likelihood of each of the models. The model with the lowest information criteria, indicating the best model fit, was the

dichotomously scored model. Even considering removal of each of the 10 items identified as containing distractors with useful stochastic information, none of the models fit the data better than the dichotomously scored model.

The reliability estimate for the 21-item assessment was in the acceptable range, with dichotomous scoring yielding slightly higher reliability (.837) than partial credit scoring (.826). Although partial credit scoring provided additional information for some items, the gains were modest and did not outweigh the increased scoring complexity for most intended uses. Overall, findings suggest that the revised 21-item CGI TKA with dichotomous scoring functions as a psychometrically sound measure of teachers' CGI knowledge.

Discussion

This study contributes to mathematics education research by providing evidence for a validated instrument that directly measures teachers' *specialized content knowledge* of CGI. The findings address an opportunity for more specialized, targeted assessment of CGI knowledge. By demonstrating that CGI knowledge can be assessed reliably and validly, this work strengthens the empirical foundation for future studies examining relationships among teacher knowledge, instructional practice, and student learning.

From a practice-oriented perspective, the CGI TKA has important implications for professional development. Scores from the assessment can be used to identify areas of strength and need in teachers' CGI knowledge, inform the design of targeted professional learning experiences, and evaluate the effectiveness of CGI-focused professional development or pre-service preparation programs over time. The assessment also offers potential for broader applications, such as supporting instructional coaching or informing decisions about advanced CGI training.

Methodologically, the use of (a) classical test theory for item and distractor analyses, (b) exploratory and confirmatory analyses, (c) item response theory analyses, (d) comparison between dichotomous and partial credit scoring, and (e) reliability analyses collectively suggest a thorough validation of the CGI TKA.

Implications and Conclusions

The CGI TKA is a methodologically robust, psychometrically sound tool for measuring teachers' *specialized content knowledge* of CGI. Its use can support both research and practice by making teachers' CGI knowledge visible and measurable. Current use of the CGI TKA supports consideration of the effectiveness of professional development and need for further support of teachers and school leaders. It is often used to assess understanding and changes in CGI knowledge among preservice teachers during teacher preparation programs and with practicing teachers during inservice professional development. However, this assessment could potentially be used for teacher certification, advancement, or endorsement purposes if so desired. Additional validation may be necessary to support use of the CGI TKA as a criterion-referenced test (Kane, 2013).

Future research should examine how CGI TKA scores relate to instructional practice and student outcomes and explore the use of the instrument with larger and more diverse teacher populations, accounting for various demographics to suggest validity for multiple groups of examinees and, therefore, reducing the potential for bias. Together, these efforts can deepen understanding of how teacher knowledge functions within CGI and how professional development can be designed to support meaningful instructional change. Additionally, an alternate form of the CGI TKA could be created and validated to provide more robust

psychometric opportunities for pre- and post-testing scenarios to demonstrate knowledge growth and program effectiveness.

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UNDERSTANDING LONGITUDINAL DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING AMONG NOVICE ELEMENTARY MATHEMATICS SPECIALISTS

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Elementary Mathematics Specialists (EMSs) require robust Mathematical Knowledge for Teaching (MKT) to effectively support student learning and teacher development. This convergent mixed methods study examined longitudinal shifts in MKT of 26 novice EMSs across the first 3 years of a 5-year EMS preparation program. Participants completed mathematics and coaching endorsement courses, coupled with ongoing professional learning communities and mentoring. MKT was measured at three time points, and interviews were conducted at the end of Year 3. Results indicated significant gains in MKT during courses, with knowledge largely sustained through Year 3. Qualitative findings associate growth with increased attention to vertical content alignment, deeper analysis of student thinking, and greater curricular flexibility. Findings show the important role of particular learning experiences in developing EMSs' MKT.

Keywords: Elementary School Education, Mathematical Knowledge for Teaching, Professional Development, Teacher Knowledge

Introduction

Elementary Mathematics Specialists (EMSs) are mathematics teachers and leaders who fill a variety of roles in elementary schools, including: (a) generalist classroom teacher, (b) specialist mathematics or STEM teacher, (c) Multilingual Learner, Special Education, Early Intervention Program, or gifted interventionist, and (d) teacher leader or coach (Association of Mathematics Teacher Educators, AMTE, 2024). To effectively serve in these roles and effect change in their settings, EMSs need well-developed mathematical content knowledge they can draw from in their work with students, teachers, and school leaders. The mathematical knowledge for teaching (MKT) framework (Ball & Forzani, 2010; Ball et al., 2008; Hill, 2010) includes *specialized content knowledge* focused on teachers' abilities to: (a) analyze and interpret students' mathematical thinking and ideas, (b) use multiple representations of mathematical concepts, and (c) define terms in mathematically correct and accessible ways. Notably, research focused on the extent to which EMS preparation programs increase MKT illuminates positive but mixed findings (Campbell & Malkus, 2014; Kutaka et al., 2018; Rigelman & Lewis, 2023).

Although prior studies have documented some increases in MKT following EMS courses or professional development, less is known about the longitudinal stability and developmental trajectory of MKT as a measured construct (Campbell & Malkus, 2014; Kutaka, et. al., 2018). This study addresses the conceptual tension between MKT as an assessable construct and MKT as situated professional knowledge, examining whether measured changes persist over time and if and how those changes align with teachers' own accounts of their learning (Hill & Chin,

2018). We examined 26 novice EMSs whose MKT was assessed longitudinally over three years during a rigorous EMS preparation program and their reflections about their learning.

Research Questions

1. To what extent did novice EMSs' MKT change across an EMS preparation program?
2. How did the novice EMSs describe changes in content knowledge, and how did that align with the shifts measured by the LMT?

EMS Preparation Program Description and Participants

Participants

The 26 EMSs' primary responsibility was teaching students, and while novice EMS teacher leaders, the participants were experienced teachers, averaging 12+ years of experience. All held master's degrees in education. They largely identified as persons of color (70%) and worked in 23 urban-situated elementary schools with underserved student populations.

Program Description

The EMS preparation program included seven university courses leading to professional endorsements in Elementary Mathematics (K-5ME) and Coaching (TCSE), with concurrent participation in professional learning communities (PLCs) and individual mentoring that extended beyond their coursework (see Table 1). Grounded in research and professional recommendations (AMTE, 2024, 2017), overall goals of the Elementary Mathematics endorsement courses were to develop: (a) ambitious and equitable mathematics instructional practices (Aguirre et al., 2024; Carpenter et al., 2015; National Council of Teachers of Mathematics [NCTM], 2014, 2020); (b) deep and broad knowledge of elementary mathematics, including *specialized content knowledge* (i.e., "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" [Ball et al., 2008, p. 399]); (c) productive mathematics beliefs; and (d) knowledge and utilization of mathematics education standards, research-informed curricula and resources, and networking opportunities. When considering MKT development, we conceptualized it as socially mediated and participation-based, drawing on communities of practice theory (Wenger, 1998). This positions teacher knowledge as evolving through engagement in shared mathematical work across courses and learning communities.

Table 1: Timeline and Project Elements Aimed at Preparing and Supporting EMSs

Year 1			Year 2		Years 3–5
Fall	Spring	Summer	Fall	Spring	
TCSE Course: Teacher Support & Coaching	K-5ME Course: Number & Operations	K-5ME Course: Data & Probability	K-5ME Course: Algebra & Rational Number *and* Coaching Internship	K-5ME Course: Geometry & Measurement *and* Math Internship	
PLCs and Mentoring	PLCs and Mentoring		PLCs and Mentoring	PLCs and Mentoring	PLCs and Mentoring

Methodology

This study used a convergent mixed methods design (Creswell, 2014), with the initial analysis considered separately and then merged to compare, contrast, or validate findings. The EMSs' MKT was measured prior to the program, upon conclusion of the related mathematics course, and at the end of Year 3. The intent was to consider if the participants' MKT significantly increased during courses and if that was maintained over time. MKT data were collected using the Learning Mathematics for Teaching (LMT), which focuses on *specialized content knowledge*, common content knowledge, and knowledge of content and students (Ball et al., 2008; Hill et al., 2004). The LMT includes three subscales—number and operations, algebra, and geometry—and person abilities are reported as IRT scores. Data were also gathered via eight individual (II) and three focus group (FG) interviews, with questions focusing on: (a) possible changes in knowledge of elementary mathematics content, and (b) how program learning experiences influenced knowledge development. Hierarchical linear modeling of the IRT person ability scores for each test occasion nested in EMSs describes shifts in content knowledge for the entire group of participants. The interview data were analyzed using the constant comparative method. Changes in LMT subscale scores for individuals were connected to the findings of the interview data for the same individuals.

Findings

Analyses of the LMT data show significant positive changes in MKT after three years of the program, with most improvement occurring in the first two years as evidenced by significant linear and quadratic hierarchical mixed models with test occasions nested in individuals (see Table 2).

Table 2: Type III Tests of Fixed Effects – Quadratic Mixed Models by Content Domain

Outcome	Effect	Num df	Den df	F (p)
Geometry	Intercept	1	35.20	1.75 (.194)
Geometry	Linear Time	1	30.23	28.97 (<.001)
Geometry	Quadratic Time	1	30.05	19.38 (<.001)
Algebra	Intercept	1	32.15	0.02 (.878)
Algebra	Linear Time	1	32.24	17.27 (<.001)
Algebra	Quadratic Time	1	29.34	8.97 (.006)
N&O	Intercept	1	39.07	5.67 (.022)
N&O	Linear Time	1	32.49	7.43 (.010)
N&O	Quadratic Time	1	32.70	6.12 (.019)

Note. Satterthwaite degrees-of-freedom approximations used throughout.

Interview data provided insights into how participants described their shifts in content knowledge as situated in practice and connected to particular learning experiences. Analyses show that participants especially attributed growth in MKT to: (a) new knowledge of vertical alignment of mathematics content across K-5 grade levels (*horizon knowledge)*, (b) more robust understanding and expectations of students' mathematical thinking and reasoning abilities (*specialized content knowledge related to student thinking)*, and (c) increased understanding of curricular materials as resources with potential need for modification *(curricular knowledge)*. From a communities of practice perspective (Wenger, 1998), these shifts reflect not only changes in knowledge but also changes in participation, expanded engagement across grade-level

boundaries, deeper involvement in collective sense-making about student thinking, and movement from curriculum implementers toward curriculum interpreters and contributors.

Relationships between the measured changes in MKT and the participants' perceptions of those shifts obtained through interview data were also explored to consider alignment among the two data sources. These are shown with positive (+) or negative (-) changes in 0.2 IRT score intervals. In most cases, the shifts described are consistent with the changes measured via the LMT, and many of the participants described their knowledge growth as situated in practice and linked to specific experiences in the EMS preparation program.

For example, when considering *horizon knowledge*, one participant stated:

> I think working with all of us as a whole group because we each play different roles at different levels in elementary education. And so being able to look at the content more K-5 really helped me understand how it builds on each other and how I can better serve the groups that I'm with. (II, ll. 50–54)

Alg: ++
Geo: ++++++++
N&O: --

A statement that reflected *specialized content knowledge related to student thinking* includes:

> … some of the different ways that students think allow me to really think further and differently about how students explain their responses … I'm able to give students the freedom to think about it and share the way they learn it, which then increases my understanding and what I'm able to truly allow them to do to demonstrate their knowledge. (II, ll. 95–106)

Alg: +++++++++
Geo: ++++++
N&O: ++++

When considering *curricular knowledge*, one participant commented:

> I think before the program, I was kind of wary to veer off from like the county plan or what I was told to do. … [The program] has just given me like the tools and the ability to deliver mathematics the way that I want to and also stand up for that decision. (II, ll. 19–25)

Alg: +++
Geo: ++++
N&O: ++++++

Conclusion

Overall, findings confirm and extend the extant literature, illuminating the complexity of changes in EMSs' MKT and associated connections with their own teaching and program learning experiences. With previous research on EMS programs showing mixed findings when it comes to MKT (e.g., Campbell & Malkus, 2014; Kutaka et al., 2018), it is optimistic that the learning experiences provided in this program supported significant growth in and retention of this knowledge.

General consistency between the participants' descriptions of the shifts in MKT with the measured changes using the LMT illuminate internalized growth in MKT. Interview data suggest preparation program components such as rigorous coursework, collaboration with colleagues across grade levels, and ongoing interaction beyond coursework supported gains and retention of MKT throughout the project, suggesting shifts in MKT may be socially mediated and situated in communities of practice as Wenger (1998) described. Continued study of these EMSs and how they draw from their content knowledge when implementing and advocating for ambitious and equitable mathematics instruction offers an important opportunity to advance the knowledge base on EMS preparation.

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MATHEMATICAL KNOWLEDGE FOR TEACHING AND DISCOURSE FACILITATION IN INTERMEDIATE CLASSROOMS

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Teachers vary considerably in their ability to facilitate productive mathematical discourse yet research inadequately explains this variation. This multiple case study investigated how mathematical knowledge for teaching and pedagogical expertise influenced discourse facilitation among four expert intermediate mathematics teachers. Drawing on Ball, Thames, and Phelps' (2008) framework, the study examined teachers' specialized content knowledge, knowledge of content and students, and curriculum knowledge through interviews, classroom observations, and fields notes. Findings revealed that teachers with deeper mathematical understanding navigated student contributions more flexibly, recognizing mathematical validity in unexpected solution paths. Teachers demonstrated curriculum knowledge understanding essential for discourse facilitation. All teachers acknowledged that highly procedural topics resist discourse approaches, suggesting that mathematical knowledge for teaching includes knowing precisely when discourse serves specific learning goals.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Classroom Discourse, Middle School Education

Objectives and Research Questions

Teacher quality represents the most significant school-based factor influencing student achievement (Rivkin et al., 2005; Wayne & Youngs, 2003). However, identifying specific teacher attributes and practices that improve student learning remains challenging. Research consistently demonstrates wide variation in teacher effectiveness, yet explicating what effective teachers know and do proves elusive (Hill et al., 2005; Stigler & Hiebert, 2009). Mathematical knowledge for teaching (MKT) has emerged as a critical construct explaining teacher effectiveness (Ball et al., 2008; Hill et al., 2005). MKT encompasses specialized mathematical knowledge beyond what typical adults possess, including: recognizing and analyzing diverse solution strategies, identifying the mathematical point in student contributions, sequencing examples appropriately, and choosing representations that illuminate mathematical structure (Ball & Bass, 2003). Research demonstrates that teachers with stronger MKT produce greater student achievement gains (Hill et al., 2005; Baumert et al., 2010).

Mathematical discourse places particular demands on teacher knowledge. When students explain reasoning, offer unconventional strategies, or propose incorrect solutions, teachers must make rapid, complex decisions: Is this mathematically valid? How does this connect to learning goals? Which student ideas should I pursue? (Stein et al., 2008; Stockero & Van Zoest, 2013). These in-the-moment decisions require robust mathematical understanding integrated with pedagogical expertise (Jacobs et al., 2010; Sherin et al., 2011). Research further suggests that productive discourse depends on teachers' capacity to elicit, interpret, and build upon student reasoning in ways that advance mathematical learning (Chapin et al., 2013; Selling, 2016).

This study investigates: How does teachers' mathematical knowledge for teaching and pedagogical expertise influence their facilitation of discourse in Grade 7 and 8 mathematics classrooms? Sub-questions explored: (1) how mathematical content knowledge shapes teachers'

responses to student contributions during discourse; (2) what role curriculum knowledge plays in discourse planning and facilitation; and (3) how teaching experience influences discourse success.

Theoretical Framework

This study draws on Ball, Thames, and Phelps' (2008) framework distinguishing six domains of mathematical knowledge for teaching: common content knowledge (mathematical knowledge any educated adult should possess), specialized content knowledge (mathematical knowledge unique to teaching), knowledge of content and students (knowledge about how students think about, know, or learn particular mathematical content), knowledge of content and teaching (knowledge about teaching particular mathematical content), horizon content knowledge (knowledge of how mathematical topics relate across curriculum), and knowledge of content and curriculum (knowledge about curricular materials and programs).

The study focuses particularly on specialized content knowledge and knowledge of content and students, as these domains prove especially relevant for discourse facilitation. Specialized content knowledge enables teachers to recognize mathematical validity in unconventional student strategies, decompose mathematical ideas for instruction, and select representations illuminating mathematical structure (Ball et al., 2008; Ma, 1999). Knowledge of content and students enables teachers to anticipate common misconceptions, interpret student thinking during discourse, and craft questions accessing student reasoning (Hill et al., 2008; Carpenter et al., 2014).

Additionally, the study employs Shulman's (1986, 1987) construct of pedagogical content knowledge, which emphasizes transformation of content knowledge into forms teachable and learnable by students. Extending this framework to mathematical discourse, researchers have identified key aspects of pedagogical content knowledge for discussion-based teaching, including knowing which mathematical topics lend themselves to student discussion, how to frame problems inviting multiple solution strategies, and how to sequence student contributions to build collective understanding (Lampert, 1990; Smith & Stein, 2011).

Methods

Participants and Context

Four intermediate mathematics teachers participated in this multiple case study, purposefully selected for their commitment to and expertise to discourse-based instruction. All participants taught Grade 7 and/or 8 mathematics in schools within a large urban district. Three participants held additional qualifications in mathematics, representing specialized professional learning beyond initial teacher certification. While participants varied in cumulative years of teaching experience, they all integrated classroom discourse into their instructional practices frequently.

Data Collection

Data collection followed case study protocols (Merriam, 2009; Stake, 1995) and included: (1) two semi-structured interviews per teacher, one conducted before and one after classroom observations; (2) two 45-minute classroom observation sessions per teacher, audio-recorded to capture discourse patterns; (3) researcher field notes documenting classroom environment, student interactions, and teacher moves; and (4) demographic data from school websites and publicly available statistics. Interview protocols explored teachers' beliefs about discourse, implementation strategies, challenges encountered, and decision-making processes. Observations captured naturally occurring discourse during mathematics instruction.

Data Analysis

Analysis followed Merriam's (2009) two-stage process for multiple case studies. Firstly, each case was analyzed individually using Kvale's (2007) five-step meaning condensation process to identify themes within teacher practice. Interview transcripts were analyzed by extracting natural meaning units and grouping them into emergent categories using NVivo software. Secondly, cross-case analysis was conducted to identify commonalities and variations across the four teachers (Merriam, 2009; Stake, 1995). Trustworthiness was established through triangulation of multiple data sources (interviews, observations, field notes) and by providing interview transcripts to participants for review and approval.

Results

Analysis revealed four major themes characterizing the relationship between teacher knowledge and discourse facilitation.

Theme 1: Specialized Content Knowledge Enables Flexible Responding

Teachers with stronger mathematical knowledge demonstrated greater flexibility responding to diverse student contributions. When students offered unconventional solution strategies, these teachers quickly assessed mathematical validity, identified the mathematical principle underlying student thinking, and formulated questions helping students refine or extend ideas. Teachers holding mathematics specialist qualifications exemplified this flexibility. During observed lessons, when students suggested approaches that differed from standard algorithms, these teachers recognized the mathematical validity and asked probing questions. For example, during an observed lesson, one participant, who held a doctoral degree in Biology, selected four student answers to use for lesson consolidation. She systematically asked students to explain their strategies, built upon their reasoning, and sequentially transitioned from one strategy to the next. She reached the final student work sample and summarized everything coherently before the end of the lesson. After the lesson, she shared that she picked the four student works and presented them to enable students to understand the range of strategies that can be used when solving math problems and that some strategies are shorter and simpler than others. Specialized content knowledge allowed her to do such consolidation at the end (Shulman, 1986).

Theme 2: Curriculum Knowledge Frames Discourse Decisions

All teachers demonstrated strong knowledge of Grade 7-8 curriculum expectations and learning progressions. They all held specialized math qualifications. This curriculum knowledge directly shaped how they planned their lesson around the use of discourse: specifically, which mathematical ideas to emphasize, what prerequisite understanding to activate, and how concepts are connected across grades. One teacher explained the importance of understanding prerequisite knowledge, recalling: "I remember when I came to Grade 7 as a new teacher, I did not know what the Grade 7s should know by [the end of] Grade 6. So, knowing all that is important." Curriculum knowledge also informed in-the-moment decisions about which student contributions to pursue and what questions to ask. Towards the end of the lesson, teachers selected student works to showcase and those that advanced learning toward learning goals (Shulman, 1986; Smith & Stein, 2011).

Theme 3: Experience Enables Anticipation and Preparation

Teaching experience correlated with discourse success. Teachers with more years teaching intermediate mathematics demonstrated superior ability to anticipate student difficulties and to prepare questions addressing common misconceptions. The participants, who are all experienced teachers, described how planning discourse-rich lessons became more efficient over time. One participant noted that "preparing such lessons [discourse-based] now took much less time

because of a well-developed repertoire of questions, anticipated student responses, and strategies to address them." Another participant described the process of trying to "anticipate what the different strategies might be, in advance" and prepare corresponding questions for each approach. Furthermore, all the participants managed classroom logistics more efficiently, as due to their experience and habit, they found it easier to form groups quickly and were more likely to facilitate a productive lesson using discourse. One teacher added that "if my students are not in the mood to talk or if they are tired, I am going to forego talk for another teaching strategy" (Interview). They all agree that teachers need experience as well as strong mathematical knowledge, suggesting that mathematical knowledge and experience work synergistically (Hiebert & Grouws, 2007).

Theme 4: Recognizing Discourse Limitations Demonstrates Expertise

Significantly, all teachers acknowledged that certain mathematical content resisted discourse approaches. Highly procedural topics requiring foundational knowledge, they argued, benefited from explicit teacher demonstration before student exploration. One teacher reflected on teaching the Pythagorean theorem: it would be "hard to teach students how to find the length of the diagonal in a right-angled triangle by asking students to talk as they have no knowledge about the existence of the Pythagorean Formula and how to use it" (Interview). This recognition that discourse does not universally serve all learning goals reflects pedagogical sophistication (Hiebert & Grouws, 2007; Lampert, 2001). Rather than ideologically committing to discourse regardless of content, teachers exercised judgment about when discourse advanced learning versus when other approaches proved more effective.

Discussion and Implications

This study illuminates the complex knowledge base underlying effective mathematical discourse facilitation. Findings confirm that mathematical knowledge for teaching matters significantly (Ball et al., 2008; Hill et al., 2005), while extending understanding of specifically how MKT enables discourse success. Specialized content knowledge enabled flexible response to diverse student contributions. These differences were visible when teachers asked probing questions to advance student thinking and when they connected student ideas to mathematical big ideas. This finding suggests that professional development targeting discourse facilitation must strengthen teachers' mathematical understanding, not just teach pedagogical moves (Ma, 1999; Stigler & Hiebert, 2009). The deeper teachers' content understanding, the more effectively they can guide students toward multiple solution pathways and deeper mathematical understanding.

Curriculum knowledge, one of Shulman's (1986) three categories of content knowledge, plays an important role in effective mathematics teaching. Teachers needed understanding of learning progressions, prerequisite concepts, and how topics connected across grades to make productive discourse decisions. This suggests professional development should explicitly address how to orchestrate productive mathematical discussions, including anticipating student strategies and sequencing contributions to build mathematical understanding (Smith & Stein, 2011). Professional development should make this full process explicit to teachers so that they understand how discourse fits and connects across concepts and grade expectations.

The finding that experience enables anticipation of students' questions and responses aligns with research on teacher expertise, which shows that teachers can prepare for classroom interactions by anticipating what students will think and do (Stein et al., 2008). However, experience must be combined with strong mathematical knowledge for developing expertise (Ball et al., 2008). Professional development facilitators are encouraged to provide teachers with

collaborative design opportunities where they map out discourse-based lessons, explicitly script questions, and anticipate student questions and prepare the responses.

The acknowledgment that discourse does not universally serve all mathematical learning goals demonstrates adaptive expertise. Rather than viewing discourse as a pedagogical panacea, teachers exercised judgment about when discourse advanced learning. This finding cautions against treating any single approach as always appropriate, but instead emphasizing the need to match instructional methods to learning goals (Hiebert & Grouws, 2007).

For teacher education, findings suggest three priorities: (1) strengthen prospective teachers' mathematical knowledge through content courses emphasizing specialized knowledge needed for teaching; (2) provide extensive practice analyzing and responding to student mathematical thinking through video case studies and clinical experiences (Sherin et al., 2011); and (3) develop adaptive expertise through opportunities practicing pedagogical decision-making in varied contexts.

Future research could investigate how teachers' MKT develops over time through discourse practice, examine the specific mathematical knowledge needed for different content domains, and explore how professional learning experiences strengthen teachers' capacity for discourse facilitation.

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TEACHERS' APPROACHES TO FRACTION MULTIPLICATION AND DIVISION IN A REAL-WORLD SETTING

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Researchers have explored how students' conceptual knowledge does not develop if students only have the opportunity to procedurally memorize algorithms, such as invert-and-multiply (e.g., Behr et al., 1992; Cramer et al., 2010; Young & Zientek, 2011). Investigations of teacher knowledge have identified challenges including struggles to utilize appropriate representations and limitations in teachers' understandings of fraction division and multiplication (e.g. Armstrong & Bezuk, 1995; Borko et al., 1992; Izsák, 2008; Post et al.,1988). This study focused on (a) whether teachers were able to appropriately answer two questions about a particular recipe and (b) what mathematical approach they used to solve each question.

Thirty practicing teachers in grades 3–9 from four states responded to both items as part of a larger project. Item 1 asked how many loaves could be made given a certain amount of sugar, while Item 2 asked how much sugar would be needed for a certain number of loaves. Interviews were conducted via Zoom with participants using a Google Jamboard to show their thinking. Interviews were transcribed verbatim. Two researchers coded each of the 60 teacher responses for correctness (remainder correct, without remainder, incorrect) with 87% agreement. Next, researchers coded the mathematical approach: multiply, reciprocal, cross multiply, numeric, visual, and repeated addition. One response did not include a solution. Video and/or Jamboards were consulted to look for disconfirming and confirming evidence.

Overall, the majority of responses (42 out of 59) were coded as Correct. Many teachers successfully used a standard algorithm, including multiplication (particularly for Item 2), reciprocal (changing a division problem to multiplying by the reciprocal), and cross multiply (Table 1). Across the majority teachers who were able to correctly solve two items, we observed an important reasoning that underlined the correctness. The correctness appeared to hinge on identifying the situation as either multiplication or division rather than on the approach. However, identifying the situation correctly did not guarantee a correct solution.

Table 1: Standard Algorithm Approaches Across Items & Correctness

Code	Total	Item 1			Item 2		
		Correct	Incorrect	No Answer	Correct	Incorrect	No Answer
Multiply	24	1	3	1	17	2	0
Reciprocal	11	6	1	2	0	2	0
Cross multiply	9	5	0	0	3	1	0

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DO GSI'S LOOK FOR SOLUTIONS SIMILAR TO THEIR OWN WHEN ASSESSING STUDENTS' ATTEMPTS AT COVARIATIONAL REASONING?

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Keywords: Undergraduate Education, Teacher Noticing, Precalculus, Cognition

Given the impact covariational reasoning—reasoning about how two quantities change together (Carlson et al., 2002)—has in supporting productive meanings in precalculus concepts, researchers have provided (undergraduate) students with repeated opportunities to engage in covariational reasoning (e.g., Moore & Carlson, 2012; Yu, 2022). Some research in professional development settings (e.g., Clark et al., 2008) and with secondary educators (e.g., Teuscher et al., 2016; Liang, 2025; Tallman & Frank, 2020) have analyzed the ways in which instructors can decenter from their own mathematical meanings to understand and support their students' mathematical meanings, including covariational reasoning. However, an understudied population is the Graduate Student Instructors (GSIs, hereafter), instructors who are often responsible for teaching hundreds of undergraduate students over their graduate program, but who often receive little pedagogical training. Thus, via a study with 11 GSIs enrolled in a teaching undergraduate mathematics seminar, I aimed to answer the following research question: Do GSIs look for solutions similar to their own when assessing students' covariational reasoning?

To answer this research question, I asked the 11 GSIs to complete an assignment that was adapted from the *Bottle Problem* (Carlson et al., 2002) (*Adapted Bottle Problem*, hereafter) and submit a video describing their solutions. They then submitted a second video describing how they would support two Applied Precalculus students who completed the same *Adapted Bottle Problem* assignment. I used two frameworks used to analyze the data (covariational reasoning (Carlson et al., 2002) and SNR (Herbst et al., 2023). The SNR framework is used to describe a student contribution's serviceability (the extent a contribution supports the lesson goal), normativity (the extent a contribution aligns with the norms of the instructional situation), and responsiveness (the extent a contribution answers the problem posed).

Prior analysis of GSI's noticing of student descriptions using the SNR framework (Herbst et al., 2023) indicated that GSIs tended to notice serviceable and non-normative aspects of students' work (Stevens, 2026). The preliminary analysis reported here compares initial results with the GSI's own descriptions of solutions to the problems. The poster will include two preliminary results. First, I provide an example of a GSI who comments on positive aspects of student work dissimilar from their own. Second, I provide an example of a GSI who comments on negative aspects of student work similar to their own. The first result indicates GSIs' capacity to decenter from their own reasoning to identify positive aspects of students' work despite reasoning and a result that was different than their own. This result shows promise that GSIs who are new to undergraduate instruction can move beyond the correctness of a students' solution (Jacobs et al., 2010) and highlight the advantages of covariational reasoning. This contributes to literature on GSI's mathematical knowledge for teaching (Ball et al., 2008). However, it does not indicate if and how this knowledge is used in practice. The second result indicates potential disparities between GSI's own work and the expectations they have for their students. Continued analysis is needed to better understand the extent to which GSI's responses to student contributions align and misalign with their own reasoning.

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MATHEMATICAL PROCESSES AND PRACTICES

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

HOW SEMANTIC STRUCTURE AND PROBLEM CHARACTERISTICS SHAPE REPRESENTATION CHOICE IN FOURTH GRADE STUDENTS' SUBTRACTION PROBLEM-SOLVING

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Research has established that subtraction is not a single operation but a set of semantic structures. However, little is known about how students' representational choices mediate their problem-solving processes across different tasks. This task-based research investigates how semantic structures and problem characteristics of subtraction context create opportunities for different representational practices, and how those representations shape students' problem-solving processes. Data was drawn from videotaped and audio-recorded interview, transcribed, and coded. Findings revealed that representational choices mediate interpretation, reasoning and strategy choice with semantic structure and problem characteristics shaping the way students interpret the task situation. The study emphasized the need to reconceptualize subtraction instruction as a problem-solving practice that privileges structural sense-making and flexible use of representations.

Keywords: Problem Solving, Elementary School Education, Mathematical Representation

Word problem tasks place distinct cognitive demands that extend beyond those required for arithmetic or algebraic tasks presented in purely symbolic form. It requires the coordination of multiple advanced cognitive skills, including language comprehension and logical reasoning, which are less central when mathematical expressions are already provided (Wong & Yip 2023). To successfully solve a word problem, student must identify the problem type, make sense of the context, and recognize its underlying semantic structure, often referred to as schema identification. They must also determine which quantity is unknown within that structure; only after completing these interpretive steps can students formulate a suitable mathematical equation and proceed to solve the problem (Carpenter & Moser, 1982/2020; Wong & Yip, 2023).

Previous studies by Jerman (1973), Cook (1973) and Searle et al. (1974) focused on classifying word problems by surface-level features such as syntax, vocabulary, and word count, highlighting how linguistic form can affect problem difficulty. This align with a synthesis of research by Andersson (2007), Mayer (2001), Nathan and Kintsch (1992), Daroczy et al. (2015), and Carpenter and Moser (1982/2020) indicating that variation in mathematical solution strategies is a function of three interrelated dimensions: linguistic factors (e.g., wording, semantic categories, propositional structure), cognitive features (e.g., schema activation, metacognition, retrieval fluency), and numerical factors (e.g., number magnitude, position of the unknown). Likewise, Lean et al. (1990) and Daroczy et al. (2015) noted that problems containing explicit semantic cues—such as conjunctions (e.g., and), adverbs (e.g., left, more than, less than), or determiners (e.g., each)—often guide students toward a particular operation.

Meanwhile, representations are key elements in building conceptual understanding, enabling students to visualize, manipulate, and connect mathematical ideas across different forms of expression. The effective use of these representations relies on a critical cognitive process and translation between systems. These include signs, symbols, diagrams, objects, images, and graphs which are fundamental tools in mathematics teaching and learning (Mainali, 2021). Research on children's subtraction problem solving shows that learners employ a variety of

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representations such as concrete manipulatives, visual models, symbolic/numeral, algorithms (vertical/horizontal) notation depending on the task context and their own strategy preferences (Barmby, 2009; Björklund et al., 2019; Boulton-Lewis et al., 1996; Bruner, 1966 cited in Presno, 1997; Center, 2019; Li et al., 2018).

Existing studies on word problem-solving often focus on outcomes, strategy selection or categorized problems by semantic structure rather than in the moment interpretive work students engage in before computation begins. Building on this body of work, this study addresses the non-obvious cognitive processes children engage in when solving subtraction word problems, linking how student's representational choices interact with semantic structure to shape reasoning, strategy development, and sense-making during subtraction word problem solving. Thus, the purpose of this study is to investigate these processes through the following questions:

1. How do semantic structures and problem characteristics of subtraction word problems create opportunities for distinct problem-solving approaches?
2. How do students' representational practices mediate their problem-solving processes across different semantic structures of subtraction problems?

Theoretical Framework

The Boulton-Lewis model (1993) provides the theoretical framework for this study. It classifies mathematical representations into three distinct categories based on the structure and associated cognitive load they placed on the learner. Material strategies or analogs involve using concrete objects or physical visualizations to represent mathematical concepts. Discipline-specific representations are formal and conventional tools adopted by the mathematics discipline, such as written algorithms, symbols, and number lines. Mental/Verbal strategies by contrast refer to internal models, and self-generated processes that children employ in preference to or alongside formal procedures to support mathematical reasoning. Within this framework of representation use, different categories of representations serve distinct pedagogical functions in learning subtraction. However, existing research has often examined representation use separately from the semantic structure of word problems. Specifically, subtraction research typically isolates strategies and representations, whereas word-problem research emphasizes schema identification, linguistic cues, and semantic structures (Boulton-Lewis, 1993; Carpenter et al., 1981; Carpenter & Moser, 1982/2020; Daroczy et al., 2015; Wong & Yip, 2023).

In this study, the framework is used as an analytical lens through which the participants' problem-solving process was examined, revealing how representational choice mediates the relationship between semantic structure and problem characteristics. Thus, create affordances or constraints in the transition from concrete modeling to abstract reasoning in subtraction.

Methods

Design and Participant

This study adopts a qualitative single case study design using a task-based interview approach to examine student's solution processes, strategy use and representation choices to subtraction problems. A single-case design is appropriate for such an exploratory analysis as it allows for a rich, contextualized examination of cognitive phenomena within a specific, bounded instance while the task-based interview protocol served as the primary research instrument (Goldin, 2000). During the interview, follow-up questions were posed to the student on all items to elicit the students' thinking and justification. The study used an eight-year-old male student (pseudonym: Ini) currently enrolled in the fourth grade at a public elementary school.

Data Sources

Two subtraction tasks were selected based on Carpenter and Moser (1982/2020) taxonomy: Task 1, represents a comparison/joining problem (involves determining the difference between two quantities) while Task 2 represents a multi-step subtraction problem (combines both a Change/Start Unknown structure and a Comparison structure). It requires that students coordinate sequential operations in which the result of the first calculation becomes the input for the second.

> Task 1: Morgan wants to buy the next book in her favorite series when it is released next month. So far, she has saved $15. The book will cost $22. How much more money does Morgan need to save so that she can buy the book? (NCTM, 2014).
>
> Task 2: Mariam will be 20 years old in 3 years. Her brother, Dahlan's age is 2 years more than Mariam. What is Dahlan's age now? (Ling & Ghazali, 2014).

Data Analysis

The interview session was both video and audio-recorded and was subsequently transcribed, reviewed, and coded. Supplementary data sources included the student's written work and researcher field notes. The analysis employed a hybrid coding approach, with criteria synthesized from the frameworks of Carpenter and Moser (1982/2020), Torbeyns et al. (2009), Rosenthal and Resnick (1974), and Boulton-Lewis (1993). Strategies were coded according to the type of operation and reasoning used, such as direct subtraction, counting up, decomposition, matching/modeling, and trial-and-error. First, I considered the semantic structure and the position of the unknown in the corresponding mathematical equation in the tasks. Secondly, representations were coded deductively using Boulton-Lewis's model of mathematical representations: categories of Material Strategies (e.g., fingers, pizza drawings), Discipline-Specific Representations (e.g., equations, number lines, standard algorithms), and Mental/Verbal Strategies (e.g., mental regrouping, verbal explanations). For example, when the student hypothetically used a number line to explain a comparison problem to a peer, this very episode was coded for representational form and later analyzed for its function in supporting explanation. The episode was then interpreted in relation to the problem's semantic structure, illustrating how representation mediated the student's problem-solving process. The task-based interview took one and a half hours, guided by the two research questions. To reduce researchers' bias, video recordings, verbal explanations, field notes, and student work artifacts were triangulated to support interpretations of the student's reasoning.

Results

Influence of Semantic Structure and Problem Characteristics on Problem-Solving Strategies

Ini initially used the standard algorithm (conventional step-by-step mathematical problem-solving process) to solve Task 1. To probe further into why he used this strategy, I asked him to explain his understanding of the task by asking what it was saying. He interpreted (summarized) the task as "*What's the answer to 22 minus 15?"*

From his responses, he identified key operations of the problem based on the linguistic clue "*how many more*," which he interpreted as signaling subtraction (minus), thus guiding his choice of strategy direction. I then asked:

> Interviewer: What about the problem helped you to decide how to solve it the way you did?

Ini: First, because you said that we're going to do subtraction numbers, and because subtraction numbers always have the bigger number on top and then when I read and fought around the question, I saw how many more, how many much more. And then I was like, oh, so this is subtraction… *That's the (unknown) number that we have to find.*

He explicitly identified the unknown number (7) as "the answer" and "the unknown number that we have to find" framing the problem as a missing addend (15 + ? = 22) while he calculated the difference using direct subtraction. This aligns with research indicating that the semantic structure of word problems often influences the strategies children select.

Ini's ability to connect the larger quantity to the minuend and the smaller quantity to the subtrahend demonstrates an understanding of subtraction as a structured relationship between quantities. This reasoning was shaped by the problem characteristics, which guided how the student identified and organized quantities prior to computation. Previous research similarly highlights that attention to problem characteristics enables students to meaningfully map contextual information onto symbolic expressions, thereby supporting conceptual understanding of subtraction (Carpenter et al., 1999; Fuson, 2003).

The second task was presented, and Ini immediately recognized it as a two-step subtraction word problem.

Ini: This is a two-step word problem.
Interviewer: Two-step word problem? How do you know?
Ini: I have to figure out what twenty minus three is. Which is? Twenty-eight, no, twenty seven and then add two more which means that, um, this is for the M is for Maria and the B is for brother, which means, um, twenty-seven plus two equals twenty-nine. How I first knew it was going to be a minus problem is that like Maria would be, that would be 20 years old in three years. So, then I have to figure out what age she is right now. And then when I figured that out, and then farther in the question it said, but her brother's two years older than what she is right now. So then I had to add, so then I had to add two to it, which he called 49 and 29.

The analysis confirms that the distinct semantic structure and cognitive complexity of the two tasks directly shaped his problem-solving and strategy choices (see Table 1).

Table 1: Summary of Semantic Structure and Problem Characteristics on Problem-Solving Strategies

Task	Sematic Structure	Key Linguistics Clue	Problem Characteristic	Strategy Selection
Task 1: Book Savings Problem	Compare/ Difference (How many more?)	"How much more"	Two quantities given (22, 15); unknown difference; one-step. Larger minuend explicit (22 > 15) Multiples of 5 embedded (15, 20, 22)	Standard subtraction algorithm; skip counting; backward jumps. Stacking with regrouping. Skip counting and number line reasoning.

Task 2: Age Problem	Change with comparison (Two-step)	"In three years;" "two years more"	Future reference; two-step structure; multiple unknowns	Sequential subtraction then addition. Bar model reasoning

The Use and Integration of Multiple Representations in Subtraction Problems

Across both tasks, the respondent engaged multiple representational forms, showing flexibility in organizing and communicating mathematical reasoning through physical, symbolic, and mental modes.

Task 1. Ini's primary representation was the symbolic/formal notation using the standard algorithm (Discipline specific) because it enabled him to efficiently use the direct subtraction strategy. This representation is visually organized by place value columns, which he felt simplified in the regrouping (borrowing) process and reduced the cognitive load of having to mentally track an imaginary one. He also noted that the convention of placing the larger number on top facilitated the process of taking away. "*I put the one over to make sure that I remember that I already borrowed 1… So, stacking (crossing it to show borrow) makes it easier.*" While engaging with the task, I observed that Ini was using his fingers alongside (see Figure 2). I asked him what he was trying to figure out by doing that.

Ini: 5 + 2, 5 and then I did. And then I was like 1, 2 and then I did 5 + 2 like 5, 6, 7 and then I was a new 7.



Figure 2: Student Using Finger to Compliment His Problem-Solving Process

Using the standard stacking subtraction algorithm, in which digits are crossed out to indicate regrouping, reduced the cognitive demand associated with holding a borrowed value in working memory by externalizing the regrouped quantity on the written record—supporting reasoning that would otherwise need to be mentally maintained when using nonstandard or mental strategies (Fuson, 2003). Also, Ini's integration of finger; a concrete strategy within a formal symbolic procedure aligns with Boulton-Lewis's (1993) observation that analogs like finger counting often support the execution of more abstract operations.

Furthermore, to know Ini's preferred representation, I asked him how he would *explain the task to his friend and what picture or tool he would use to help them understand thereby* shifting the focus of the task cognitive demand from solution execution (solving) to mathematical explanation. His response was "'*A bar model*.' *So, if I like got squared. And they each have 5 tallies. And I gotta make my 15. And then there's my 15. And then I'll say, and then I'll put a plus sign and then make like…"* (see Figure 3).

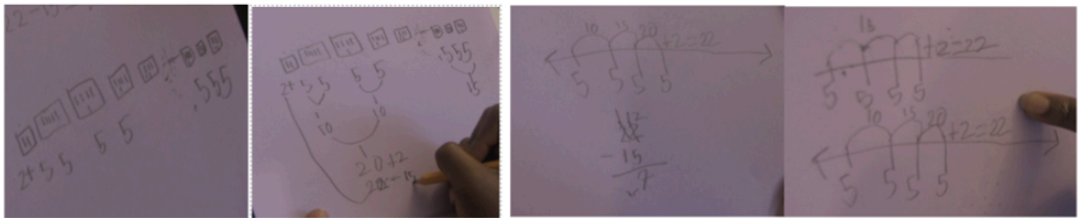


Figure 3: Student Engaging with Various Representations

The bar model (a semi-structured visual analog) was used to show groups of fives composing the whole (22) and then conceptually subtracting the (15) by grouping tallies into units of five and used labels like 5, 5, 5, 5, 2. He connected the whole (22) to its parts through repeated addition/skip counting. The elements (groups of five) made the numbers easier to decompose and manipulate. He was further prompted to represented the problem differently, he used the number line, a discipline-specific visual tool to model skip counting up to 22 and later showed how the number line could represent subtraction by jumping backward from 20 to 15, then adding the remaining two, to find the difference, leading to the result of seven thereby connecting the representation to both additive and subtractive reasoning. These further probe the awareness of representations as communicative tools (doing math to teaching math) and how representation mediates conceptual understanding.

Subsequently, I prompted Ini to recontextualize the task to determine whether his reasoning was driven by the surface story context, underlying semantic structure or if context influences his representational choice. Thus, uncovering the entry point into the problem-solving process.

Interviewer:	So, did the story or the situation make you think about a problem in a certain way? Imagine if this task was like a balloon task and let's say pizza cause that's your favorite, and not buying books, do you think you would have solved it in a different way?
Ini:	Yeah, well, then I would have just made the slices in fractions like that (for Task 1 again).

He drew four whole pizzas, partitioned into five slices each, plus a fifth pizza partitioned into halves, totaling 22 slices (whole Pizza). He then shaded 15 of these slices to model the known savings, visually identifying the seven unshaded slices as the needed amount. This pictorial model (see Figure 4) functioned as a concrete, enactive demonstration of "taking away," translating the abstract numerical comparison into a tangible, part-whole scenario (Boulton-Lewis, 1993).

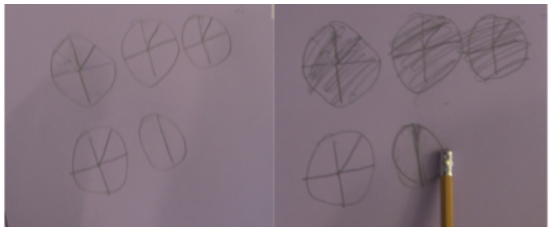


Figure 4: Student Using Picture to Model Fractional Part of Subtraction

Task 2. Ini primarily relied on formal notation (equations) to execute the two steps, labeling the steps "M" and "B" (see Figure 5) and writing corresponding equations (M: 20 – 3 = 27; B: 27 + 2 = 29). His preference for formal notation can be explained by the semantic structure, cognitive complexity, and the problem characteristics of the task, and this direct arithmetic approach (standard procedures) is common once a problem is defined. Research shows that multi-step word problems increase working-memory demands, prompting students to externalize their thinking through symbolic notation to organize and stabilize intermediate results (Soneira et al., 2023; Wang et al., 2016). From a cognitive load perspective (Boulton-Lewis, 1993), noted that formal notation reduces processing demands by allowing the student to offload memory requirements—labeling steps ("M" and "B") and recording equations helps track dependencies between quantities. The two-step characteristics initially constrained Ini's ability to generate alternative visual models.

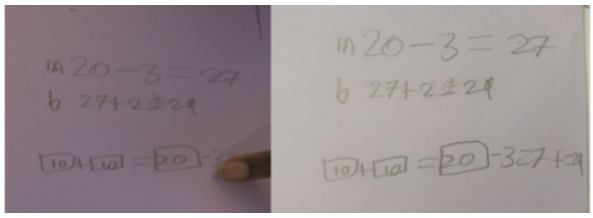


Figure 5: Student Using Bar Model to Show Two-Factor Subtraction

Ini's overall representation choices, as summarized in Table 2, revealed a patterned interplay employed in all three representational categories described in the Boulton-Lewis (1993) model.

Table 2: Student's Use and Integration of Multiple Representations Across Tasks

Boulton-Lewis Representation Category	Representation Used	Task 1: Book Problem	Task 2: Age Problem
Material Strategies (Analogs)	Concrete groupings: finger counting; pictorial analogs (e.g., pizza slices, tallies)	Used	Limited use
Discipline-Specific Representations	Standard subtraction algorithm; symbolic expressions; number. Bar model	Used	Used
Mental / Verbal Strategies	Mental regrouping; verbal explanations; internal tracking of quantities	Used	Used

Discussion

This study examines the problem-solving process of a fourth-grade student using two analytical lenses. It focuses on the semantic structure and problem characteristics of the subtraction tasks, alongside the student's representational choices. Findings indicate that the two tasks grounded in location of the unknown (whether the missing quantity appears as the starting amount, change amount, or difference within the problem structure) shape the student's strategy choice and the representational affordances made available during the problem solving. The

student's initial choice of standard algorithm for subtraction was triggered by the surface-level linguistic cue "how many more," which he interpreted as a direct signal for the minus operation. Also, his ability to identify the minuend and the subtrahend based on the problem characteristics reflects a conceptual understanding of subtraction as a relationship between quantities. This aligns with research highlighting how children's early strategy selection is often cued by salient verbal features before a deeper structural understanding develops (De Corte & Verschaffel, 1987, as cited in Calabrese et al., 2024). However, the student's reliance on a linguistic cue for "minus" underscores the caution raised by Boulton-Lewis's (1993): without explicit instructional support to connect such cue to the underlying mathematical structure of the problem (e.g., the comparison relationship), strategy use may remain superficial.

Furthermore, the problem characteristics strongly shaped the representations the student selected and constructed. For Task 1, the explicit "how many more" linguistic characteristic guided him toward both direct subtraction (standard algorithm) and complementary additive strategies (counting up/skip counting), which he effectively modeled using bar models, number lines, fingers. These types of representations are highly effective because they made the difference perceptible as a visible gap between quantities, enabling the student to leverage the inverse relationship between addition and subtraction. Meanwhile, the need to determine a present value from a future age and then adjust for a relational comparison appeared to privilege formal notation (Symbolic Representation) as a means of keeping track of the steps involved in the second task. Notably, the bar model was the only visual representation he identified as applicable across both tasks. Thus, representations are not neutral supports; they actively shape how students think, the strategies they employ, and how they transition from situational reasoning toward abstraction (Mainali, 2021). At the same time, Boulton-Lewis (1993) cautions that the instructional value of representations depends on their coherence and systematic variation within a small set of tools are more effective for discerning mathematical structure: an overabundance of tools, or representations that are poorly aligned with the task structure, may increase cognitive load and obscure rather than clarify mathematical relationships.

Conclusion

The study highlights how subtle differences in problem structure can shape how students engage with and represent mathematical ideas. As the student tackled different problem types, his shifts in representational forms revealed how semantic structure and problem characteristics influenced active cognition, and the implication for mathematics education is profound. While the student progress toward symbolic fluency, it underscores a central challenge that procedural skill is accompanied by adaptive, relational thinking. This requires a pedagogy that is responsive, discourse-rich, and designed to make the structures of mathematics visible, thereby empowering students to make intelligent and flexible strategy choices across all problem types. Likewise, teachers must deliberately guide students through three distinct pedagogical steps: recognizing the structural foundations of subtraction problems, pairing these structures with appropriate visual or physical models, and utilizing these tools to build efficient mental math skills. However, the absence of physical manipulatives during the interview constrained the student's range of visible strategies, forcing him into symbolic methodologies and obscuring his full problem-solving potential. Consequently, future research may address these gaps. By positioning mathematical representations as a core practice, this research enriches ongoing discussions regarding how elementary students build strategic and adaptive reasoning in word problem solving.

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PRESCHOOLERS' NOTICING, WONDERING, AND POSING PROBLEMS ABOUT MATHEMATICS IN CHILDREN'S BOOKS

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In this study, we focused on preschoolers' noticing, wondering, and posed problems during individual read-aloud sessions. In particular, we share findings for 12 participating preschoolers in two different classrooms with two different picture books. Initial insights highlight that preschoolers' noticings, wonderings, and problems posed emerged from the researcher's prompts as well as on their own. Based on the picture books' contexts and illustrations, preschoolers noticed various mathematical elements: numbers, space, measurement, and geometry. We determined the ways in which these noticings were related to their wonderings and problems posed depending on whether the noticings functioned as problem situations.

Keywords: Early Childhood Education, Instructional Activities and Practices, Number Concepts and Operations, Measurement

Posing problems to solve and using problem solving skills to solve them are important mathematical practices (Kilpatrick, 1987; Siller et al., 2025). Problem posing is generally defined as generating a new problem from a given situation or reformulating a given problem (Silver, 1994). Although problem posing is often overlooked in mathematical explorations, improvements in students' problem posing are associated with better problem-solving ability (Bonotto & Dal Santo, 2015; Cai & Hwang, 2002; English, 1997; Singer & Voica, 2015). As an important practice, research on problem posing is relatively new (Cai et al., 2015). Further, most research attention has focused on third grade through high school students (e.g., Bonotto, 2013, Bonotto & Dal Santo, 2015; Cai, 2003; Ellerton, 1986; English, 1997, 1998; Silver & Cai, 1996; Wilkie, 2024) and college or preservice teachers (e.g., Baumanns & Rott, 2024; Crespo & Sinclair, 2008; Silber & Cai, 2017). Research with children under the age of six is rare (Carmona-Medeiro et al., 2024; Martin-Diaz et al., 2026; Pálmer & van Bommel, 2020) and has not focused specifically on the process of helping young children pose problems.

For preschoolers, play is an important avenue for posing and exploring problems (Fosse et al., 2020; Parks, 2015). Stories can encourage students to try creating new block creations (Casey et al., 2004, 2008) and picture books, specifically, can provide a broad context and allow students to focus on what they notice and wonder mathematically (Hintz & Smith, 2022). Picture books' openness allows preschoolers to play with potential mathematics and is likely to encourage them to create new problems as well (Ginsburg, 2006). Responding to Cai et al.'s (2015) call to better understand how students interpret problem situations to pose problems, we investigate what mathematics preschoolers notice in picture books and how these noticings relate to what they wonder about or problems they pose based on the contexts.

Theoretical Framework

Children's literature is an influential instructional tool that supports children's mathematics learning, interest and mathematical concept development (Young et al., 2018) and mathematical language (Purpura & Reid, 2016), motivates students (Luedtke & Sorvaag, 2018), and links mathematics with children's experiences (Marston & Mulligan, 2012) and emotions (Griffiths & Clyne, 1991). During read-aloud sessions, prompts used by readers are necessary and essential for stimulating their mathematical thinking (Elia et al., 2010). In particular, questioning, giving feedback, and identifying practical applications are crucial for developing strong mathematical skills (Jacobi-Vessels et al., 2014). Hintz and Smith (2022) outlined a process called *open notice and wonder reads* for mathematizing children's picture books, regardless of whether the books have an explicit or implicit mathematical focus. Central to the focus of this instructional move is that children have the space to engage with stories and share their thinking. The teacher stops at predetermined pages that have rich illustrations and invites children to explore them by asking, "What do you notice?" Children might focus on the context and illustrations (Hintz & Smith, 2022). Noticing includes observations that one makes and is used to refer to a momentary shift in focus or a form of attention (Mason, 2003). Therefore, preschoolers' mathematical noticings in picture books could involve spontaneous attention to relevant aspects of visual or narrative elements such as quantities, relationships, shapes, sizes, patterns, etc. They might also focus on situations they consider noteworthy, even if not mathematical.

Next, the teacher encourages children to explore the story and visuals further by prompting, "What do you wonder?" During this process, the teacher records children's thoughts to make them visible. In brief, children's noticing and wondering process provides an opening for them to share their observations and curiosities. This approach also creates a mathematical opportunity for teachers to think about posing problems based on their noticing and wondering in the story (Hintz & Smith, 2022). Likewise, children can turn their noticings and wonderings into problem posing situations, which are the context and data offered to problem posers for generating problems (Cai et al., 2022). Thus, in this study, we used open notice and wonder reads, encouraging preschoolers to mathematize children's literature and build their background knowledge (Crespo & Sinclair, 2008). We then used their noticings and wonderings to encourage the development of new problems (Marston, 2014). Hence, we investigated the following research questions:

1. What do preschoolers notice mathematically in individual read-aloud sessions based on the stories' contexts and illustrations?
2. What do preschoolers wonder and pose problems about mathematically in individual reading aloud sessions?
3. In what ways do preschoolers' noticings correspond to their wonderings and problems posed?

Methodology

Participants and Setting

The data presented in the current study come from a larger qualitative study approved by our university's IRB in which we recruited 23 preschoolers (ages 2.5–5) from four different classrooms at a midwestern preschool in the US. In the larger study, in session one, each classroom participated in a notice and wonder read. Then, in sessions two and three, participants participated in two small-group notice and wonder reads with two new books. In this paper, we focus on results from the final session with 12 preschoolers (ages 3–5) who participated in an

individual notice and wonder read with one of the researchers. All names are pseudonyms. In particular, they listened to one of two picture books: *Ten Apples Up On Top* by Theo. LeSieg (eight preschoolers listened to this one) or *I'm the Biggest Thing in the Ocean* by Kevin Sherry (four preschoolers listened to this one). During the notice and wonder read, the researchers invited each preschooler to share what they noticed and wondered on set pages from the book. The researcher then further probed for what questions they could pose to trick them and gave an example as needed. The researchers modeled wondering and posing problems both in previous sessions and within these individual sessions using language such as, "What do you wonder? What do you want to know more about? What questions do you have? What math questions could you ask to try and trick me?"

Analysis

We coded students' responses into the two main categories highlighted by Hintz and Smith (2022): noticing and wondering. However, we split the wondering code into wondering and problems posed. Wonderings were broader than problems (e.g., "Why are there so many apples in that big basket?" versus "How many apples are in the container?"). Under these three main codes, we coded students' responses for their general mathematical domain and wrote descriptors for specific topics within those domains (see Table 1). If the noticings, wonderings, or problems did not relate to mathematics, we noted if they related to the book's context, visuals, or other elements not found in the book. The coding process involved two rounds. First, each researcher coded one of the interview transcripts deductively based on the mathematical domains (or non-mathematical) and the noticings, wonderings, and problems posed. In the second phase, we discussed the codes and refined them collaboratively.

Table 1: Codes For Noticing, Wonderings, and Problems Posed

Codes	Description
Math-Number	Referencing numbers of things
counting	Counting numbers of things
quantity	Talking about more/less
chunks	Identifying chunks of numbers (like subitizing)
addition	Identifying an addition problem with objects
Math-Measurement	Referencing measurement-related aspects
length/height	Talking about how long/small, tall, or wide something is
size	Talking about how generically how big or small something was
time	Talking about how much time something took
weight	Talking about how heavy/light something is
Math-Geometry	Referencing shapes or shape properties
Math-Space/Location	Referencing locations with words such as in, on, near, etc.
Math-Patterns	Referencing patterns
Non-Mathematical	Referencing non-mathematical aspects
visuals	Noticing objects in the visuals not related to mathematics concepts
context	Using the context of the story to notice beyond the page
not related to book	Talking about sticky notes we had on the book

For each book page where we elicited preschoolers' responses, we grouped the noticings and wonderings/problems posed according to each category to look for patterns and to identify which

domains of mathematics were referenced. We also linked wonderings/problems posed with any responses or possible answers preschoolers provided. Finally, we identified whether their wonderings and problems posed arose out of similar noticings or if they were unrelated to what they had been discussing prior to that.

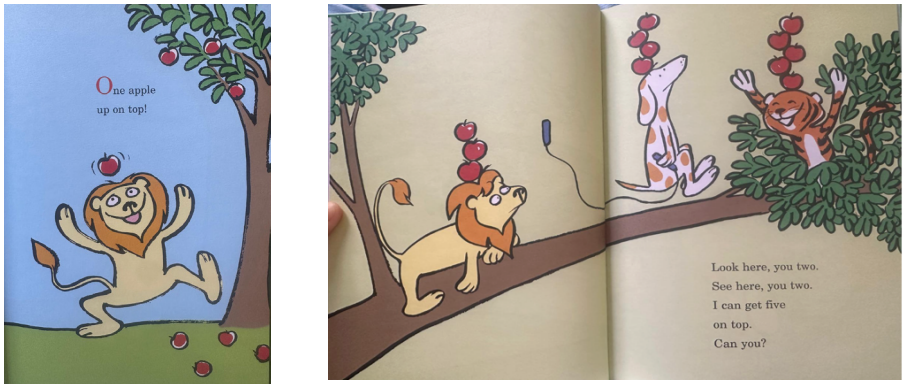
Results

In this section, we present the preschoolers' noticings, wonderings, and problems posed for each book according to mathematical domains and nonmathematical ones. While describing these data, we gave particular attention whether their responses were spontaneous or prompted.

Findings Related to the *Ten Apples Up On Top* Picture Book

Across the eight preschoolers who engaged with *Ten Apples up on Top* during their individual session and the five pages we stopped on, they shared 181 noticings. Of these, 70% were mathematical. Further, they had 32 wonderings and posed 44 mathematics problems. Next, we discuss these further in relation to the mathematical domains they discussed.

Number-related noticings, wonderings, and problems posed. Preschoolers were most likely to have number-related noticings, wonderings, and problems with this story. Sometimes, these noticings occurred in response to the researcher's prompts or sometimes emerged without any prompts. For instance, on the first page we stopped on (see Figure 1a), when the researcher modeled posing the problem, "How many apples do you see on top of the lion?" Waldon answered, "One (apple on the ground)," but then continued to notice, "There are three apples on the tree" and "There are four apples on the floor." Subitizing these three sets of apples was common and sometimes related to their problems posed. For example, on the same page, without prompting, Wiley posed a problem, "How much apples are on the tree?" and immediately answered his own problem as "Three," directly relating his noticing to his problem. Then, he posed another problem, "How much apples are on the ground?" and answered, "Four." The next time we stopped at a page (see Figure 1b), Whitney noticed that the apples on the animals' heads were in counting order. She then asked two non-mathematical questions, so the researcher probed, "Do you have any math questions about this page?" The student then posed the problem, "How much apples are there altogether, if we put 3, 4, 5 all together?" to which she counted to find the answer. Later in the story, the researcher asked Wayne, "If this bird eats one of these apples, how many will the tiger have?" Wayne also answered it by counting them and stated, "Nine."



(a) (b)

Figure 1: (a) First Page and (b) Second Page That Preschoolers Talked About

Space and location noticings, wonderings, and problems posed. Space-related noticing was the next most common and predominantly occurred embedded within number-related noticings rather than as an independent focus. For example, when asked about the first page (see Figure 1a), "What do you notice on this page?" Weston noticed, "Apple *up on* this animal." The preschoolers frequently talked about the location of the apples: "… on the cat," "… on the dog," "… on top," and "… on his head." Related to the second page we stopped on (see Figure 1b), the preschoolers also had space-related wonderings, such as, "Why are they *on* that stick [branch]?" and "Why is the tiger *in* the fruits?" These were also categorized as context-related wonderings. Although space-related noticings emerged as the second most frequent mathematical focus, the preschoolers did not pose space-related problems and did not relate their space-related noticings to their space-related wonderings.

Measurement noticings, wonderings, and problems posed. Measurement-related noticings were less frequent compared to the numerical and space domains. Yet, preschoolers occasionally mentioned length, size, volume, time, and weight, particularly after the researcher's prompts. For instance, on the second page we stopped at, the researcher asked, "How long is the lion's tail?" and Wiley responded to this question stating, "This long (showing the length with his fingers), one of this long." The researcher invited, "Do you have any questions about this page? … You can trick me with any questions." Then, the preschooler posed the question, "How long is the tree?" and he answered his own question, "Five inches." Winnie didn't have any measurement-related noticings, but she posed multiple questions about size: "How big is this tree?", "How big are the branches?", "How big is the doggie?", and "How big are the letters?" Preschoolers posed measurement problems second in frequency to number problems, but in all but the one case mentioned, their measurement problems were not related to their noticings.

Geometry and pattern noticings, wonderings, and problems posed. Geometry-related and pattern-related noticing were least common and only occurred when the researcher asked them, such as, "What shapes do you see on this window?" The only shape mentioned (by two preschoolers) was a square. There weren't any wonderings or problems mentioned, highlighting that this mathematical domain was not a mathematical focus within this book. Further, one student provided a pattern-related noticing on the second page we showed. The student saw the tiger and said, "Orange, black, orange, black, orange, black," describing the pattern with the tiger's stripes.

Non-mathematical noticings, wonderings, and problems posed. Beyond mathematics, preschoolers generally noticed visual elements, naming characters ("It's a tiger and a dog and a lion"), actions ("The lion looks like it's crawling"), emotions ("The cat is angry"), and objects ("There's a fridge") and less frequently made comments about the context, e.g., "Because that's just pretend." Yet, these noticings rarely led to nonmathematical wonderings. The preschoolers wondered about particular aspects of the visuals, such as when Wiley asked, "And why does the dog's tail look like a marshmallow?" On the other hand, Whitney stated her context-related wonder, "How could they reach up with those (long forks)?"

Findings Related to the *I'm the Biggest Thing in the Ocean* Picture Book

Across the four preschoolers who engaged with *I'm the Biggest Thing in the Ocean* during their individual session and the four pages we stopped on, they shared 77 noticings. Of these, 65% were mathematical. Further, they had eight wonderings and posed 10 mathematics problems. Next, we discuss these further in relation to the mathematical domains they discussed.

Number-related noticings, wonderings, and problems posed. As with the previous book,

the preschoolers were most likely to make number-related noticings, and they also posed more number-related problems than those from other domains. On the first page we stopped at (see Figure 2a), the shrimp stood out to the preschoolers, but they all responded in different ways when asked what they noticed. Aria just pointed to the shrimp, using a gesture rather than words; Alfio said, "There's lots of shrimp;" while Alvin immediately started counting them (from 1 to 22). On the other hand, Austin immediately posed the problem, "How many of these are there?" On the third page we stopped on (see Figure 2b), the teeth were prominent to the preschoolers. One of them counted the teeth while another first asked, "How many teeth are there?" and then counted them. Alfio made an interesting comparison observation, "He (whale) has more teeth than him (squid). He (squid) has zero teeth." Of their numerical problems posed, the problems were related to their numerical noticings just over half the time.



(a) (b)

Figure 2: (a) First Page and (b) Second Page That Preschoolers Talked About

Space and location noticings, wonderings, and problems posed. Overall, across the four preschoolers, only two of them referred to location, and that was only when prompted. On a page showing the fish inside the whale, the researcher asked where the other turtles went, and Alvin replied, "Right there." With another student, the researcher asked what was going to happen, and Alfio replied, "Get out." Surprisingly, no other students made comments about the fish being inside the whale (even though they previously saw the fish outside of the whale).

Measurement noticings, wonderings, and problems posed. Across the data, measurement-related noticings were second highest after the numerical ones. From the data coded, the mathematical measurements were related to length and size comparison. For example, on the first page we stopped at (see Figure 2a), the researcher asked if the shrimp were shorter or longer, and Alvin responded, "Long," even though they were smaller than the squid. On the next page we stopped at, the researcher asked if the squid was bigger than the turtles, and the preschooler said, "These (turtles) are bigger than that (squid)" subtly indicating that he was considering the group of turtles as one large group. Similar reasoning might account for why he thought the shrimp were longer than the squid. By the third page we stopped on (see Figure 2b), Austin noticed several aspects related to size and size comparison and posed a measurement problem without prompting. For example, Austin said, "This whale is bigger" and later, "Look how small he (squid) is compared to this (whale)." One of his noticings, "Look how wide it is from my hand," directly related to the problem he posed, "How big his mouth is?" He also posed and answered a question about which tentacle was longest on a previous page.

Geometry and pattern noticings, wonderings, and problems posed. Compared to the apple book, preschoolers noticed some geometry in the book with less prompting. While one student needed prompting to mention shapes, Austin brought up shapes without prompting,

noticing that one of the tentacles made a "circle" (see Figure 2a) and pointing to an eye while describing it as an "oval." On the second page we stopped at, the researcher paused, and Aria pointed to the backs of the turtles and indicated there were "rectangles" on the shells. On the third page we stopped on, Austin posed, "What shape is this when he is like that?" and then answered, "It's half of a circle." Both geometry-related problems that the preschoolers posed were directly related to measurement noticings. Overall, while not a content focus of the story, shapes were more prominent to these preschoolers in this book than for the preschoolers with the apple book.

Non-mathematical noticings, wonderings, and problems posed. As with the apple book, the preschoolers noticed visual elements in the story, naming and describing characters, (e.g., "A whale" and "They're white"), actions ("Yelling"), and objects ("Those are shells") and less frequently made comments about the context, e.g., "If they're (turtles) scared, they will go back in the shell." Even though they made fewer context-related noticings, they wondered about the context more than any other domain except for number. For example, regarding the page in Figure 2b, they wondered, "Is he gonna get, eat them?" In some cases, when asked what they wondered, they phrased their wondering like a prediction, "I think the jellyfish will sting the whale" or "The whale will die." These responses indicate that for some preschoolers, posing questions or wonderings is not as natural as providing statements.

Discussion

Drawing on Hintz and Smith's (2022) notice-and-wonder reads for mathematizing literature, we determined preschoolers' noticings, wonderings, and posed problems during individual read-aloud sessions with two picture books. Across both books, our findings indicated that children stated a range of mathematical noticings, wonderings, and problems along with nonmathematical ones. Not surprisingly, preschoolers were more likely to talk about what they noticed than what they wondered or could pose problems about. However, some students became enthusiastic about posing questions or wonderings and did so spontaneously. Throughout the sessions, researchers' modeling and prompts also supported extending children's noticing into mathematical inquiry (i.e., wondering and posing mathematical problems).

Preschoolers particularly expressed number-related noticings, including numbers, counting, quantities, comparisons, etc., and mentioned number related wonderings, and posed problems. As known, children naturally tend to count (Saxe et al., 1987), and their tendency to focus on numbers might be crucial to their mathematical development (McMullen et al., 2020). Across both books, preschoolers were likely to notice number-related aspects at the same rate. Based on the examples described in the results, the number-related noticings were strongly related to problem posing. In various instances, preschoolers immediately responded to their own problems, revealing a relational cycle of problem posing and/or wondering. Hence, picture books can provide meaningful contexts in which these inherent numerical elements become visible, are supported, and can be used as problem situations. At the same time, preschoolers expressed more mathematical wonderings and posed more problems about numbers with the *Ten Apples Up On Top* book compared to the second book. This result could be due to the explicitness of the number-related context in *Ten Apples Up On Top*.

Furthermore, in our study, preschoolers noticed plenty of space-related elements in the *Ten Apples Up On Top* picture book partly due to the space element being built into the text of apples being *on top* of the animals' heads. Even so, preschoolers generally stated the number-related noticings more than space-related ones, unlike van den Heuvel-Panhuizen and Elia's study (2013) where preschoolers talked about characters coming or going out of the doctor's office and

Elia et al.'s study (2010) where preschoolers talked about position and location. Our choice of which pages we stopped at could account for this difference as children in the other studies talked about each page. The fewer references to space-related elements in *I'm the Biggest Thing in the Ocean* also points to the importance of book choice, as this story does not lend itself to talking about position or location much.

Similar to the spatial findings in the second book, geometry-related elements were limited, largely due to the fact that the books did not have a geometric focus. In fact, there were few shapes that even stand out as being important in the story. Largely, students noticed shapes when prompted to look for them, aside from noticing the prominent eyes or teeth on the ocean creatures in the second book. Likewise, wondering about shapes in terms of the books would not be expected given the focus of the stories.

Regarding measurement, there were elements especially prominent in *I'm the Biggest Thing in the Ocean*, offering a learning environment where preschoolers had a chance to notice diverse measurement-related elements. One of the preschoolers needed prompting to respond to measurement-related ideas; however, two of the others continued talking about measurement after an initial suggestion. In terms of wonderings and problems posed, especially in *Ten Apples Up On Top* picture book, preschoolers' high number of problems and wonderings might be due to the picture book's context and visuals or more elaborate story compared to the second book. These results suggest that further investigating the features of stories in relation to what children notice and wonder would be important. The findings also highlight the significance of picture book selection. Explicit mathematical illustrations (e.g., compositions of shapes) in strategically chosen books by teachers (i.e., books where those mathematical elements in illustrations are important to the story) may better encourage focus on specific mathematical topics and concepts (e.g., geometry, shape composition).

Considering in what ways preschoolers' noticing, wonderings, and posed problems varied, number-related noticings and sometimes their measurement noticings were often related to their wonderings and problems posed. However, their geometry and space-related noticings did not result in problem posing. Again, this is likely due to those areas not being a focus of the stories. Further studies should continue this process with a wider range of stories to better capture the potential of this method for encouraging students to wonder and problem pose. Consistent with the teacher's dialogic approach (Björklund & Palmer, 2020; van den Heuvel-Panhuizen & Elia, 2013), for children to derive meaning from picture books (Björklund & Palmer, 2020), the findings of our study highlighted that the noticing-and-wondering process encouraged children to express their thinking with the researcher's prompts. Further, these results provide important evidence that even preschoolers can pose mathematical problems that are relevant and solvable (Martin-Diaz et al., 2026).

Overall, the findings of our study indicate that picture books provided rich contexts and illustrations integrating various mathematical elements. Concurrently, preschoolers' spontaneous noticings and wonderings were shaped by the researcher's purposeful guidance and prompts or formed independently of it, and some of these noticings served as problem situations. This interaction supports preschoolers' contextual and visual understanding of explicit mathematical understandings and encourages them to pose problems.

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LEARNING THROUGH OTHERS: CONCEPTUALIZING DECENTERING FOR MATHEMATICAL LEARNING ACROSS PEER, STUDENT-TEACHER, AND HUMAN-AGENT INTERACTION

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Mathematical learning is often propelled by engaging with others' thinking. Drawing on studies across diverse contexts, including students working with computational agents, students collaborating with peers, and teachers interpreting student thinking, this proposal conceptualizes how mathematical knowledge is modified and enriched when learners take up, coordinate, and extend perspectives that are not their own. We argue that a common cognitive thread is decentering: the mental process of temporarily setting aside one's own thinking to adopt and reason from the perspective of another. Synthesizing findings from two empirical studies, we articulate six cognitive mechanisms of decentering-to-learn, organized into three types: (1) representational, (2) coordinative, and (3) epistemic decentering. We also clarify how decentering is triggered and sustained by interactional conditions in different contexts.

Keywords: Cognition, Learning Theory, Communication

Background and Motivation

Engaging with others' thinking is a powerful driver for mathematical learning (Rasmussen et al., 2020; Webb et al., 2023). Among the many ways learners engage with others' thinking, *decentering*, the mental process of temporarily setting aside one's own thinking to adopt and reason from the perspective of another, is central to this engagement. The construct originates from Piaget's account of *egocentrism* and the developmental emergence of perspective taking. In the Three Mountains Task (Figure 1), children demonstrate egocentrism when they assume that a viewer at Position C can see the same objects they see from Position A (Piaget & Inhelder, 1948/1967). The difficulty lies in remaining bound to one's own viewer-centered reference frame rather than constructing the perspective of others. Related challenges appear in technology-rich mathematics learning, such as Logo geometry, where students are required to adopt the agent's perspectives and make sense of its actions to program effectively (Abelson & diSessa, 1981; Geva & Cohen, 1987).

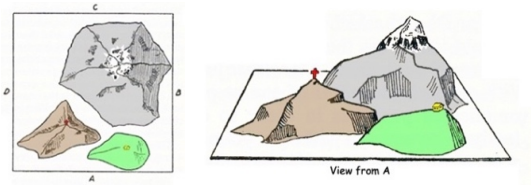


Figure 1: The Three Mountains Task, Reprinted with Permission from Piaget and Inhelder (1967), p. 211

Educational research has extended decentering to characterize peer interaction and student-teacher interaction. Hackenberg (2005) framed teacher decentering as an epistemic disposition, one of orienting to students' ways of operating and "mak[ing] space to see realities that may

differ significantly from their own" (p. 47). Subsequent work has specified the mental actions and observable manifestations of teacher decentering in mathematics classrooms (Baş-Ader & Carlson, 2022; Carlson et al., 2024; Teuscher et al., 2016). Building on this line, Liang (2021, 2025) documented the mathematical consequences of teacher decentering, showing how teachers enrich their personal mathematical knowledge through decentering to student thinking. Complementary constructs appear in science education, where *epistemic empathy* is described as teachers' capacity to understand and appreciate students' cognitive and emotional experiences (Jaber et al., 2018, 2024). Warren (2014) likewise emphasized *perspective taking* as an intellectual component of empathy, alongside empathic concern.

Decentering has also been examined in peer interaction. When students engage in reciprocal construction of understandings with each other, they are poised to incorporate peers' perspectives or argue against perspectives they seek to challenge or reject (Moon et al., 2017; Rasmussen et al., 2020). Together, these studies provide strong evidence that decentering could be productive for learning. However, the field still lacks a systematic conceptualization of how decentering becomes consequential for disciplinary learning, that is, the cognitive mechanisms through which decentering yields knowledge growth.

This proposal addresses this gap by conceptualizing a set of mechanisms of mathematical learning through decentering that cut across educational contexts. We go beyond conceptualizing the decentering act itself to ask: *How do individuals learn mathematically through adopting and coordinating perspectives of others, that is, through decentering*? By foregrounding learning outcomes, we develop a taxonomy of decentering grounded in the kinds of mathematical reorganization that emerge from its enactment.

Throughout, we use "others" broadly to include peers, students, and computational agents. We distinguish the *learner* from the *other*, using *learner* to denote the individual who is engaging in and learning through decentering. When students are the learners, the other may be a peer or a computational agent; when teachers are the learners, the other is typically a student, and we examine how teachers learn by constructing and reasoning with students' perspectives.

Method

We draw on two empirical studies with distinct purposes, populations, and research settings. Study 1 examined primary students' programming-enhanced mathematics learning in Scratch, a block-based programming environment. We asked students to work as a pair because it can offer a supportive environment in which they collaboratively encounter and resolve challenges through interaction (Hannay et al., 2009). For this proposal, we focus on data of one focal pair from eastern China: Seymore (Grade 5) and Selina (Grade 6). Both students had prior experience with Scratch. They completed five hour-long task-based clinical interviews (Goldin, 2000; Hunting, 1997) organized around a series of Scratch-based mathematical tasks. Study 2 involved three in-service mathematics teachers (Mrs. Winn, Mrs. Kelly, and Mrs. Fang) who conducted task-based clinical interviews with students and then reflected on students' mathematical thinking in video-stimulated recall interviews (Calderhead, 1981; Tuckwell, 1980). Mrs. Winn and Mrs. Kelly are Asian American teachers with less than one year of teaching experience; Mrs. Fang is a Chinese teacher with seven years of teaching experience. Mrs. Fang and Mrs. Kelly participated in five interview-reflection cycles, and Mrs. Winn participated in two.

We selected these studies for synthesis because participants enacted different mechanisms of learning through decentering across peer interaction (Study 1), human-agent interaction (Study 1), and student-teacher interaction (Study 2). We conducted thematic analysis of the video data for each case, first identifying critical moments illustrating participants' decentering activities

and then identifying emergent themes that characterized how these activities are generative to their mathematical learning.

Mechanisms of Mathematical Learning Through Decentering

Based on the kind of cognitive work in relation to decentering, we specify six mechanisms organized into three decentering types: (1) representational decentering, (2) coordinative decentering, and (3) epistemic decentering.

Representational Decentering

Representational decentering is concerned with a learner's interpretation of a situation when adopting a different perspective. The enactment of decentering enables the learner to re-interpret the situation such that its properties are distinct from adopting the learner's own perspective.

Shifting reference frames. One mechanism under this type of decentering involves a learner shifting from the observer's reference frame to that of the other (the observed) in order to represent ideas faithful to the other's reference frame. In our work involving human-agent interaction, representational decentering is evident as learners adopt the perspective of a computational agent and represent actions undertaken by the agent in order to understand its behaviors. This decentering act is crucial for the learners to generate effective commands on the agent in order to achieve particular goals. For example, Study 1 addressed a Polygon Task that asks students to use Scratch to draw an equilateral triangle (Figure 2a). Seymore initially struggled to explain why 120 degrees instead of 60 degrees worked to generate an equilateral triangle. It was not until he adopted the sprite's reference frame and modeled the behaviors of the sprite that he realized the turning angle is the external angle of the polygon (Figure 2b). Early work on Logo geometry echoed this finding and revealed students' persistent difficulty with specifying the angle referent of the numerical input (Geva & Cohen, 1987). We view Seymore's learning to be non-trivial and entailed decentering because he had to reconceive the polygon-drawing task to be about movement and turning of the sprite rather than "the shape I want." The external angle thus becomes visible to him when he needs to adopt the sprite's reference frame and model the dynamic generation of the polygon as if he were the sprite.

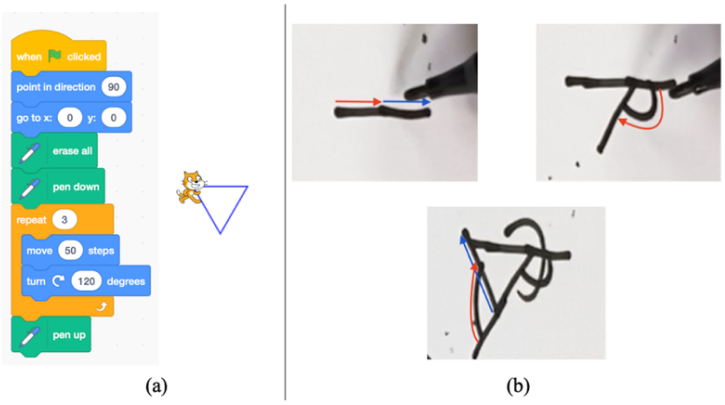


Figure 2: (a) The Normative Solution and (b) Seymore's Work, Reprinted with Permission from Liang et al. (2026), p. 22

Shifting representational systems. Another mechanism of representational decentering is representing the other's perspective with a representational system distinct from that used by the other. This involves the learner internalizing the other's thought and representing it in their own

way. In Study 2, Mrs. Kelly was working with her student, Taylor, on a linear programming task (Figure 3a). Taylor solved the problem with a purely algebraic method (Figure 3b): he first selected a series of value pairs that satisfy the 10,000-square-foot constraint (e.g., 1000–9000, 2000–8000, and 5000–5000) and then calculated the total cost of furnishing for each combination. Eventually, he concluded that 5000–5000 was the best solution because it is "the only place where you are spending all of your money and using up all of your space.

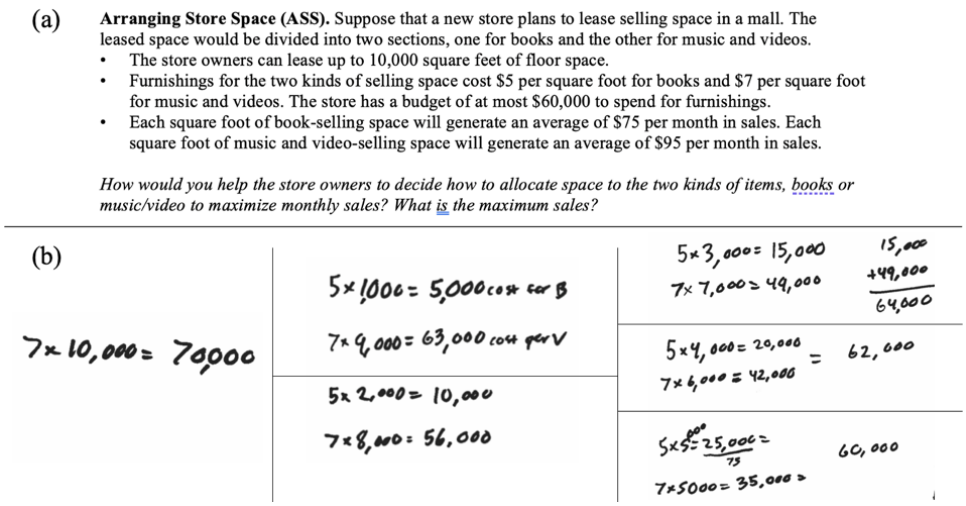


Figure 3: (a) The Linear Programming Task and (b) Taylor's Solution, Reprinted with Permission from Liang (2025), p. 22

When discussing Taylor's thinking in the stimulated recall, Mrs. Kelly contrasted his method with the conventional graphical method. She showed that while the conventional method considers all feasible points in the shaded area in Figure 4a, Taylor's method is "more efficient" because it only considers a subset of feasible solutions (see the segment highlighted in blue in Figure 4b). We view this to be a form of representational decentering because Mrs. Kelly represented Taylor's thinking (a purely algebraic method) with a graphical representation.

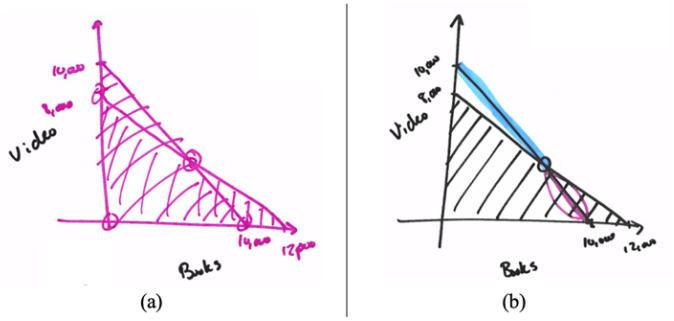


Figure 4: Mrs. Kelly's Graphical Representations of the (a) Conventional Method and (b) Taylor's Method, Reprinted with Permission from Liang (2025), p. 24

In both cases, decentering changes what counts as the relevant description of the situation to the learners. For Seymore and Selina, decentering involves recasting the mathematical object

within the agent's reference frame and reconstructing representations from the agent's perspective. Decentering to the agent's frame reorganizes their interpretation of the mathematical idea under study (e.g., turning angle) and makes certain ideas salient (e.g., exterior angle). For Mrs. Kelly, decentering means introducing a novel representation that makes another's perspective visible and intelligible. In both cases, learning occurs as a result of adopting different representational frames so that others' perspectives become coherent and sensible to the learner.

Coordinative Decentering

Coordinative decentering is concerned about the relationship between multiple perspectives and involves a learner coordinating others' perspectives with their own. The cognitive work may involve comparing, combining, and connecting different perspectives.

Integrating perspectives. One mechanism of coordinative decentering involves a learner integrating elements and aspects of their own thinking with those of another. When the two perspectives entail different strategies, this integration often yields a composite strategy repertoire. In Study 1, as previously mentioned, Seymore used the "turn 120 degrees" block to draw an equilateral triangle. Selina, by contrast, used the "point in direction" block, considering the absolute directional angles for drawing each side. She first input 90 as the initial direction (rightward) and estimated –135 for the second side, but she struggled to determine the third direction that would return the sprite to its starting point and close the shape (Figure 5a). As Seymore engaged with Selina's approach and attempted to support her debugging, he worked to compute the directional angles precisely. He produced the drawing shown in Figure 5b, where he leveraged the 120-degree turning angle to infer that the remaining directional angles should be –30 and –150 degrees (Figure 5c). We interpret this as coordinative decentering by which Seymore not only decentered to Selina's thinking but also coordinated two angle conceptions and reference systems: (a) angle as an absolute direction in Selina's thinking (with upward as 0 degrees), and (b) angle as a relative turn from the current heading in his own thinking.

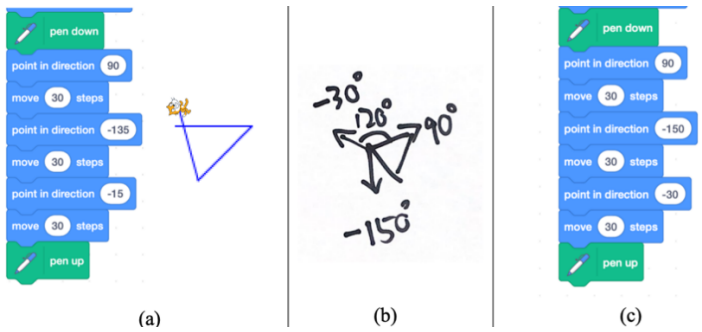


Figure 5: (a) Selina's Solution and (b-c) Seymore's Solution, Reprinted with Permission from Liang et al. (2026), p. 16

Building coherence. This involves constructing a translation between two perspectives and establishing a mapping that explains why both perspectives work and how they relate. In Study 2, Mrs. Winn's students worked on a task on dilation: they were given an image rectangle $A'B'C'D'$ and asked to construct the preimage $ABCD$ with a scale factor of 2.5 from the center of dilation O (Figure 6a). In explaining her own solution, Mrs. Winn represented 2.5 as "two wholes and a half" and emphasized its equivalence to five halves (Figure 6b). She reasoned that because the length OD' is measured as 2 inches, one-half should be 0.4, and thus one "whole" is 0.8. In comparison, Carlee approached the problem by undoing the scale factor: she calculated

$OD = 2 \div 2.5 = 0.8$ and placed Point D at 0.8 inches from O; she also represented "2.5" on the segment by partitioning it into two and a half sections (Figure 6c). In stimulated recall, Mrs. Winn reported that she could not make sense of why division was appropriate and noted that she "should have asked for Carlee's explanation." When prompted to explain, Mrs. Winn invoked the "N (units per group)–M (numbers of groups)–P (product)" model of multiplication/division (Izsák & Beckmann, 2019), mapping the scale factor to the number of groups ($m = 2.5$), the image-center distance to the product ($p = 2$), and the preimage-center distance to units per group ($n = p/m = 0.8$) (Figure 6d). She noted that n is one "whole" and p is equivalent to "taking the whole and enlarging it by 2.5." By coordinating Carlee's division-based method, her own partitioning method, and the N–M–P model, Mrs. Winn constructed a translation between these perspectives that was not initially apparent, and she expressed excitement at making these connections. We interpret this as learning through building coherence because Mrs. Winn reorganized her knowledge so that each account became intelligible in terms of the other, and the meanings for division, partitioning, and dilation became all connected.

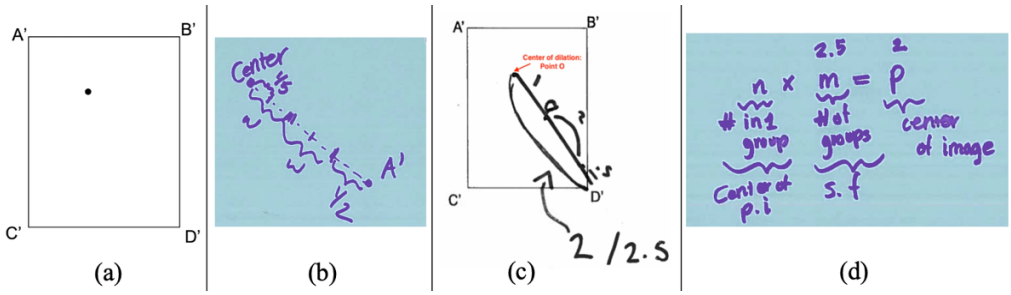


Figure 6: (a) The Dilation Task, (b) Carlee's Work, and (c–d) Mrs. Winn's Work

Both integrating perspectives and building coherence are forms of coordinative decentering, but these mechanisms foreground different observable learning outcomes. Integrating perspectives involves incorporating one perspective into another while working within the constraints of the other. Building coherence entails deeper conceptual reorganization such that the perspectives become mutually explanatory—each can be used to interpret, justify, and make sense of the other. In both cases, the common mental act is engaging with others' thinking through coordinating it with additional perspectives and articulating relations among them.

Epistemic Decentering

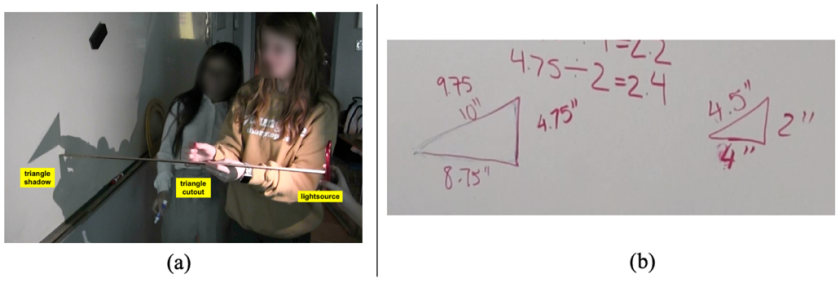
Epistemic decentering refers to decentering that shifts what one takes as assumed, thereby reorganizing one's existing knowledge or generating new knowledge. It concerns the status, grounds, and assumptions of knowledge, for example, how ideas are defined, what is treated as given, and what counts as an adequate explanation.

Clarifying implicit thinking. This mechanism involves making implicit premises, assumptions, definitions, or warrants explicit in light of alternative thinking. Because others may operate with different assumptions, decentering to their perspectives can reveal invisible presuppositions in one's own knowledge. This is especially consequential for teacher learning, since curricular tasks and language often hide definitional choices (e.g., defining similarity in terms of length rather than other quantitative attributes), choices that teachers must surface and clarify in instruction. In Mrs. Winn's case, she worked with her students to create a shadow of a triangular cutout that was "twice as big" as the original (Figure 7a). She traced the outlines of the shadow and the cutout on the whiteboard (Figure 7b), intending to engage the students in a

discussion on the proportional relationship between the dimensions. Before she began, however, Carlee remarked that the shadow looked much bigger than a doubling of the cutout. During stimulated recall, Mrs. Winn realized that Carlee may have been comparing areas rather than lengths, an interpretation under which the shadow can appear more than a doubling even when linear dimensions are scaled by two. Reflecting on this, she noted:

> I never think about area when I think about dilation. I think about dimensions … just take it as a given … I think it goes back to the language, because if you say "something is twice as big as the other," what does that mean? It could mean area … or it could mean ... like the dimensions are twice as big. I think it is just that language is very confusing.

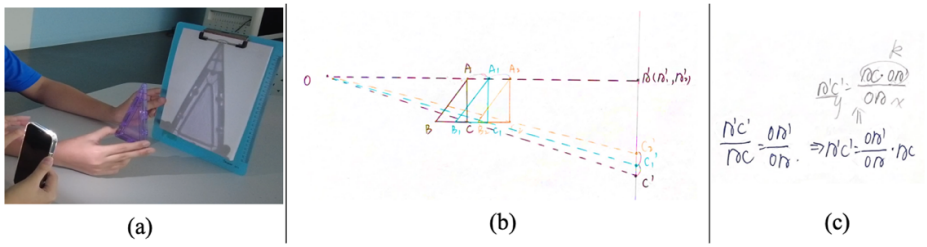
Here, Mrs. Winn was coordinating two competing perspectives of dilation: dilating length versus area. Only through working with students and decentering to Carlee's thinking did she realize that her taken-for-granted conception of dilation-as-length was not trivial but an implicit choice that needed to be made explicit. This episode is epistemic rather than representational or coordinative because the learning hinges on surfacing and revising what is presupposed, that is, which quantitative attribute "twice as big" refers to, and more broadly, what counts as an appropriate basis for multiplicative comparison in dilation contexts. Decentering thus functions as an epistemic cue: encountering the student's perspective exposes what Mrs. Winn had been treating as given and brings previously implicit assumptions in her reasoning into the foreground.



(a) (b)

Figure 7: (a) The Shadow Task and (b) Mrs. Winn's Drawing

Generating new knowledge. Decentering to another's perspective can also prompt learners to revise their initial understanding of a situation, resulting in the generation of new knowledge. In these moments, the learner does not simply adopt the other's view; rather, the other's ideas catalyze new cognitive work that yields insights that go beyond either initial perspective.



(a) (b) (c)

Figure 8: (a) The Triangle Shadow Task and (b–c) Mrs. Fang's Work

In Study 2, Mrs. Fang worked with her student Jasper to explore the relationship between the dimensions of a triangle and those of its shadow (Figure 8a). Jasper proposed that if the triangle

moves away from the light for equal amounts, the side length of the shadow would increase by equal amounts. This claim intrigued Mrs. Fang, but she was uncertain how to respond and not convinced that a linear relationship was warranted. During the stimulated-recall interview, she took Jasper's claim as a hypothesis to test, producing Figure 8b to simulate the situation and showing that the relationship is inverse proportional and thus nonlinear (Figure 8c). Mrs. Fang's decentering is evident in her sustained effort to pursue Jasper's proposed relationship and to formalize it mathematically. We interpret this as epistemic decentering because Jasper's perspective reorganized Mrs. Fang's understanding of what counts as a plausible underlying relationship in the situation and motivated her to construct a newly warranted explanation.

Discussion and Conclusion

In summary, we identify six mechanisms of learning through decentering, organized into three types: representational, coordinative, and epistemic decentering. These mechanisms link decentering to diverse learning outcomes (e.g., shifting frames of reference and representational systems, coordinating methods and ideas, foregrounding implicit assumptions). These mechanisms are not mutually exclusive: a single episode can involve multiple forms of decentering. For example, Mrs. Kelly's case can be characterized as both shifting representational systems and building coherence as she related Taylor's method to a conventional graphical method. Likewise, when directional-angle and turning-angle perspectives are coordinated, the work of integrating perspectives also positioned Seymore to recognize their equivalence, supporting building coherence. Treating these mechanisms as overlapping enables us to capture different dimensions of a learning situation and supports multiple, complementary accounts of how learning unfolds through decentering.

Our contribution builds on prior research that has established the importance of decentering and perspective taking and offers a proof-of-concept that decentering can be theorized not only as a stance or interactional feature, but also as a set of cognitive mechanisms through which disciplinary learning becomes possible. We also extend existing accounts by shedding light on what mechanisms are at play in certain interactional contexts, specifying interactional conditions that make decentering more likely to be triggered, sustained, and consequential.

In human–agent contexts, representational decentering is often central, with many activities requiring learners to adopt an agent's frame to generate effective commands. We note that such opportunities are occasioned in task designs, in which the "other" is intentionally engineered to constrain and redirect learners' reasoning. The emergence of AI agents further motivates design work on feedback and prompts that foster productive decentering of AI agents' perspectives.

In peer interaction, decentering is frequently occasioned by multiple strategies, making coordinative decentering salient as students compare, test, and reconcile different methods. Productive peer talk can also support epistemic decentering as students press for warrants and surface hidden assumptions. Future research on peer interaction can continue to adopt the lens of decentering to investigate the productive role of peer decentering in learning.

In student–teacher interaction, asymmetric roles and epistemic responsibility often foreground building coherence (coordinative decentering), as teachers interpret students' informal accounts while connecting them to disciplinary explanations, and can also support epistemic decentering, as teachers notice differences in languages, referents, definitions, and criteria across student thinking and curriculum. This points to the need for design-based research in teacher education to develop learning opportunities and structural support for teachers' learning through decentering.

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SECOND GRADERS' MOVES THROUGH EMERGENT MATHEMATICAL MODELING LEVELS IN BLOCK PUZZLE GAMEPLAY

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This study examines how second graders' gameplay behaviors reflect their progression through emergent modelling levels during block puzzle play. Guided by two research questions, it explores the types of moves students make and how these moves align with four emergent modelling levels. Data were collected through six video-recorded interview sessions with twelve students working in pairs. Thematic analysis was used to identify and categorize students' moves, which were then mapped onto the emergent modeling framework. Findings revealed two main move types: exploratory moves, most common at the situational level, and strategic moves, which appeared primarily at the referential and general levels, with a few at the formal level. This study extends emergent modeling to a digital context and provides insights into how gameplay can support early mathematical reasoning.

Keywords: Elementary Students, Emergent Modeling, Mathematical Reasoning, Gameplay, Digital Tool, Student Moves

Understanding how children develop mathematical reasoning is a critical area of research in mathematics education (Laurens et al., 2017). Early experiences that engage children in problem-solving and reasoning with mathematical concepts provide a foundation for later success in formal mathematics learning (Bang et al. 2023; Cobb et al. 2021). These early opportunities for mathematical reasoning are essential, as they support children in building connections between intuitive ideas and more abstract mathematical concepts (Sarama & Clements, 2009).

Puzzle-based digital games have increasingly been recognised as valuable tools for supporting this developmental trajectory (Bang et al. 2023; Polinsky et al. 2022). Such games provide interactive, low-stakes environments in which children can experiment, strategise, and reflect on their decisions, fostering mathematical agency and exploration (Bane et al., 2023; Bofferding & Aqazade, 2023). These environments also make children's reasoning visible, offering researchers and educators rich opportunities to observe how spatial understanding evolves through digital play and interaction (Kim et al., 2017). Despite the widespread integration of digital technologies in primary classrooms (Bang et al. 2023; Hussein et al. 2022), there remains limited research on how specific gameplay behaviours connect to children's reasoning processes, particularly in relation to reasoning during mathematical modeling (Sengupta et al., 2014).

Research in early-grade mathematical modeling shows that engaging children in activities such as identifying patterns (English, 2003), using numbers to represent mathematical objects (Lesh et al., 2013), exploring two- and three-dimensional shapes and figures (Stohlman et al. 2016), and applying optimization strategies on a grid area (Ozturk, 2025) can significantly support the development of spatial reasoning. While these studies highlight promising pathways for fostering modeling competencies, much of this work has been carried out using hands-on materials rather than technology-based environments (Stohlmann & Albarracín, 2016). More recently, studies have begun to explore how digital tools can enhance modeling opportunities for young learners (Hussein et al., 2022; Sengupta et al., 2014). However, there remains a need for a deeper understanding of how children's strategies in digital gameplay contexts can reflect and

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

promote their progression in mathematical modeling (English, 2021; Stohlmann and Albarracín 2016).

This study addresses this gap by investigating how second graders' moves during block puzzle gameplay reflect their progression through emergent modeling levels. Emergent modeling, as described by Gravemeijer (1999), provides a framework for tracing the development of students' reasoning as they move from intuitive, context-bound strategies toward generalized and formal mathematical reasoning. This framework has been instrumental in examining how learners shift from informal problem-solving to formal mathematical representations (Van den Heuvel-Panhuizen, 2003). By adapting emergent modeling approach to a digital gameplay context, this study connects two growing areas of research: early mathematical reasoning and technology-enhanced learning environments. The present study is guided by two research questions: *(1) What kinds of moves do second graders make while engaging in block puzzle gameplay? and (2) How can these moves be related to the levels of emergent modeling?* These questions are designed to illuminate how young learners' in-game moves can serve as indicators of their mathematical reasoning.

By situating gameplay moves within the emergent modeling approach, this research contributes to the field in two keyways. First, it extends the application of the emergent modeling framework to digital learning environments, offering methodological insights for researchers examining mathematical reasoning through technology-mediated tasks. Second, it provides educators with a nuanced understanding of how different types of gameplay moves—ranging from exploratory to strategic—may reflect distinct stages of mathematical reasoning. In doing so, this work responds to calls for context-rich, developmentally appropriate approaches to early mathematics education that bridge children's informal reasoning with the more formal practices of mathematical modeling in elementary classrooms (English, 2021).

The Framework: Emergent Modeling and Its Four Levels of Activity

Emergent modeling refers to the process through which students' context-specific strategies and representations develop into general mathematical tools for reasoning (Gravemeijer, 2007). At its core, an emergent model is not simply given to students but rather arises from their own activity. Initially, students work with informal representations that are closely connected to the task at hand. Over time, these representations are reorganized and generalized, evolving from a *model-of* the situation into a *model-for* broader mathematical reasoning (Doorman & Gravemeijer, 2009). The aim of this process is to support students in developing increasingly sophisticated ways of reasoning (Cobb et al. 2021; Gravemeijer 1999). In this sense, an emergent model is both a product of students' activity and a catalyst for moving from context-bound strategies towards formal mathematical understanding (Van den Heuvel-Panhuizen, 2003).

The theoretical foundations of emergent modeling lie in Realistic Mathematics Education (RME), which conceptualizes mathematics as a constructive human activity rather than a fixed body of knowledge to be transmitted (Freudenthal 1991; Verschaffel et al. 1994). RME emphasises that students should engage with problems that are experientially real, whether these stem from everyday life, play, or meaningful classroom contexts (Laurens et al., 2017). Within this framework, emergent modeling serves as a design heuristic for bridging the gap between students' informal reasoning and formal mathematical concepts (Doorman & Gravemeijer 2009; Van den Heuvel-Panhuizen, 2003).

Gravemeijer (1999) articulated four levels of activity through which emergent modeling progresses. At the *situational level*, students' reasoning remains tied to the immediate problem context. At the *referential level*, their representations begin to function as *models of* the situation,

capturing mathematical relationships within the context. Progressing to the *general level*, these representations evolve into *models for* mathematical reasoning, enabling their application across different contexts. Finally, at the *formal level*, students' activity is characterized by the use of symbols and abstract reasoning that are no longer dependent upon the original situation. These levels informed our analysis in this study. We adopted this framework as it offers a systematic lens through which to examine how second graders' moves during gameplay demonstrate transitions from intuitive, context-bound strategies towards increasingly generalized mathematical reasoning.

Research Setting, Participants and The Game App

This exploratory qualitative study was conducted with 12 second-grade students (seven girls and five boys) who were enrolled in an after-school program at a public primary school in the Midwestern United States. The students worked in pairs during a single 45-minute interview session. Each pair was interviewed individually and provided with a touchscreen iPad to engage with the free *Block Puzzle Game App*. The iPad was connected to a projector, allowing the gameplay to be displayed on a larger screen. Each interview was video recorded from an angle that captured both the students' interactions and their moves during gameplay.

The game requires players to arrange block shapes, each composed of unit squares, onto an 8×8 grid with the aim of forming complete horizontal or vertical lines. At the beginning of each turn, three new blocks appear at the bottom of the grid, and players must strategically place them within the available spaces. Whenever a line of eight consecutive squares is completed, it disappears, enabling players to earn points and create additional space for new blocks. Success in the game depends on players' ability to anticipate future moves, position the given set of blocks effectively, and maximize the number of cleared lines. The game concludes once no remaining blocks can be placed on the grid. Importantly, it is age-appropriate for second graders and aligns with their curricular standards, which highlight reasoning with shapes, partitioning rectangles into equal squares, analyzing attributes, and exploring equal groups of objects as a foundation for multiplication.

Data Collection, Coding and Analysis

No formal instructions were provided to the students, as the purpose of the study was to examine how their gameplay moves illustrated progression through emergent modeling levels. A total of six interviews were conducted, each structured into three phases. During the initial 10 minutes, students engaged in free play, allowing them to explore the app's features and familiarize themselves with the objectives of the game. The subsequent 25 minutes focused on strategic play, during which students began to formulate and apply strategies, identify patterns, and practice spatial reasoning. The final 10 minutes comprised a reflection phase, where students responded to prompts such as, "What was difficult or easy about the game?" and "How did you decide your next move?" These reflective questions were designed to elicit students' reasoning about their decision-making processes and strategies. Although the participants were familiar with puzzle games and educational applications from their regular mathematics lessons, none had prior experience with the specific app used in this study.

The data comprised four and a half hours of video-recorded interviews, which were transcribed and analyzed alongside students' move data saved within the app. In this study, a *move* is defined as a single, observable action or decision made by students during gameplay, accompanied by the corresponding verbalization or reasoning that explains that action. The analysis focused on categorizing the types of moves students employed during gameplay and

examining how these moves aligned with the levels of emergent modeling. Specifically, the process of students placing the three newly generated blocks during each turn, thereby creating a new scenario for completing a horizontal or vertical line on the grid, was conceptualized as an *emergent model.*

To address the first research question, we employed thematic analysis (Clarke & Braun, 2013) to systematically identify and categorize the types of moves students made during gameplay. Two researchers repeatedly reviewed the transcripts of gameplay sessions to establish familiarity with the data, after which initial codes were generated inductively to capture distinct types of moves (e.g., trying out fits). To address the second research question, these coded moves were then interpreted through the emergent modeling framework. Within this framework, students' moves with blocks were categorized as emergent models and situated within specific modeling levels. Moves were assigned to levels based on the nature of students' reasoning and the extent to which their decisions remained tied to the immediate gameplay context or were generalized into broader mathematical ideas. In this way, thematic analysis produced a coding scheme of students' moves, which was subsequently mapped onto four levels to capture students' progression in emergent modeling.

To strengthen the trustworthiness of the coding and analysis, the initial coding scheme was developed through close readings of all transcripts and iterative discussions between the two researchers over approximately four hours. The codes were then collaboratively refined before moving to the second phase of analysis, in which mapping students' moves onto the four framework levels required an additional three hours. Throughout the process, the researchers met regularly to apply the coding scheme across the full dataset, ensuring consistent application of coding and enhancing the reliability of the findings.

Findings and Discussion

Our analysis identified two overarching types of moves—exploratory and strategic—with exploratory moves comprising five subcategories and strategic moves comprising six subcategories. We first describe the characteristics of these two types of moves before examining how they occur across the modeling levels. Tables and illustrative examples are provided to highlight distinctions between exploratory and strategic moves within the same modeling level.

Exploratory moves reflect students' efforts to test, notice, and compare possibilities in the moment. We identified five subcategories of these moves. *Trying Out Fits* involves experimenting with multiple placements before settling on one, as when a student drags an L-shaped block around the board while saying, "it could go here… no, maybe here." *Observing Line Effects* occurs when students notice the immediate outcome of completing a line and attempt to reproduce it, such as realising that "it disappears!" when a line is cleared. In *Counting*, students use numerical reasoning to check spatial fit, for example, "This one is three long—one, two, three—so it fits in this row." *Recognition of Patterns* reflects students' awareness of recurring shapes or effects in the grid, as when they identify a "T-shape." *Comparison* entails weighing immediate alternatives without long-term planning, such as deciding, "If I put it here, there's more space left than if I put it over there."

Strategic moves reflect students' forward-looking decisions aimed at managing space and optimizing future possibilities on the grid. We identified six subcategories of these moves. *Prioritising Corners* refers to positioning blocks in the corners to keep multiple pathways open, as one student noted, "I'll tuck it in the corner, so we still have room on the board." *Keeping the Centre Clean* reflects efforts to preserve open space for larger pieces, with students' reasoning, "Don't put it there! We need the middle open for the big piece."

In *Managing Junk Blocks*, students proactively place oversized blocks to avoid unfillable gaps, saying, "Let's get rid of this big block now before it messes us up later." *Watching Out for Single Squares* involves monitoring one-square spaces that are difficult to place and can block future moves, as a student said, "Wait, that will leave just one little square, and nothing fits there." *Checking Horizontal and Vertical Lines Simultaneously* refers to placing blocks to complete rows and columns at the same time, which maximises points and opens up more space on the grid, as one student said, "If we put it here, it clears both across and down!" *Delaying Placement (Resisting the Immediate Fit)* demonstrates students' willingness to forego an immediate fit to preserve better long-term options, as captured in the reasoning, "This T-shape fits here, but let's save that space for the little one square."

Occurrences of Moves Across the Emergent Modeling Levels

Analysis of the 138 identified moves revealed that students engaged more frequently in exploratory activity (84 moves) than in strategic activity (54 moves). Table 1 illustrates the distribution of these moves across the four emergent modeling levels, highlighting that exploratory moves primarily occurred at the situational level, while strategic moves were concentrated at the referential and general levels, with only a few extending into the formal level.

Table 1: Types of Moves and Their Frequencies of Occurrence at Modeling Levels

Moves (Total: 138)	Levels of Emerging Modeling			
	Situational Level	Referential Level (Model-of)	General Level (Model-for)	Formal Level
Exploratory Moves (84)	63	21	-	-
Strategic Moves (54)	-	39	13	2

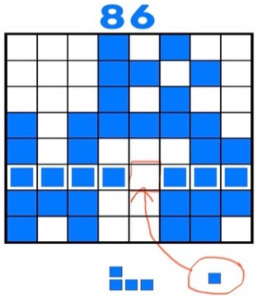
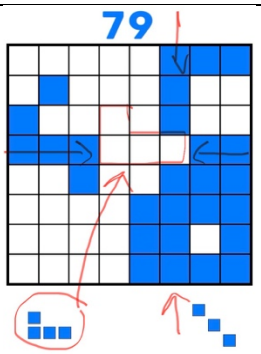
Exploratory moves were the most observed at the situational level**,** where students relied on immediate fits on the grid, and to a lesser extent at the referential level, where they began to notice and describe emerging block patterns within the game. No exploratory moves were observed at the general or formal levels, indicating that this type of activity remained closely tied to the task setting. In contrast, strategic moves showed a wider distribution across levels. Mostly observed at the referential level, where students developed strategies directly connected to their observations of the game environment, such as avoiding single-square gaps or placing blocks in corners. Thirteen strategic moves were observed at the general level, reflecting the use of broader strategies such as delaying placement to optimize future options. Notably, only two strategic moves were observed at the formal level**,** where students abstracted beyond the game context to articulate foundational general principles of reasoning with shapes.

Example of Exploratory Versus Strategic Move at the Referential Level

At the Referential level, students begin to develop *model-of* their activity within the task setting. At this stage, they notice, describe, and interpret block patterns or game effects that emerge from their moves. While their reasoning remains closely connected to the concrete situation, they start to form representations of how the game functions. In contrast to the situational level, where students simply place blocks wherever they fit, those at the referential level begin to articulate observations about the game's rules and outcomes—for instance,

recognising that certain shapes consistently fit into similar spaces. Table 2 illustrates this distinction by providing examples of one exploratory move and one strategic move, together with the corresponding emergent models and students' dialogues, demonstrating how these two types of moves differ even within the same modeling level.

Table 2: Example Students' Moves, Emergent Models, and Students' Dialogues

Referential Level (Models-of)	Sub-Category of Moves	Emergent Model	Students' Dialogues
Exploratory Move	Observing Line Effects		A: "Let's move that little one [the circled block] here." B: "See, it [the row] disappeared!"
Strategic Move	Checking Horizontal and Vertical Lines		C: "Look over that space. If I place it [the circled block] here, it will make the lines on both ways disappear." D: "Yes, we need to keep the center empty."

Observing Line Effects, as an exploratory move at the referential level, shows that students are beginning to notice and describe the immediate outcomes of their moves, such as recognizing when completed rows disappear. While this demonstrates growing awareness, the focus remains on interpreting visible outcomes rather than planning ahead. In contrast, *Checking Horizontal and Vertical Lines Simultaneously,* as a strategic move, reflects a more advanced form of reasoning. Here, students not only notice patterns but also apply this reasoning to make forward-looking decisions during gameplay. As illustrated in Table 2, a single move that clears multiple lines at once indicates deeper engagement with the game's structure, where students' observations begin to guide decision-making and the development of emerging strategies. This comparison highlights how growing awareness of block patterns and game outcomes informs more deliberate and purposeful moves.

Conclusion and Implications

This study examined how second graders' moves during block puzzle gameplay reflect their progression through four emergent modeling levels. The findings revealed that exploratory moves were dominant at the situational level, whereas strategic moves emerged more prominently at the referential and general levels. These patterns demonstrate how gameplay can

serve as a rich context for observing the transition from informal problem-solving to early forms of mathematical abstraction. This progression echoes prior literature on emergent modeling, emphasizing the importance of context-based tasks that allow mathematical reasoning to develop organically from students' actions (Doorman & Gravemeijer 2009; Gravemeijer 1999; Van den Heuvel-Panhuizen, 2003).

The implications of this study extend to both research and practice. From a research perspective, and in line with Sengupta et al.'s (2014) findings, our results highlight the value of using the digital puzzle game as a methodological tool for examining young children's mathematical reasoning during modeling activity. By mapping students' gameplay moves onto emergent modeling levels, researchers can gain deeper insights into how specific contexts support or constrain the development of higher-order reasoning (Cobb et al., 2021). From a practical standpoint, teachers can utilise such activities to actively engage students while simultaneously making their mathematical reasoning visible, thereby providing opportunities for formative assessment (Doorman & Gravemeijer, 2009). Future studies could build on these findings by exploring how different game designs or instructional supports shape young children's progression through emergent modeling levels, deepening our understanding of how they build mathematical knowledge through play.

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TOWARDS DEVELOPING A HUMANISTIC-COGNITIVE FRAMEWORK FOR MATHEMATICAL MODELING: THE CASE OF GWEN AND ARYA

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The cognitive perspective of mathematical modeling cannot be viewed in isolation from the influence of students' personal experiences, beliefs, and emotions. Hence, we developed a set of competencies as part of a (re)humanizing mathematical modeling framework. Through a qualitative single-case study approach, we analyzed one pair of pre-service teachers' modeling work for the presence of (re)humanizing modeling competencies (RMC) and cognitive modeling competencies (CMC). Findings were that personal connections and expose and engage injustice were most prevalent in systematizing and validating. In contrast, RMC were least prevalent in working mathematically, interpreting, and mathematizing. From our results, we introduce a humanistic-cognitive framework, showcasing humanistic ways of reasoning as an integral part of students' modeling processes, especially when reasoning within the real-world domain.

Keywords: Modeling; Cognition; Affect, Emotions, Beliefs, and Attitudes

When situated in students' personal, cultural, and social context, mathematical modeling (modeling hereafter) can (re)humanize mathematics (Gutiérrez, 2018). Through humanizing mathematics, we can positively shift students' mathematical disposition and support their development of "mathematical power"—the confidence, competence, and agency to engage in complex mathematical tasks—and critical ability to "read the world" (Gutstein, 2003, p. 44–46).

To understand how modeling can be humanizing, it is necessary to examine when and how humanistic ways of reasoning emerge in students' thinking, the extent to which they align with cognitive modeling competencies (CMC), and the ways they influence students' modeling decisions. Within the modeling education field, modeling cycles have been designed to examine student's cognitive thinking (e.g., Borromeo-Ferri, 2007). As such, they do not explicitly account for affective or contextual dimensions of learning, despite the interrelationship between affect, cognition, and modeling (Chamberlin, 2019). Thus, it is imperative to view students as embodied thinkers—individuals whose cognitive processes are shaped by their emotions, personal experiences, and social contexts (Bandura, 1986; 1999)—as a holistic perspective will lead to insights that can better support students' modeling sense-making.

The goal of this paper is to present our conceptualization of (re)humanizing modeling competencies (RMC) and to examine the interplay between humanistic ways of reasoning and modelers' cognition. We seek to answer *how do (re)humanizing modeling competencies emerge and shape students' cognitive modeling competencies during modeling?* Based on the modeling work of one pair of pre-service teachers (PSTs), we propose a humanistic-cognitive framework. This framework extends existing cognitive modeling perspectives and cycles by integrating the affective and contextual dimensions into the modeling process.

Empirical and Theoretical Background

In this report, we position our work as a humanistic-cognitive perspective of modeling. Drawing inspiration from Bandura's (2007) socio-cognitive theory, we operationalize the humanistic-cognitive perspective on modeling as modelers' personal factors (e.g., cognitive and affective factors) and contextual factors working together to construct mathematical

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representations for real-world situations. In this way, we define modeling as the cognitive *and* affective processes involved in constructing a mathematical representation for a given context (i.e., task or situation). The cognitive component of modeling is informed by theories from the cognitive perspective on modeling (e.g., Kaiser, 2017). The humanistic component of modeling is informed by theories from rehumanizing mathematics, social justice-oriented modeling, and culturally oriented modeling (e.g., Gutiérrez, 2018; Jung & Magiera, 2021; Rosa & Orey, 2010).

Cognitive Perspective on Modeling

Within the cognitive perspective on modeling, the process of modeling is schematically depicted by modeling cycles. A typical modeling cycle consists of CMC that connect one stage of model construction to another. According to Blum and Leiß's (2007, p. 225–226), a modeler *understands* the given real situation, engages in *systematizing* efforts to construct a "real model," applies mathematical knowledge and tools during *mathematization*, *works mathematically* to arrive at a "mathematical result," *interprets* this result to produce contextualized "real results," and *validates* the results against real-world constraints.

The cognitive perspective on modeling has mostly involved investigating students' cognitive strategies as they construct and validate their models. In particular, research has mainly focused on documenting students' modeling routes (Borromeo-Ferri, 2007), investigating the blockages students experience while modeling (Jankvist & Niss, 2020; Klock & Siller, 2020; Stillman et al., 2010), analyzing the CMC elicited as students engage in modeling tasks (Czocher, 2016; Czocher, 2018), and examining how students' reason quantitatively as they developed mathematical models of real-world scenarios (Kularajan et al., 2024; Czocher et al., 2022). More recently, Wickstrom and Jung (2024) developed Modeling Decision Maps as a tool to analyze how K-8 pre-service teachers "constructed and refined" their models (p. 182). While research on students' thinking during modeling has yielded robust descriptive and, at times, explanatory frameworks for students' modeling activities, these accounts typically focus on cognitive processes. This is due to existing frameworks not being designed to capture affective and contextual dimensions. Hence, there is a need to integrate cognitive and humanistic domains (e.g., affect) within modeling cycles to develop authentic and comprehensive explanatory accounts of why a student might have engaged in a certain modeling decision.

Humanizing Perspective on Modeling

Frameworks that humanize mathematics situate modeling within social justice or cultural contexts, such as Aguirre et al.'s (2019) social justice and modeling cycle or Rosa and Orey's (2010) ethnomodeling diagram. Across these approaches, students have been observed to draw on their "ways of thinking" as they engage in CMC, connecting mathematics to their personal experiences, cultural knowledge, and world understanding (Anhalt et al., 2018, p. 321).

For instance, in an adaptation of the Flint Water Task (Aguirre et al., 2019), Flavin and Hwang (2024) tasked their PSTs to consider the food and water needs of Flint children. The authors observed that the cultural and educational backgrounds of PSTs impacted their modeling approaches and decisions. They found that U.S. PSTs calculated bottle sizes with ounces while their Korean counterparts used litres. They also found that Korean PSTs assumed 365 days when estimating the number of schools, with one group considering the Chinese Lunar calendar, while U.S. PSTs estimated either 110 or 180 days after accounting for vacation days. Kaygısız and Şenel (2023) found that understanding and interpreting competencies could be attributed to a students' ability to associate the task with prior knowledge and previous experiences. This influenced their achievement in attaining the desired CMC level during a model eliciting activity in the context of planting a field. These works reveal how a students' background and prior

knowledge are important contributors to their engagement of modeling tasks, highlighting modeling as a reflective and highly cognitive experience.

Influenced by the humanizing perspective on modeling, we propose a (re)humanizing mathematical modeling perspective, which extends the cognitive perspective on modeling to include affective and contextual dimensions. The (re)humanizing modeling perspective values and encourages students to freely express and use their full selves in mathematics through engaging in (re)humanizing modeling competencies (RMC). We use the term "rehumanizing" to pay homage to Gutiérrez's (2018) call for rehumanizing mathematics. RMC refers to students' ability to draw upon their emotions, attitudes, beliefs (affective), and social and situational (contextual) factors while de-contextualizing the real-world situation (i.e., transitioning from the real-world domain to mathematical domain) and re-contextualizing their mathematical results (i.e., transitioning from the mathematical domain to the real-world domain). We operationalize RMC as consisting of the following facets:

- *Personal connections*: students leverage their personal experiences while modeling due to the mathematics being situated in their personal context.
- *Emotions:* students experience mathematics beyond just the cognitive process but also as an experience where they can draw from themselves as living, feeling bodies (e.g., emotion, intuition) while engaging in mathematical reasoning and thinking.
- *Self-reflexivity:* students reflect on their own perspectives and biases during modeling, thereby fostering awareness of how doing mathematics is influenced by their identities.
- *Expose and engage injustice*: students critically examine social, cultural, and environmental injustices at macro- and micro-levels, thereby building empathy and empowering students to see mathematics as a tool for advocating change.
- *Collaborative sense-making*: students construct knowledge through integrating multiple perspectives and knowledge systems in a community-oriented modeling approach.

Methods

Research Design

We followed a case study research design (Yin, 2009) to explore how RMC inform CMC. Data for this study were drawn from a larger project in which two pairs of PSTs engaged in seven task-based cognitive interviews designed to investigate how their modeling-oriented noticing informs their modeling strategies (Kularajan et al., 2026). This illustrative, explorative analysis uses Gwen and Arya's work on the Hospital Task.

The Case: Gwen and Arya and the Hospital Task

The PSTs. The participants consisted of Gwen and Arya, two elementary PSTs from a public university in the Western United States. They were recruited from a mathematics content course. We chose Gwen and Arya's work on the Hospital Task as (i) our PSTs were familiar with the geographical location of the task context—an important part of modeling of the task—and (ii) we hypothesized that consideration of "best" would allow for RMC to emerge. Moreover, these PSTs were comfortable vocalizing their thoughts and experiences as they constructed and reflected on their model, making this case a meaningful selection for providing empirical evidence regarding the interplay of RMC and CMC.

The Hospital Task. The Hospital Task (Zbiek & Connor, 2006) was presented in two scenarios. In Scenario 1, the PSTs determined the best location for a high-powered hospital that serves Boise, Idaho; Helena, Montana; and Salt Lake City, Utah (see Figure 1). In Scenario 2,

the PSTs were given census data (see Figure 2) and again asked to determine the best location for a hospital. PSTs' work on the task lasted 33 minutes and 22 seconds.

> Boise (Idaho), Helena (Montana), and Salt Lake City (Utah) are three large cities in the northwestern part of the United States. The map shows the locations of the three cities as well as other large cities in the area. Determine the best location for a high-powered hospital.

Figure 1: The Hospital Task (Abridged Version)

What if the additional information is also given to you:

City	Census
Boise	232,000
Salt Lake City	199,000
Helena	32,000

Figure 2: Population of the Cities Given 13 Minutes into the Task

Data Analysis

The transcribed interview was analyzed using MAXQDA 2024 (Verbi Software, 2021). The unit of analysis was segments of text (e.g., single or multiple utterances) that conveyed a single idea. For analyzing the CMC, we used a deductive coding procedure based on a prior framework (see Czocher, 2016). This included using sub-codes (e.g., making assumptions, writing math representations) for identifying CMC of *understanding, systematizing, mathematizing, working mathematically, interpreting,* and *validating.* For coding the RMC, we coded instances related to the RMC facets, described previously. We then used inductive strategies to identify student actions that belong under each RMC (i.e., sub-themes). Table 1 shows this analytical framework.

Table 1: (re)Humanizing Modeling Competencies with Sub-themes

Sub-theme	Definition
Personal connections	
Drawing on lived experiences	Students reference *firsthand* stories or acknowledge the absence of such experiences while constructing and reasoning about their model.
Drawing on learned knowledge	Students reference *acquired* knowledge through *secondary sources* or acknowledge the absence of such knowledge while constructing and reasoning about their model.
Emotions	
Expressing affect	Students express emotions (e.g., frustration, excitement, curiosity, uncertainty) while constructing and reasoning about their model.
Intuitive decision making	Students make decisions or draw conclusions based on their intuition or gut instinct. This is often done fast and emotionally with little regard for logic (for more information, see Kahneman, 2011).

Self-reflexivity	
Emotional reflexivity	Students reflect on how their emotions influence their modeling approaches and decisions.
Drawing on internal beliefs	Students may explicitly or implicitly express using their personal values, ethical beliefs, or principles as reasons that influenced their decisions, approaches, and interpretations.
Expose and engage injustice	
Ethical consideration	Students propose and reflect on approaches, decisions, or solutions based on introspection Examples of ethical principles that could be considered include fairness, justice, and equity.
Data consideration	Students consider biases in the data, mathematics, or situation and describe how the model accounts for or addresses such injustices.
Collaborative sense-making	
Asking questions	Students engage one another by asking questions that promote reflection, dialogue, and critical thinking to support the construction and refinement of a model.
Supporting dialogue	Students support and validate each other's thoughts. Examples include using affirmative language or building on one another's ideas.

In both analysis process, the first author (L. R.) completed the initial coding. This was then reviewed by the second author (S. K.). Discrepancies in the coding were noted; afterwards, both authors met and discussed each code individually until a 100% intercoder agreement was reached. We used MAXQDA's quantizing feature to generate code frequencies and corresponding percentages. We used the Code Relations Browser to identify co-occurrences between CMC and RMC.

Findings

Analysis revealed that the most frequent CMC were systematizing (11.8%) and validating (21.8%). For RMC, they were sense-making (15.8%), personal connections (10.9%), and expose and engage injustice (10.9%). The five most co-occurrences were: validating and collaborative sense-making (19.4%), validating and personal connections (12.9%), systematizing and collaborative sense-making (12.2%), validating and expose and engage injustice (10.8%), and systematizing and personal connection (9.3%). The least frequent co-occurrences include any RMC with working mathematically (0.7%), interpreting (5.8%), and mathematizing (6.5%). Due to space constraints, we illustrate on two co-occurrence episodes.

PSTs Embraced Personal Connections as They Simplified the Model

Personal connections with systematizing occurred 13 times, accounting for 9.3% of code co-occurrences.

In Scenario 1, the PSTs were asked to determine the best location for a hospital. After reading the task, Gwen initially suggested Pocatello as the best location. She commented, "it's the biggest town kind of in between all of them [and] it's kind of close to this water source too. So, you could use that for power for the power grid. I'm sure they have a dam somewhere." This was agreed by Arya, leading the pair to "box that one [i.e., Pocatello]." Here, systematizing was identified four times with Gwen as she noted the non-existence of numerical information,

estimated the size of Pocatello, commented on a water source as an important consideration, and drew a box around a potential answer. Her certainty of the presence of a dam reflects her background knowledge that a dam is connected to a power source, contributing to the personal connections theme. These considerations led the PSTs identifying Pocatello as an initial location.

As the pair progressed in the task, they began measuring the distance between towns, finding Rexburg to be about two inches from each city. In considering Rexburg, Arya introduced additional parameters, such as driving time and the curviness of the road. For drive time, Gwen commented, "I have no idea", while Arya remarked, "I know from Logan to Rexburg, it's about three-ish hours, … So, what is Logan from Salt Lake?" Gwen then used the reference point mentioned by Arya to estimate that "Logan from Salt Lake is like an hour and a half." Here, systematizing was identified three times and personal connections twice. While Gwen did not have experience with this drive, Arya's familiarity assisted the pair in estimating the drive time from Logan to Salt Lake. Thus, the PSTs integrated their real-life experience to approximate distance, reinforcing the importance of personal connections for systematizing decisions.

In Scenario 2, the PSTs were given population data of the cities (see Figure 2). Based on this new information, the pair engaged in the following conversation:

> Gwen: My first thought again goes to Pocatello because it's closer to Boise… and it's also still close-ish to Salt Lake City and all that. A part of me wants to go a little bit closer to Boise, but at the same time, I don't know how big of towns Twin Falls and Blackfoot are. I've never been…
>
> Arya: Blackfoot's small. I mean, most towns in Idaho are pretty small.

This exchange highlights how lived (or lack of lived) experience shaped PSTs' modeling decisions. Gwen's uncertainty about the size of Twin Falls and Blackfoot due to not having been to those places limited her ability to refine their model; she had to rely on her co-modeler's insights. Arya possessed the necessary background, recalling Idaho's town as small. This guided the pair's decision not to consider Blackfoot as a location for a hospital. Through these examples, we illustrated how drawing on personal connections was integral in systematizing the situation model. PSTs' personal connections informed their thinking as they made decisions (e.g., ruling out certain locations), estimated values (e.g., driving time), and identified real-world considerations for modeling parameters (e.g., power source).

PSTs Exposed and Engaged Injustice as They Validated the Model

Expose and engage injustice with validating occurred 15 times, accounting for 10.8% of code co-occurrences.

In Scenario 1, at a first glance, Gwen suggested finding an equidistant location. This approach, we interpret, highlights the inherent consideration of equity. In other words, we suggest Gwen's inclination for using equidistance reflected her implicit belief that a shared hospital means placing it in a central position so that there would be equal travel distance between any city. This, she thought, would ensure a fair solution. PSTs, therefore, showed how justice ideals (i.e., sharing, accessibility)—whether explicitly or implicitly stated—influence modeling decisions (i.e., focusing on equidistance). Below, Gwen explained their reasoning as:

> Gwen: Just the general idea of sharing these three cities is sharing this, so it should be of equal use to all of them ideally … So, if they're assuming that they're all putting in the same time and resources to this, they all want to have equal access to it, I'm guessing.

In Scenario 2, PSTs integrated more social justice perspectives as they shifted their thinking towards considering human needs. They determined that "we still want it pretty close to all, so it's still accessible, but maybe just closer to Boise." Gwen justified their decision to choose Boise by stating, "with the populations, I'd adjust it to be just a little bit closer to the places that have higher populations" because "places with the higher populations are probably going to have a higher amount of medical emergencies." The PSTs no longer focused on equidistance as described in Scenario 1 but began exploring and using "population ratios" as a determining factor. Ultimately, after considering population and other "humanizing" factors (see Figure 3), Pocatello was chosen for the hospital in Scenario 2. The PSTs reasoned Pocatello as being closer to Boise and Salt Lake (cities with higher populations) but more accessible to the people of Helena than Twin Falls. Here, PSTs engaged with justice issues for validating their models through considering city populations.

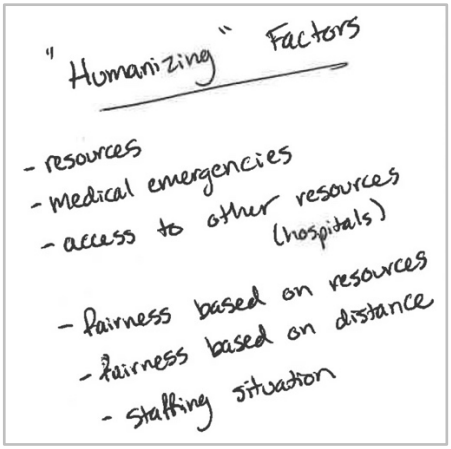


Figure 3: PSTs' "Humanizing" Factors

Even after formulating a solution for the hospital placement, RMC continued to emerge as the PSTs reflected on whether their answer addressed ethical considerations. For example, Arya introduced the idea that "maybe Boise already has a pretty big hospital, so maybe they don't need it as much. Helena is more in need." This statement illustrates an important purpose of (re)humanizing mathematical modeling, which is to challenge assumptions and re-evaluate solutions within broader justice contexts. An example of this is presented in the excerpt below:

Gwen: Okay. So, if I followed the actual, the math, then it would say, okay, seven times closer to Boise would be right there, which I mean, theoretically makes sense. But then, if you're taking into account the people, Helena still has a lot of people that need help. So, think of all the medical emergencies and the people who could die on the way because of the distance.

Arya: It's a hospital for all of them.

This exchange illustrates how the PSTs raised ethical concerns as well as inherent questions that are important when engaging with justice issues, such as: *Who benefits from this decision*? *Who is excluded*? and *What are the trade-offs of prioritizing one outcome over another*? Although Helena and Montana have a smaller population, the PSTs recognized that everyone should have access to a hospital. This led them to grapple with and consider how equity, fairness, and the needs of people could be represented in their model.

Through these examples of validating their model, Gwen and Arya moved away from relying on numerical calculations to determine a solution. Instead, they began looking at modeling through an ethical lens. The PSTs reflected on real-world justice issues and used those concerns in their modeling (e.g., travel distance, population). Moreover, their collaboration revealed an awareness of how different modeling choices can impact communities, thereby exposing and engaging injustice. As Gwen eloquently stated, "there's a big difference between the mathematically correct way of making sense versus the ethically just way."

Discussion and Conclusions

Gwen and Arya demonstrated that cognitive and humanistic ways of reasoning can occur in modeling. Based on our findings, we propose a Humanistic-Cognitive Modeling Framework, developed from iterative analysis of the PSTs' modeling activity and the systematic interactions observed between RMC and CMC. This framework emphasizes student thinking as multilayered. By integrating multiple dimensions—cognitive, affective, and social—into student thinking, we position students as complex individuals with thoughts, emotions, values, and beliefs that play an instrumental role in their modeling-oriented thinking.

In this framework, students enter the beginning phases of modeling (i.e., *understanding, systematizing*) by drawing from their lived experiences and knowledge. This is often done through questioning and refining results based on their perception of the world. Thus, personal connections impact modeling decisions and, therefore, students' engagement with CMC. In adding personal connections, we contribute to a more holistic view and understanding of student thinking while modeling. Expose and engage injustice is placed at the end of the modeling process (usually, *validating*). Our PSTs not only asked whether their solutions were reasonable but also reflected on how their answer could impact certain populations and communities. They were exposing and engaging injustice throughout their modeling, but particularly while validating. Our framework highlights expose and engage injustice as an influencing factor in how students validate their thinking.

Our study has implications for research and education. For the modeling education field, our study has two contributions. First, we proposed, developed, and tested the feasibility of an analytical framework for identifying RMC. This can be used by researchers to develop comprehensive accounts of students' modeling activities. We have ensured its feasibility by providing detailed definition of (re)humanizing mathematical modeling and their sub-themes to support the identification and categorization of RMC. For education, this framework can help educators recognize that modeling can support both cognitive and affect competencies simultaneously—an insight that could potentially lead to more purposeful and integrated modeling implementation (e.g., social justice, culture-based).

We recognize that our study has limitations. The framework used to describe RMC may not consider all possible humanistic elements. While the elements were informed by literature and emerged from iterative analysis of our data, it is also constrained by those. Investigating RMC in a different research setting may yield different outcomes. Moreover, the modeling task used in this case study was local and therefore, contextually familiar to the students. This familiarity means that our study cannot make conjectures about how RMC may appear in tasks that are not local to students. However, the existence of feasibility, as demonstrated by our analysis, is sufficient for theoretical adaptations and modifications. The strength of the framework can be evaluated by testing it against a larger dataset to gain more insights into the prevalence and relationships between RMC and CMC. Additionally, future research could investigate whether tasks explicitly designed with RMC influence the co-occurrences of RMC and CMC.

In this work, we have shared our reconceptualization of the modeling competencies to include not just *what* students do mathematically (i.e., CMC) but *why* they do it (i.e., RMC). The re(humanizing) mathematical modeling perspective is not intended to capture all modes of humanistic reasoning, but it is intended to be an analytic lens for identifying RMC within CMC. Our theory development, although incremental, is a significant step towards understanding the cognitive and humanistic domains of modeling and their integration. We call for continued research to develop approaches that view modelers' mathematical work through a comprehensive lens, where modelers' personal experience plays an active role in model construction and interpretation. An integrated approach like this can lead to a shift in how modeling is conceived and implemented within the modeling education field.

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HOW PRESERVICE TEACHERS EXPERIENCE INTEGRATED COMPUTATIONAL THINKING AND MATHEMATICS: A SYSTEMATIC LITERATURE REVIEW

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This systematic review investigates how preservice teachers (PSTs) experience the integration of computational thinking (CT) and math. Across 27 studies, most research appears outside math education journals despite 50% being math-related courses. The review shows structural differences in where PSTs encounter CT: elementary PSTs in math methods and Tech/CS courses and secondary PSTs in math content courses. This points to the need for mathematics education researchers to more intentionally shape PST's CT experiences.

Keywords: Computational Thinking, Preservice Teacher Education, Computing and Coding

Computational thinking (CT) has become a central focus in contemporary educational reform as countries worldwide integrate it into K-12 curricula (Bocconi et al., 2022; Hsu et al., 2019). Research indicates that purposeful CT integration can deepen mathematical understanding and contribute to students' developing identities as mathematical thinkers (Weintrop et al., 2022; Mills et al., 2024), though the potential of this relationship remains only partially explored (Li et al., 2019). Because PSTs rarely encounter CT in their preparation, they need meaningful opportunities to engage with CT themselves to understand what it is and how it can support mathematical thinking in the classroom (Waterman et al., 2020). Most research on CT in the context of K-12 education focuses on K-12 students and practicing teachers. In this systematic literature review, we investigate what is known about how preservice teachers learn to integrate CT and mathematics by answering the following research questions: *RQ1:* Where is research on the integration of CT and mathematics in preservice teacher education being published, and are there differences between elements of CT emphasized based on the journal area? *RQ2:* How do documented approaches in integrating CT and mathematics into PST preparation programs align or differ across grade level preparation and course context?

Theoretical Perspectives

CT is generally understood as a set of thinking practices that support information processing and problem solving across a variety of contexts, with or without computers (Cansu & Cansu, 2019; Navarro & de Sousa, 2023; Palop et al., 2024). Researchers operationalize CT as a set of interdependent cognitive and representational practices. Some of the most common practices associated with CT across the literature include *decomposition*: (breaking complex problems into smaller manageable parts), *abstraction* (identifying essential aspects of the problem and reformulating it), *pattern recognition* (identifying trends, similarities or repeated structures in data or problems), *algorithmic thinking* (creating step-by-step procedures, instructions to solve a problem), *debugging* (testing solutions to find errors and improve performance or validating),

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

and *iterative development* (refining through repeated improvements; Kallia et al., 2021; Sherwood et al., 2024; Shute et al., 2017; Lehman, 2025; Tang, 2020). This literature review draws on Weintrop et al. (2022) and distinguishes between CT (pattern recognition, abstraction, decomposition, and algorithms) and programming skills (iterative development and debugging).

Methods

We constructed search strings using keywords from four categories where one keyword from each group had to be found within the title or abstract: teachers (*teacher**), mathematics (*math**), computational thinking (*coding, programming, computing*, or CS*) and coursework (experiment*, classroom*, course*, or intervention*). Searches were conducted in EBSCO Host, Web of Science, JSTOR, Supplemental Education Index, and Sage Journals during September 2025 and results were limited to peer-reviewed articles published in English or Turkish since 2006, aligning with Wing's (2006) seminal paper on CT. Any duplicate papers were removed, and this resulted in 4507 papers. All search results were exported to Rayyan for screening and management. The screening process occurred in three stages: initial title screening to remove duplicates and irrelevant items (3435 articles), abstract screening according to the inclusion and exclusion criteria which is ongoing (58 articles), and from that set a full-text review to determine final eligibility (27 articles to date). For each included study, we extracted journal focus, research questions, geographic information, grade band, course context, CT and programming elements, computing environments, and content domain. For clarity, content domains were grouped into number/quantity, algebra/functions, geometry, and data/statistics which included measurement.

Results

The 27 studies included in this analysis were located within five different journal types. The distribution of studies across journal categories were educational technology and computer science journals (11 studies), general education journals (8 studies), mathematics education journals (7 studies), and social science journals (1 study). These numbers suggest that roughly 25% of the published work on integrating CT and mathematics appears in education journals in comparison to the 40% for EdTech/CS journals. The missing conversation from mathematics education was also seen in the research questions of each article, as most focused on the affective elements of the integration, computational skills, integration into lesson planning, or other areas. Only one article had a research question that centered on the mathematical learning of the students and how CT could bring about a better mathematical modeling understanding for PSTs.

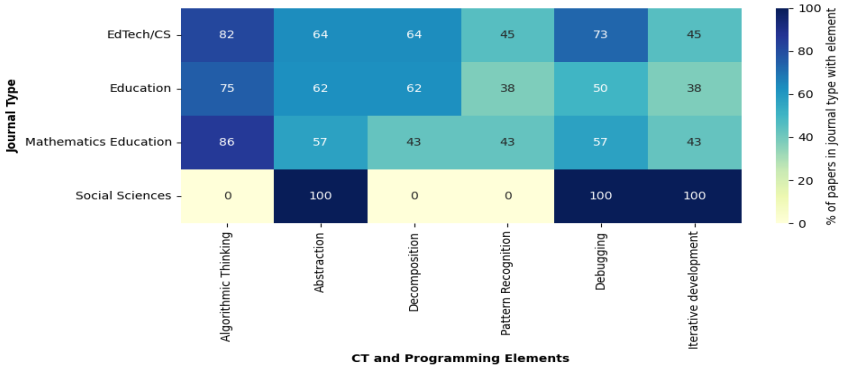
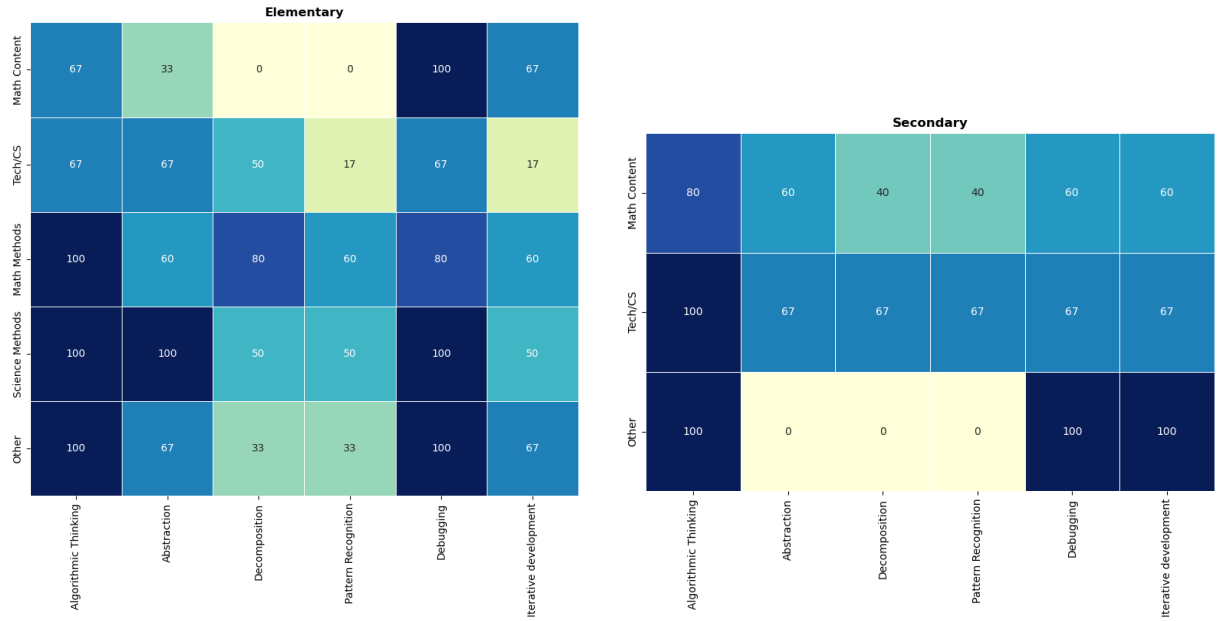


Figure 1: Percentage of Papers within Journal Types that Include Each Computational Thinking and Programming Element, with Values Normed within Journal Type

Table 1: The Distribution of Where CT and Mathematics are Integrated in PST Education, Split by Grade Band and Course Type

	Math Content	Math Methods	Science Methods	Tech/CS	Other
Elementary	3	5	3	6	3
Middle	1	2	0	3	0
Secondary	5	0	0	3	1

Figure 1 displays the percentage of each journal type that includes the specific CT and programming elements described within the theoretical framing. EdTech/CS journals had a broad distribution of CT elements, and a high percentage of studies included algorithmic thinking (82%), debugging (73%), abstraction (64%), and decomposition (64%). Similarly, education had a broad distribution across these four elements, but slightly lower frequencies. Mathematics education journals had the strongest emphasis on algorithmic thinking, which appeared in 86% of studies, yet only 43% of articles referenced decomposition and pattern recognition. Across all journals algorithmic thinking was the most consistently represented CT element.



(a) (b)

Figure 2: Percentage of Articles in Each Course Context that Addressed Specific CT and Programming Elements at the (a) Elementary and (b) Secondary Grade Bands

Mathematics content, mathematics methods, science methods, and Tech/CS courses were the main CT math integration courses (Table 1). At the elementary and middle grade level, most studies were situated in math methods and Tech/CS courses with fewer studies in math content courses. Notably, there were no reported secondary methods courses with this integration. Rather, secondary PSTs primarily encountered integration in math content courses, followed by Tech/CS. The mathematical domains represented also varied by grade band. Elementary studies

were concentrated in geometry (n = 8) and data/statistics (n = 5), with smaller numbers in number and quantity (n = 4) and algebra/functions (n = 3). Middle-grades studies were limited overall but appeared in algebra/functions (n = 2) and geometry (n = 2). Secondary integration spanned all domains, with geometry (n = 4) and data/statistics (n = 3) most prevalent. Although there was integration across PST grade bands, Figure 2 shows that which elements of CT or programming were emphasized split by course context and grade band. Notably elementary math content courses did not have a documented focus on decomposition or pattern recognition, but all emphasized debugging. In comparison, elementary math methods courses had a more balanced distribution of CT elements. Within secondary math content courses, decomposition and pattern recognition were the least mentioned elements of CT and the programming foci of debugging and iterative development were of greater emphasis.

Discussion and Conclusion

Despite PSTs most commonly encountering the intentional integration of CT and math in their math content and methods courses, most of the research is occurring outside of mathematics education journals. Further, the absence of work is not isolated to journal focus but also in the very research questions asked. Only one of 27 studies centered PST mathematical learning and understanding in comparison to typical programming skills or affective elements. If PSTs don't have the chance to center CT as a mathematical sense-making tool or explicit focus on CT supporting mathematical thinking, then early conceptions of CT may solely emphasize how to code or procedural steps rather than mathematical processes such as abstraction, decomposition, and pattern recognition. This could position CT as something tangential to math rather than as a set of practices which reveal structure and deepen conceptual understanding in math.

Structural patterns across course types and grade bands further bring this noticing to light. Elementary PSTs most often encounter CT through math methods and Tech/CS courses, yet in math content courses they primarily experience programming practices such as debugging, with little documented emphasis on decomposition or pattern recognition, which are two practices tightly aligned with mathematical thinking. Secondary PSTs, in contrast, encounter CT almost exclusively within math content courses, but with no documented integration in secondary mathematics methods courses. This means that many future secondary teachers may never have opportunities to develop pedagogical content knowledge to bring CT into math classrooms. Across all grade bands, algorithmic thinking dominated the reported CT elements. While algorithmic thinking is important, its overwhelming emphasis raises concerns that CT math integration may be framed primarily as procedural or solution driven rather than exploratory or sensemaking. This could lead to narrow views of what CT is and what it can offer to mathematics learning.

Together, all these patterns highlight an urgent need for mathematics education researchers and teacher educators to engage more centrally in shaping how PSTs come to understand CT. If CT is to play a meaningful role in mathematics instruction, the mathematics education community must contribute research that examines how CT supports mathematical reasoning, conceptual understanding, modeling, and problem solving. This includes designing and studying learning experiences that foreground the CT elements aligned with mathematical practices, and investigating how PSTs' mathematical understandings as well as their beliefs, identities, and instructional commitments shift when CT is presented not merely as coding, but as a way of engaging with mathematics.

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OBSERVING THE TRANSITION FROM EMPIRICAL EVIDENCE TO CONJECTURE IN A DGS ENVIRONMENT

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This study investigates the boundary between empirical observation and conjecture formulation using dynamic geometry software. Employing a paired think-aloud protocol and screencasting technology, the researchers captured authentic, non-linear mathematical reasoning in a classroom setting. Findings suggest that existing frameworks for inductive reasoning are too linear; the authors propose expanding the model to include "messy" but productive stages such as attempted validation, conjecture rejection, and attempted justification. Ultimately, the study highlights how screencasting captures the essential, non-linear reasoning typically lost in final mathematical products.

Keywords: Reasoning and Proof, Geometry and Spatial Reasoning, Technology

In a collegiate course on mathematical problem solving with technology, we ask our students to explore mathematics with dynamic geometry software (DGS). They use sketches to conduct empirical experiments with geometric objects to explore patterns, pursue conjectures, and test the verity of claims. We routinely ask, "under what conditions might something like this happen?"

As students engage with tasks, we have observed the tipping point where enough empirical evidence accumulates and a conjecture emerges. This boundary is clearly demarcated in the literature (cf. Cañadas & Castro, 2007) and warrants more attention. Specifically, we focus on the way *thinking aloud in action,* where the language is messy, hesitant, and full of contradictions (Maine & Čermáková, 2023) supports the conjecturing process. In a classroom, moments of pattern-finding and where conjectures emerge are fleeting and often hidden within a non-linear path of mathematical behavior and creativity.

Historically, reasoning and process has been documented by Piagetian-style clinical interviews (Cañadas & Castro, 2007) or think-alouds (e.g., Schoenfeld, 1985). Ericsson & Simon (1998) claim that for a think-aloud protocol to be valid, the subject must perform the task (such as a math problem) without trying to explain or justify their actions to the researcher. It is rare, but possible to study the formation of conjectures in a classroom or small group setting. Maine & Čermáková (2023) video recorded and transcribed whole-class and small group conversations in elementary classrooms and were able to capture the moment a child pivots from a vague observation to a specific conjecture. Here, researchers are not a direct audience for the classroom actions of explanation or justification, though they still lurk with a camera as a proxy.

Screencasting offers additional insulation between the participant and the researcher. Cox et al. (2018) tested the potential of screencasting as a tool to capture small group problem solving and found it was capable of capturing moments of authentic creativity within the context of the mathematics classroom. The combination of recording on-screen behavior and verbalized narration enabled the researchers to preserve the authenticity of student reasoning within a classroom culture.

In this study, we used a paired think-aloud structure inside a classroom where students are working within a DGS environment. DGS has been shown to provide feedback to users and encourage iterative design (Laborde et al., 2006). We sought to use the DGS environment as a collaborative work tool, provoking conversation and enabling in-the-moment tweaks and edits,

giving immediate access to the results (Beatty & Geiger, 2010). So, supported within a DGS environment and using screencasting technology, we asked, (1) What reasoning is present at the boundary between observing and organizing particular cases and formulating conjectures when working in a DGS environment? and (2) What are the potentials and limitations of using screencasting to better understand this boundary in a DGS environment?

Theoretical Framework

Stylianides (2008) defines a conjecture as "a reasoned hypothesis about a general mathematical relation based on incomplete evidence," (p.11). This definition encapsulates the tension between a thought that is non-arbitrary (reasoned) and yet uncertain (hypothesized) or as Harel and Sowder (1998) put it, "[a] conjecture is an observation made by a person who has doubts about its truth" (p. 241). Cañadas and Castro (2007) describe a seven-stage framework describing the inductive and deductive reasoning processes surrounding the act of making conjectures (Table 1). These stages provide nuance when examining the boundary between inductive and deductive reasoning.

Table 1: Framework for Inductive and Deductive Reasoning Processes

Stage	Description
Observation of particular cases	Experiences with particular cases of the problem posed
Organization of particular cases	Use strategies to systematize and facilitate the work with particular cases
Search and prediction of patterns	This pattern is not general, but based on specific observed cases to generate new ideas about possible cases
Conjecture formulation	Making a statement about all possible cases based on particular ones, but with an element of doubt (Reid, 2002)
Conjecture validation	Attempt to validate conjectures for new cases, but not in general.
Conjecture generalization	Hypothesizing that the conjecture is true beyond observed cases
General conjecture justification	Giving reasons that explain the conjecture with the intention of convincing another person that the generalization is justified.

Methodology

We asked seventeen PSMTs (working in small groups within a class period) to consider the question, *What conditions have to be satisfied for the perpendicular bisectors of the sides a quadrilateral to meet at a point?* We collected eight screencasts ("sessions") of partners working with DGS, which, along with transcriptions, makes up the corpus of our data. These sessions are typically 12–15 minutes long. For each session, we prepare a detailed narrative (Clandinin & Connelly, 2004) documenting *episodes of conjecturing* within the session. An episode of conjecturing is defined as a period of time during which there is sustained focus on one mathematical idea, even if that period does not include a *conjecture formulation* event. Breaks between episodes can be marked by a shift in focus or rejection of an idea/stated conjecture. Once we have established episodes, researchers began writing a narrative including descriptions of conjecturing behavior (CB) within. These drafts are reviewed and revised until consensus is

reached on the team. Final narratives are complex and include a transcript of the conversation, descriptions of visible, on-screen actions within the DGS environment, and identified CB linked to the framework above (Table 1).

The Case of Elijah and Grant

While a full analysis of all eight sessions is under way, the case of Elijah and Grant is worthy of intermediate attention because it contains five distinct episodes and opportunities to observe CB not included in the original framework. These episodes and the observed stages are summarized in Table 2. Stages in bold represent newly described CB.

Table 2: Observed Stages of Inductive Reasoning

Episode	Stages of Inductive and Deductive Reasoning Processes
Episode 1: Dragging Point D 1:40–3:49 minutes	Observation of particular cases (n = 6)
Episode 2: Visualizing and Exploring Angle Bisectors 4:12–5:55 minutes	Observation of particular cases (n = 2) Conjecture formulation *Attempted conjecture validation*
Episode 3: Perpendicular Bisectors/Side Length 6:00–7:19 minutes	Observation of particular cases (n = 2) Conjecture Formulation *Conjecture rejection*
Episode 4: Perpendicular Bisectors and Angle Measure 7:19–11:08 minutes	Observation of Particular Cases (n = 2) Search and Prediction of Patterns Observation of Particular Cases (n = 1) Conjecture Formulation Observation of Particular Cases Search and Prediction of Patterns Conjecture Validation Conjecture Generalization *Attempted conjecture justification*
Episode 5: Unfixing A,B, and C 11:13–13:30 minutes	Observation of particular cases (n = 4) Conjecture generalization

As Elijah and Grant worked through the task, the DGS environment supported multiple opportunities to observe particular cases and formulate conjectures. Further, the pair moved through five of the established stages; some stages multiple times. (For this paper, we will focus on Episode 4.) In this episode, the pair construct the perpendicular bisectors of all four sides and measure the four angles in the quadrilateral. They observe two cases and begin their search for patterns with two observations. First, the pair recalls that point C is fixed and angle C is a constant size (88.6 degrees). Second, there is a moment when Grant questions whether the intersection has to be "at the middle" of the quadrilateral. In this moment, there is a tacit acknowledgement that the intersection point was inside of the quadrilateral in both observed cases and Grant's question suggests that he isn't sure this would always hold true. He sees evidence of a pattern and questions whether it will hold up with additional examples. As they

identify one more example (see Figure 1), Grant begins to formulate a conjecture, with help from Elijah, which culminates with a confident generalization of the conjecture by Grant. The remaining time in the episode is spent trying to find technological solutions to pin the intersection point so that they can move D more confidently in the environment. Ultimately, they are unsuccessful, but remain confident in the statement of their conjecture.

Grant:	So maybe this one always has to be 91 [Angle D]… potentially because these were always 90. I think in… in at least our case. Yeah, I think it kind of always was around 91.
Elijah:	For it to be around 91 for it to be intersecting. [Drags point D in such a way to maintain the point of intersection and observe how angle D behaves.]
Grant:	[Confidently] Yeah. For it to intersect, it's got to be around 91 degrees.

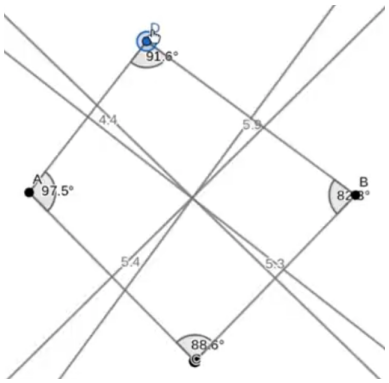


Figure 1: Transcript and Geometric Example from Episode 4

Findings and Implications

The methodological approach used in this study proved essential to the integrity of our findings, and this carries meaningful implications for researchers studying conjecture-making in technology-mediated environments. DGS, combined with Screencasting, enabled us to utilize a think-aloud protocol, linking conjecture to observed cases without researcher intervention or direct observation. The resulting data is an authentic representation of how students create, observe and interact with existing cases to formulate, generalize, and even reject conjectures. Even further, the process captures essential non-linear reasoning typically lost in final mathematical products.

While Cañadas and Castro's (2007) framework provides a rich description of the reasoning processes surrounding conjecture-making, our findings suggest that it may benefit from expansion to account for the full range of student experiences, particularly those involving invalid conjectures. In our data, we observed three stages not represented in the existing framework: *Attempted conjecture validation*, *Conjecture rejection*, and *Attempted conjecture justification* (see Table 2). These stages reveal that students frequently engaged in productive mathematical reasoning even when their conjectures were not valid. The current framework implicitly traces a successful path towards generalization and justification, but student reasoning is rarely so linear. When a student attempts to form or validate a conjecture and instead discovers a counterexample, this act of rejection is not a failure to progress through the stages. Instead, we have captured a meaningful reasoning event that may cycle back to earlier stages, shift in focus, or generate entirely new conjectures. By omitting these experiences, the framework does not illustrate a significant portion of students' mathematical activity. Future work might consider the iterative nature of inductive and deductive reasoning in authentic mathematical practice.

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IMPLEMENTING SUPPORT BEYOND THE SCHOOL DAY: A CASE STUDY OF MATH INTERVENTION IN A DETRACKED CONTEXT

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Detracking promotes equity but creates wide achievement ranges, prompting outside-the-school-day (OSD) math interventions. This qualitative case study examines OSD implementation in an urban Canadian school board using participant interviews. Wave 1 findings reveal four themes: recruitment/accessibility challenges, structural constraints, enabling factors, and professional learning needs. While small-group peer learning benefits students, implementation faces barriers like high ratios, limited training, and unclear communication. Resource limitations constrain impact despite creative adaptations

Keywords: Detracking, Instructional Practices, Equity in Education, Learning Trajectories

Introduction and Purpose

This study examines the implementation and effectiveness of the Outside the School Day (OSD) program for Grades 7–10 students at risk of not meeting provincial math expectations. Detracking aims to extend opportunities to all students (Mehan, 2015), but the broader range of abilities in these classrooms presents complex challenges (Parekh et al., 2021), revealing disparities in student readiness and hindering engagement and progress for students lacking foundational knowledge (Ree et al., 2023).

OSD programs aim to strengthen numeracy foundations, enabling students to better access and succeed in their math courses (Alegre et al., 2019). This study explores how these programs are implemented to meet student needs and support equitable outcomes. How do educators at one school site implement and adapt an OSD program to address math learning gaps in a detracked context? What implementation factors do educators identify as facilitating or constraining the OSD program's effectiveness?

This study is part of a broader Grade 9 mathematics study in a large urban district involving approximately 40 schools and 60 teachers. That study revealed widespread student learning gaps, prompting closer examination of OSD implementation through semi-structured interviews with a teacher, principal, and district learning coach.

Literature Review

Research demonstrates that well-designed OSD programs can significantly improve math achievement for struggling students. Key features associated with positive outcomes include small student-to-teacher ratios, frequent sessions, curriculum alignment with in-school instruction, and well-prepared teachers who build strong student connections (Nickow et al., 2023; Gallagher et al., 2022; White et al., 2023; Alegre et al., 2019). Early intervention and summer intensive sessions yield particularly strong results (Knopf et al., 2015; Chappell et al., 2015), suggesting OSD programs can serve as strategic equity tools in detracked settings where classroom heterogeneity complicates targeted instruction (Gortazar et al., 2023).

However, research remains heavily weighted toward whether programs work while providing limited insight into how they are implemented in practice (Munter et al., 2015). Notably, the existing OSD literature has largely developed in tracked or mixed contexts where students receiving supplementary support have already been identified. OSDs developed in response to detracking present a distinct challenge: they must address the full range of foundational gaps that detracking surfaces, serve a larger and more heterogeneous population, and align with grade-level curriculum all students are now expected to access. Whether design features associated with effective OSDs in the existing literature translate to this detracked context remains an open question this study addresses.

Theoretical Framework

Program Implementation Theory (Boylan, 2025; Munter et al., 2015) provides a framework for examining how design, delivery, and contextual adaptations shape implementation processes. Key constructs include implementation fidelity (adherence to program design), local adaptation (modifications to fit context), and the tension between these competing demands. This tension is particularly salient in detracked settings, where OSDs must simultaneously address heterogeneous learning gaps while remaining aligned with grade-level curriculum — a dual demand that programs designed in tracked contexts were not built to meet.

Methods

This ongoing study employs a qualitative case study design with data collection across two time points during the 2025-26 school year, exploring the perspectives of a learning coach, principal, and teacher at the implementation site. Each participant is being interviewed twice: once shortly after program commencement (completed Fall 2025) and again at year-end (scheduled June 2026), for a total of six semi-structured interviews of approximately 45-60 minutes each. These roles represent different implementation levels: school-based delivery (teacher), school leadership (principal), and system-level support (learning coach). Complementary document analysis will capture formal structures guiding implementation. Ethics approval was obtained from the University of Toronto and the participating school board. Trustworthiness is being enhanced through data triangulation, member checking, and maintenance of an audit trail.

Data Analysis

Interviews were transcribed and imported into Nvivo 12 for analysis. Following an inductive thematic approach, data were open-coded to identify emergent themes related to program implementation, coaching practices, and perceived effectiveness. Consistent with Program Implementation Theory, the coding framework attended to three constructs: implementation fidelity, local adaptation, and fidelity-adaptation tension, applied deductively alongside inductive open coding. Trustworthiness was enhanced through triangulation of interview and document data and member checking with participants.

Results

Analysis of Wave 1 interviews revealed four major themes: (1) recruitment and accessibility challenges, (2) structural constraints, (3) enabling factors, and (4) professional learning and resource needs.

Recruitment and Accessibility Challenges

Getting students to participate emerged as a significant barrier. The teacher described intensive outreach: "I had to make many phone calls to parents to entice them and to promote the program. With persistence, more students joined" (Interview, October 14, 2025). The learning coach noted system-level difficulties expanding beyond current sites. Accessibility barriers further limited participation, with the learning coach emphasizing the need for virtual and evening options for students with transportation, caregiving, or work responsibilities. That those with the greatest needs face the greatest barriers suggests structural features are undermining fidelity to the program's equity goals.

Structural Constraints

While OSD classes were smaller than regular instruction, ratios remained high for intensive remedial work, with "a minimum of 15" (Learning Coach, September 29, 2025), limiting individualized attention. Resource limitations constrain impact, and program purpose confusion further complicated implementation: "The parameters are to fill numeracy gaps…this is not a homework club" (September 29, 2025). These findings suggest structural fidelity (appropriate ratios, resources, and shared understanding) is a prerequisite for the instructional fidelity the program requires.

Enabling Factors

Despite constraints, participants identified several factors supporting student learning. The teacher reported gains through small-group peer learning and noted that students not attending day school regularly were attending the OSD program. A local bakery provided breakfast to support attendance, and a community volunteer extended instructional capacity. These grassroots adaptations illustrate how school-level actors supplement formal structures when full fidelity is not feasible, a dynamic central to the fidelity-adaptation tension.

Professional Learning and Equity Considerations

Teacher preparation emerged as a critical fidelity risk, with the learning coach recommending math Additional Qualifications to strengthen instructional understanding (Interview, September 29, 2025). Equity considerations shaped enrollment, with priority given to students with the greatest need. However, the absence of systematic data tracking limits the program's ability to demonstrate impact and justify future funding. Across participants, a tension emerged between program potential and implementation reality.

Conclusion

Preliminary findings show that while OSD programs show promise, implementation faces significant barriers echoing concerns in the broader literature. High student-to-teacher ratios undermine the individualized attention Nickow et al. (2023) and Alegre et al. (2019) identify as central to effective supplementary math programs. Teacher preparation gaps align with Munter et al.'s (2015) observation that implementation research underestimates the instructional capacity interventions require. Community-level adaptations reflect what Boylan (2025) describes as school-level actors compensating for design shortfalls. These challenges are inseparable from the detracked context: by expanding the population requiring support while removing existing identification structures, detracking places demands on OSD design the existing literature has not yet addressed. Wave 2 data will examine whether these barriers persist as the program matures. While generalizability is limited by the single-site design, this study contributes critical insights into how supplementary interventions function within detracked systems, pointing toward design principles the field has yet to develop.

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FUNDS OF KNOWLEDGE IN MATHEMATICS EDUCATION: A SCOPING REVIEW OF APPLICATION IN INSTRUCTION

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Funds of knowledge provide an avenue for bridging the gap between students' lived experiences and school mathematics. In this study, we examined research trends on funds of knowledge in mathematics education, focusing on empirical studies. A total of 63 articles were identified through ERIC and Google Scholar, screened using established criteria, and analyzed thematically. Findings indicate that these studies primarily center on task and curriculum design, community and family engagement, and teacher capacity building. Studies involving classroom instruction were mainly conducted at the elementary and middle school levels and commonly used qualitative, participatory, and community-based approaches.

Keywords: Culture and Asset-Based Pedagogies; Equity, Inclusion, and Diversity

The mathematics classroom in the United States is increasingly diverse with the growth of K–12 student populations (US Government Accountability Office, 2022). However, instruction remains standardized, creating a disconnect between school mathematics and students' lived experiences and contributing to persistent achievement gaps (Goethe & Colina, 2018). When teachers do not understand students' contexts, they may struggle to address diversity, affecting student performance (Clement et al., 2020; Insorio, 2024). The funds of knowledge framework (Moll et al., 1992) addresses this challenge by centering home and community (Moll et al., 1992; Velez-Ibanez, 1998; Moje et al., 2004). In mathematics education, it supports engagement by linking school mathematics to these resources, including everyday practices such as budgeting and cooking (Civil, 2007; Turner & Drake, 2016; Hurtado, 2018; Napoli & Purpura, 2018). Scholars note that such connections can strengthen participation and potentially improve achievement through more meaningful home–school integration.

Although widely studied in general education, with Moll et al. (1992) highly cited (Denton & Borrego, 2021) and contributions such as Gonzalez (2005), mathematics education lacks a focused synthesis. Existing scoping reviews cover education and STEM broadly (Denton & Borrego, 2021; Llopart & Esteban-Guitart, 2018) but not mathematics. Given its relevance to learning (Hurtado, 2018; Napoli & Purpura, 2018), this study conducts a scoping review of empirical studies to examine how funds of knowledge is applied and identify research gaps. It is guided by the question: What are the characteristics of studies that have applied the funds of knowledge framework in mathematics education, including those situated specifically in classroom instruction?

Funds of Knowledge

The concept of funds of knowledge emerged from work in multilingual, transnational, and borderland communities in the United States. Rodriguez (2013) noted that it was proposed to challenge deficit thinking, racial discrimination, and unequal power relations in education by

recognizing the strengths of multilingual communities. Velez-Ibanez (1988) defined funds of knowledge as the skills and information households use for survival, while Moll et al. (1992) described it as "historically accumulated and culturally developed bodies of knowledge and skills" (p. 134). These resources are embedded in households and communities. Gonzalez (2005) further conceptualized funds of knowledge as cultural capital developed over time, including labor skills, family interactions, and shared experiences from formal and informal activities.

Methods

This study adopts the scoping review framework proposed by Arksey and O'Malley (2005). Scoping reviews are used to map literature on emerging research topics (Mak & Thomas, 2022) and to examine the range and nature of research activities within a field (Arksey & O'Malley, 2005). In this study, the framework was used to investigate the characteristics of funds of knowledge research in mathematics education. Arksey and O'Malley (2005) outlined five stages of the process: (a) identifying the research questions, (b) identifying relevant studies, (c) selecting studies, (d) charting the data, and (e) collating, summarizing, and reporting the results.

Identification and Selection of Studies

Studies were *included* if they were (a) empirical, (b) grounded in the funds of knowledge framework, (c) focused on mathematics, (d) published between 2015–2025, and (e) published in peer-reviewed journals or conference proceedings, to capture recent developments in mathematics education research. Studies were *searched* through ERIC and Google Scholar using the keywords "funds of knowledge" and "mathematics," and the first 20 results from each database were *screened* by title and abstract due to practical constraints in manual screening. While this approach may not capture the full scope of relevant literature, it will be expanded in later phases of the study.

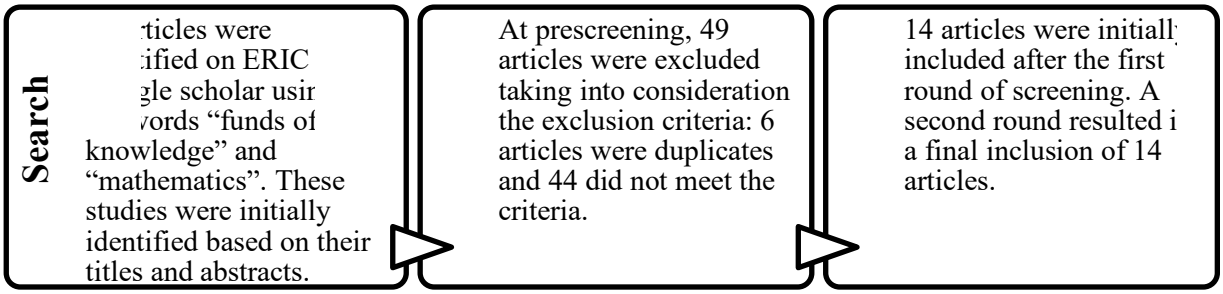


Figure 1: Article Search and Selection Procedures

Data Charting and Analysis

Consistent with Arksey and O'Malley's (2005) scoping review framework, each of the 14 included articles was read in full and open-coded by the research team. Codes captured multiple dimensions, including study purpose, theoretical framing, participants, methodological approach, mathematical content or tasks, and key findings. These codes were then compared and iteratively grouped into broader themes aligned with the exploratory aims of the scoping review.

Results

Fourteen studies met the inclusion criteria for the scoping review. Thirteen were journal articles and one was a conference proceeding, and all were analyzed in relation to the research questions.

Characteristics of Studies Applying Funds of Knowledge in Mathematics Education

To address the research question, we examined studies applying the funds of knowledge framework in mathematics education. The purpose, theoretical framing, methods, analysis, and findings of the 14 studies were reviewed, yielding three themes: (a) mathematics tasks and curriculum design, (b) family and community engagement, and (c) teacher capacity building.

Mathematics tasks and curriculum design. One major characteristic of studies applying funds of knowledge in mathematics education focused on mathematics tasks and curriculum design, with seven studies aligned to this theme. These studies connect school mathematics to students' lived experiences and cultural backgrounds by embedding learning in familiar contexts rather than treating mathematics as culturally neutral or abstract. Ma (2016) contrasted traditional instruction with "disruptions" that bridge home and school knowledge by taking geometry outdoors, where students used everyday resources such as language and body movement to explore geometric shapes. Similarly, Hunter and Sawatzki (2019) developed a contextualized problem-solving task for Pacific Islander students involving sharing the cost of a family meal, integrating division with cultural practices of fairness and decision-making. Turner and Drake (2016) emphasize curriculum flexibility to support multiple strategies drawing on students' out-of-school experiences. Overall, these designs make mathematics more inclusive and highlight its relevance in real-world contexts.

Family and community engagement. Instead of viewing family and community as outsiders to school mathematics, some studies position family, parents, and community members as partners in the learning process. Five studies highlight this role, strengthening home–school connections by linking home activities to classroom mathematics. Suh and Calabrese (2025) illustrate this by having students bring family photographs and artifacts as starting points for mathematical questions and discussion, an example of mathematizing everyday family life. Gutiérrez et al. (2024) similarly used workshops with Latina mothers to engage community knowledge, grounding tasks in caregivers' funds of knowledge such as crafts, cooking, and budgeting. These workshops showed how family practices can be explicitly connected to mathematical practices (Suh & Calabrese, 2025).

Teacher capacity building. Another important theme in the literature is capacity building for mathematics teachers. Many teachers struggle to integrate family, cultural, or community funds of knowledge into instruction. Turner and Drake (2016) found that incorporating students' cultural and community experiences supports diverse learners' participation and helps teachers better address classroom diversity. Kalinec-Craig and Zavala (2019) further support teacher capacity through the community mathematics exploration (CME), which uses videos and texts reflecting students' cultural practices to help teachers design lessons grounded in learners' community experiences.

Characteristics of Studies Applying Funds of Knowledge to Mathematics Instruction

The broader thematic patterns indicate that funds of knowledge research in mathematics education is closely aligned with teaching and instruction, particularly through mathematical task and curriculum design. Results from research question one suggest these studies primarily focus on task design and curriculum and are strongly connected to mathematics instruction. Given the potential of funds of knowledge to improve instruction and performance (Hurtado, 2018; Napoli & Purpura, 2018), we further examined this context by analyzing a subset of 10 classroom instruction studies, organized into three categories based on methodological approach, participants, and mathematical examples used.

Methodological approaches. Seven studies used qualitative, community-based designs, including photovoice (Hunter, 2022), home math interviews (Williams et al., 2020), and community mathematics walks (Kalinec-Craig & Zavala, 2019; Ma, 2016). Six studies used participatory approaches that positioned families and students as active contributors, and five incorporated classroom artifacts and student-centered tasks grounded in authentic, real-world contexts. These modeling activities drew on learners' mathematical knowledge and community norms in decision-making (Bennett et al., 2023; Gutiérrez et al., 2024). Most studies used qualitative methods, with only two employing mixed-methods designs (Suh & Calabrese, 2025), enabling examination of how students' funds of knowledge are contextualized in mathematics classrooms.

The participants. The funds of knowledge literature in mathematics education spans multiple populations, but most studies focus on elementary and middle school contexts. Four studies involved elementary learners (Bennett et al., 2023; Hunter, 2022; Suh & Calabrese, 2025; Williams et al., 2020), and four included middle school students (Hunter, 2022; Hunter & Sawatzki, 2019; Ma, 2016; Williams et al., 2020). Studies also examined preservice and in-service teachers (Hunter & Sawatzki, 2019; Kalinec-Craig & Zavala, 2019; Kelley, 2020; Suh & Calabrese, 2025; Gutiérrez et al., 2024), while some involved parents (Williams et al., 2020; Gutiérrez et al., 2024). Overall, the concentration of studies at the elementary and middle school levels highlights the importance of community and family experiences in supporting mathematics learning during these years.

Mathematical examples or activities and their content domains. The studies incorporated mathematical tasks grounded in learners' family and community practices, including financial decision-making (e.g., budgeting and shopping), food preparation, and fair-share or proportional reasoning (Bennett et al., 2023; Hunter & Sawatzki, 2019; Ma, 2016). These tasks positioned students' cultural and household experiences as authentic contexts for mathematical learning. Numbers and operations were most common, appearing in nine of ten studies, followed by ratio and proportional reasoning (seven), measurement (six), and data modeling (five). Only one study addressed geometry explicitly (Ma, 2016), though visual representations appeared in studies using artifacts and photovoice methods (Hunter, 2022; Kalinec-Craig & Zavala, 2019). No study focused explicitly on algebra, despite implicit algebraic reasoning.

Discussion and Implication

The findings of this review indicate that mathematics education studies using funds of knowledge primarily focus on mathematical task and curriculum design, community and family engagement, and teacher capacity building. This aligns with Llopart and Esteban-Guitart (2018), who identified curriculum contextualization and family–school relationships as key applications of funds of knowledge in general education, suggesting mathematics education has largely adapted rather than developed discipline-specific approaches. In contrast to Denton and Borrego (2021), teacher capacity building is more prominent in mathematics education.

Regarding classroom instruction, the dominance of qualitative, participatory, and community-based methods mirrors broader research but limits evidence for achievement claims (Hurtado, 2018; Napoli & Purpura, 2018). Mixed methods studies would strengthen future work. The concentration at elementary and middle levels, with limited algebra-focused research, highlights a gap that may hinder students' transition to abstract reasoning, making expansion into secondary contexts important. Although teacher capacity building is common, it often relies on isolated interventions such as community mathematics exploration models (Kalinec-Craig & Zavala, 2019; Williams et al., 2020), indicating a need for more scalable approaches.

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MAPPING COMPUTATIONAL THINKING IN MODULAR ARITHMETIC PROBLEM-SOLVING: EPISTEMIC NETWORK ANALYSIS

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Using Epistemic Network Analysis, this research examined how problem-solvers (prospective elementary school teachers) integrate computational thinking (CT) practices in their solutions to modular arithmetic tasks. Analysis of written solutions and accompanying explanations (n = 22) revealed three statistically distinct profiles of CT engagement: Synergistic and Abstract (associated with optimization-oriented strategies), Integrated and Procedural (associated with verification-oriented strategies), and Narrow and Fragmented (associated with trial-and-error oriented strategies). Results demonstrate clear links between solution sophistication and the patterns of interaction among CT practices. The study offers a new framework for scaffolding CT in mathematical problem-solving, suggesting that moving students towards the mathematical efficiency of solutions requires fostering synergies between abstraction and pattern recognition.

Keywords: Computational Thinking, Problem-Solving, Cognition, Undergraduate Education

Introduction

Computational Thinking (CT) constitutes one of the fundamental problem-solving competencies for participating in today's society (Li et al., 2020; Shute et al., 2017; Tariq et al., 2025). Wing (2006) initially introduced CT as a problem-solving approach that draws on practices from computer science. Since then, CT has been recognized as encompassing a broad range of skills and ways of thinking shared across STEM disciplines and applied to everyday problem-solving, rather than a computer science-specific competency (Kallia et al., 2021; Listjaji & Molnar, 2025; Weintrop et al., 2016).

In recent years, mathematics education researchers have begun to generate an understanding of the relationship between CT and mathematical problem-solving processes. Some researchers examined CT practices that problem-solvers frequently draw on during mathematical problem-solving (Cui et al., 2023; Hong et al., 2021; Ng & Cui, 2021). Others (Labusch et al., 2019; Lehmann, 2024, 2025; Wu et al., 2024; Ye et al., 2024) focused on CT practices that problem-solvers activate during different phases of problem-solving, such as problem understanding, planning, carrying out the plan, looking back, or in conjunction with different problem-solving heuristics (e.g., drawing a diagram, working backward, examining special cases).

While past research provides insights into CT practices in problem-solving contexts, it falls short of explaining how problem-solvers connect the CT practices they draw on as they generate problem solutions. To date, limited research has examined how different CT competencies, such as problem decomposition, abstraction, algorithmic thinking, pattern recognition, and debugging, interact and support one another in the problem-solving process. Ye and colleagues (2024) argued that a robust understanding of the relationship between CT and mathematical thinking in problem-solving requires an understanding of possible links between CT and different solution strategies. This study responds to this call by utilizing Epistemic Network Analysis (ENA) to explore patterns of problem-solvers' CT engagement in the problem-solving context. Drawing on written task solutions, the following question was addressed: *What does the analysis of the*

strategies problem-solvers use to solve contextual modular arithmetic tasks reveal about their engagement in CT?

Theoretical and Analytical Frameworks

Epistemic Frame Theory

Epistemic Frame Theory (Shaffer, 2006) builds on the concept of a *frame*, an organizational principle introduced by Goffman (1974) that reflects how individuals filter information and organize their understanding of the given situation to guide their future actions. Shaffer expanded the concept of the epistemic frame using it as a window into an individual's thinking in domain-specific situations. He subsequently developed Epistemic Network Analysis (ENA), an analytical tool for studying thinking processes. ENA shifts the focus of analyzing one's cognitive engagement from examining isolated pieces of information about one's thinking to exploring the structure of connections among identifiable cognitive actions. It allows the identification and quantification of cognitive processes and the visualization of their relationships (Shaffer, 2017; Shaffer et al., 2016). The method has been widely used in educational research to explore how knowledge is constructed or shared (e.g., Bressler et al., 2019; D'Angelo et al., 2020; Orill & Brown, 2023; Su et al., 2023; Ye & Zhou, 2022).

ENA models connections between the identified data features (codes) by representing their co-occurrence, both visually and statistically. As such, ENA reveals the underlying structure of cognitive engagement within a dataset. This modeling approach is particularly useful for studying engagement in CT because it captures the dynamic integration of multiple CT practices as they unfold during the problem-solving activity.

Computational Thinking

The analytical framework for this research draws on existing CT literature to operationalize CT as an interdependent set of practices (Grover & Pea, 2012; Shute et al., 2017; Weintrop et al., 2016). These include abstraction and problem reformulation, problem decomposition, algorithmic thinking (algorithm design), pattern recognition, and debugging. These aspects of CT are central to mathematical problem-solving, encompassing processes such as generating procedures, recognizing patterns, evaluating strategies, and constructing generalizable rules (Lehmann, 2025; Weintrop et al., 2016). Table 1 defines the CT dimensions examined in this study.

Table 1: Dimensions of CT

CT Dimension	Evidence in problem-solving
1. Abstraction/Problem Reformulation (ABS)	Identifies essential conditions and converts problem information into a mathematical structure.
2. Decomposition (DEC)	Breaks the problem into smaller steps (subproblems).
3. Algorithmic Thinking (ALG)	Develops a repeatable, logical strategy or incremental solution plan (rule, sequence of steps).
4. Pattern Recognition (PAT)	Identifies helpful regularities.
5. Debugging (DEB)	Checks the consistency of the solution, corrects, and revises as needed.

Method

Study Context and Participants

The participants were 22 undergraduate students (all non-STEM majors) enrolled in the mathematics *Problem Solving and Reasoning for Teachers* course at a Midwestern University. They were all in their first or second year at the University and volunteered their participation. The course was structured to provide the students with problem-solving experiences across a broad range of problems and problem-solving strategies.

Modular Arithmetic Problems

Contextual modular arithmetic problems provide a revealing context for the study of CT because they uniquely prompt engagement with CT practices by requiring decomposition of the problem context, abstraction, algorithm development, monitoring one's reasoning, identification of errors, and iterative refinement of strategies (Cripa et al., 2025). Solutions to contextual modular arithmetic problems demonstrate not only problem-solvers' engagement with different CT practices but also their understanding of mathematical structure. Furthermore, these types of problems can also expose conceptual gaps when the algorithms developed to solve them break down. Figure 1 shows the problem used in this study.

> **Corporation Meeting**. Dayna was in charge of a large corporation meeting. She wanted everyone to work in groups, and she really wanted all the groups to be the same size. She started out by organizing everyone into groups of 6, but one person was left out and didn't have a group. So she started over again and organized everyone into groups of 7, but again, there was one person left out. So she tried again, this time organizing everyone into groups of 8, and this time, there were five people left out. So, finally, she organized everyone into groups of 5, and nobody was left out. There were less than 100 people at the meeting. How many people were there?

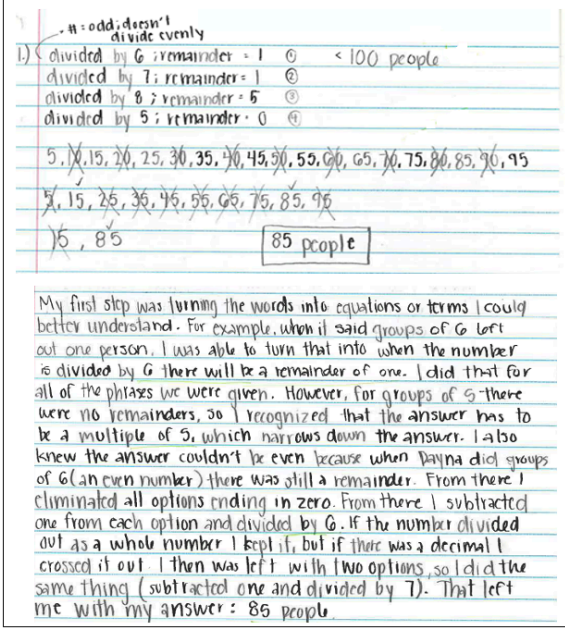
Figure 1: Modular Arithmetic Task

Data and Data Analysis

The data consisted of written responses (problem solutions and accompanying explanations, $n = 22$) to the task presented in Figure 1. The first stage of data analysis was grounded in qualitative content analysis and analytical inductive methods (Miles et al., 2018). Using open coding, each response was annotated to delineate the overall solution strategy. Each solution was also coded for the correctness of the answer. The author and the research assistant discussed their independent observations until they reached 100% agreement on the overall solution strategy profiles identified across the data. These solution profiles were then labeled as (a) *optimization-oriented strategies,* (b) *verification-oriented strategies,* and (c) *exploration-oriented strategies*. Each solution profile is later described in the results section.

The second stage of the analysis focused on creating and validating ENA models to illustrate and describe the patterns of interactions among the CT practices documented in each response. This stage involved utilizing the moving-stanza-window method (Shaffer et al., 2016) and parsing each response into segments (stanzas) that reflected distinct problem-solving moves. Each segment corresponded to a single step in the solution process or a distinct mathematical claim. Using a priori defined codes consistent with the analytical framework (Table 1), each segment was coded for evidence of CT practices. Figure 2 provides a sample response and coding to illustrate this stage of the analysis. Across the entire data corpus, 127 stanzas were

delineated and coded. Finally, the ENA web tool (Marquart et al., 2018) was used to create and compare the models of CT engagement for problem-solvers with different solution profiles. The unit of analysis for this step was the solution profile, that is, the group of participants with the same strategy profile. In group model development, the ENA conversation was defined as all lines of code associated with each participant within the given solution profile (optimization-oriented, verification-oriented, or exploration-oriented). Models were developed by quantifying code co-occurrence within conversations, producing weighted co-occurrence networks and associated visualizations for each unit of analysis in the data. CT network difference graphs were constructed (by calculating differences in the weights of corresponding connections across solution profile groups) to provide a visual comparison of CT engagement for each solution profile group. The Mann-Whitney *U* test was used to examine differences among the CT networks for optimization-oriented, verification-oriented, and exploration-oriented groups.



Response Segments	Coding	Justification
1. My first step was turning the words into equations I could better understand. For example, when it said groups of 6 left one person, I was able to turn it into when the number is divided by 6, there will be a reminder of 1...."	ABS	**Abstraction:** Translates contextual information to a mathematical description using mathematical symbols
2. However, for groups of 5 there were no reminders, so I recognized that the answer has to be a multiple of 5.	ABS DEC	**Abstraction:** Identifies a global property that describes the numerical result (number has to end in 0 or 5) **Decomposition:** Narrows the problem space to a subset of multiples of 5.
3. I also knew that the answer couldn't be even because, when Dana did groups of 6, there was still a reminder.	ABS PAT DEC	**Abstraction:** Further considers the global property of the answer (answer can't end in zero) **Pattern:** Recognizes that the answer will end in 5 **Decomposing:** Eliminates unnecessary entries and further reduces the problem space to odd multiples of 5
4. From there, I subtracted 1 from each option and divided by 6...	ALG DEB	**Algorithm:** Identifies and applies a specific iterative sequence of steps (subtract 1, divide by 6) **Debugging:** Uses a whole number vs. a decimal as a self-check. Checks the final two results against the remaining conditions.

Figure 2: Sample Response and Coding (Optimization-Oriented Strategy)

Results

Solution Profiles

The analysis revealed three distinctive solution strategy profiles across the responses: (a) *optimization-oriented*, (b) *verification-oriented*, and (c) *exploration-oriented*.

Problem-solvers whose solutions were categorized as *optimization-oriented* ($n = 8$) developed the most computationally efficient strategies. This group of problem-solvers drastically reduced the problem space early in the process by strategically reasoning about the properties of potential solution candidates and generating new rules. By doing so, they rapidly advanced towards the final answer. All solutions within this group were correct.

Problem-solvers whose strategies were categorized as *verification-oriented* ($n = 8$) utilized a highly organized procedural approach. They documented and validated the problem constraints, revisited assumptions, and recalculated results to ensure accuracy of their work at every step. They consistently checked intermediate results to maintain internal consistency and ensure

alignment of their work with the problem constraints. Also in this group, all solutions were correct.

Exploration-oriented strategies (n = 6) were characterized by brute-force or trial-and-error approaches. Problem-solvers in this group documented their iterative reasoning but often lacked a cohesive framework to guide their work. For example, some participants examined numbers against problem constraints (e.g., tested numbers from a list of 1–99). Other participants generated separate lists of multiples of 6, 7, 8, and 5 and examined them independently, checking each constraint without making connections across their lists. Their solutions included evidence of strategy adjustments and changes in the representations used only after discovering an error or a dead end in their arithmetic calculations. Within this group of responses, three included the correct answer, and three did not.

Engagement in CT Practices

The ENA analysis revealed statistically significant differences in CT engagement among problem-solvers with different solution profiles. As shown in Figure 3, these differences are indicated by the non-overlapping 95% confidence intervals surrounding the centroids, which represent normalized mean CT networks for problem-solvers in each solution profile group. The horizontal dimension of ENA space (SVD1, 71.9% variance) explains the shift in co-occurrence of CT practices from optimization-oriented to exploration-oriented solution strategies. The vertical dimension (SVD2, 9.3% variance) captures co-occurrences of CT practices that distinguish the verification-oriented group from both the optimization and exploration-oriented groups.

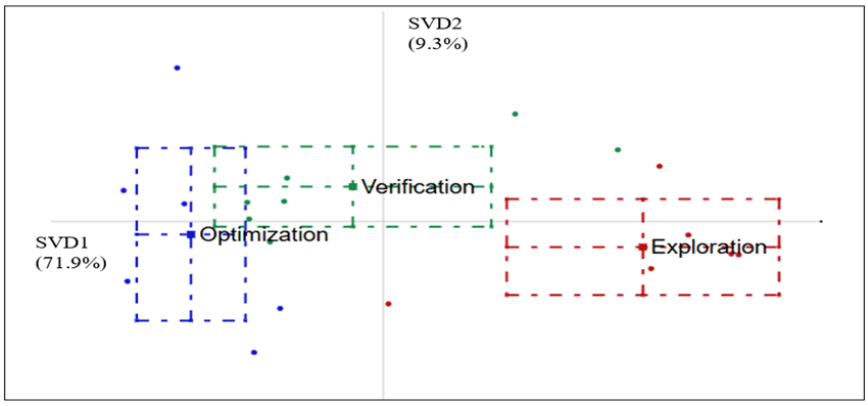


Figure 3: Network Centroid Plots of CT Engagement for Optimization (Blue), Verification (Green), and Exploration (Red) Solution Profile Groups

The patterns of CT engagement among problem-solvers with optimization-, verification-, and exploration-oriented solution strategies are further illustrated in Figure 4 through their respective CT engagement networks. The nodes of each network graph correspond to the analytical codes (Table 1), and the edges reflect the relative frequency of co-occurrence (connections) between the codes within each solution-profile group. The strongest co-occurrences among the five CT practices are visualized by edge thickness, with the thickest lines representing the most frequent connections.

The ENA revealed that the *optimization-oriented* group (Figure 4a) used a highly connected network of CT practices, dominated by abstraction, pattern recognition, decomposition, and debugging. This group of problem-solvers likely drew on abstraction to discover new patterns

that helped them "simplify" the solution process. By leveraging synergies among the CT practices in their network, this group of problem-solvers bypassed the frequent reliance on algorithmic steps, enabling them to generate computationally efficient strategies.

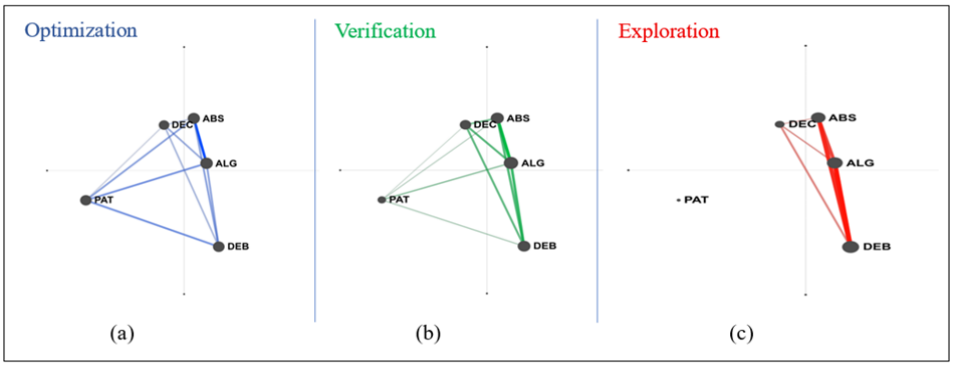


Figure 4: CT Networks for Each Solution Profile Group

The *verification-oriented* group (Figure 4b) also engaged in a range of highly integrated CT practices (abstraction, decomposition, algorithms, and debugging). Their network, however, was largely disconnected from pattern recognition, as evidenced by weaker links between pattern recognition and the remaining four practices. The weaker link between pattern recognition and abstraction in this group suggests that while they may have used abstraction to define problem constraints, they did not use it to discover any new rules within the problem. Consequently, their systematic solutions stemmed from careful implementation of procedures and a systematic focus on result validation.

Finally, the ENA revealed that the *exploration-oriented* group used a narrower and more fragmented set of CT practices (Figure 4c) that primarily included abstraction, algorithmic thinking, and debugging. The strongest co-occurrence within their network was between abstraction and algorithmic thinking, and between algorithmic thinking and debugging. As illustrated in Figure 4c, within this group, connections among decomposition and abstraction, algorithmic thinking, and debugging were weak, and their CT network was entirely disconnected from pattern recognition. The absence of pattern recognition and a limited focus on decomposition might have led to a trial-and-error or brute-force approach, given that this group was not associating any new rules with the problem space.

Group Comparisons of CT Engagement

Figure 5 includes an ENA-generated comparison networks that visualize differences in CT engagement between the optimization and verification and the optimization and exploration-oriented groups. The engagement in CT practices within the optimization group (Figure 5a) was dominated by abstraction, pattern recognition, decomposition, and debugging. The strong connection between abstraction and pattern recognition suggests that the problem-solvers in this group likely used abstraction to discover new patterns, which helped them 'simplify' the problem space rather than merely represent the problem mathematically. By leveraging synergies among the CT practices in their network, they bypassed the frequent reliance on algorithmic steps, as documented by the lower density edges to a node representing algorithmic thinking in Figure 4a and the absence of a connection to algorithmic thinking in Figure 5a. In contrast, the engagement in CT practices within the verification-oriented group was dominated by algorithmic thinking

and debugging (Figure 5a). The lack of a strong link between pattern recognition and abstraction in the verification group suggests that these problem-solvers may have relied on abstraction to define problem constraints but did not use it to discover any new rules within the problem. Consequently, they focused on systematic procedures and careful verification of results rather than finding logical shortcuts.

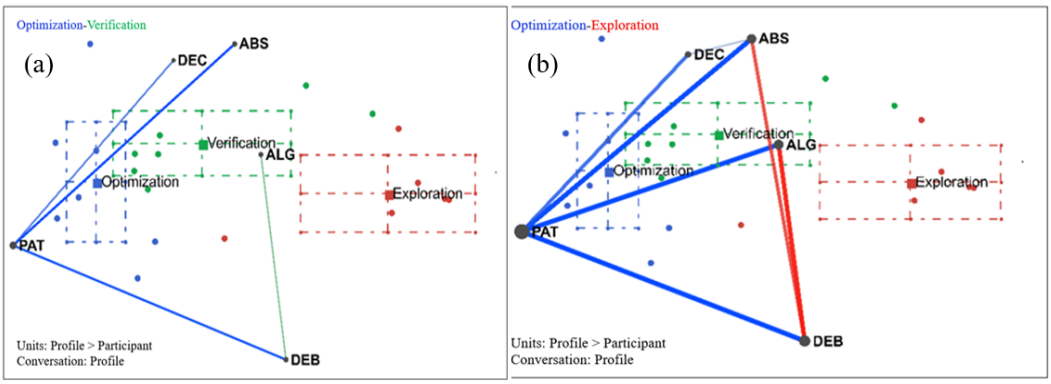


Figure 5: Comparison Graphs Illustrating Differences in the Co-Occurrence of CT Practices Between the (a) Optimization (Blue) and Verification (Green), and (b) Optimization (Blue) and Exploration-Oriented (Red) Groups

A comparison graph of engagement in CT practices between the optimization- and exploration-oriented groups (Figure 5b) shows that the exploration group showed no engagement in pattern recognition. As illustrated in Figure 4a, in the optimization-oriented group, pattern recognition served as a central hub, mediating connections to abstraction, decomposition, algorithmic thinking, and debugging. In contrast, the network illustrating CT engagement for problem-solvers in the exploration group (Figures 4c and 5b) was limited and clearly disconnected from pattern recognition. This structural network difference indicates that, unlike the optimization group, problem-solvers with the exploration-oriented strategies predominantly used abstraction to represent the problem constraints (mathematize the verbal description of the problem) rather than as a tool for generating ideas, deriving new rules and shortcuts, and gaining insights into the problem structure. The lack of engagement in pattern recognition likely explains this group's reliance on guess-and-check and brute-force approaches in the absence of the structural anchors needed to move beyond guessing-and-checking.

A visual comparison of engagement in CT practices between the verification and exploration groups is shown in Figure 6. Pattern recognition remained strongly connected to all other CT practices in the verification group (Figure 4b and Figure 6), but, as discussed earlier, the problem-solvers in the exploration-oriented group did not engage in this practice (Figure 4c and Figure 6).

The Mann-Whitney *U* test confirmed that the co-occurrence of CT practices differed significantly at the 0.05 level across all group comparisons. Significant differences were between the optimization-oriented (Mdn = –0.26, $n = 8$) and verification-oriented (Mdn = –0.12, $n = 8$) groups; $U = 5.00$, $p = 0.01$, $r = 0.82$. The same was true when comparing the co-occurrence of CT practices between the optimization-oriented and exploration-oriented (Mdn = 0.41, $n = 6$) groups; $U = 0.001$, $p < 0.01$, $r = 1.00$, and between the verification-oriented and exploration-oriented groups; $U = 2.00$, $p < 0.001$, $r = 0.90$. In each case, the effect size was large, indicating

substantial differences in CT engagement patterns among problem-solvers across these three solution profiles.

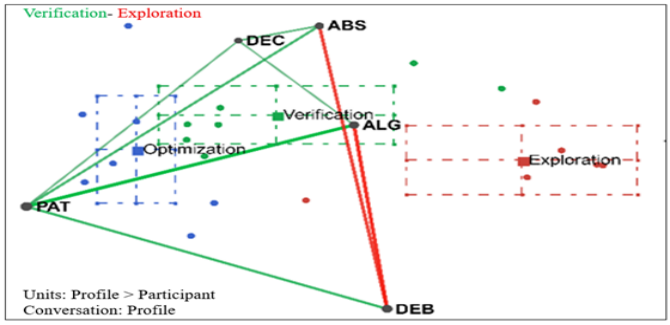


Figure 6: Comparison Graph Illustrating Differences in the Co-Occurrence of CT Practices Between the Verification-Oriented (Green) and Exploration-Oriented (Red) Groups

Conclusion

This study shifts the focus of past research from the examination of individual CT competencies that problem-solvers demonstrate to the examination of these competencies as interdependent networks, and offers insights into the potential trajectories of CT engagement in problem-solving contexts. This study also identifies the mechanisms that might support problem-solvers in increasing the efficiency of their strategies during complex problem-solving.

The results show that the optimization-oriented group's efficiency, specifically their ability to eliminate much of the computational labor, can be explained by a unique synergy among abstraction, pattern recognition, decomposition, and debugging. In their CT network, pattern recognition served as a central hub, providing cognitive shortcuts and structural insights into the problem, thereby supporting the computational efficiency of solutions. The verification-oriented group achieved accuracy through a highly connected synergy of algorithmic thinking and debugging, which served as the primary driver of their work. The exploration-oriented group's network of CT practices was narrower and lacked engagement in pattern recognition, likely leaving these problem-solvers without a robust framework for their work.

While limited to one type of problem and a small sample, the results point to a critical synergy between the strength and nature of connections across CT practices and the problem-solving strategies. Further research should use ENA to investigate whether the identified network profiles of CT engagement remain stable when problem-solvers work on tasks of varying complexity or in different mathematical domains. Future research should also apply ENA to real-time transcripts of problem-solving sessions (think-aloud protocols and observations) to further visualize how problem-solvers' engagement in CT evolves across different stages of the problem-solving process. Despite these limitations, the current study serves as a critical step towards understanding how problem-solvers' CT capacities are networked and provides a bridge to understanding CT integration in mathematics education.

Acknowledgments

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BRIDGING WORDS AND GRAPHS: THIRD-GRADE STUDENTS' MATHEMATICAL REASONING WITHIN MULTIMODAL TEXTS

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This study investigates how third-grade students interpret and integrate verbal and graphical information in a multimodal mathematical text. Grounded in theories of dual-coding, data display comprehension, and question-answer relationships, this research focuses on students' processes for answering questions using information from text, a pictograph, or both. Analyses of eye tracking data combine verbal protocols, fixation counts, dwell time, and visual pathways. Results show most students relied primarily on the pictograph and answered correctly when information was duplicated across modalities. However, students who depended solely on text were often incorrect even when adequate information was available. While most students successfully extracted data from the pictograph, few integrated information from the key, indicating limited information integration.

Keywords: Elementary School Education, Mathematical Representations, Eye Tracking, Multimodal Text

Objectives or Purposes of the Study

Multiple representations are an important part of mathematics learning, providing opportunities to apply skills to information in multiple embodiments. Lesh et al. (1987) highlighted verbal and pictorial representations alongside other mathematical representations embodying "critical features of mathematical constructs and actions, such as drawing diagrams and using words to show and explain the meaning of fractions, ratios, or the operation of multiplication" (National Council of Teachers of Mathematics [NCTM], 2014, p. 24). When students interpret and relate mathematical information presented in verbal and visual representations, their understanding deepens (NCTM, 2014). Further, text and trade books written for children, which can enhance engagement with mathematics (Larson & Rumsey, 2018; Radebaugh, 1981; Russo & Russo, 2018a, 2018b; Shatzer, 2008; Strain, 1969) are typically multimodal with verbal and visual modalities. Sometimes information is shared through one modality or the other, meaning that neither can be ignored if the full meaning of the text is to be constructed. In other cases, the same or complimentary information is shared through both modalities, so dual-coding of information is needed to support both comprehension of, and memory for text (e.g., Paivio, 1991; Sadoski & Paivio, 2013). In this study we examine students' mathematical understandings of multimodal text, asking: What sources and processes do third-grade students use to respond to mathematical questions about an informational text that can be answered using information from text, data display, or both?

Background and Conceptual Framework

Data displays are one important form of visual representation in mathematics education and present an interesting case because they can be viewed as a whole (e.g., combining information

from a table with that of the written text to answer a question) or as a compilation of information (e.g., combining information from different data points on a graph to draw a conclusion). When categorizing the kinds of questions that graphs can be used to answer, independent of text, Friel et al. (2001) reported three levels of data display comprehension: "an elementary level focused on extracting data from a graph (i.e., locating, translating); an intermediate level characterized by interpolating and finding relationships in the data as shown on a graph (i.e., integrating, interpreting), and an advanced level that requires extrapolating from the data and analyzing the relationships implicit in a graph" (p. 130). These levels coincide with Curcio's (1987) three levels: reading the data, reading between the data and reading beyond the data. Alongside these levels, we draw on Raphael and Au's (2005) Question–Answer Relationship (QAR) framework involving (i) right there, (ii) think and search (iii) author and me and (iv) on my own questions. From the QAR framework, we concentrate specifically on "think and search" questions, in which the information needed to construct an answer must be located in multiple parts of the text and synthesized. Parallel to this, we employ Curcio's notion of "reading between the data," which involves making comparisons or identifying relationships within a graph.

Methodology

Participants

Participants in this study were 31 third-grade students across three classrooms in a suburban, midwestern school. The school is generally considered to be academically strong, with 73% of third-grade students demonstrating proficiency on statewide math assessments (38% is the state average) and 59% doing so for English language arts (40% is the state average). The school is ethnically and culturally diverse; 47% of students identify as Asian or Pacific Islander, 35% as White, and the remaining 18% as Black, Hispanic, or two or more races. Prior to data collection, a warm-up task on sharing their thinking out loud was completed to ensure that all students were comfortable and understood how to engage in verbal protocols.

Materials

Materials used in this study included a modified two-page spread from the book *One Well: The Story of Water on Earth,* by Rochelle Strauss and Rosemary Woods (Illus.) (2007). The pages in Figure 1 were shown to students on a laptop screen, equipped with eye tracking and audio recording software to capture eye movements, fixations, and verbal explanations.

Data Collection

Data was collected using the portable EyeLink Duo eye tracker. Binocular gaze data were recorded at a sampling rate of 1000 Hz, with measurement error below 0.4%. The system recorded students' gaze behavior as they read and provided verbal protocols. Quantitative measures included the number, sequence, and duration of fixations across designated interest areas. Qualitatively, the software allowed us to merge audio and video to play back a session and watch the participant's eye gaze as they read and explore the 2-page spread. During each recorded one-on-one session, participants first explored the page spread for two minutes while thinking aloud about what they were viewing and reading. They then answered seven questions about the spread, two of which are analyzed in this study (i.e., 1. Which of the places mentioned here uses the most water? How did you figure that out? 2. What do you notice when you compare the average daily water use by people in India and North America?). If not provided spontaneously, students were also asked to explain their reasoning and sources for their answers.

Data Analysis

Data analysis started with transcribing deidentified audio recordings and separating students' verbatim answers (noting accuracy) from their explanations. Quantitative eye-tracking metrics

(e.g., fixation counts and dwell time in milliseconds) were extracted for each question using Data Viewer in the EyeLink Duo software. We also reviewed each eye-tracking video to document students' visual pathways, including areas of focus and viewing sequence. For triangulation, we began by analyzing the verbal protocols, which included both participants' answers to the questions and their explanations of their thought processes when doing so. Using this data, we drew preliminary conclusions about how participants answered the questions, then confirmed, refined, or revised those conclusions using eye-tracking videos showing where they looked while responding. We then compared our conclusions to the quantitative data (dwell time and fixation counts). The research team collaboratively analyzed all data to strengthen rigor and reliability, resolving disagreements through systematic review to reach full consensus.

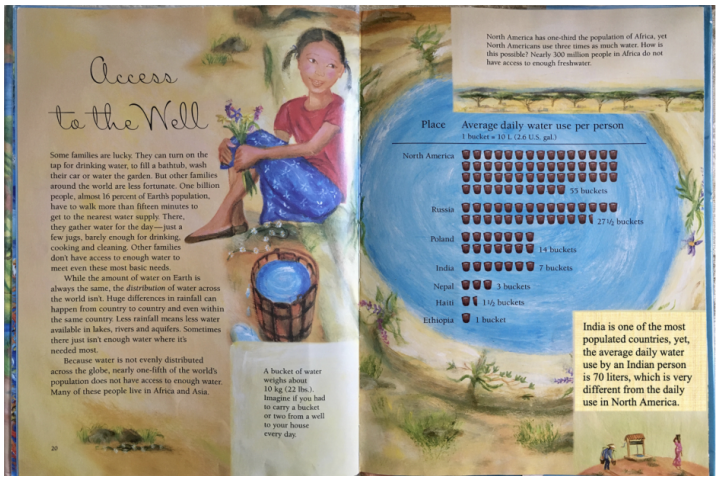


Figure 1: Modified material from One Well written by Rochelle Strauss and illustrated by Rosemary Woods is used by permission of Kids Can Press Ltd., Toronto. Text © 2007 Rochelle Strauss, Illustrations © 2007 Rosemary Woods

Results and Discussion

Question 1: Which of These Places Mentioned Here Uses the Most Water?

Students could use text or pictograph to provide an answer to this question. In either case, the question could be a "right there" question; the correct answer did not require integration of information from multiple places. Twenty-seven out of 31 responses were correct; one was partially correct, and three were incorrect. Twenty-five students who answered correctly relied primarily on the pictograph ("reading the data," fixating on North America portion in the pictograph, $n = 262$ out of 734 fixations, see Table 1), indicating that the pictograph effectively supported accurate interpretation of the data. One student used both the text and the pictograph to answer correctly with 33 fixations on the pictograph and 10 fixations on the running text. Explaining their thought process, the student first referenced the pictograph, correctly comparing water usage across regions (e.g., North America using 55 buckets compared to lower usage in Russia, Poland, India, Nepal, Haiti, and Ethiopia) and concluded by explaining, "North America is definitely the majority, and I know Africa isn't because, I know it doesn't mention Africa but it couldn't." The student then referenced the text, explaining why it couldn't be Africa because "many of these people live in Africa and Asia that have a low water supply." One student used the pictograph and prior knowledge, and described their thought process as, "there are lots of buckets and I have seen people in North America use lots of buckets." In contrast, several

students referenced only the text, none of whom gave the correct response, suggesting that the textual information may have been confusing. From this, we conclude that, when information is available in both text and graphic form, most students will defer to the graphical form.

Table 1: Question 1 Quantitative Data

	Fixation Count	Dwell Time (ms)
Graph North America	262	96077
Graph Total	734	274678
Running Text	50	14358

Further, when the information is readily available in a pictograph, most students are able to successfully locate it quite quickly. Their eyes moved directly to the pictograph one the question had been asked, most often to the top of the pictograph, as the water usage was depicted in descending order. When students did not refer to the pictograph to answer, it was either because they looked at the pictograph, but were not sure how to read it ("I have no idea what the answer is") or technically answered correctly, using thought processes that included largely irrelevant information from the text ("North America because we have more people and other places have less people … [I found that information] on the next page").

Q2: What Do You Notice When You Compare the Average Daily Water Use by People in India and North America?

Most students responded to this question by comparing the number of buckets shown in the pictograph ("reading between the data"), as evidenced by reporting either the number of buckets (seven for India, 55 for North America) or giving the mathematical difference (48 buckets). Ten students responded only that North America used more or India used less without providing an explanation; while an additional three gave the same response and gave an explanation not provided in the text (e.g., related to population or land size). Only three students compared liters as opposed to buckets (the pictograph key stated that each bucket represented 10 liters), describing their thought processes in ways such as "so like, the difference? They use 48 more buckets than India has 70 liters; so that means it has maybe 1000 [exaggerating the difference] liters probably." These findings suggest that most students used the pictograph to make a reasonable comparison. They were able to extract data from the pictograph to respond to the question, but only three demonstrated the skill of approaching the question as a "think and search" question, integrating information from across the representation (information in the key and pictographic information). Eye tracking data confirms this: all but one student fixated at the pictograph (n = 1,021, see Table 2), while only 13 fixated on the key (n = 41), specifically.

Table 2: Question 2 Quantitative Data

	Fixation Count	Dwell Time (ms)
Graph North America	345	123753
Graph India	27	8611
Graph Key	41	14474
Graph Total	1021	386868
Text Caption	145	47210

Conclusions

Mathematics, as a discipline, is integrated into many content areas, and multiple forms of representation can be beneficial when information is provided in different embodiments. In this study, mathematical understanding was required for students to understand the full content of the text. While text comprehension is generally thought to be about understanding the words, this study highlights the importance of mathematical understanding complementing text comprehension. Related implications include the necessity of teaching, both in PK-12 and teacher education programs, that mathematical and literacy content knowledge need to be used concurrently, at times, as well as specific strategies for doing so. While the first of these is widely known (if inconsistently practiced), the second merits additional research. The connection between content knowledge (here, mathematical) and overall reading comprehension is well-established (e.g., Cabell & Hwang, 2020), however this study highlights not only the necessity of mathematical understanding, in general, but also that mathematical sense-making could be text-specific and can vary considerably between students, even those with nearly identical academic histories (i.e., same teachers, same curricula).

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STUDENTS' JUSTIFICATION OF LOGARITHM RULES FROM AN APOS PERSPECTIVE

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Investigations have been conducted into how students conceptualize exponents and logarithms, utilizing APOS theory, particularly regarding how students justify exponent rules. Williams' (2011) thesis presents a set of codes for identifying students' conceptions of logarithms. However, a gap in the literature remains on how students justify the corresponding logarithm rules. In this paper, I present some results of such an investigation to extend her work.

Keywords: Precalculus, High School Education, Algebra and Algebraic Thinking, Advanced Mathematical Thinking

This report draws on data from my dissertation regarding how students in a complex analysis course leverage their knowledge of real-valued exponents and logarithms to reason about the complex-valued analogs. Part of my dissertation aims to describe college students' conceptions of real and complex exponents and logarithms and to compare their similarities and differences using APOS theory. In my literature review, I have come across studies examining how students justify various logarithm rules, particularly Williams (2011). Part of Williams' (2011) claim is that responses to questions about logarithm rules can be used to characterize their process or object conceptions of logarithms. However, her thesis does not report on such responses. The findings in this report contribute to the larger goal of extending Williams' (2011) framework on exponents and logarithms from an APOS perspective and provide an answer to the following research question: How do students justify various properties of logarithms, and how might their justifications differ?

In addition to investigating this research question, I also demonstrate the utility of my methodology. Because the literature on reasoning about exponents was necessary for my dissertation but was lacking in the wider research literature, I found it appropriate to extend the codes for logarithms from Williams (2011) to analogous codes for exponents. I contend that this methodology can be productive in a wide variety of APOS studies beyond the one presented in this paper.

Theoretical Framework

APOS theory is an adaptation of Piagetian notions of reflective abstraction to the context of undergraduate mathematics education by Dubinsky (1991), Asiala et al. (1997), and Dubinsky and McDonald (2001). These three papers define actions, processes, objects, and schemas and detail the mechanisms by which individuals construct them. Dubinsky (1991) refers to five methods of constructs as reflective abstractions that "Piaget found in the development of children's logical thinking" (p. 10) and adapts them to "the context of advanced mathematical thinking" (p. 11). These five are interiorization, coordination, encapsulation, generalization, and reversal. However, this last action is Dubinsky's contribution and not part of Piaget's reflective abstraction (p. 8).

Actions, processes, objects, and schemas, as the name suggests, constitute the basis of APOS theory, which concerns the construction of schemas and mental objects. The theory posits that mental objects result from sufficient reflection on processes or the thematization of schemas

(Asiala et al., 1997). According to Asiala et al. (1997), an action "is a transformation of objects which is perceived by the individual as being at least somewhat external," (p. 7) a process is the result of reflective abstraction on an action or the reversal of an existing process, and an object is a result of either "when an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that actions or processes can act on it, and can construct such transformations," (p. 8) or schemas via thematization, which is defined as a schema's inclusion in other schemas. Dubinsky (1991) and Asiala et al. (1997) define a schema as a coherent collection of objects and processes and argue that the main concern of mathematics education researchers should be the construction of schemas. Although direct observation of schemas, objects, and processes is impossible, they serve as the entry point to reflective abstraction (Dubinsky, 1991).

Literature Review

To best situate my investigations within the existing literature, I adopt findings from studies that employ APOS Theory. I am primarily drawing on Williams' (2011) framework, which outlines codes for an object, process, and functional level of understanding of logarithms. Relevant to this study are the various codes she develops to characterize her participants' APOS conceptions, particularly those she deems relevant for justification tasks. Williams (2011) arrives at a set of codes for different levels of understanding of logarithms. She has five codes for an object understanding, five codes for a process understanding, and seven codes for a function understanding of logarithms. The five codes for object understanding are O-def, O-num, O-rule, O-not, and O-part. The first of these codes describes a student who understands the logarithm expression $log_a(b)$ as the unique number where $a^{log_a(b)} = b$. The second code, O-num, characterizes a student who is comfortable leaving logarithmic expressions alone and does not need to write them as decimals. O-rule is a description that applies to students who can use logarithm rules to change the form of logarithmic expressions, and O-not refers to students who adhere to notational conventions when reasoning with logarithms. Finally, the code O-part describes students who understand that logarithmic expressions have a base and argument.

Regarding process understanding, the relevant codes are P-def, P-est, and P-exp. The code P-def refers to when "the student can think of $log_a(b)$ as a process of taking the argument b, converting it to the form of a^x, and writing as the answer" (p. 42). The P-est conception refers to students' ability to use the P-def conception to estimate the value of logarithmic expressions. The P-exp conception describes students who use exponents to evaluate logarithmic expressions. Also relevant to my data are her function codes, prefixed with an "F," which describe the various ways students can conceptualize logarithmic functions. For this study, the key concept is the F-inv conception of logarithms, which conceptualizes logarithms as the inverses of exponents. Although she does not describe them as such, I also take these codes to describe object conceptions.

I extended Williams' codes to apply analogously to exponents, as these codes did not apply to exponents and there is a lack of APOS-related codes for exponents in the literature. The basis for my analogization was the inverse relationship between exponents and logarithms. At times, the analogization was straightforward. For example, an O-num conception of exponents describes students who do not need to represent exponential expressions as decimal numbers. Other times, the analogization required additional mathematical justification, such as the P-def conception of exponents. The original P-def conception concerns transforming logarithmic expressions into an exponential expression to compute them. Rather than analogizing the P-def

conception to exponents by describing a transformation into a logarithmic expression, I instead describe it as a transformation into a multiplicative expression, as that is the more standard definition of an exponent.

I also draw from the findings of Gol Tabaghi (2007), Pitta-Pantazi et al. (2007), and Weber (2002). According to Pitta-Pantazi et al. (2007) and Weber (2002), an action conception of logarithms is evidenced by calculating logarithms via explicit multiplication. According to Gol Tabaghi (2007), an action conception is also characterized by participants' use of a guess-and-check method for solving logarithmic equations. A process conception is characterized by participants who use a general rule to determine algorithms for calculating values and can solve algebra problems involving logarithms. They can interpret the various parts of the logarithm expression and use them to justify the algorithms they employ. An object conception is indicated by participants' ability to add, subtract, or compare logarithmic expressions. Finally, a conceptual understanding of the relationship between exponents and logarithms that allows them to use logarithm laws to justify exponent laws is evidence of a schema level.

Methods

As stated above, I am drawing from data for my dissertation. I approached students in a complex analysis course at a large Southern research university and offered them compensation for their participation. For each student, I conducted four interviews: two individual and two paired. I had also disclosed this interview structure to the students and asked them to name partners they would like to collaborate with. Priority was then given to pairs of students who named another volunteer to be interviewed with. The data I am analyzing for this report comes from the first individual interview. The individual interviews were semi-structured and conducted on an iPad via the Notability application. The screen and microphone recorded the participants' scratch work and utterances, and I used a webcam to record their physical actions.

The protocol for the first interview was designed to detect their conceptions of exponents, logarithms, and complex arithmetic and consisted of 7 questions. The third question of my protocol asks the participants to justify the logarithm rules $log(ab) = log(a) + log(b)$, $log(a/b) = log(a) - log(b)$, and $log(a^r) = r\, log(a)$, and the fourth question asks the participants to justify the exponent rules $(ab)^c = a^c b^c$, $(a^b)^c = a^{bc}$, and $a^{b+c} = a^b a^c$. Within the context of the wider study, these questions are designed to calibrate my analysis of their second interview and to compare how they apply the rules of exponents and logarithms in the real and complex cases. However, this paper primarily reports on their responses to questions about real logarithms in answering my research question.

I recorded the interviews using Zoom's auto-transcription feature and reviewed the audio to clean up the transcript, as the software was still imperfect at the time. The coding process consisted of multiple rounds. I first coded the student work using a codebook that synthesized multiple studies on students' understanding of logarithms, exponents, and complex analysis, primarily Williams (2011), Pitta-Pantazi et al. (2007), and Gol Tabaghi (2007). Next, I reviewed the interview transcripts and video and coded relevant utterances. In a spreadsheet, I summarized their responses to the tasks, included relevant interview excerpts, and determined which codes were relevant to the description. Finally, based on the relevant codes, I identified the highest concept that emerged at that point in the interview.

Participants

Although my data involves six participants in think-aloud interviews (Montague & Applegate, 1993), I am reporting only on the results from two students, Jim and May. Jim was a second-year math major minoring in Engineering Science and Mechanics, but by credits has

senior standing, and May described themself as "a senior double majoring in Chemistry and the traditional Math degree who has been a part of undergrad research since freshman year and doing math research since spring of junior year."

Analysis

One of the most insightful codes about May and Jim comes from Gol Tabaghi (2007). According to her, relating the exponent and logarithm rules indicates a schema that includes conceptions of both exponents and logarithms. Thus, both May and Jim's schemas of logarithms and exponents accommodated each other. This, along with the variety of conceptions on display, suggests that the conceptions at different points were heavily context-dependent and may not indicate a lack of knowledge on their part.

Jim's Conceptions

Jim first demonstrated proficiency in simplifying logarithmic expressions. When Jim was asked to simplify log_{15} (20), his immediate strategy was to rewrite 20 as $15 \cdot 4/3$, and then use a logarithm rule to rewrite the original expression as $1 + log_{15}$ (4) $- log_{15}$ (3). When pressed to simplify further, he wrote a final expression of $1 + \frac{ln4}{ln15} - \frac{ln3}{ln15}$, but indicated that he believed that the summands in the prior expression were sufficiently simplified. Williams' (2011) framework suggests that Jim demonstrated an object conception of logarithms. His flexibility in manipulation is characteristic of an O-rule conception, and his metric for appropriateness of a particular form is characteristic of an O-num conception. Furthermore, Jim's instantaneous evaluation of other logarithms suggests a process conception of logarithms according to Weber (2002).

Later, he solved the logarithmic polynomial equation $log_3 (x + 1) + log_3 (x) = 1$. His logic was the following: Because logarithms and exponents are inverses, if raising a power to the one logarithm expression yields the argument of the other, the two sides are equal. Then, he can transform the expression into a polynomial and then determine which possible values are in the domain of the logarithm to narrow down possible solutions. This implicitly uses the argument that if $a^b = a^c$ then $b = c$. Implicit use of this fact could be related to the O-def conception from Williams (2011) and is further evidence of his object conceptions for logarithms.

Jim's justifications for the logarithm rules $log\ (ab) = log(a) + log(b)$ and $log\ (a/b) = log(a) - log(b)$ were similar. In both cases, he took the sum expression, i.e., the right-hand side, and raised a number c to the power of it. By taking $c^{log(a)+log(b)}$, he deferred to properties of exponential expressions and split it into factors, giving $c^{log_c a} c^{log_c (b)}$, and stated, "because it's in the power, you can split it over addition." He then reasoned that each factor, by definition, was a and b, so the product was ab, "but then this gives you a log base c of ab from there," concluding the proof. Despite his abuse of notation, which might suggest $ab = log\ (ab)$, His utterances made it clear that he was using equal signs to indicate implication. According to Williams (2011), his appeal to exponential properties to justify logarithm rules could indicate a P-def and P-exp conception of logarithms. Thus, his responses were characteristic of a process conception of logarithms.

Despite the relative ease with which Jim justified the product rule for logarithms, his conceptions did not yield a similarly smooth proof for the rule $log(a^r) = r \cdot log(a)$. His initial strategy consisted of setting $r \cdot log(a)$ equal to some number c, and subsequently raising 10 to the power of both sides, giving the equality $10^{r \cdot log(a)} = 10^c$, and then splitting this up into $10^r \cdot 10^{log(a)}$, leading to the expression $10^r \cdot a = c$. He determined that the equality $10^{r \cdot log(a)} = 10^c$ was incorrect and wrote two expressions: $(10^r)^{log(a)} = 10^c$ and $(10^{log(a)})^r = 10^c$. From the

second expression, he simplified to get $a^r = 10^c$, and then "you can take log of both sides again, giving $log\ (a^r) = 10^c$, and that should do it." He remarked, "Because of how this is structured, I think I could go backward, but one direction flows easier if that makes sense." Notably, he used an explicit base to reason about logarithm rules, but used a general base in his prior justification. This addresses the part of my research question concerning how students justify the logarithm rules. In general, Jim used his F-inv conception of logarithms to reduce the justification of logarithm rules to a justification concerning exponent rules, which is further evidence toward a general object conception of logarithms, although his last justification suggests his object conception

Next, Jim's justifications of exponents relied on combinatorial interpretations of exponents, and he would try to reduce the general version of exponent rules to the original combinatorial interpretation. For example, when justifying $(a^b)^c = a^{bc}$, he described the left side as a^b c times, and then a^b as a multiplied by itself b times, so described a multiplied by itself bc times, which meets the definition of the right-hand side. However, when generalizing this property to rational powers, Jim did not appeal to a definition of roots because "I don't know if that relies on knowing that the root of a power can be split up." His concern about building up fully rigorous arguments for exponent properties may indicate that his schema of exponents and logarithms treats exponents as more fundamental, hence his relative lack of concern in using exponent properties to justify logarithm properties. By appealing to exponent rules and combinatorial interpretations of exponential expressions, his proofs for exponent rules indicated an operationalization of an O-rule and O-def conception of exponents. Thus, he was operating with primarily object conceptions of exponents.

May's Conceptions

May's conceptions of logarithms were varied to a similar degree as Jim's. In calculating log_{10} (1000), they determined the value to be 3 because "it's 100 times 10." Similarly, when calculating log_3 (81), they said, "Three, so twenty… seven times 3" while calculating the product of 27 and 3. When they evaluated this product as 81, they determined it would be 4 "because 3 to the 4 is 81." When asked to simplify log_{15} (20), they tried to estimate it using two approaches. The first was looking for a solution to $15^x = 20^x$, and the second attempt resulted in estimating the value to be around 5/3 by using the equalities $log(20)/log(15) = log\ (20 - 15) = log(5)$. Similar to Jim, they identified the value of $log_b\ (a)$ as the power of b that gives a and, notably, they iterated multiplication to compute logarithm values. Because their method of calculating logarithms involved explicit multiplication, their conception of logarithms would fall under an action conception, according to Weber (2002) and Pitta-Pantazi et al. (2007). However, their estimation of logarithm values falls within Williams' (2011) P-est conception of logarithms, so they had at least a process conception of logarithms.

May's justification for $log\ (ab) = log(a) + log(b)$ was similar to how Jim justified $log(a/b) = log(a) - log(b)$, but used a base of 10 instead of c, indicating an F-inv object conception of logarithms. They differed most in their justification of $log(a/b) = log(a) - log(b)$. When faced with the expression, May similarly identified $log(a/b)$ as "z such that $10^z = a/b$, and likewise identified $log(a)$ and $log(b)$ as "x, y such that…" and wrote $10^x = a$ and $10^y = b$. Continuing the proof, they realized that a/b could be rewritten as ab^{-1}, and setting $z = log(a/b)$, reasoned that $10^z = a/b$, and "applying log to both sides would give you $log(a/b)$, and then this would be $log(a/b)$ and this would be $log(ab^{-1})$." They finished the proof by saying "you could just split this up" into $log(a) + log(b^{-1})$ and then "assuming we can use the rule of powers" yielded $log(a) - log(b)$. Similar to Jim, their appeal to exponential properties to justify

the logarithm rules could indicate a P-def and P-exp conception of logarithms, rather than a general process conception.

May used induction to justify the rule $log(a^k) = k\ log(a)$. They reasoned, "that one's pretty obvious…I would just do the base case, which would be $log\ (a)$ equals, well, one times $log(a)$. Obviously, this is just 1." In justifying the inductive step, they assumed that the equality held for all natural numbers up to k, took the expression $log(a^{k+1})$, broke it up into $log(a^k \cdot a)$, and using a prior established logarithm rule, took this to be $k\ log(a) + log(a)$. Finally, they collected like terms and got the expression $(k+1)\ log(a)$, completing the proof. The ability to act on logarithmic expressions via induction is further evidence toward an object conception of logarithms. Furthermore, they operationalized logarithmic expressions as satisfying some exponential expression, which falls under an object conception according to Weber (2002) and Pitta Pantazi et al. (2007).

Some of May's justifications of exponent rules displayed object conceptions of exponents, but others relied on logarithm properties. When justifying $(a^b)^c = a^{bc}$, they used a combinatorial argument similar to Jim's, indicating similar object conceptions of exponents. But when justifying the rule $a^b a^c = a^{b+c}$, they applied the logarithm to both sides, yielding the expression $log(a^b) + log(a^c) = log(a^{b+c})$. They then used another logarithm rule to get the expression $b\ log(a) + c\ log(a) = (b+c)\ log(a)$, and then canceled out $log(a)$ to 1. When asked about the differences in justifications, they said, "If you had asked these in a different order, I might have given a different answer," suggesting a stronger relationship between their schema for logarithms and exponents. The combinatorial reasoning for the former rule and their calculation of logarithm values suggests that they can reason about the other exponent rules combinatorially and were strongly influenced by the task order. This response also addresses the part of my research question concerning how students justify the logarithm rules. Broadly speaking, the use of the inverse relationship of exponents and logarithms demonstrates a P-exp conception of exponents and logarithms.

Discussion

This analysis of May and Jim's responses provides an answer to my research question: How do students justify various properties of logarithms, and how might their justifications differ? In summary, May and Jim's conceptions of logarithm and exponent rules were similar in terms of what conceptions were present, but differed considerably in how they were organized. May and Jim demonstrated similar process and object conceptions of exponents and logarithms in a unified schema until they were asked to justify the properties. Whereas May justified logarithm and exponent rules in terms of each other, Jim sought to justify exponent rules combinatorially and with an explicit interpretation of exponents as iterated multiplication. More broadly, both May's and Jim's schemas for exponents and logarithms have coalesced into a single schema. So a description for a college student's conceptions of logarithms is very likely incomplete without accompanying descriptions for exponents. My interview protocol, which addressed both logarithms and exponents, provided me with a broader context for their justifications. In particular, May's sensitivity to task order seemed to confirm my hypothesis that participants' thinking could be significantly influenced by the ordering of the tasks in the interview. Their circular reasoning may indicate an exponent/logarithm schema that contains an object conception of logarithms independent of exponents. Contrast this with Jim, who was less affected by the order of tasks but was compelled toward a more fundamental justification of exponent rules over logarithm rules. Thus, his object conceptions of logarithms were built on top of and attached to his conceptions of exponents.

More specifically, the differences in proofs indicate some variations in their respective schemas. Jim relied solely on the interpretation of iterated multiplication to justify exponent rules. Later in the interview, he attempted to reduce the justification of fractional exponent rules to the case of integer exponent rules. He likely operationalizes definitions when justifying properties of exponents and logarithms. He will likely justify the properties of the complex exponent and logarithm by referring to their definitions and to the inverse relationship between the exponent and logarithm. In contrast, May's use of logarithm rules to justify exponent rules suggests that in the complex setting, they are less likely to refer to exponents when justifying properties of the complex logarithm.

Broadly, the different ways May and Jim engaged with the justification tasks may have broader implications for instruction. It is worth considering the different affordances of each approach. While Jim's approach may qualify as more "rigorous," I find it noteworthy that his insistence on such strict adherence limited some of his willingness to perhaps reason more powerfully. In comparison, May may not adhere to the strictest standards of "rigor," but their flexibility enabled them to have a comparatively easier time justifying real logarithmic rules. This contrast between May and Jim raises questions about the role of rigor in different types of mathematical discourse. Particularly in a teaching context, it is worth considering when rigorous thinking is prioritized over more powerful thinking, and how students' expectations fall on this spectrum.

In terms of research implications, this phenomenon suggests that similar conceptions of exponents do not necessarily imply similar conceptions of logarithms, with both research and pedagogical implications. The implication for research is as follows: Among college students, the schema for exponents and the schema for logarithms are likely unified into a single schema. However, conceptions relating to exponents are likelier to be homogeneous compared to conceptions for logarithms. Thus, to fully describe a student's conceptions of logarithms, it is necessary to describe their conceptions of exponents as well.

Conclusion

In Williams' (2011) study, she described the limitations of her codes through a participant, Mark, who demonstrated that her framework "does not necessarily help separate good understanding from exceptionally good understanding" (p. 63). She attributes this to a variety of factors, most notably his job as a math tutor. He had a proof of the product/sum rule as follows:

$$log(ab) = log(10^{log(a)} \cdot 10^{log(b)}) = log(10^{log(a)+log(b)}) = log(a) + log(b)$$

Like May and Jim, Mark presents a straightforward proof, but unlike Jim, he moves from multiplicative expressions to additive expressions. That is, he reasoned in the opposite direction from what Jim stated: "flows easier."

In this investigation, I have made two discoveries. The first concerns the nature of college students' cognition when justifying exponent and logarithm rules, and more generally, their cognition regarding high school mathematics. The second discovery concerns the fruitfulness of extending pre-existing APOS codes to larger domains to address gaps in existing literature.

Regarding the first discovery, I have found that advanced college students approach precalculus questions in at least two ways. The first is emblematic of Jim's behavior, in which he was careful to make rigorous arguments and to rely on a proof template. The second is emblematic of May's behavior, in which they were willing to use a variety of proof techniques, including induction. I attribute both Jim's and May's response patterns to the higher standards of college mathematics, despite the stark divergence between the two. I conjecture that college mathematics impressed upon Jim a certain standard of mathematical rigor, while for May, it

impressed upon them the far-reaching utility of proof techniques. In pedagogical terms, this suggests that students may be receptive to multiple styles of mathematical presentation. It is worth considering the utility of strict rigor in undergraduate mathematics if it misaligns with student thinking.

Regarding the second discovery, my findings suggest that one way to address Williams' (2011) limitations is to describe students' conceptions of exponents. The two college students' schemas for logarithms and exponents were closely related. It was necessary to explore their conceptions of logarithms and exponents to model their justification of various properties of these operations, as they were using reasoning about exponents to justify logarithm rules. It was productive enough to extend pre-existing logarithm codes by creating analogous codes for exponents.

More generally, I have demonstrated the viability of extending APOS codes of a particular mathematical concept to other domains, despite a lack of existing literature. In general, when describing possible conceptions of operations that are mutual inverses, for every conception of one operation, it may be viable to have an analogous conception for the inverse operation as an a priori conception. Examples include multiplication and division, as well as derivatives and integrals. I am also adopting another method for analogizing existing APOS codes. As stated earlier, this investigation was initially motivated by an interest in understanding how college students understand complex exponents and logarithms. I am analogizing Williams' (2011) codes to these mathematical objects and finding them similarly useful. Thus, potential APOS studies for subjects such as linear algebra over the real numbers have immediate implications for linear algebra over the complex numbers, and investigations into students' cognition of a given topic may have implications beyond their immediate subject.

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AUTHORSHIP AS CONTRIBUTION: CIRCULATING STRATEGIES THROUGH FACILITATION

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This study examines how facilitation moves within the How Many routine supports collaborative mathematical thinking in a third-grade classroom. Drawing on positioning theory, the analysis highlights how the facilitator structured opportunities for students to share, extend, and take up one another's ideas. A central contribution is the identification of facilitation practices that establish authorship to student work by publicly recognizing and sustaining students' recurring strategies as reusable classroom resources. Through this practice, student-generated ideas became collectively available and supported shifts from unitary counting toward more sophisticated composition strategies. Together, these practices repositioned authorship from individual ownership toward shared intellectual contribution.

Keywords: Positioning Theory, Authorship, Facilitation Moves

Supporting students in developing mathematical understanding requires more than eliciting correct answers—it involves cultivating classroom communities where student ideas become the primary resource for learning. In discussion-rich environments, students learn by collectively building on one another's reasoning (Chapin et al., 2009; Forman et al., 1997; Michaels et al., 2008). These pedagogical practices contrast with traditional school mathematics, where knowing is defined by speed and rule-following, and primary intellectual authority resides with the teacher and textbook, limiting students' opportunities to generate or critique ideas (Clement, 1991; Kooloos et al., 2022; Lampert, 1990). As a result, whole-class discussions have too often been confined to "show and tell" formats, where students present correct solutions in isolation, with minimal attention to conceptual connections or collective sensemaking (Stein et al., 2008). Research on mathematically significant pedagogical opportunities emphasizes the importance of leveraging student thinking as a resource for collective learning (Leatham et al., 2015; Van Zoest et al., 2017).

This study draws on positioning theory (Harré & Van Langenhove, 1991; Herbel-Eisenmann et al., 2015) to explore how teachers position students during classroom discourse to foster authorship and the circulation of mathematical ideas. Positioning theory conceptualizes social interactions as dynamic processes in which individuals are ascribed or take on particular roles, rights, and responsibilities (Drageset & Ell, 2024; Herbel-Eisenmann et al., 2015). These positions are not fixed; rather, they are negotiated through language, actions, and broader discursive patterns. In mathematics education, this perspective highlights how teachers' facilitation moves can shape who is recognized as a knowledge contributor (Tait-McCutcheon & Loveridge, 2016), whose ideas are amplified, and how ideas travel across the classroom. Authorship can thus be understood as a socially positioned phenomenon: students are positioned as authors when their ideas are treated as valuable, novel, and when their intellectual ownership is publicly recognized.

Building on this conceptualization, the exploratory study investigates how facilitation moves within a visual number-sense routine support the development of authorship by publicly recognizing and sustaining students' mathematical strategies. Prior research has conceptualized authorship in mathematics classrooms as students generating original ideas and using their voices

to question and make sense of what they are learning, framing authorship as exclusive (Povey, 2003). In these accounts, authorship is closely tied to individual expression or intellectual ownership. Extending this work, the study views authorship as a socially constructed classroom practice, in which it is not only about generating original ideas but also about how those ideas are encouraged for further development and uptake. Accordingly, this study addresses two research questions: (1) How does teacher facilitation of mathematical discussions promote authorship of mathematical strategies? and (2) How do students take up ideas authored by their peers?

Methods

Context, Participants, and Design

The study was conducted in four rural Midwestern classrooms—two second-grade and two third-grade classrooms. The intervention included one week of pre-interviews with 32 students, three weeks of instructional sessions, and one week of post-interviews. Gwen, a mathematics educator and researcher, facilitated three 30-minute lessons per week in each classroom, resulting in 36 total lessons (nine per classroom). The *How Many* instructional routine used in the intervention consisted of three components: (1) displaying an image and asking students, "How many do you see?," (2) providing think time, and (3) facilitating a whole-class discussion that highlighted a range of strategies and justifications. Each lesson incorporated one or two images from the Multiplicity Lab (www.multiplicitylab.org) designed to elicit multiple solution paths, and three images were used during student interviews.

Data Sources and Analytic Approach

Data included video recordings and transcripts of whole-class discussions and pre- and post-interviews. Although the project included four classrooms, this paper focuses on one third-grade classroom to examine how facilitation moves positioned students as authors of mathematical ideas and how peers took up and circulated those ideas. Three lessons (1, 2, and 7) were selected for in-depth qualitative analysis based on their potential to address the research questions.

Analysis proceeded in three phases. Initial coding focused on (1) teacher prompts and facilitation moves, (2) student strategies and reasoning processes, and (3) how ideas were extended, revoiced, or collectively elaborated. Codes were then iteratively refined to identify patterns in how participation structures and mathematical reasoning co-evolved across lessons. Particular attention was given to moments when a strategy shifted from an individual contribution to a publicly recognizable and reusable classroom resource. Through this process, a focal case was identified: Ian's "making 5s" strategy. This case was selected because it illustrated how a student-authored idea surfaced in discussion, was publicly recognized through facilitation, and was taken up by peers. The strategy was then traced across subsequent lessons to examine how it contributed to collective mathematical sensemaking over time.

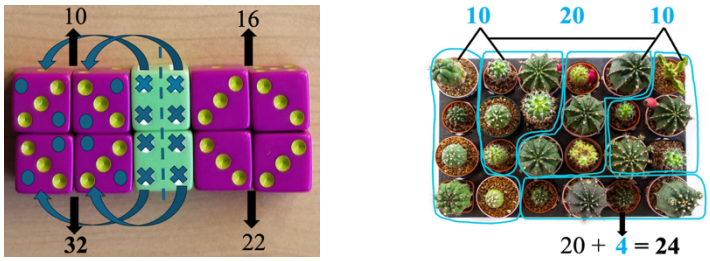
Findings

This study found that Gwen's facilitation positions students with authorship through three key moves: (1) treating partial ideas as resources and physically positioning students as contributors, (2) naming and attending to unexpected strategies as opportunities for sensemaking, and (3) reinforcing students' ownership of strategies through anticipation and public attribution.

First, the teacher positions students as authors by framing their developing ideas as valuable while maintaining their public presence in the discussion. Early in Lesson 1, Gwen began to set in motion a routine in which incomplete ideas were opened for collective work rather than evaluated for correctness. When Yahaira began grouping doubles—pairing equal dots, such as 3 and 3—on an image of multiple dice without computing the total, Gwen did not finish the

strategy herself or shift attention elsewhere. Instead, she invited a student, Steven, to help extend the reasoning toward a solution while keeping Yahaira at the board: "Stay up here … so we're using your thinking and your strategy." This facilitation move accomplished two things. It treated Yahaira's partial reasoning as a resource worth building on, and it maintained her spatial and intellectual presence as Steven contributed. Gwen further reinforced this positioning by asking Yahaira to evaluate how a classmate extended her idea. In this exchange, mathematical progress unfolded as a connected sequence of contributions rather than an individual performance. By preserving both attribution and physical positioning, Gwen established a pattern in which student ideas could be shared without becoming detached from their originators—laying the groundwork for later moments when strategies traveled across tasks and time.

In the second facilitation move (Lesson 2), Gwen named and elevated an unanticipated strategy as a learning opportunity. Gwen overheard Ian say that he had "made fives;" although she had planned to transition to a new image, she paused and told the class she was "kind of curious" about his thinking. This instructional pivot signaled her willingness to let a student-generated idea redirect the lesson. At the board, Ian explained how he split the two green dice and redistributed the dots to form groups of five—restructuring the image in a way that was neither immediately visible nor an intended feature of the task (see Figure 1a). His strategy reflected an early example of decomposition, as he adjusted quantities to make the computation more manageable. This reorganization allowed him to count more efficiently, moving clockwise around the image to reach a total of 32. Gwen responded, "But by making fives … that's really cool," publicly marking the strategy as both novel and mathematically powerful. By attending to student thinking, Gwen positioned Ian as an author of a mathematical idea and created space for classmates to explore, question, and potentially adopt the approach. Her explicit naming of "making fives" turned it into a shared focus that ultimately circulated within the classroom community, serving as a bridge to more sophisticated mathematical reasoning.



(a) (b)

Figure 1: Ian's Making 5s Strategies

Finally, building on the visibility of Ian's strategy, Gwen enacted a facilitation move in Lesson 7 that reinforced his ownership through anticipation and public attribution. When Ian was called to the board to show his thinking for the cacti image (see Figure 1b), Gwen deliberately drew the class's attention to an emerging pattern she had noticed across previous lessons—that Ian consistently organized quantities by making 5s. Before Ian began, she asked the students to predict what he might do, "based on what he's done before." Students confidently responded, "He's going to do fives," signaling that Ian's strategy was widely recognized and had entered the classroom's collective repertoire, positioning him as its author. During the cacti task, Ian organized the image into four groups of five—starting with the "L"-shaped cluster on the left and

forming a final group of four at the bottom (see Figure 1b). Classmates reacted enthusiastically, calling out "Fivesss!", "Viola!", and clapping. Gwen affirmed these reactions and later summarized, "A lot of friends already predicted that Ian will look for fives—he likes to look for fives." Through this public recognition, Gwen explicitly established the association between Ian and the strategy, while signaling that the approach remained available for collective use. At this moment, authorship functioned relationally: students learned not only a mathematical strategy, but also how to notice, remember, and flexibly apply it across contexts—from the original dice tasks to a more real-world array.

Student Engagement with Authored Ideas

Students responded to this culture of authorship in two primary ways. First, when prompted by the facilitator, they extended one another's ideas. Gwen's invitation to build on Yahaira's partial strategy was not an isolated event; across the dataset, students completed one another's thinking multiple times. Second, students independently took up and adapted a peer-authored strategy in their own work in subsequent lessons. Ian's "making 5s" approach became increasingly visible across activities. In one instance, a student named Angel composed quantities into fives—"four plus one equals five," "three and two equals five"—while annotating a dice image. Classmates immediately recognized the structure of the approach, calling out, "He's going to use 5s!" and describing it as a smart idea. Importantly, toward the end of Angel's explanation, Gwen reframed the discussion to emphasize the reuse of strategies as a productive norm, explaining that discussing strategies allowed the class to "pick up on other ways" and try them later. This moment illustrated a key mechanism of diffusion: once a strategy became publicly recognizable, peers could identify, name, and take it up as a shared intellectual resource.

Discussion

The teacher in this case study used a suite of facilitation moves to position students with authorship, ultimately supporting a learning environment in which students could build on and take up one another's thinking. These moves repositioned authorship from *authorship-as-exclusivity* to *authorship-as-contribution*. Authorship-as-exclusivity positions ideas as belonging solely to an individual—fixed, private, and disconnected from the classroom's collective work. In contrast, authorship-as-contribution frames ideas as both attributable and shareable: students are recognized as the originators of strategies, yet their ideas circulate, evolve, and become resources for supporting efficiency, flexibility, and creativity in others' reasoning. Not all ideas circulated in the same way; although doubles were more commonly used overall, Ian's "making 5s" strategy circulated more widely, which may be due to its novelty drawing more collective attention than familiar approaches like Yahaira's grouping of doubles. Importantly, this diffusion was not accidental, Gwen paused during the transition to acknowledge Ian's strategy, positioning it as worthy of collective attention. Such moments illustrate the importance of teachers discerning which student ideas to center and amplify (Tait-McCutcheon & Loveridge, 2016; Van Zoest et al., 2017). The impact of this positioning extended beyond the instructional routine itself: in post-interviews, several students referenced Ian by name and demonstrated increased strategic flexibility, suggesting that authorship had become socially recognized and instructionally consequential. Future research should investigate how positioning students' unexpected ideas with curiosity and intentional authorship shapes strategy development and fosters ongoing innovation.

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IMPLEMENTING PROBLEM OF THE WEEK WITH ELEMENTARY LEARNERS

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Keywords: Elementary School Education, Identity and Agency, Informal Education, Problem-Based Learning

Inspired by Michael Golomb's long-running Problem of the Week (POW) initiative at Purdue University, which engaged students and faculty and drew international participation, the first author launched a reimagined Problem of the Week at elementary school settings. While inspired by the original model, our initiative intentionally shifted the design to better serve K–5 learners. Specifically, we grounded our work in the principle of Low Floor, High Ceiling tasks (Boaler, 2015): problems with multiple entry points that are accessible to all students while allowing for increasingly sophisticated strategies and extensions.

Study Design and Context

Each Problem of the Week is designed to elicit a range of strategies. Students are encouraged to select the tasks and extensions with which they wish to engage, reinforcing agency and mathematical identity. Importantly, the problems are intentional designated without being tied to a specific grade level. A single task might be solved through various strategies, such as direct modeling, number sentences, table creation, or pattern recognition. This structure allows students across grade levels to engage meaningfully with the same core task (Zhou, 2025).

Beyond solving the problems, we provide open-ended prompts to further facilitate mathematical discourse (National Council of Teachers of Mathematics [NCTM], 2014). The prompts are used, such as: "What do you notice? What do you wonder?" "How are they alike and different?" "Which one doesn't belong?"

In the focused elementary school, problems are distributed every other week. On alternating weeks, student work is displayed publicly in a central campus location that all students pass multiple times daily. The display emphasizes diverse strategies rather than "right answers," and student names are prominently featured to honor authorship and thinking.

Data Source and Preliminary Findings

We maintain a participation spreadsheet to track submissions and distribute recognition pins. Participation has increased consistently since the implementation of the redesigned model in 2023–2024. Average submissions per problem increased from 35 in 2023–2024 to 41 in 2024–2025 and 55 in 2025–2026. Similarly, the proportion of students participating school-wide rose from 47% in 2023–2024 to 59% in 2024–2025 and 63% in 2025–2026. These trends suggest increased student engagement, particularly among younger learners.

We have also collected family feedback, which further suggests that the Problem of the Week engages students who do not typically identify strongly with school mathematics. For instance, one parent described her child as a "spatial thinker". Preliminary evidence suggests that the reimagined Problem of the Week fosters broad participation, strengthens mathematical identity, and builds a visible culture of collective mathematical brilliance. Public celebration of varied strategies, multilingual access, and low-stakes entry points appear to contribute to increased engagement across grade levels and learner profiles.

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DESIGNING UNCERTAINTY TO SUPPORT PROCEDURAL FLEXIBILITY IN MATHEMATICS LEARNING

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Reasoning and Proof; Cognition

Objectives and Purpose of the Study

Grappling with uncertainty in mathematics is an important but under-explored aspect of students' participation in complex mathematical tasks such as error analysis or mathematical debugging. Uncertainty has traditionally been considered a hindrance to effective learning in mathematics education. However, uncertainty can be a productive resource that facilitates deep mathematical reasoning, provided it is intentionally designed and pedagogically supported (Baker et al., 2020; DeBellis & Goldin, 2006; Zaslavsky, 2005). From this perspective, the aim of this poster is to propose a developing conceptual framework that describes how uncertainty is intentionally designed into error analysis tasks to support mathematical learners' development of procedural flexibility. Specifically, this poster revolves around the following two questions:

- RQ1. How can uncertainty be intentionally designed into math error analysis tasks?
- RQ2. In what ways can learner interactions with designed uncertainty support procedural flexibility development?

Theoretical Framework

Zaslavsky (2005) conceptualized uncertainty as a key element of a mathematically rich task and showed that situations involving conflicting arguments, ambiguous conclusions, and results that are difficult to immediately verify lead learners to engage in higher-order mathematical thinking. In this sense, uncertainty acts as a catalyst for shifting procedural knowledge from mechanical performance to adaptive and flexible mathematical reasoning (Radatz, 1979; Star, 2005).

Error analysis tasks become a key means of designing this uncertainty. Radatz (1979) argued that mathematical errors should be understood as meaningful products generated in the learner's cognitive process, not as simple mistakes. Purposeful inclusion of uncertainty requires learners to check and reconstruct their reasoning processes rather than rely on formalized procedures and automated solutions. For example, presenting learners with two different solutions to the same problem and asking them to determine which is correct, or whether both are valid, implies uncertainty that is difficult to verify immediately and allows the learner to deeply explore the meaning of the procedure. Rushton (2018) reported that students who participated in error analysis-oriented classes showed higher achievement in delayed post-testing than those who participated in traditional classes.

Conclusions and Implications

Uncertainty can be designed through error analysis tasks, and this designed uncertainty functions as a medium that leads learners to develop flexible mathematical reasoning beyond performing mechanical procedures. We posit—and intend to explore in future studies—how these mechanisms can be realized through the role of teachers in planning and coordinating students' work with uncertainty and error analysis.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

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ARBITRARY TO RATIONAL COORDINATION: A TYPOLOGY OF MATHEMATICAL CONVENTIONS

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Keywords: Communication, Cognition, Mathematical Representations, Reasoning and Proof

Introduction

Hewitt (1999, 2001a, 2001b) characterized the mathematics curriculum as divided into two domains: (1) "the arbitrary"—the conventional names, notations, and agreed usages enabling consistent communication about mathematics—and (2) "the necessary"—the rules, relationships, properties, and procedures that can be logically deduced or derived. Here I build on this delineation by proposing that the first category of conventions is not homogeneous: some are largely arbitrary choices; others persist for their efficiency, simplicity, or coherence within existing systems or practices, stabilized through processes of *rationally coordinated equilibria*. I call this the *Arbitrary-Rationally Coordinated Mathematical Conventions Gradient.* Additionally, I propose a typology of mathematical conventions that differentiates by function and by which the gradient of arbitrariness can be superimposed.

Typology of Mathematical Conventions

The typology identifies six disciplinary categories and a seventh, schooling-specific category.

I. Notational and Symbolic: symbol choices (x, y, z; π; $\sum$), abbreviations ("sin," "log"), and structured notation (fractions, exponents) settled through communal use.

II. Representational: graphs, coordinate systems, and diagrams governed by spatial norms (e.g., horizontal abscissa, anticlockwise rotation, planar diagrams of knot crossings) that mediate access to mathematical structure.

III. Measurement and Quantification: units, derived quantities (slope as "rise over run", velocity), and zero points/scales (e.g., Celsius vs. Kelvin) that afford intersubjective comparison.

IV. Procedural and Computational: order of operations and algorithmic layouts (e.g., long division), which persist as culturally entrenched even where alternative methods exist.

V. Linguistic and Discursive: disciplinary phrases ("without loss of generality") and cultural naming practices (e.g., Pascal's vs. Yang Hui's Triangle).

VI. Epistemic: proof methods and negotiated standards of rigor that govern what counts as acceptable justification.

VII. Schooling-Specific Pedagogical: locally produced conventions (e.g., PEMDAS, FOIL, "T-charts") that scaffold recall and classroom communication but are absent from global practice; conflating these with disciplinary conventions can contribute to student confusion (Karp et al., 2014; Ristroph, 2025).

Contribution and Implications

Classroom practice often treats conventions as mathematical truths "fallen from the sky" rather than as arbitrary human decisions or as socially rational, stabilized choices (Ristroph, 2025). This proposal offers a typology that makes conventions visible and differentiable, giving educators and researchers a tool for identifying mathematical and pedagogical conventions and aiding in the explication of rationales.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

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NUMBER CONCEPTS AND PROPORTIONAL REASONING

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DALTON AND ANGELA'S ITERATING AND PARTITIONING WHEN SOLVING WHOLE NUMBER AND FRACTION TASKS

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In this year-long constructivist teaching experiment session, we examine how two fourth-grade students with intensive supports in mathematics reorganize their whole number understandings to construct fraction understandings. We consider how each student's iterating and partitioning change to indicate how each student demonstrated their progression. Dalton mainly developed his whole number and fraction understandings in tandem with each other, as he did not have a units structure to fully reorganize. Dalton's ability to map fractions iteratively and utilize his halving and doubling provided him opportunities to reason more robustly in fraction tasks. Angela primarily reorganized her whole number activity with each shift in her iterating and partitioning development. This allowed her to refine and reverse her whole number activity when developing fraction understandings in a multiplicative manner.

Keywords: Number Concepts and Operations, Rational Numbers and Proportional Reasoning, Students with Disabilities, Elementary School Education

Children, including those with Intensive Supports in Mathematics (ISM), iterate and partition when solving fraction tasks, which can be leveraged to support future fraction development (i.e., Hunt et al., 2016). Preliminary work has also showcased that children with ISM develop partitioning through their iterating and vice versa when developing fraction activity from their whole number activity (MacDonald et al., 2025).Yet, it is still not clear how children with ISM reorganize their whole number iterating and partitioning activity to inform their fraction development over long periods of time. The focus of this article is to explore the year-long conceptual development of two fourth-grade students with ISM as they engaged in whole number and fraction iterating and partitioning.

Theoretical Framework

We framed our study around children's development of constructs that make up their mental models or schemes (von Glasersfeld, 1995). Modeling children's activity with scheme theory allows attending to changes in children's understandings (accommodation of schemes) and identifying connections between children's understandings (assimilation of schemes), giving us insight into utilizing children's actions (e.g., partitioning; iterating) and goals when describing children's mathematical development (von Glasersfeld, 1995).

We use scheme theory to explain the units coordination that student with ISM construct,

accommodate, and assimilate when solving whole number and fraction tasks (cf. Boyce et al., 2024). Within scheme theory, a *scheme* includes an experiential situation, a goal, anticipated actions, and expected results (Steffe & Olive, 2010). When children reorganize one or more of these components, accommodation reflects a qualitative shift in how they coordinate units, rather than simply a change in procedure. This framing allows us to examine how students with ISM reorganize their unit relationships over time. We know that through their activity and experiences children first develop units by abstracting a one-ness (Ulrich, 2015), which marks the beginning of children treating quantities as units with the potential to iterate rather than perceptual collections. To coordinate their units, children organize relationships between units they construct through the composition and distribution of units. When children assimilate their schemes, they develop structures of units, resulting in anticipatory actions (cf. Tzur, 2007).

Steffe (2001) found that children reorganize their unit relationships and unit structures to construct fraction understandings. This reorganization of units provides resources for children as they begin to construct similar types of units in their fraction activity and abstraction. Fraction schemes require coordinating a whole, a fractional part, and the relationship between them simultaneously, often demanding more sophisticated levels of unit coordination than whole-number activity. As children reflect on the results of their engagement with fractions, they accommodate or assimilate their schemes, allowing them to begin developing units in units (in units) organizations (Boyce & Norton, 2017).

This reorganization of units draws on children's iterating and partitioning of units. Iterating and partitioning are not merely external procedures but are central to conceptualizing number, quantity, and arithmetic operations (Empson & Levi, 2011). For instance, engaging in partitioning allows children to make sense of division, fair sharing situations, and unit fractions by constructing equal partitions. Similarly, to mentally reconstruct wholes from unit fractions or smaller units, children need to engage in and make sense of iterating. According to Steffe and Olive (2010), the process of partitioning and iterating becomes operative when children act on mental images of these quantities by anticipating both the number and size of parts before enacting (performing) partitioning or iterating. Moreover, at this stage of development, children can also reverse the iterating and partitioning process to reconstruct the whole. Furthermore, Hunt and colleagues found iterating and partitioning are insightful activities to focus on when examining fraction development that students with ISM demonstrate (Hunt et al., 2023; see also Silva et al., 2023). Similarly, research in children's early whole-number development demonstrates that composing and decomposing quantities (Baroody, 1987), conceptual subitizing number (Bajwa, et al. 2023), and counting-on reflect emerging iteration and partitioning of composite units.

Methods

This paper utilizes a Constructivist Teaching Experiment Methodology (Steffe & Thompson, 2000), examining the activity evidenced by two fourth-grade children with ISM, Dalton and Angela (both pseudonyms), engaged in 14 teaching experiment sessions involving whole number and fraction tasks. We employed a teaching experiment methodology due to our interest in children's interactions with both the interviewer and the features of each task. These interactions provided insight into how particular mathematical behavior (i.e., physical activity, gestures, language) reflected Dalton's and Angela's whole number and fraction scheme relationships. As such, before each interview session, we refined tasks in response to data analyses of prior interview sessions to better make visible the conceptual resources each student was constructing in relation to their developing whole number and fraction scheme relationships.

We purposively selected Dalton and Angela because they were currently receiving Tier 3 mathematics instruction. Additionally, through discussions with the classroom and special education teachers at a participating school, we purposively recruited both children due to their ISM eligibility and Angela's working memory differences (the school district indicated Angela scored significantly below the norm as measured by the Woodcock-Johnson IV Cognitive Abilities assessment; Woodcock et al., 2003). Angela's working memory differences were further verified using a working memory assessment that measured participants' ability to orally repeat sequences of numbers forwards and backwards (i.e., forwards and backwards digit spans). Dalton demonstrated relatively more success in repeating the sequences than Angela. Whereas Angela struggled to repeat sequences in the order provided and was unable to repeat them in reverse. At the time of the study, Dalton and Angela were in the fourth grade, enrolled in a small rural school in the midwestern portion of the United States.

Over the course of the 2023–2024 academic year, the second and fifth author worked with the children as a pair for sessions 2–8, and 10–16, with sessions 1, 9, and 17 conducted individually to assess each student. Each 30-minute session took place throughout the school day and aligned with topics and content taught previously and/or concurrently. Each session was video/audio recorded with two cameras, each focusing on a child's engagement with the task. Each of the sessions included one whole number task and one fraction task. This structure allowed children to engage with tasks and, in particular, engage and reveal their whole number and fraction reasoning within a comparable timeframe. We drew from research-established tasks (i.e., Tzur et al., 2021; Wright et al., 2011) to ground the tasks and activities for each session. We designed tasks on a weekly basis, following our analysis of each session (conceptual analysis) as well as cross-session discussions during retrospective analysis.

Immediately following each session, a graduate student transcribed the data, which we reviewed and edited prior to analysis. We then conducted conceptual analyses and adapted subsequent tasks in response to our interpretations of children's conceptual resources, including their operation development, emerging schemes, and interpretation of each task.

Our conceptual analysis also examined transcripts for changes to their schemes to understand possible activity-effect relationships and degree of abstraction (participatory and anticipatory). Excerpts in this manuscript indicate the child by their pseudonym, "Dalton" or "Angela," and the Researcher working with the child(ren) as "Researcher."

Results

To explore how Angela and Dalton evidenced particular iterating and partitioning relationships over time between their whole number and fraction understandings, we first examine how each student developed their iterating and partitioning activity in both whole number tasks and fraction tasks over the course of the teaching experiment sessions.

Relationships Between Dalton's Whole Number and Fraction Understandings

Early in the teaching experiment sessions, Dalton demonstrated capabilities to organize units with multiple gestures (i.e., pointing, head nodding) and physical activity (i.e., finger counting, counting with manipulatives). For instance, in Teaching Experiment Session 3, Dalton was working with Angela to solve the task, "*Here are 3 plates of 6 cakes. Now look away.*" [Place out four more plates of cakes under a screen]. "*There are 4 more plates of 6 cakes under here. How many cakes are there altogether?*" Both participants were first asked how they might solve this task without any materials before being given cubes and paper plates to solve this task. At first, Dalton counts the visible cakes on the computer screen by pointing with his fingers and then continues by looking at a number strip hanging in the classroom and keeping track of the

groups and number in each group with his fingers. This results in Dalton losing track of the number of groups, pressing him to start over and recount. To alleviate any frustration, both researchers provide him and Angela with cubes and paper plates to check their solutions. In the following excerpt, Dalton uses physical activity to distribute groups of six across four plates before counting the total number of cubes.

Researcher:	And what did you say?
Dalton:	[Counting his cubes.] Twelve.
Researcher:	Okay. You whispered it to yourself, but… mhm… Dalton, can you tell me what you're doing?
Dalton:	So, I did six groups, and then I put those together, and that was 18.
Researcher:	Okay.
Dalton:	And then I am doing six more groups, and I am going to do, and then I'm going to add all those up…
Researcher:	Okay, so when you say you make six groups, why are you making six groups?
Dalton:	Because there are six on each plate.
Researcher:	Okay. So, what does this represent?
Dalton:	One. One plate.
Researcher:	Okay, perfect. And how many total plates are you going to have?
Dalton:	[Dalton is grouping his cubes.] Four, so yeah four. One, two, three, six. One more. One, two, three, four, five, five, six. One more…

As Dalton solves this task, he utilizes multiple means to count cubes and fingers and keep track of groups of six by grouping cubes. This count all is quite rudimentary but allows him access to groups through his physical counting and reflection on the groups and the number of groups he constructed. Moreover, when Dalton is counting and grouping his cubes, he counts the first group with some halving/doubling activity (i.e., "One, two, three, six"), suggesting he is beginning to partition through some halving activity and then iterate through some doubling activity to develop the groups of six.

In the same session, Dalton is given the task, "*Estimate the size of one fair share if the strip is shared among ten people*." In Session 2, the students were given the same task but asked to share the strip of paper among five people. So, at the beginning of this session, we provided these shares to the students and asked both students how they could use the share they created for five people to help them create a share for ten people. Angela simply understood that the share should be smaller for ten people, but could not tell the researchers how much shorter, and suggested they use a trial-and-error type of strategy. Dalton struggled with the question until given a fair-share context for the question: "*If I gave you a KitKat bar and you had to split that with 5 people would you get the same amount then if you split it with 10 people?*" Dalton agreed with Angela that it would not be the same amount and that the share would be smaller. However, like Angela, he struggled to explain how much smaller this share should be for 10 people. Upon seeing Angela test out the share she creates by iterating this across the paper strip (representing the whole), Dalton explains that he is going to do "basically the same thing". As such, in the following excerpt, Dalton then iterates his share across the paper strip, counting each iteration. Halfway across the strip of paper, he pauses.

Dalton:	One, two, three, four [iterating his share across the paper strip].
Researcher:	Okay. So, looking at this, Dalton, do you think you're going to be able to

	…, you said how many are here? Okay, so how many more do you need?
Dalton:	five, … four, ten, five. [Dalton is counting the shares he iterates as he draws them on the paper strip.]
Researcher:	Five more? Do you think so? You're looking at this [referring to the paper strip Dalton is distributing his share across]. Do you think you're gonna have enough space for five more?
Dalton:	Yeah.

Here, Dalton is reasoning quantitatively about half of the paper strip and possibly doubling the five to make sense of what space he has left of the paper strip to iterate five more to create ten shares. This echoes what he has done in the whole number task, as he was halving through his partitioning activity before doubling through his iterating. In many ways, this activity also allows his to create two equal groups of five, similar to his creation of four equal groups of six in the whole number activity.

In Teaching Experiment Session 13, Dalton and Angela are given the fraction task, "*So at the fudge factory, I have this big, long slab of fudge."* [Researcher puts a strip of paper in front of both students]. *"Okay?"* [Dalton is moving his head up and down]. "*One slab of fudge, and they're poured into cooling trays. Once the fudge is cooled, the workers slice the slab of fudge into an equal number of parts and put those pieces into the display cases in the store. Carlos came into the store and selected* $\frac{2}{8}$ *of a piece of fudge. Okay. So here's the* $\frac{2}{8}$ *slab of fudge.*" [Researcher points to the piece of paper cut to represent $\frac{2}{8}$.] "*So here's my slab,*" [Researcher gestures to the slab], "*and it was cut, and this is* $\frac{2}{8}$ *of the whole, but he realized he actually wanted* $\frac{3}{8}$*, not just* $\frac{2}{8}$*. So, use this* $\frac{2}{8}$ *piece of fudge to create a* $\frac{3}{8}$ *piece. Okay?*" Dalton first takes the $\frac{2}{8}$ share and decides how to partition it to create a share.

Researcher:	Okay. And then Dalton, where's your $\frac{3}{8}$. [Dalton taps his fudge strip]. Okay. And why would it be there?
Dalton:	Well, so I would have to figure out $\frac{1}{8}$ [Dalton picks up the $\frac{2}{8}$ piece and taps it] the length of $\frac{1}{8}$, because I'm cutting at least $\frac{1}{8}$ over here [Dalton uses his hand to imitate a cutting motion on his fudge slab], but I need to figure out where to cut. [Dalton halves the $\frac{2}{8}$ share and cuts along the partitioned line to create two $\frac{2}{8}$ shares. He then iterates this three times along the larger fraction stip.]
Researcher:	Okay. So, where's your $\frac{3}{8}$ now?
Dalton:	Right here. [Dalton points at his fudge strip.]
Researcher:	Right here? Just that piece?
Dalton:	No, all of this is $\frac{3}{8}$. [Dalton gestures to three shares he created on his fudge strip.] That's just the line to stop.

Here, Dalton uses the halving and doubling he seemed to begin developing in Teaching Experiment Session 3 with both his whole number and fraction tasks. Differences that indicate changes to his schemes for whole number and fraction activity are his ability to anticipate how to halve and double when partitioning and iterating (cf. Tzur, 2007). For instance, when Dalton first engages in the halving partitioning and doubling iterating, he seems to do this in the moment and does not elicit this to make this strategy more sophisticated but seems to draw on this activity

implicitly. Moreover, he does not describe this halving and doubling activity as a strategy that would help him solve the task, but he seems to find himself using this when asked to reflect on his strategies. The activity not only develops and connects whole number and fraction activity but eventually allows Dalton insight into how he can solve complicated tasks more efficiently.

Relationships Between Angela's Whole Number and Fraction Understandings

Similar to Dalton, Angela also drew from halving and doubling activity when she solved whole number and fraction tasks. Interestingly, she uses this explicitly but often struggles to describe exact solutions, as she finds this activity alone to limit her capabilities. For instance, in Teaching Experiment Session 5, both students are given the task, "Show the card above (see Figure 1). Explain that there are 15 dots altogether, and that each row has five dots. How many rows are there? And how do you know?" Dalton first explains that he thinks there are five rows, each with five dots. Angela disagrees, explaining that having five rows "would pass" fifteen because "five and five is ten."

Angela:	And those are just two times. [Holds up two fingers in her right hand, pointer and middle fingers.]
Researcher:	Okay.
Angela:	And then, like you have to do it three more times. [Holds up three fingers on her left hand], and it'll be like twenty or twenty-one. It would be past fifteen.

Angela's doubling allows her to judge the reasonableness of Dalton's solution and provide a logical reasoning as to why his solution would not make sense. Limitations around this activity are Anegla's inability to ultimately answer the question posed to her. She could double five once but struggles to distribute a unit of five along her number sequence beyond ten, as she describes five and five making ten, but then is unable to explain the result of how ten and one more five can be composed. In fact, she may even be doubling again, considering "twenty or twenty-one" as a reasonable solution for five groups of five.

By Teaching Experiment Session 11, Angela begins distributing groups of five more multiplicatively, allowing her access to more efficient strategies when solving whole number tasks. For instance, the task posed to both students was, "*There are 4 teams in a school. Each team has 5 players. Joy said there are 35 players in all, because she skip-counted by 5 (Joy said, 5-10-15…" and kept going to 35). Is Joy correct? If not, how many teams did Joy count?"* Angela decides that Joy is not correct and explains that Joy is counting too many teams.

Angela:	So, he's trying to do six [holds up five fingers on her left hand and her pointer finger on her right hand to have six total], but it actually needs to be four. [Angela puts down fingers until she has four fingers left on her left hand]. So, five, ten, fifteen, twenty, [Points with her right pointer finger to her left pinky, ring, middle, and pointer finger] because I think it was twenty because there was four and was skipping by five's. [Angela gestures to her four fingers and does hand motions while talking].
Researcher:	Four what? So what do your fingers represent?
Angela:	There's four teams.

Here, Angela is coordinating groups of five beyond her original doubles activity, allowing her to now not only reason logically why Joy's solution does not make sense, but also to explain what the correct answer should be.

In the final handful of sessions, Angela is beginning to reason multiplicatively and engage in fraction tasks in relatively more sophisticated ways. For instance, in Teaching Experiment Session 14, both students are given the fraction task, "*At a fudge factory, the workers pour hot fudge into long rectangular trays and let it cool into slabs. Once the fudge has cooled, the workers slice the slab of fudge into an equal number of parts and put those pieces into display cases in the store. Carlos selected ⅜ of the slab of fudge. Alex saw Carlos's piece and said that it is three times more than what I want. What size piece of fudge does Alex want?*" Angela immediately multiplies three times eight and explains that the answer is "one-twenty-fourth".

Researcher:	Okay. Alright. So what was your reason for one-twenty-fourth? What do you think?
Angela:	I think one-twenty-fourth, because if you did eight times three would be twenty-four, and we kept the one, so it'll be one-twenty-fourth.
Researcher:	Oh, okay. Can you show me where that one-twenty-fourth looks like on here, like Dalton did for his? (Angela slides the one-eighth piece along the fudge strip three times). Okay, so the same, you're going to say the same size piece, but you're going to call that one-twenty-fourth.
Angela:	Yeah.

Here, Angela is using her multiplicative reasoning to solve this task and distributes a $\frac{1}{8}$ share across the paper strip three times, resulting in a $\frac{3}{8}$ size piece. However, she is using her multiplicative reasoning to also name this piece $\frac{1}{24}$, suggesting that she has not yet formalized her understanding of what $\frac{1}{8}$ three times can be represented as. Hence, we posit this understanding is participatory, meaning Angela does not understand structural relationships that she can anticipate in future activity.

By Teaching Experiment Session 16, Angela is given the task, "*At a fudge factory, the workers pour hot fudge into long rectangular trays and let it cool into slabs. Once the fudge has cooled, the workers slice the slab of fudge into an equal number of parts and put those pieces into display cases in the store. Carlos selected 5/8 of the slab of fudge. Alex saw Carlos's piece and said that it is five times more than what I want. What size piece of fudge does Alex want?*" In many ways, this task asks Angela to engage in the inverse of her activity when partitioning $\frac{5}{8}$ to conceptualize $\frac{1}{8}$ as five times smaller than $\frac{5}{8}$. After hearing the task, Angela talks aloud to conceptualize what the task is asking, stating, "Is it like they're trying to get 5 times smaller than 5 times?" and then, "So 5 times 5 is 25, but you don't times it because he wants less." Following these moments of her conceptualizing what the task is asking, Angela begins developing a strategy where she decides she needs to create the whole before "minusing". As Angela is partitioning the $\frac{5}{8}$ to create $\frac{1}{8}$ pieces, she reverts to whole number counting (one, two, three, four, five) and iterating to determine the size of the $\frac{1}{8}$ pieces in the $\frac{5}{8}$ share. In fact, in the end, she struggles to understand the new relationship between the parts and whole. This suggests that her ability to multiplicatively distribute fractional units across the whole and the composite fractional unit was limited based on her ability to inversely engage with multiplicative relationships.

Conclusion

Both students developed fractional activity from their whole-number activity; however, Angela appeared to engage in greater reorganization than Dalton, as reflected in her coordination of unit structures. In contrast, Dalton often developed fractional knowledge by mapping his

whole-number activity directly onto fractional situations. At times, Dalton also extended his whole-number activity through his fractional activity, suggesting that the two sets of relationships developed in tandem.

In consideration of both students' iterating and partitioning development across their whole number and fraction strategies, it seemed that both students began with and often revisited a trial-and-error type of iterating and partitioning. This allowed them to construct and reconstruct part-whole relationships while refining their reasoning to more robustly understand inverse part-whole relationships between their whole number and fraction understandings. Also, both students developed access to particularly challenging whole-number and fraction relationships through halving and/or doubling. This initially emerged as part of their strategy development before becoming an intentional part of their activity for accessing early forms of multiplicative relationships and fraction relationships. Finally, Angela began developing multiplicative connections between her whole-number and fraction understandings. As she developed these relationships, she struggled with the inverse reasoning required to transition from whole number multiplicative reasoning to fraction multiplicative reasoning. Exploring these forms of iterating and partitioning with whole-number and fraction tasks can better inform the literature as to why and when it is beneficial to leverage the iterating and partitioning strategies students with ISM demonstrate.

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EXPLORING THE ASSOCIATION OF ELEMENTARY AND MIDDLE SCHOOL STUDENTS' NUMBER SENSE AND MULTIPLICATION FLUENCY

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Students with mature number sense understand numbers and operations, use reasoning to notice patterns, and flexibly select the most effective and efficient problem-solving strategies (McIntosh et al., 1997; Yang, 2005). Despite being highlighted in national standards and policy documents (CCSS, 2010; NCTM, 2000), the association between elementary and middle school students' number sense and other important outcomes in mathematics education is not well specified. To begin to address this, we analyzed 3rd–8th grade students' (N = 1305) scores on a measure of mature number sense and their fluency with single-digit multiplication facts. We found that students' multiplication fact fluency was significantly associated with their mature number sense, controlling for school and individual demographic factors, but the strength of this association differed significantly by students' grade-level.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education, Middle School Education

One goal of mathematics education reforms in the United States is to help students develop number sense, which involves making sense of numbers and operations, using flexible reasoning to notice patterns, and selecting the most effective and efficient problem-solving strategies (CCSS, 2010; National Council of Teachers of Mathematics [NCTM], 1989, 2000; National Research Council [NRC], 2001). For example, students exhibiting number sense understand $8 \times 6 = 48$ as a logical extension of $8 \times 5 = 40$ by representing 48 as 40 with an additional group of 8. Rather than seeing them as disparate facts, students with number sense have a connected understanding of numbers and operations. Despite being a central goal in mathematics education, there is limited evidence on the typical progression of students' number sense across elementary and middle school, and it is unclear if there is an association between students' number sense and other outcomes in mathematics education, such as knowledge of single-digit arithmetic. In this study, we aimed to begin addressing these questions on how 3rd–8th-grade students' number sense relates to their fluency with single-digit multiplication facts and whether the strength of this association differs based on students' grade or number sense component.

Background and Theoretical Framework

In the late 1980s, government-commissioned reports (NRC, 1989) and national curriculum frameworks (e.g., NCTM, 1989) highlighted "number sense" as a core objective of K-12 mathematics education. However, there has been a significant divergence in how researchers

operationalize ‘number sense’ and examine it with students (Berch, 2005). This paper draws upon Whitacre et al.’s (2020) use of the adjective “mature” to differentiate the construct studied here from both approximate (e.g., Dehaene, 2001) and early number sense (Jordan et al., 2022). We define *mature number sense* in line with McIntosh and colleagues (1992) as the ability to use an understanding of both numbers and operations to flexibly and efficiently solve problems. Importantly, this is not to imply that maturity in number sense naturally develops with age, that the number sense of 3rd graders is *fully mature*, or that our construct of interest is different from prior number sense work in mathematics education (e.g., Markovits & Sowder, 1994). Rather, it is solely to avoid a case of polysemy in the literature.

Perhaps to facilitate operationalization in empirical research, mature number sense has often been subdivided into components (NCTM, 1989; Reys et al., 1999; Yang & Lin, 2015). Building from this prior work and in consultation with teacher educators, researchers, and mathematics teachers, we specified a four-component framework of mature number sense (Kirkland, Cheng, et al., 2024; Kirkland et al., 2025) with mature number sense as the overarching latent construct. Student response data from our assessment reflect this framework. A bifactor model with each component as a specific factor best fit the response data over theoretically related models.

1. Understanding fundamental number concepts and number magnitude: Students demonstrating strong mature number sense understand fractions, decimals, whole numbers, and their corresponding magnitudes. Students efficiently estimate results using concepts such as unit fractions and the value of digits within a number.
2. Using multiple representations of a number: Students demonstrating strong mature number sense are able to translate among multiple representations of rational numbers efficiently and flexibly to solve problems. Students use this understanding to convert between rational numbers represented pictorially, as fractions, or as decimals, to make sense of numerical situations.
3. Understanding the effect of operations on numbers: Students demonstrating strong mature number sense recognize how the four core operations affect whole numbers as well as fractions and decimals. They understand that patterns observed in operations with whole numbers may not hold true for fractions and decimals between 0 and 1. Students use this understanding to efficiently estimate computational results and ensure the results make sense given the relationship between operations and rational numbers.
4. Understanding mathematical equivalence: Students demonstrating strong mature number sense understand the equal sign as a relational symbol, reflecting that the two sides of an equation are equal and interchangeable. Students who understand equivalence recognize patterns across the equal sign and use this relational thinking to flexibly and efficiently solve problems (c.f., Jacobs et al., 2007). Students use the properties of the four core operations, such as commutativity and distributivity, to generate equivalent expressions.

Mathematics educators care about improving students’ mature number sense, in part, because they argue it lays a foundation for better learning of more advanced mathematics (Boaler, 2019; Yang & Wu, 2010). However, there is limited evidence that mature number sense is positively correlated with other constructs (summarized in Kirkland, Guang, et al., 2024). In this study, we focused on a key component of upper elementary and middle school students’ mathematical proficiency: their fluency with single-digit multiplication facts.

Rationale for Current Study

Multiplication is traditionally introduced to US students in 3rd grade, with the goal that students know all products of single-digit factors from memory by the end of that year (CCSS, 2010). This retrieval of facts from memory is often referred to as *multiplication fluency* or *automaticity.* As a subset of students' broader arithmetic fluency (i.e., McNeil et al., 2025), we define *multiplication fluency* as the efficient and accurate mental retrieval (i.e., correct response in ~3 seconds) of products of single-digit numbers. The importance of students' multiplication fluency has been debated recently across special education and mathematics education (Codding et al., 2023; NCTM, 2023). Amidst these debates over how best to *teach* arithmetic facts, there is consistent evidence that students' *knowledge* of single-digit multiplication facts is a significant predictor of their future mathematics learning. Nelson et al. (2016) found that students' fluency with multiplication predicted later math success, even extending beyond the grade levels where these facts are formally taught. Hansen et al. (2015) found 6th grade students' multiplication fluency significantly predicted their proficiency with fraction computation across the four arithmetic operations. Additionally, McGlaughlin et al. (2005) found that for college students with identified mathematics disabilities, fact fluency performance significantly predicted their success in a college algebra course. Thus, supporting students in their development of multiplication fluency is a crucial outcome in K-8 mathematics (McNeil et al., 2025).

The association between students' mature number sense and their multiplication fluency is inconclusive. Recently, we found that 3rd–5th grade school students' multiplication fact fluency was strongly positively correlated ($r = 0.66$) with their mature number sense (Kirkland et al., 2023). However, the within-grade-level sample sizes ($N = 47, 53, 53$ respectively) limit the generalizability of these findings across grade levels, and there are significant developmental differences in upper elementary. After being introduced to single-digit multiplication in 3rd grade, students are expected to apply this knowledge to more complex operations in 4th and 5th grade, including multidigit multiplication and division (e.g., long division and partial quotients), generating equivalent fractions, finding a common denominator for fraction comparison, factoring, and place value understanding (CCSS, 2010). Third grade is also when students are first introduced to division and fractions as numbers. Therefore, we aimed to collect a larger sample in this study to more fully explore whether the association between students' mature number sense and their multiplication fluency changes across these grade levels. In addition, the association between middle school students' multiplication fluency and mature number sense has yet to be examined in the literature. As noted above, prior work has found advanced fluency with multiplication facts to be predictive of later math proficiency, beyond the grade levels when the facts are formally taught (e.g., Nelson et al., 2016). Therefore, we aim to test how mature number sense relates to students' fluency with multiplication at both the upper elementary level, when students are just learning their multiplication facts, and in middle school, when students should be fluently using those facts to solve more advanced mathematical problems.

To start to address this gap in the literature, we aimed to study if the correlation between mature number sense and multiplication fluency was moderated by students' grade level, controlling for both individual and school level demographic factors. Our research questions were as follows: (1) How does 3rd–8th-grade students' mature number sense relate to their fluency with multiplication? and (2) Does the strength of this association differ based on students' grade level or the components of mature number sense?

Methods

Participants

In August 2025, our research team sent invitations to teachers and school leaders in our local and national network of educators to participate in an ongoing longitudinal study. For the primary analysis of interest in that longitudinal study, our power analysis indicated that a minimum of $N = 125$ for each grade level or a total sample of $N = 750$ students was necessary. Thirty-five teachers from eleven different schools across the United States agreed to have their math classrooms participate. For the timepoint 1 data presented here, 1305 students from these classrooms participated (45% self-identified as female; 44% self-identified as White, 11% Hispanic or Latino, 9% Multiracial, 8% Black, 2% Asian, and 26% as unlisted or unsure).

Procedure and Measures

Teachers administered the assessments described below in the context of their usual mathematics classrooms. Students completed the measures at a time determined by the teacher via a Qualtrics survey on school-provided devices (e.g., Chromebooks) in a set order: assessment of mature number sense, then multiplication fluency. All assessments for timepoint 1 were completed between mid-August and the first week of October.

Brief assessment of mature number sense (Kirkland et al., 2025). This is an electronic, 24-item multiple-choice test of students' mature number sense, aligned with our theoretical framework. Items are designed specially to assess students' number sense rather than general mathematics achievement. Each item has a time limit of 60 seconds, and students are not allowed to use paper or pencil to discourage using algorithms. Two different assessments were used: one for elementary school (3rd–5th grade) and another for middle school (6th–8th grade). There were eight common linking items across the two forms, allowing for a vertically scaled score (see Figure 1). Scores were calculated as the total number correct (sum score) as well as a latent trait estimate of students' number sense using an expected a-posteriori (EAP) method with a multivariate normal prior distribution (Chalmers, 2012) with a mean of 0. Using the mean and sigma linking method (Embretson & Reise, 2000) the latent trait estimates for 3rd–5th grade students' mature number sense are transformed onto a common scale with 6th–8th grade students (i.e., a score of 0 reflects performance at the mean level for middle school students). This transformed latent trait estimate score is used in this study to allow for comparisons across the two forms. The validation analyses (Kirkland, Cheng, et al., 2024; Kirkland et al., 2025) provide strong validity evidence for the intended use of student scores supported by expert reviews, factor analyses, student retrospective think-aloud protocols, and item response theory analyses.

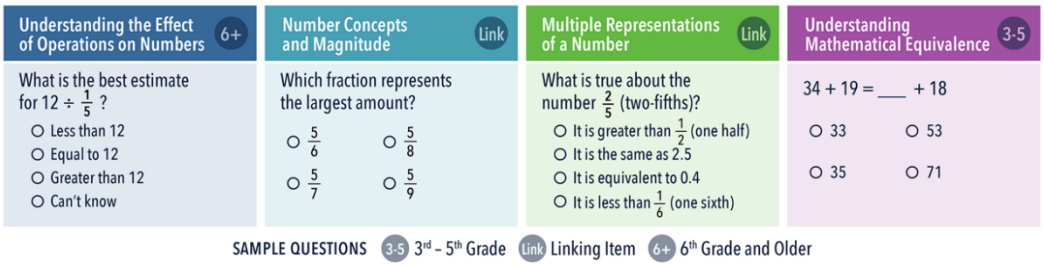


Figure 1: Sample Questions from Brief Assessment of Mature Number Sense

Multiplication fact task (Burns et al., 2015; Nelson et al., 2013). Participants were given one minute to complete this measure, which includes all combinations of single-digit

multiplication facts using the digits 1–9. Students received 1 point for each correct response, with possible scores ranging from 0–81 (observed range of 0–39 in this study). The order of the facts was randomly predetermined and then kept standard for all participants.

Data Analyses

To address Research Question 1 on how students' mature number sense relates to their multiplication fluency, we first calculated the overall Pearson correlation between student scores on the two measures. We then ran a series of regression models, with students' mature number sense as the outcome variable, testing if students' multiplication fluency is significantly predictive of their number sense, controlling for demographic factors. The first model was a simple linear regression with multiplication fluency as the only predictor. The second model included all demographic variables (gender, race, grade, school) and was a Hierarchical Linear Model (HLM) or mixed-effects model to best model the naturally occurring nesting structure for the participants within the 11 different schools (Raudenbush & Bryk, 2002). For the demographic variables, the largest group was coded as reference, except in the case of grade-level. For grade-level, 3rd grade was coded as the reference to examine changes developmentally.

To address Research Question 2 on whether the association between the two target constructs differs by grade level and component of mature number sense, we first calculated the zero-order correlations within each individual grade level and within the two developmental levels (elementary vs. middle school). To test if the association is significantly different by grade-level, we tested for a significant categorical moderation effect of grade level on fluency. We then probed the moderation effect and tested the pairwise comparisons of each grade level's slope, adjusted for multiple comparisons using Tukey's HSD (Tukey, 1953).

To examine if the association between multiplication fluency and mature number sense differs by component, we first calculated sum scores for each component (NCM: Number Concepts and Magnitude, MR: Multiple Representations, EO: Effect of Operations, EQ: Equivalence). We could not use the latent transformed score for this analysis as the latent subscores have not been shown to be reliable given the small number of items per component. We calculated the zero-order correlations between component subscores and multiplication fluency for both elementary and middle school. We then ran a linear regression model with multiplication fluency as the outcome to allow for all four components to be predictors. From this analysis, we compared each regressor's semi-partial correlation coefficient to analyze each component's importance to multiplication fluency (Darlington & Hayes, 2016).

Results

We first begin with overall student performance, summarized by grade level in Table 1. Students solved on average 11.65 (49%*)* items correctly on the brief assessment of mature number sense (SD = 5.21). On their respective forms, upper elementary students solved fewer correctly (M = 9.82, SD = 4.6) compared to middle school students (M = 13.04, SD = 5.22). Overall, students' mean transformed latent estimate of their mature number sense was –1.22 (SD = 1.65). On average, students solved 12.55 (SD = 8.08) single-digit multiplication problems correctly. Once again, upper elementary students solved fewer correctly (M = 7.19, SD = 6.11) than middle school students (M = 16.63, SD = 6.93).

To begin to address Research Question 1, we calculated the zero-order correlations between the two constructs in the whole sample. As predicted, students' mature number sense and multiplication fluency were strongly positively associated (r = 0.73), and this was significant, ($t(1303) = 38.12$, $p < .001$). We then conducted the base linear regression to further explore the

association. An increase of one correct multiplication fact is associated with a 0.15-point predicted increase ($b = 0.15$, $p < .001$) in a student's mature number sense latent trait estimate. This significant association holds in the full mixed-effects model with all relevant demographics included, ($b = 0.06$, $p < .001$). In the whole 3rd–8th grade sample, students' multiplication fluency and mature number sense are significantly related.

Table 1: Correlation Between Number Sense and Multiplication Fluency by Grade

Grade Level	*N*	Mature Number Sense		Multiplication Fluency		*r*
		M	*SD*	*M*	*SD*	
3rd Grade	159	–3.38	0.78	1.92	2.58	0.49
4th Grade	222	–2.78	1.07	6.61	4.35	0.34
5th Grade	183	–1.89	1.38	12.47	5.82	0.42
6th Grade	345	–0.22	0.87	15.96	6.24	0.51
7th Grade	165	0.06	0.92	18.13	6.59	0.49
8th Grade	231	–0.13	0.90	16.57	7.95	0.49
Overall	1305	–1.22	1.65	12.55	8.08	0.73

To address Research Question 2, we then examined how this association may differ by grade level. The grade-level specific correlations are summarized in Table 1. Descriptively, we see the lowest correlation among 4th-grade students ($r = 0.34$) and the highest with 6th grade students ($r = 0.51$). To test if any grade-level differences are significant, we ran a mixed-effects model with students' mature number sense as the outcome of interest, their multiplication fluency, grade level, and school as predictors, and a grade level by multiplication fluency interaction term. Each grade level's (4th–8th) interaction term with multiplication fluency was significant (model summarized in Table 2). The full model with the interaction term fit significantly better than the base model, $\chi(5) = 29.2, p < .001$, suggesting that the strength of the association between students' multiplication fluency and their mature number sense is moderated by grade level. That is, there are grade-level differences in the strength of the association between the two constructs. To probe this interaction, we calculated and compared the slope term for students' multiplication fluency at each grade level (see Figure 2). We tested all of the pairwise comparisons for grade-level slopes (e.g., 3rd vs. 4th, etc.) and corrected for multiple comparisons using Tukey's HSD. In this analysis, the slope for 3rd grade students' fluency was significantly more positive than 4th grade ($p < .05$), 6th grade ($p < .01$), 7th grade ($p < .01$), and 8th grade ($p < .001$). For 5th grade students, the slope of their fluency was significantly more positive than 6th ($p < .05$) and 8th ($p < .01$) grade students. No other pairwise comparisons were significant.



Figure 2: Multiplication Fluency and Mature Number Sense by Grade

Table 2: Summary of Regression Model Predicting MNS with Grade-level Interactions

Variable	*B*	*SE*	*t*	*p*
(Intercept)	–4.00	0.18	–22.60	<0.001
Multiplication Fluency	0.15	0.02	6.26	<0.001
4th Grade	0.55	0.13	4.25	<0.001
5th Grade	0.86	0.16	5.31	<0.001
6th Grade	2.83	0.15	18.65	<0.001
7th Grade	2.78	0.21	13.39	<0.001
8th Grade	3.02	0.16	18.87	<0.001
Fluency * 4th Grade	–0.08	0.03	–2.87	<0.01
Fluency * 5th Grade	–0.06	0.03	–2.06	0.03
Fluency * 6th Grade	–0.10	0.03	–3.78	<0.001
Fluency * 7th Grade	–0.09	0.03	–3.56	<0.001
Fluency * 8th Grade	–0.10	0.03	–4.11	<0.001

To analyze if the association observed above differs by the component of mature number sense, we calculated the zero-order correlations between component subscores and students' multiplication fluency. Since the items across components differ in the two forms, we analyze elementary school students first and then middle school students. For elementary students, the descriptively highest correlation was with multiple representations ($r = 0.51$), followed by number concepts and magnitude ($r = 0.49$), equivalence ($r = 0.39$), and effect of operations ($r = 0.34$). In middle school, a different pattern was observed. The correlations were all more similar, with equivalence ($r = 0.46$) the descriptively largest, followed by number concepts and magnitude ($r = 0.45$), multiple representations ($r = 0.44$), and effect of operations ($r = 0.41$).

To probe the relative strength of each component's association with multiplication fluency, we ran linear regression models for each developmental level. For elementary students, all four components were significant predictors in the model. To compare them directly, we calculated the semi-partial correlation for each component with multiplication fluency. Similar to above, we found that, for elementary students, multiple representations ($sr = 0.23$) and number concepts and magnitude ($sr = 0.20$) were the two components most strongly linked with multiplication fluency. Equivalence ($sr = 0.11$) and effect of operations ($sr = 0.08$) were lower. With middle school students, all four components were significant predictors, but equivalence ($sr = 0.17$) was the strongest related of the four. Multiple representations ($sr = 0.12$) and number concepts and

magnitude ($sr = 0.11$) were next, with effect of operations ($sr = 0.07$) the lowest. Overall, we see that the components differ in strength of association with multiplication fluency between upper elementary and middle school students, with multiple representations the strongest association in upper elementary and equivalence the strongest for middle school.

Discussion

From the evidence presented here, students' mature number sense and multiplication fluency are strongly positively associated across upper elementary and middle school ($r = 0.73$ in the whole sample). This strong positive association held when controlling for both school-level and individual-level demographic factors. Within grade levels, however, the strength of this association differed significantly. Students' multiplication fluency and mature number sense were most positively associated in 3rd grade and then in 5th grade. In terms of the components of mature number sense, surprisingly, students' understanding of the effect of operations on numbers was the least positively related to their multiplication fluency. For elementary students, their use of multiple representations of numbers correlated most strongly with their fluency. For middle school students, it was their understanding of mathematical equivalence. Together, this all suggests that there is a strongly positive association between 3rd–8th grade students' mature number sense and their knowledge of single-digit multiplication facts, but there are developmental and component-specific differences in the strength of that association.

Across 3rd–8th grade, the patterns observed with 3rd grade students' mature number sense and multiplication fluency were distinct from those of other grades. Perhaps not surprisingly, 3rd grade students had the lowest level of mature number sense ($M = -3.38$) and multiplication fact fluency ($M = 1.92$ correct). As discussed above, 3rd grade is a unique time in students' mathematical development. Students are first introduced to multiplication, division, and fractions as numbers. It may be that students need the full 3rd grade year to acquire new mathematical knowledge in these domains, consolidate this knowledge, and connect it with their initial number sense understanding with whole numbers. The results presented here reflect start-of-the-year knowledge, so 3rd grade students may not yet have begun practice with single-digit multiplication. Despite this, the strength of the association between the two constructs was strongest in 3rd grade. We interpret this as reflective of how few 3rd grade students displayed multiplication fact knowledge. Of the 159 students in 3rd grade, only 20 (13%) correctly answered more than three multiplication facts. For this group, their mean number sense was –2.55, almost a full standard deviation above the 3rd grade mean. Their early knowledge of multiplication as an operation may have allowed them to access more items on the number sense assessment, leading to stronger performance. Further analysis of this association longitudinally across the rest of 3rd grade will shed light on the observed findings in this study.

Among the four components of mature number sense, students' understanding of the effect of operations on numbers had the weakest association with their multiplication fluency. This was surprising to us given many effects of operations items directly assess multiplication (e.g., "What is the best estimate for $26 \times \frac{3}{4}$?", "What is true about 18×12?"). However, many of these items include operations with fractions and decimals, rather than solely whole numbers. It may be that students with higher multiplication fluency have had more exposure to repeated traditional arithmetic practice and are, therefore, more susceptible to whole number bias. The effect of operations items directly assess several common "rules that expire" from elementary (e.g., Karp et al., 2014), such as "multiplication makes a number bigger." The evidence here suggests that,

while positively associated, strong retrieval of single-digit multiplication facts does not guarantee an understanding of multiplying and dividing rational numbers (e.g., Siegler & Pyke, 2013).

Overall, we have evidence here on the association between 3rd–8th grade students' mature number sense and their knowledge of single-digit multiplication facts. We see that the strength of this association differs by grade level and by mature number sense component. Though evidence from only one timepoint with online measures, this is an important first step in more fully understanding how these two important constructs in mathematics education develop in upper elementary and middle school. Future work can extend this analysis longitudinally to map their natural progression over the course of a school year. By continuing to characterize mature number sense's natural progression, researchers can better identify the pivotal developmental windows for supporting student sense-making with numbers and operations.

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STUDENTS' MULTIPLICATIVE REASONING IN DECIMAL MULTIPLICATION

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Current research on decimal multiplication suggests that students rely on memorized rules to apply the Decimal Placement Rule, with little to no scholarship examining the role of their multiplicative reasoning on such tasks. This study explored students' engagement with decimal multiplication tasks and how their actions corresponded with their multiplicative reasoning. Eye-tracking enabled clinical interviews were conducted with 28 fifth and sixth grade students. Findings indicate students' multiplicative reasoning corresponds to both their application of the Decimal Placement Rule, and their engagement with place value in decimal multiplication.

Keywords: Elementary School Education; Number Concepts and Operations

Multidigit multiplication is an understudied topic (Fuson, 2003; Hickendorff et al., 2019), with multiplication of decimals receiving even less attention in the research literature. Current scholarship on multidigit multiplication of whole numbers suggests that students understanding of partial products, which involves a place value based application of the Distributive Property, is a conceptually difficult concept for students (Gill et al., 2025; Hickendorff et al., 2019; Izsàk, 2005). Interestingly, others have observed that place value is an important factor in students' decimal multiplication as well (Larsson et al., 2017; Lortie-Forgues & Siegler, 2017), but we have been unable to identify literature examining the relationship between these. Those who have studied children's decimal multiplication have found that students often follow rotely learned procedures, with many errors occurring when decimal problems aren't presented in recognized formats (Hiebert & Wearne, 1985; Larsson et al., 2017). Even when students demonstrate strong whole number multiplicative reasoning, they have difficulty with decimal multiplication (Larsson et al., 2017). What is less understood is how students' varying multiplicative reasoning with whole numbers associates with their decimal multiplication, and how such multiplicative reasoning is associated with students' reasoning about decimal multiplication. Given the role of place value in both whole number and decimal multiplication, there is a need to understand how students' reasoning with each intersects. Thus, the purpose of this study was to explore how fifth and sixth grade students, with different levels of multiplicative reasoning, reasoned through decimal multiplication tasks.

Theoretical Perspective

Multiplicative Reasoning & Whole Number Multiplication

Multiplicative reasoning emerges when individuals can conceptualize a number as being both one unit and an iterated composite of multiple units (Hackenberg, 2010; Kosko, 2019; Nunes et al., 2010; Tzur et al., 2021). We discuss this unit coordination through Scheme Theory with particular focus on Hackenberg's (2010) description of three multiplicative concepts. The first multiplicative concept (MC1) describes children's ability to anticipate one level of units and construct the second level. A child at MC1 may solve 6×6 by iterating 6 six times until reaching

36. Such children may also use doubling or repeated addition as they learn to coordinate units in more nuanced ways (Ambrose et al., 2003; Hickendorff et al., 2019; Lampert, 1986). In solving 16×6, the same student may iterate 16 six times (Gill et al., 2025). At the second multiplicative concept (MC2), a child anticipates two levels of units and constructs the third level in activity. Such children may solve 16×6 by partitioning 16 into 10 and 6 and multiplying each by 6; what we refer to as meaningful use of two partial products (Gill et al., 2025; Izsàk, 2005). Students learn to work with partial products by integrating their understanding of 10s multiples with the distributive property to break apart a two-digit number and multiply it by a single-digit number (Ambrose et al., 2003; Flores et al., 2020; Izsàk, 2005).

At the third multiplicative concept (MC3), children can anticipate three levels of units (Hackenberg, 2010). It is at this point that a child can meaningfully use four or more partial products (Gill et al., 2025). Many conjecture that by using four partial products, a child can generalize to partial products with more place-value components (Flores et al., 2020; Hickendorff et al., 2019). A child at MC3 may solve 16×56 by coordinating four separate expressions (10×50 + 10×6 + 6×50 + 6×6). Key in distinguishing between these levels of multiplicative reasoning is that children can be at a lower level and solve a more complicated task using a rotely learned procedure. Thus, they may be able to fluently use an algorithm rotely, but may not be able to explain the place-value meaning of certain operations or relate the operations to a visual or concrete representation (Kosko, 2024; Gill et al., 2025).

Decimal Multiplication

The few studies of children's decimal multiplication often focus on students' procedural, not conceptual, understanding. Hiebert and Wearne (1985) conjectured students learn decimal multiplication procedurally and disconnected from underlying concepts. They found that "students' procedures are indeed syntactically based" (p. 200), with decimal operations viewed by students as moving symbols around on paper. Yet, there were several instances where students deviated from Hiebert and Wearne's (1985) predicted error types. Specifically, students used the Decimal Placement Rule inconsistently, leading Hiebert and Wearne (1985) to state "we cannot explain the conditions under which such a rule is applied" (p. 193). More recently, Larsson et al. (2017) reported case studies of two students followed from fifth to seventh grade. Despite both students' competence with multidigit whole number multiplication, they consistently had difficulty explaining decimal multiplication. Indeed, one student described decimal multiplication as "a completely different system not susceptible to distributivity" (Larsson et al., 2017, p. 10). Others have observed children's difficulty with decimal multiplication (Greer, 1987; Lortie-Forgues et al., 2017; Tian et al., 2021) but provide similarly limited descriptions of children's multiplicative reasoning therein. Rather, the literature suggests that place-value is an issue, but there is no indication of differences in how children coordinate multiplicative units in such contexts. The central goal of the current study is to examine children's approaches and reasoning with multidigit whole number and decimal multiplication when these children are at different documented levels of multiplicative reasoning.

Overview and Research Questions

This paper explores the intersection of two understudied areas of scholarship: multidigit multiplication (Hickendorff et al., 2019) and decimal multiplication (Larsson et al., 2017). Scholars generally agree that many students apply the Decimal Placement Rule rotely (Hiebert & Wearne, 1989) and that students' difficulties with decimal multiplication are often associated with their conceptions of multiplication and place value in decimal multiplication (Larsson et al., 2017; Lortie-Forgues & Siegler, 2017). Given that children's ability to engage with partial

products is facilitated by their multiplicative reasoning (Gill et al., 2025), we sought to better understand how children's ability to engage with partial products corresponded to how they engaged with their multiplication of decimals. Thus, we sought to answer the following question: *How do students reason mathematically when engaged in decimal multiplication tasks, and how do their actions correspond to their multiplicative reasoning?*

Methods

Participants & Protocol

Participants included 28 fifth and sixth grade students enrolled in different schools in a Midwestern US state. Participants were part of a larger study of 1,451 students across grades 4-6. Students completed grade-level specific written packets that included the Multiplication Reasoning Assessment (MRA) to assess which multiplicative concepts students operated. The MRA includes substantial validity evidence, including analysis of students' written work and interviews relating their scores to their reasoning with multiplication and division tasks (Kosko, 2019). The packets also included 10 multiplication tasks that had four common items across grade levels. Fifth and sixth graders completed: 21×4; 19×16; 37×34; 86×49; 789×382; 81×0.36; 24×0.32; 5.7×93. Fifth graders also completed 457×21 and 2083×35 while sixth graders completed 1.43×0.006 and 0.15×0.85. We used MRA scores, as well as written responses to canonical multiplication tasks, to purposefully sample a cross-section of 18 fifth and 10 sixth grade students. Participants wore Pupil Neon eye-tracking glasses to capture eye gaze data. Students were asked to solve a set of tasks one at a time and then to explain their reasoning for how they solved each task. Fifth grade (14×2, 8×42, 43×38, & 0.54×25) and sixth grade students (8×42, 43×38, 0.54×25, & 0.81×2.4) solved both whole number and decimal multiplication tasks. Follow-up questions focused on students' procedures, the place-value meaning of certain digits, and/or how the child might visually represent their procedures.

Analysis & Findings

We implemented a transformation variant mixed methods design where qualitative themes were transformed into ordinal data for quantitative analysis (Creswell & Plano Clark, 2018). This study emphasizes the qualitative findings, while the quantitative analysis provided statistical support for observed trends in qualitative analysis.

Qualitative Analysis & Findings

We used a descriptive-interpretive qualitative approach to examine interview data from video recorded with mobile eye-tracking headsets (Elliott & Timulak, 2021). Specifically, we examined students' approaches to solving multiplication problems, focusing on observable actions that included writing/drawing, discourse, gesture, and eye-gaze behavior. Each author conducted multiple rounds of open coding before the research team discussed observations and developed initial themes based on this analysis. Four dominant themes emerged across all 28 participants: Arithmetic Decimal Operator (n = 10); Rote Decimal Counter (n = 11); Multiplicative Decimal Operator (n = 5); and Meaningful Decimal Counter (n = 2). Below we describe four brief cases of each theme focusing on Josefa, Irven, Hunt, and Faith.

Arithmetic decimal operator. Ten students relied on arithmetic approaches to solve decimal multiplication problems and in placing the decimal in the product. Quite often, this involved multiplying numbers by column, doubling, or repeated addition before placing the decimal in the product directly below where it was aligned in the multiplication problem. Such actions mimicked rules used in decimal addition. Josefa, a sixth grade student operating at MC1, provides a useful case of students demonstrating this theme. At MC1, Josefa could anticipate one level of units and construct the second level in activity. For example, early in the interview

Josefa used doubling approach to solve 8×42 by writing four pairs of 42 and adding each pair until eventually finding a sum of 336. The approach demonstrated an ability to iterate the composite 42 eight times but through a process of partitioning the multiplier into four 2s. However, Josefa did not demonstrate an ability to partition the multiplier by place value, which is typically associated with students at MC2 (Gill et al., 2025). Instead, for tasks where multiplicands included more than one digit, she typically would stack the numbers vertically and multiply within a column. This was evident in her decimal multiplication. For 0.54×25, Josefa placed 0.54 above 25, aligning the 4 and 5 digits on the right (see Figure 1) and multiplied the digits within each column. Josefa confirmed this approach explaining "…Five times four is twenty so you put the two over the five. Then, five times two [points to the 5 and 2 in column] is ten plus two is twelve.". When a decimal was present in both multiplicands, Josefa would align by decimal in both her written work and in the interviews. Across the decimal tasks, Josefa's eye gaze consistently moved along the column structure of tasks (see Figure 1). For example, for 0.81 ×2.4, Josefa multiplied 1×0 (inferring a 0 below the 1), 8×4 within that column, and so forth. Her eye-gaze follows the same pattern when placing the decimal between the 5 and 2 in her product. Josefa explained her decimal placement stating, "…they are always supposed to line up." Based on Josefa's and other students' interviews, we infer this approach to be a direct application of rules learned for decimal addition and subtraction. Interestingly, the approach was most common amongst pre-multiplicative and MC1 students.

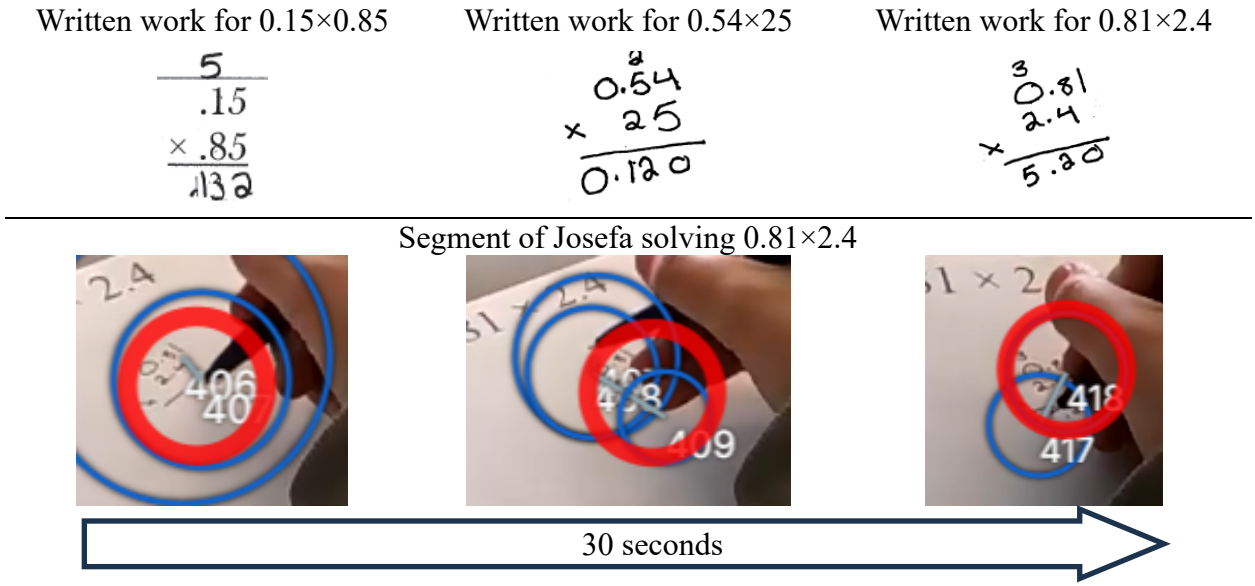


Figure 1: Written Work and Segments from Interview for Josefa (MC1)

Rote decimal counter. This theme describes students' actions wherein they used the Decimal Placement Rule (whether such actions were mathematically correct or not) but could not provide a rationale for it. Irven, a sixth grader, was one of 11 students demonstrating this theme. He operated at MC2, meaning he anticipated two levels of units and constructed three in activity. For example, early in the interview when multiplying 8×42 with the Box Method, he was able to break 42 into 40 and 2 to multiply each by 8 via disembedding by place value. This was evident in his explanation for why 8×40 allowed him to "add a zero" for 320. With 40 aligned by place value directly above 320, he stated "it is like a 10, but you don't really see it…"

As he stated this, his eye-gaze quickly went between the two 0s which we inferred was evidence of him disembedding 40×8 as (10×4)×8 because the 40 was simultaneously described and treated as four 10s as well as 40 without the need to construct the relationship in activity. This allowed Irven to also solve whole number problems with four partial products by operating on one set of two partial products at a time. When solving the decimal multiplication tasks, Irven solved these tasks as he did whole number multiplication tasks, until it was time to place the decimal point. Irven's eye gaze shifted to the expression 0.81×2.4 as he focused first on the 81 in .81 and then on the 2 in 2.4. As he did this, his eye gaze shifted back and forth between the given expression and his calculated product. He then counted from left-to-right and paused his pen and eye gaze between the 4 and 4in 1944 before counting from right-to-left and placing a decimal for an answer of 1.944. When asked about why he chose the latter of the two places, he indicated it was a rule that he had learned. When pressed why the rule worked, he said, "I just know you're supposed to do this" and motioned with his pen three counted spaces from right-to-left on the digits 1944. Thus, Irven knew the Decimal Placement Rule, but was not consistent in the directionality for counting. We infer this to mean he did not understand the role of place value in the Decimal Placement Rule but did understand the role of place value for partial products.

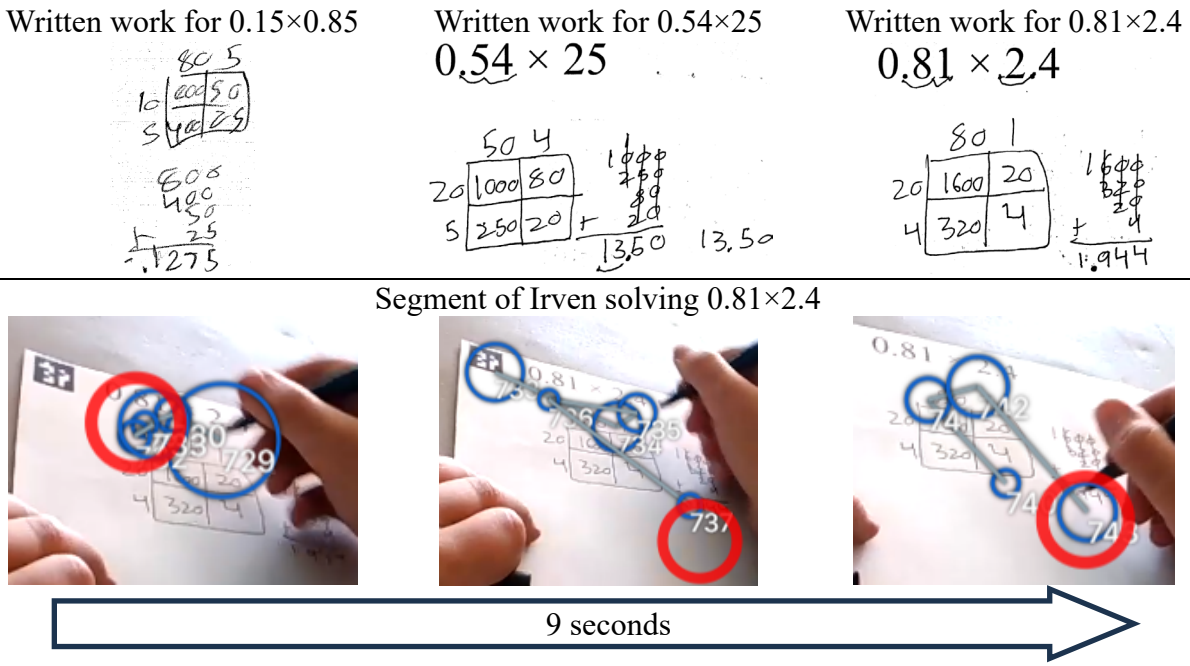


Figure 2: Written Work and Segments from Interview for Irven (MC2)

Multiplicative decimal operator. Students demonstrating this theme did not rely on counting the number of decimal places to determine the position of the decimal point, but instead attempted to solve the problem using place value. Hunt, a fifth grader operating at MC3, was one of five students demonstrating this theme. Thus, Hunt was able to anticipate three levels of units. For example, when solving 43×38, Hunt disembedded 38 into 30 and 8, and 43 into 40 and 3, according to their place values, and then wrote down the expressions for each partial product. As he explained this procedure in the interview, his eye-gaze shifted between the corresponding digits in 38 and 43, indicating he considered the 3 digit in 38 as 30 and the 4 in 43 as 40,

simultaneously. When solving the decimal multiplication tasks, Hunt solved tasks as he did whole number multiplication tasks, until it was time to place the decimal point. For example, when solving 0.54×25, he wrote four partial products: 0.50×20, 0.04×20, 0.50×5, and 0.04×5. When focusing on the partial product 0.04×5, he quickly wrote down 20 and spent about 3 seconds (see Figure 3) focusing on the particular location of the digits in relation to the decimal's location before placing the decimal before the 2 for 0.20. Important here is that Hunt's gaze did not move in a counting motion but instead moved between the whole number product produced (20) and the location of decimals in the other multiplicand. In explaining his work, Hunt referenced the whole number place value of the product as a rationale for where the decimal would be placed, which appeared similar to approaches used by Rote Decimal Operators, but with a focus on the place value meaning of multiplication.

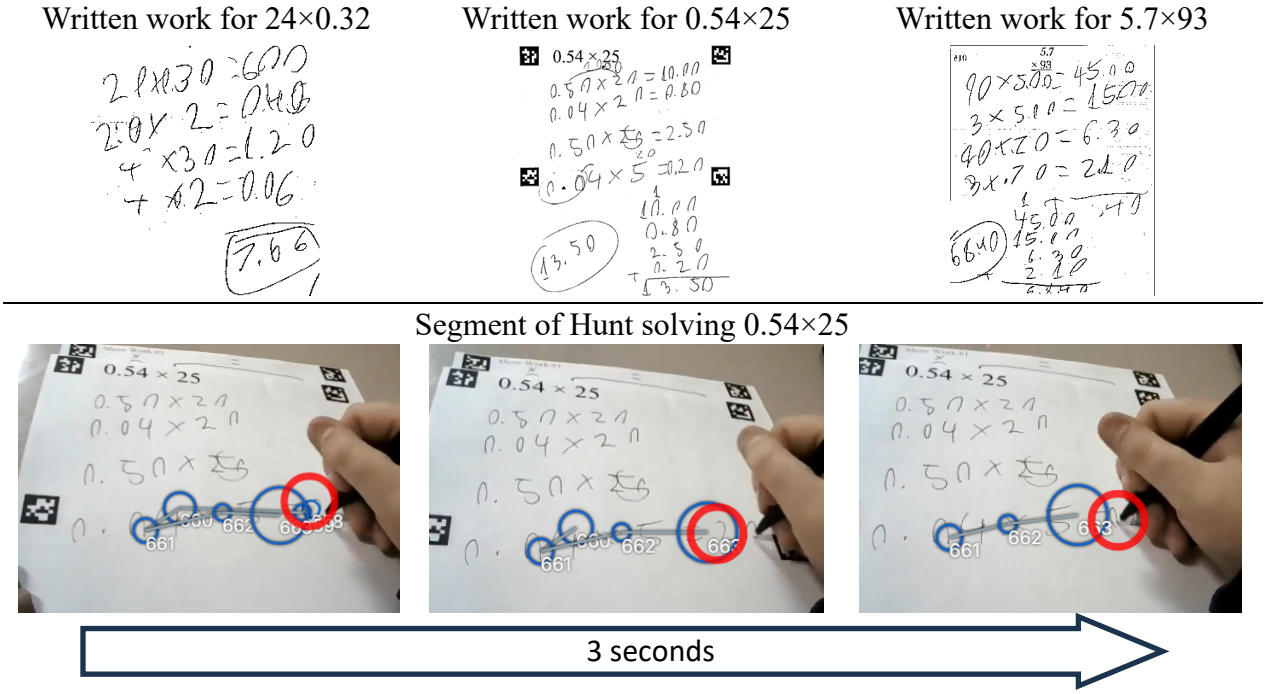


Figure 3: Written Work and Segments from Interview for Hunt (MC3).

Meaningful decimal counter. This theme describes students' actions in which they used the Decimal Placement Rule, but were able to provide a place value related rationale for it. Faith, a sixth-grade student at MC3, was one of only two students demonstrating this theme. At MC3, Faith could anticipate three levels of units and demonstrated this earlier in the interview when solving 43×38 using the standard algorithm. In explaining her approach, she consistently referenced the place value meaning of all digits without the need to explicitly write or construct them. For example, she explained "8 times 3 is 24. So you regroup the 20 and put it up there" to explain what the regrouped 2 meant, and then later explained "then you add a 0 [for the second row of products] because it's not a 3 [you're multiplying by], it's a 30..." While Faith's MRA scores also suggest she operated at MC3, we inferred her actions here as confirmatory because she could describe the place value meaning within the algorithm and identify all four partial products within the two rows of products. This latter aspect is evidence of simultaneous

disembedding of relating two partial products (43×8 & 43×30) to four (40×8, 3×8, 40×30 & 3×30). Faith was able to articulate the same reasoning and procedures for decimal multiplication, while consistently applying the Decimal Placement Rule. For example, solving 0.54×25, Faith's eye-gaze shifted to the 0.54 and moved just ahead of her pen as her eyes moved between the 4 and 5 indicative of counting each digit in 0.54 before doing so in the product. When probed about this, she explained that after summing her partial products "…you would get 1,350. But since it's two decimal – it's a hundredth and it's not in the one's place. You'd have to add the two decimals or…I don't know…It's divided by 100 sort of." We infer from Faith's actions and explanations that she understood that place value served a role in why the Decimal Placement Rule worked, and this was evident of a meaningful understanding of it.

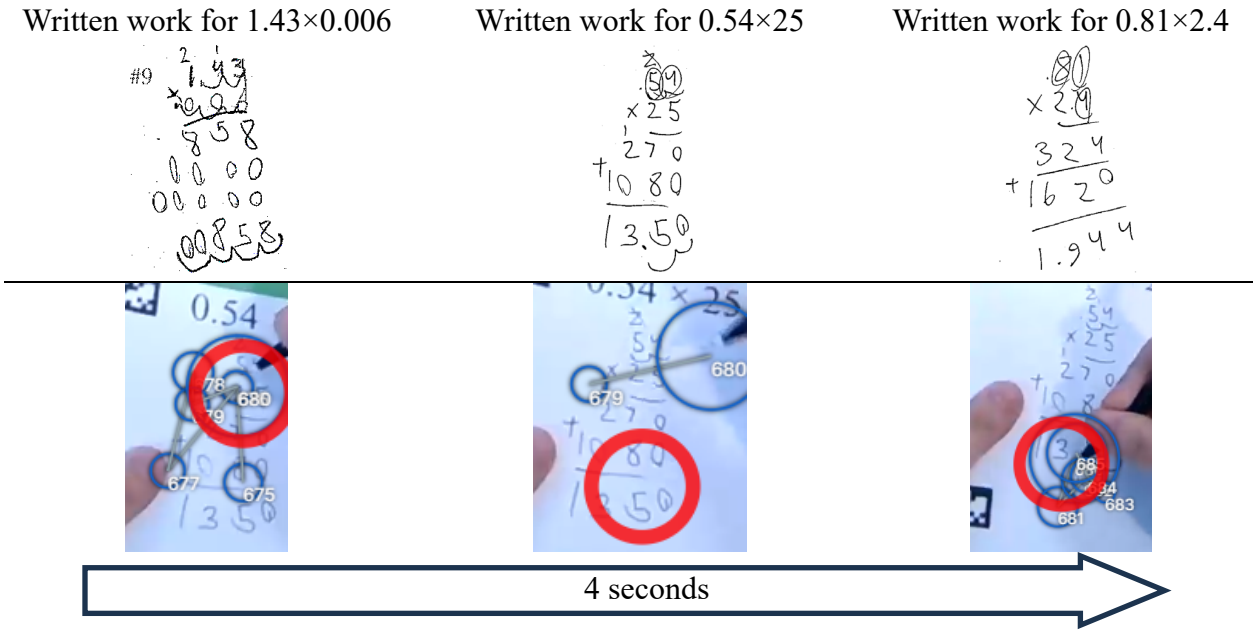


Figure 4: Written Work and Segments from Interview for Faith (MC3)

Quantitative Analysis & Results

Themes from qualitative analysis were quantified into ordinal data (1 = Arithmetic Decimal Operator; 2 = Rote Decimal Counter; 3 = Multiplicative Decimal Operator; 4 = Meaningful Decimal Counter). Qualitative descriptions of children's reasoning with decimals provided the necessary rationale for this order. Specifically, Arithmetic Decimal Operators focused neither on partial products nor the multiplicative meaning of the decimal, whereas Rote Decimal Counters typically focused on the meaning of partial products, but not on the multiplicative meaning of the decimal. Multiplicative Decimal Operators focused on partial products, while often having to construct the multiplicative meaning of decimal placement. Only Meaningful Decimal Counters provided place value explanations related to the Decimal Placement Rule. Next, we calculated a Gamma Statistic, which is a nonparametric measure of association between two ordinal variables (Siegel & Castellan, 1988). The calculated Gamma of .805 ($p < .001$) was both strong and statistically significant, suggesting that knowing either the level of multiplicative reasoning or decimal multiplication actions would reduce one's error in prediction by 80.5% (see Table 1).

Table 1: Contingency Table for Themes and Multiplicative Concepts

	Arithmetic Decimal Operator	Rote Decimal Counter	Multiplicative Decimal Operator	Meaningful Decimal Counter	Total
PreM	2	1	0	0	3
MC1	6	0	0	0	6
MC2	1	3	1	0	5
MC3	1	7	4	2	14
Total	10	11	5	2	28

Discussion

Most of the scholarship on decimal multiplication has focused on students' written procedures or task format (Hiebert & Wearne, 1985; Lortie-Forgues & Siegler, 2017; Tian et al., 2021), with relatively few examinations of children's reasoning through such tasks (Larsson et al., 2017). For example, Hiebert and Wearne's (1985) seminal study noted that students applied the Decimal Placement Rule inconsistently but could not explain why. Findings reported in the present study provide four distinct ways in which students engage with decimal multiplication, and these are strongly associated with their multiplicative reasoning. Initially, students engage with decimal multiplication in ways similar to decimal addition/subtraction, but with multiplication of digits within columns. As students begin to understand partial products (MC2 & MC3 reasoning), they begin to engage in the latter three themes.

Although there is some variance in alignment between multiplicative reasoning and decimal multiplication themes (i.e., a pre-multiplicative student can learn the Decimal Placement Rule by rote), findings suggest students at MC2 and higher are more likely to apply the Decimal Placement Rule. Yet, the meaning behind this rule is still being constructed. Irven could apply the rule, but sometimes mixed up the direction he counted the decimal places, and this inconsistency may be what Hiebert and Wearne (1985) observed in their own interviews. Multiplicative Decimal Operators seldom used the Decimal Placement Rule, but did focus on the place value meaning of multiplication with decimals. Although Hunt demonstrated correct approaches, other Multiplicative Decimal Operators demonstrated meaningful reasoning that led to incorrect solutions. Notably, we observed only two students able to provide a conceptual explanation of the Decimal Placement Rule and both of these students were at MC3. Findings here are preliminary but highly significant, as they go beyond models of examining students' rote procedures (Hiebert & Wearne, 1985) and extend models of their reasoning across a range of multiplicative reasoning (Larsson et al., 2017). Additional analysis is needed to both confirm and extend observations here, but these findings provide important insight into how children reason with decimal multiplication.

Acknowledgments

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PRESCHOOL AND KINDERGARTEN CHILDREN'S SUBITIZING AND NUMBER STRUCTURING WHEN SHOWN SIX DOTS

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Subitizing supports early number development, yet we know little about how children coordinate rapid quantity apprehension with visual structuring across representations. Using a procept lens, we analyzed preschool and kindergarten children's responses (N = 20) to brief presentations of six-dot in random, growing, canonical array, structured, structured and growing, canonical array and structured, and ten-frame layouts (fivewise, doublewise). Coding 159 instances across eight tasks identified nine activity forms from counting and estimating to conceptual subitizing and number structuring. Random displays more often elicited estimation and unstable drawings; whereas, structured arrays and ten-frames supported explicit part–whole composition. Many children reorganized random dots into canonical groupings, and some showed "groups of groups" and reversible reasoning, suggesting early pathways toward multiplicative thinking.

Keywords: Number Concepts and Operations, Early Childhood Education, Learning Theory, Learning Trajectories and Progressions

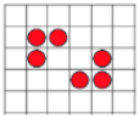
The literature is incomplete in describing when and how children subitize and structure items visually in mathematics classrooms and how this can be leveraged to support early forms of number development. *Subitizing*, a quick apprehension towards numerosity when shown a small set of patterned items (Kaufman et al., 1949), has been found to explain early forms of children's counting development (Wilkins et al., 2022). Sarama and Clements (2009) distinguished between perceptual subitizing (subitizing patterned stimuli with verbal number associations) and conceptual subitizing (subitizing and composing units conceptually). Ellemor-Collins and Wright (2009) describe children's subitizing with base-ten frames as number structuring (conceptualizing number relations). Given this, we wondered how these constructs could further explain how varying arrangements of six items shown to children may relate to their number understanding. The focus here is on the different ways preschool and kindergarten students subitize when shown a random six, an array of six (with/without structure), a staircase of six (with/without structure), and a ten-frame of six (fivewise and pairwise; see Table 1).

Theoretical Framework

To examine young children's number understandings concerning their subitizing activity, we examine number abstraction through the lens of a procept (Gray & Tall, 1993; Gray et al., 2000). Fundamental to a procept is the notion that "abstraction is drawn not from the object that is acted upon, but from the action itself" (Piaget, 1970, p. 16). Gray et al. (2000) define a *procept* as children's reification and coordination of their processes to form mental objects. A procept characterizes flexible reasoning, explaining how children move back and forth between their processes and mental objects. In this study, we considered processes children used when

describing parts of a whole and mental objects they used to conceptualize parts and their respective wholes. We considered children's re-presentations (i.e., drawings, finger patterns) as foreshadowing potential future processes and object development.

Table 1: Arrangements of Dots Shown to Participants

Task No.		Type of Arrangement	Interview Session
1		Random	Interview 2
2		Growing	Interview 2
3		Canonical - Array	Interview 2
4		Structured	Interview 3
5		Structured and Growing	Interview 3
6		Canonical - Array and Structured	Interview 3
7		Ten-Frame - Fivewise	Interview 4
8		Ten-Frame - Doublewise	Interview 4

Subitizing and Number Structuring

Kaufman et al. (1949) first named and operationalized subitizing, providing educators constructs to examine in children's reasoning and to inform curricula (Sarama & Clements, 2004), pedagogical practices (Parrish, 2010; Sun et al., 2018), and children's entertainment (e.g., https://www.ncetm.org.uk/classroom-resources/ey-numberblocks-support-materials/). Resulting from Kaufman et al.'s work, scholars distinguish subitizing as operationally unique from estimation, as individuals evidenced significantly faster and more accurate response times when subitizing items one through three versus when they estimated four or more items.

Perceptual subitizing, first defined by Clements (1999), refers to the process by which children develop simple perceptual associations between canonical patterns (e.g., with dots, fewer than three or four) and number words. When perceptually subitizing, children often describe the shape made by the dots or identify subgroups without identifying the total group (MacDonald & Wilkins, 2016). Comparatively, *Conceptual Subitizing* describes children as uniting several perceptually subitized groups of items to form larger groups (e.g., "I saw five because I saw two, two, and one!"), especially with canonical patterns. This activity has been found to explain more sophisticated activity and requires children to partition the canonical pattern and develop part-whole additive relationships between the subitized units (Clements et

al., 2019). *Number Structuring* or Spatial Structuring explains networks of units, often after children are quickly shown arrays (Bajwa et al., 2023; Ellemor-Collins & Wright, 2009). MacDonald and Wilkins describe many constructs preschool children may utilize when subitizing (i.e., ascending towards the group, descending from the group, perceptual subgroup subitizing), but does not include varying aspects of number structuring. We will partially frame subitizing with these constructs to explain procept development. In this proceeding, we aim to utilize these constructs to explain children's access to varying number and operation understandings when subitizing unstructured dot arrangements, canonical patterns, ten frames, and arrays (Bajwa et al., 2023).

Methods

Part of a larger study, this paper examines the type of subitizing activity of 20 children (9 preschool students; 11 kindergarten students) evidenced across three 15-minute interview sessions. Participants were enrolled in schools in the midwestern portion of the United States. Each participant engaged with tasks that drew from random, canonical, unstructured, structured, growing, and ten-frame arrangements of dots (see Table 1). Dots were shown in a span of three-second intervals. Because preschool participants were a year younger, they were shown arrangements with relatively fewer dots than the kindergarten participants' tasks. However, in this paper, we present results on dot patterns of six that all participants saw.

For broad data organization, the authors watched and transcribed the videos before organizing the data in Excel. For each participant, we placed transcriptions in rows (a new row per speaker change). We then included columns for notes, images of participants' drawings, descriptions, and/or images of their finger use, descriptive codes (i.e., describing groups and then the total), and constructs (i.e., ascending towards the group) used to operationalize procepts. Through this initial data analysis, we wondered how children re-presented arrangements for six in different ways. As such, we organized in a new Excel spreadsheet children's language around groups used and drawings related to different arrangements for six shown to them.

Results

From the eight tasks across three interviews, the 20 students had the potential to evidence 160 types of activity, with 159 instances documented. Given the space of this proceeding, we describe a portion of these instances. From our analysis of these instances, we found nine themes (*counting, estimating, perceptual subitizing, perceptual subgroup subitizing, ascending towards the group, descending from the group, counting and subitizing, conceptual subitizing,* and *number structuring*), evidencing relations between subitizing and procept development.

Counting

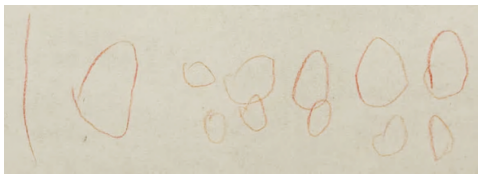
When one preschool student, Bluey, used counting, he seemed to either re-present the shape shown to him or the number of items he counted. After showing Bluey a random arrangement and asking him to draw what he saw, Bluey started drawing a vertical column of six circles (see Figure 1a), while verbalizing, "Two, three, four, five, six" as he drew each additional circle. Bluey is capable of producing six and abstracts the six drawn items with his written symbol "6."

When shown a typical canonical pattern of six dots (an array of two rows of three dots), Bluey immediately started counting to six as the picture was taken away and then declared it to be 10. Bluey seems to be focusing on the image shown to him and considers this primary and his counting secondary. This is evidenced when Bluey draws an array of two rows of five dots (see Figure 1b), suggesting his use of number is dependent on the shape shown to him. For instance, he drew his first dot at the right side of the paper, saying, "One," and then drew three additional

dots to the left while counting them, "two, three, four." He drew his fifth dot, "five" above the fourth one, and then drew the next three to the right of that one, "six, seven, eight," making an array of two rows of four dots each. He continued drawing "nine" to the right of his eighth dot and paused. When asked if that's what he saw, he reiterated, "Well, it WAS ten," and he drew another dot, "ten," under the ninth dot. When counting, Bluey used a remembered image similar to what was shown to him, allowing him access to all the items to count. This suggests that Bluey was relying on actions for future mental processes and objects (Gray & Tall, 1994).



a) Random Arrangement, Task 1



b) Canonical Array, Task 3

Figure 1: Bluey's Drawings, When Shown Arrangements in Tasks 1 and 3 (see Table 1)

Estimating

Kaufman et al. (1949) describe *estimating* as when individuals express a relatively lower confidence level, degree of accuracy, and speed in response. We found that participants estimated more often when shown random arrangements (40% estimation) than canonical patterns (5–10% estimation) (Moscoso et al., 2020). We coded students' activity as estimating when they did not describe groups, arrangements, or operations that supported their response. As the students estimated, we found that they were, at times, accurate (four instances) and at times, inaccurate (seven instances). When students drew what they saw, they often re-presented arrangements shown to them in a horizontal arrangement (see Figures 2b–2d). This suggests they may have been engaging in an Approximate Number System (Norton & Alibali, 2019; Dehaene, 2011) because they may have been keeping track of their number sequence as they drew the items, and stopped once they reached an implicit number, informed by the shape and/or the space taken up by the arrangement.



a) Random Arrangement, Task 1



b) Random Arrangement, Task 1



c) Growing Pattern, Task 2



d) Ten-Frame, Five-wise, Task 7

Figure 2: Drawings Evidencing Estimating (see Table 1)

Two students, one a preschool student (see Figure 2a) and one a kindergarten student, drew the arrangement in a shape dissimilar from the one they were shown. This may indicate that, even though they did not explicitly mention shape in their response, they were implicitly relying on the movement their eyes engaged in when estimating (cf. MacDonald et al., 2012).

Perceptual Subitizing

When students relied heavily upon shape to accurately describe the number six, we described this as perceptual subitizing. Sarama and Clements (2009) describe children transitioning

towards conceptual subitizing when shown six items. However, MacDonald and Wilkins (2016) describes some young children relying on perceptual activity when shown six items. As such, when participants described seeing six items and seemed to explicitly rely on the shape shown to them or a shape associated with their understanding of six, we considered this to be perceptual subitizing. For instance, even though three students (one kindergarten student and two preschool students) perceptually subitized the random arrangement of items, they all seemed to draw from their association with six in an array format (similar to Task 3, see Table 1). This suggests that when perceptually subitizing, participants may have been implicitly drawing on a grouping structure to reorient the items in this way, but could not yet explicate this. We also observed participants re-presenting a similar arrangement shown to them, suggesting that the features of the shape (i.e., structures, canonical groups) allowed them access to not yet explicated groups and/or patterns, which in turn allowed participants to access the number represented by the arrangement of items (see Figure 3). Finally, one student, Manesh, considered the ten-frame pairwise arrangement (see Task 8, Table 1) as a character from a television show he had seen, Numberblocks (see Figure 3f). He drew the character next to the arrangement that he had seen, but was not able to explain the subgroups that made this arrangement six. This suggests that previous experiences can inform perceptual subitizing activity and inform opportunities for future mental objects to develop (Gray & Tall, 1994).



a) Random Arrangement, Task 1 | b) Random Arrangement, Task 1 | c) Growing Pattern, Task 2 | d) Growing Pattern, Task 2 | e) Growing Pattern, Task 2 | f) Ten-Frame, Doublewise, Task 8

Figure 3: Drawings Evidencing Perceptual Subitizing (see Table 1)

Perceptual Subgroup Subitizing

Perceptual Subgroup Subitizing fundamentally explains a participant's ability to partition a group of items and subitize each subgroup, but not use these subgroups to accurately describe the total number of items (MacDonald & Wilkins, 2016). For instance, when shown structured arrays or dots in ten frames, Moana, one of the preschoolers, identified subgroups and drew the two groups of three (see Figure 4c), before putting up three fingers on each hand and counting her fingers to determine there were six dots. Likewise, when shown the structured and growing arrangement of six, she called it "stairs" and accurately drew one dot, then two dots in the next column, and then three dots in the final column (see Figure 4d). However, she once again needed to count them to name the group as six. We found 13 instances of when participants engaged in perceptual subgroup subitizing (see Figure 4). Students demonstrated perceptual subgroup subitizing when they could subitize subgroups and connect this to a total group developed through their counting and/or estimating. By allowing participants opportunities to re-present the items shown to them in subgroups, they were able to count or estimate to determine the total number of items shown to them. When students engaged in perceptual subgroup subitizing, they often developed partitioned patterned images for future procept development (Gray & Tall, 1994).

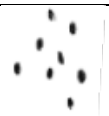
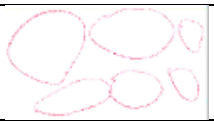
				
A: Random Arrangement, Task 1	B: Canonical Pattern, Task 3	C: Structured, Task 4	D: Structured Growing Pattern, Task 5	E: Canonical - Array and Structured, Task 6

Figure 4: Drawings Evidencing Perceptual Subgroup Subitizing (see Table 1)

Ascending and Descending Towards/From the Group

When participants used groups in their subitizing activity, they either described the subgroups and then the total items in the group (*ascending towards the group*) or described the total number of items and then described subgroups (*descending from the group*; MacDonald & Wilkins, 2016). The distinction between this activity and conceptual subitizing was that the participants did not seem to concurrently understand the subgroups and the total, but could either subitize the subgroups and then the total or vice versa. This suggests participants were capable of conceptualizing part-whole numerical relationships, but could not anticipate and coordinate these relationships when quickly shown the arrangement of items.

In particular, when ascending towards the group (see Figure 5), we observed two instances, I which kindergarten students described subgroups and then explained the total number of items in the group. It seemed two students changed the number of items in each of the subgroups each time, until determining the subgroups and then using this activity to construct the total number of items in the group. John (see Figure 5a) explains he saw five and two, and then as he draws the circles, he changes his response to five and one before describing the total amount as "six."



A: Growing Pattern, Task 2



B: Structured, Task 4

Figure 5: Drawings Dvidencing Ascending Towards the Group Subitizing (see Table 1)

When descending from the group, Olasa, one of the preschoolers, saw the six in the ten-frame doublewise and quickly stated that she knew it was six. She was unable to say how she knew it was six; instead, she drew a grid with several empty boxes. Although her grid was eight by three, she still drew her six dots in pairs (vertically instead of horizontally, see Figure 6), noticing, "There's a two and a two, and a two." Her responses indicate that she only considered the subgroups of the dots after considering her drawing. This evidences participants' flexible activity between objects and actions, informing future procept development (Gray & Tall, 1994).

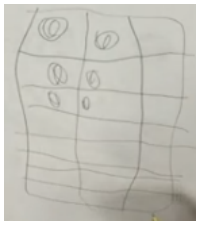


Figure 6: Olasa's Drawing of Six When Shown the Arrangement from Task 8 (see Table 1)

Counting and Subitizing

There were also three instances when participants coordinated their sequential activity through their counting with their grouping structures through their subitizing. For instance, when shown the growing pattern in Task 2 (see Table 1), Finn (a kindergarten student) quickly states that he saw "three … six", suggesting his grouping structures for three were distributed along his number sequence. This is very different from stating that he saw three and three and this is six, as he may not have been explicitly aware of two groups of three. After showing Bluey the arrangement from Task 8 (see Table 1), he first says, "one, two, three, four, five, six." He then draws the image he saw, while drawing the eight additional dots outside of the rectangle he drew to show what had been "eaten." It seems Bluey is experiencing conflict as to which to attend to: his grouping structure and the shape shown to him, an experience where he is "eating something," or his counting. Hence, students coordinated their past experiences with their grouping and counting structures or switched between them. Here, participants drew from counting actions to further inform relationships between their mental objects for "six" and their processes related to "six" (Gray & Tall, 1994).

Conceptual Subitizing

There were 50 instances where participants demonstrated conceptual subitizing. Given the space of this proceeding, we only focus on three different types of conceptual subitizing (with random arrangements, with patterned arrangements, and explicit operations). The preschoolers we worked with demonstrated conceptual subitizing only when the dots were structured in grids or ten frames. When shown random arrangements, four kindergarten students demonstrated conceptual subitizing, suggesting relatively greater reliance on more abstract operations (Moscoso et al., 2020). Finally, both kindergarteners and preschoolers explained some groups in the form of an operation ("three plus three equals six").

Cinderella talked about the arrangements in terms of lines, indicating that the ten-frame fivewise was a "straight line [of five] and one down" (see Figure 7b). Scarlet also described this arrangement as "one group of five and another group of one." Then, when shown Task 8 (see Table 1), Cinderella explained the two rows of three as "two lines" with "two at the top, four at the bottom" (see Figure 7a). Here, her drawing matched her explanation, not the visual.



A: Ten-Frame, Pairwise, Task 7 B: Ten-Frame, Doublewise, Task 8 C: Random Arrangement, Task 1 D: Canonical Pattern, Task 3

Figure 7: Drawings Evidencing Conceptual Subitizing (see Table 1)

When four different kindergarten participants were shown the random arrangement of six (Task 1, see Table 1), they seemed to describe or utilize groups when subitizing. This was often re-presented in their drawings, where John (see Figure 7c) drew two groups of three and described these groups. This suggests that some students organized items shown to them in a way that allowed them to anticipate subgroups coordinated with the total number. We also found that eight participants drew from addition operations when shown arrays, growing patterns, and structured patterns. For instance, when shown a canonical pattern of six as an array, Bella (see

Figure 7d), a kindergarten student, described seeing six because she knows "three plus three equals six." Participants develop procepts when conceptually subitizing, allowing them resources to develop structural relationships in future activity (Gray & Tall, 1994).

Number Structuring

When students discussed and/or demonstrated groups of groups understandings, we described this as number structuring, explaining early forms of more complex relations fundamental to multiplicative reasoning (Bajwa et al., 2023; Ellemor-Collins & Wright, 2009). Of the 18 instances of accurate number structuring, one student, Stephen, described the groups of groups (i.e., "three twos") while also anticipating the sections he would need to draw (i.e., "I will need to draw two lines for this") to re-present these structures (see Figure 8a, 8b, 8e, 8f). Three of these instances described structuring that utilized reversible operations. For instance, Bella (see Figure 8i) used the seven items shown to her beforehand to subtract one from, explaining that she "took one from the one before." Finn used the three-by-three array arrangement of nine items shown to him beforehand to subtract three from the nine, saying, "It was minus three," and it was "like the nine … three times three…. minus three." These instances explain how participants engaged in reversible activity, as they reversed their reasoning to operate within this structure and make sense of the subsequently shown dot arrangement. When structuring number, participants were developing structural relations with procepts, further informing their capabilities to develop axiomatic concepts (Gray & Tall, 1994).

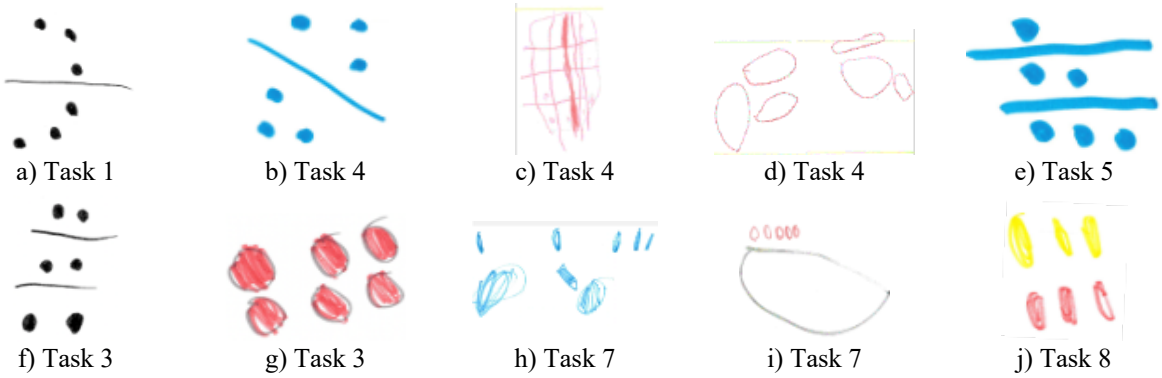


Figure 8: Drawings Showing Accurate Number Structuring (see tasks from Table 1)

Conclusion

Random displays more often elicited estimation and unstable drawings, whereas structured arrays and ten-frames supported explicit part–whole composition. This echoes Moscoso et al.'s (2020) findings, as they explained more sophisticated subitizing is grounded in individuals' ability to draw from operations such as addition and multiplication versus features of the arrangements. Many children reorganized random dots into canonical groupings, and some showed "groups of groups" and reversible reasoning, suggesting early pathways toward multiplicative thinking. As students demonstrated relatively rudimentary activity (i.e., counting, estimating, perceptual subitizing) and coordinated this activity (i.e., perceptual counting subitizing, ascending towards groups, descending from groups), they demonstrated opportunities for procept development through their conceptual subitizing and number structuring (Gray & Tall, 1994).

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SUPPORTING PROPORTIONAL REASONING IN DIGITAL PLATFORMS: THE ROLE OF MODELS AND CONTEXT

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Digital learning platforms, while prevalent, often lack practice problems closely aligned with research on how visual models and contextual features support conceptual understanding in mathematics. In this study, we investigated which modified models and/or contexts may better support Grade 6 students' proportional reasoning. We conducted one-on-one cognitive interviews in which 63 students solved 24 variants of ratio and proportion items on the ASSISTments digital platform. We present results on the proportions of responses exhibiting different forms of reasoning across model and contextual variants. Preliminary analysis indicates that students were more likely to demonstrate proportional reasoning with context modifications combined with informal or formal iconic models. Results will be used to generate items for interventional purposes.

Keywords: Mathematical Representations, Rational Numbers and Proportional Reasoning

Introduction and Theoretical Framework

Digital learning platforms have emerged as prominent tools for delivering digital curriculum in mathematics education (Doan et al., 2022). As these platforms evolve, it is important to ensure practice problems they provide not only target CCSS-M standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), but also are closely aligned with research on how visual models and contextual features support students' conceptual understanding (e.g., Fyfe & Nathan, 2019; Küchemann et al., 2011).

This may be especially important in proportional reasoning, a content area widely recognized as challenging for students. Their understanding often begins with unsophisticated strategies, like non-coordinated units reasoning or an incorrect additive approach instead of the necessary multiplicative relationships (Carney et al., 2022; Carney & Crawford, 2016; Misailidou & Williams, 2003; Steinthorsdottir & Sriraman, 2009). With support, students can progress to developing composed unit reasoning and applying increasingly sophisticated strategies for scaling ratios. The goal is for students to progress to an "operating within ratio" strategy, indicating they recognize both the scalar and functional relationships. However, it is unknown how proportional reasoning practice items offered in digital learning platforms are designed to support this conceptual progression.

Mathematics education researchers have identified various models that effectively support students' proportional reasoning (e.g., Booth & Koedinger, 2012; Karplus et al., 1983; Küchemann et al., 2011; Lamon, 2007; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985). These models exhibit varying degrees of abstraction (Fyfe & Nathan, 2019), with different models potentially affording different types of reasoning. For example, informal iconic models, like pictorial and bar models, represent relationships visually. Formal iconic models, such as tape diagrams and double number lines, progressively emphasize iteration and

multiplicative relationships. Whereas, ratio tables and formal symbolic models highlight scalar and functional relationships using mathematical notation.

Research has also identified two contextual aspects of problems that may influence students' understanding of proportional situations: (a) the imaginability of the contextual setting, and (b) the chosen number sets. For example, the imaginability of proportional relationships can make tasks more accessible by grounding mathematics in known experience (Fyfe & Nathan, 2019). Furthermore, number sets can influence students' abilities to reason proportionally based on whether the numbers and multipliers in the task are integers or non-whole rational numbers (Carney & Crawford, 2016; Lamb et al., 2018; Tian & Siegler, 2018).

In this study, we selected and modified four ratio and proportional reasoning practice problems from the ASSISTments digital learning platform to increase their alignment with the research findings noted above. The purpose of this study was to investigate the effects of these modified digital practice problems, enhanced with a variety of models, accessible number sets, and imaginable contexts, on Grade 6 students' proportional reasoning.

Method

Item Development

We identified four practice items from the ASSISTments platform that involve ratio and proportional reasoning for Grade 6 students. For each item, we developed a range of models (e.g., informal iconic, formal iconic, and informal symbolic) and modified the context (e.g., by simplifying the number set or adding context to a bare problem). By crossing model types with contextual variants (including the original item), we generated eight possible variants per item. From these eight, we selected six variants for each item, resulting in 24 total variants.

Participants

We recruited Grade 6 teachers from three schools (one rural, one suburban, one urban) in the Western U.S. to offer the variants to their students (63 of whom participated). We developed a planned rotation of implementation in which each participant solved one of the six variants of each of the four items, yielding 10–11 responses per variant.

Data Collection and Analysis

Using cognitive interview procedures, students individually solved each of the four problems assigned to them and described their thinking to an interviewer. Video recording equipment was set up to capture students' written work. Our study produced codable videos of students solving 241 problems.

Our coding protocol included 24 deductive codes describing correctness, strategy use, reasoning, and explanation features. This report presents inferences about proportional reasoning–evidenced by the codes "multiplicative comparison" or "composed unit,"–and non-proportional reasoning–evidenced by the codes "non-coordinated units" or "none" (i.e., guessing or using a memorized algorithm). Six coders iteratively refined descriptors in the codebook across nine videos. Interrater reliability was established on four additional videos, yielding 75% agreement across all 24 codes and Krippendorf's alpha = 0.82. After coding, frequencies were tabulated for reasoning types across model types, context modifications, and their combinations.

Results

Student Reasoning Across Model Types and Modified Contexts

As shown in Figure 1, responses were coded according to the types of mathematical reasoning students used in the presence of different types of contexts and models. When comparing context modifications, students evidenced proportional reasoning (i.e., multiplicative

comparison or composed unit) more often in response to contexts that had been modified (71%) compared to their original form (57%). When comparing model types, we found that students evidenced proportional reasoning (i.e., multiplicative comparison or composed unit) at the same rate (66%) for problem variants that offered (a) informal iconic models, (b) formal iconic models, and (c) no models. In contrast, informal symbolic model variants elicited fewer responses evidencing proportional reasoning (56%), and the only model with no evidence of eliciting multiplicative comparison. These findings suggest that informal symbolic models may be less likely to elicit proportional reasoning than other types of models, whereas context modifications may be more likely to elicit it for some students.

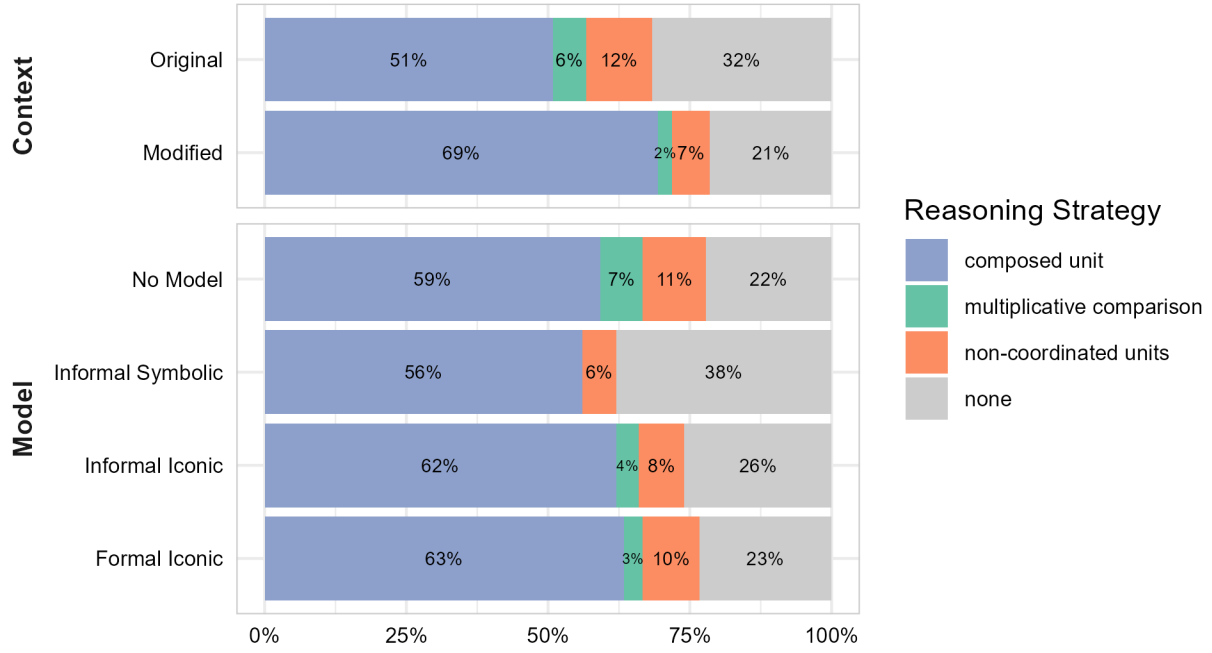


Figure 1: Student Reasoning by Context and Model

Student Reasoning Across Model and Context Combinations

As seen in Figure 2, students evidenced proportional reasoning (i.e., multiplicative comparison or composed unit) at varying rates across the eight problem variants that combined model and context modifications, ranging from 40% (original context with informal symbolic models) to 80% (modified contexts with formal iconic models). These findings corroborate those noted above but also highlight a potential advantage of formal iconic models that was not previously identified. Additionally, modified context variants with models showed the fewest non-coordinated unit responses (3–5% compared to 10–15%), possibly indicating that models and contexts supported unit coordination.

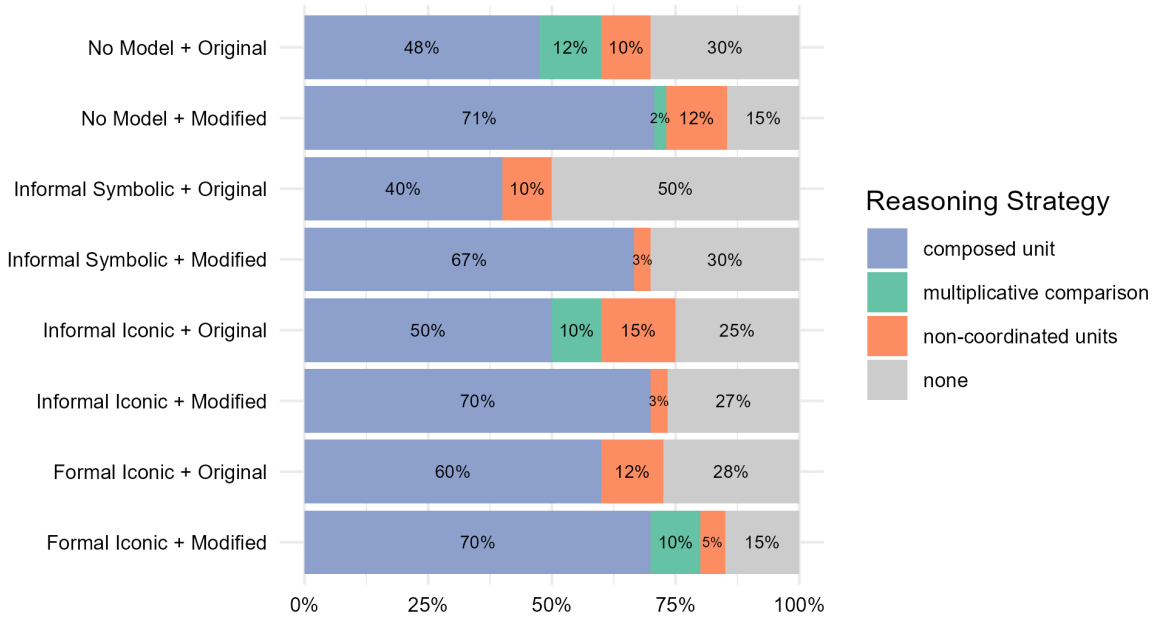


Figure 2: Student Reasoning by Pairings of Model and Context

Discussion and Conclusion

Our preliminary results suggest that model and context modifications influence students' proportional reasoning. Modified contexts consistently elicited higher rates of proportional reasoning than original contexts, and the combination of modified contexts and formal iconic models achieved the greatest success (80%). This suggests context modifications combined with informal or formal iconic models may be most likely to elicit proportional reasoning for some students. This aligns with research indicating that imaginable contexts and appropriate visual representations can support conceptual understanding (Bruner, 1966; Freudenthal, 1983; Fyfe & Nathan, 2019). However, the lower instance of proportional reasoning with informal symbolic models was unexpected, particularly when combined with original contexts (40%). This may indicate that Grade 6 students require more scaffolding or instruction to effectively use these representations, or that the symbolic nature of these models creates additional cognitive demands that interfere with proportional reasoning for some learners. Furthermore, the substantial reduction in non-coordinated units reasoning within modified context variants with models (3–5% vs. 10–15%) suggests these modifications may help students move beyond additive thinking toward multiplicative comparison and composed unit strategies. This finding supports the design principle of aligning practice problems with students' learning trajectories (Carney et al., 2022).

These results suggest digital learning platforms might better support proportional reasoning by incorporating imaginable contexts and formal iconic models into practice problems. However, the variability across model types indicates that careful alignment with research on student reasoning progressions remains essential. Future work should examine whether these patterns persist across different problem structures and with larger, more diverse samples.

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FIFTH-SIXTH GRADE STUDENTS' ALGORITHM USE, RESPONSES, AND ERRORS IN MULTI-DIGIT MULTIPLICATION

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This study reports empirical evidence of the algorithms commonly used by 814 fifth- and sixth-grade students. The goal was to understand what algorithms children use for multi-digit multiplication and how they engage with them. This study examined fifth-grade (n = 618) and sixth-grade (n = 196) students enrolled in 36 different teachers' classrooms in a Midwestern US state. The results showed that the Box Model, Partial Products Algorithm, and Standard Algorithm were the most prevalent. Moreover, the Standard Algorithm recorded the most correct answers across the three algorithms. Meanwhile, arithmetic, multiplication, and partial-product errors were common across all algorithms, though some were more prevalent in certain algorithms. Implications are discussed, but a call is made for future analysis and examination of these trends.

Keywords: Elementary School Education, Number Concepts and Operations

Many scholars argued that multi-digit multiplication is an important and cogent component of upper-elementary mathematics simply because it requires complex mathematical skills, such as students' ability to coordinate place value, number decomposition, as well as multiplicative reasoning (Egbedeyi & Kosko, 2025; Carpenter et al., 1996; Young-Loveridge & Mills, 2009). Despite significant scholarship on other arithmetic operations, such as multi-digit addition and subtraction, research on multi-digit multiplication has been relatively limited, especially regarding the use of algorithms (Egbedeyi & Kosko, 2025; Fuson, 2003; Harrison, 2013; Hickendorff et al., 2019). Evidence of students across grades using different algorithms to solve multiplication has been properly documented in research, for instance, counting strategies (Lampert, 1986), arrays (Izsàk, 2001; Barmby et al., 2009), the Area Model (Izsàk, 2001), the Lattice Algorithm (Speiser et al., 2012), and others (Hickendorff et al., 2019). For instance, Egbedeyi & Kosko (2025) reported in their study on fourth-grade students that the Standard Algorithm (Alanazi, 2024) was the most frequently used strategy, followed by the Box Model (Kosko, 2024) and the Partial Products Algorithm (Flores et al., 2020; Speiser et al., 2012), while repeated addition and doubling algorithms were infrequently used.

Previous studies comparing multiplication algorithms have shown mixed but informative patterns regarding students' correctness. For instance, both the early and later quasi-experimental studies revealed that fourth-grade students who used the Lattice Method were more likely to obtain correct answers than those who used the Partial Products Algorithm (Hughes & Burn, 1975; Speiser et al., 2012). Furthermore, research indicates algorithms grounded in concrete or pictorial area models, such as the Box Model and Partial Products, can support students in producing correct solutions by making multiplicative structure explicit (Izsák, 2001; Flores et al., 2020). However, Javornik and Lipovec (2020) reported that there were no statistically significant differences in fifth-grade students' performance when using the U.S. Standard Algorithm, the Box Model, or a combination of algorithms, including the Lattice Method. Although Egbedeyi and Kosko (2025) compared the three algorithms on partial products in fourth grade, there is a need for additional empirical evidence from a larger dataset to determine which algorithm tends to produce correct multi-digit answers among fifth- and sixth-grade students.

One way of examining which algorithms may best scaffold students' learning about multi-digit multiplication is to examine and compare the types of errors often associated. It is important

to note that, in doing so, we are not using these errors to make judgments about the students, but about the algorithms they are exposed to. Errors in multi-digit multiplications are considered to be deviations from accepted strategies and/or algorithms, which likely, though not necessarily, result in an incorrect answer. According to Egbedeyi and Kosko (2025), the deviations include mistakes, misunderstandings, or conceptual errors. For instance, Englehardt (1977) describes inversion errors when students "carry" the wrong number for regrouping and grouping errors, as when students do not add the correct number of 0s in their rows for addition at the bottom of an algorithm. Most of the available literature on multi-digit multiplication errors centered on the standard algorithms, despite the fact that other algorithms, such as the Box Model and the Partial Products Algorithm, are also prone to students' errors. For instance, Egbedeyi and Kosko (2025) found that fourth-grade students' use of the Standard Algorithm, regardless of task type, led to more observed partial product errors than expected by chance. By contrast, the other two common algorithms did not show such a trend, except for the Partial Products Algorithm on tasks with four partial products. Therefore, Egbedeyi and Kosko (2025) argued that the introduction of the Standard Algorithm to students should be delayed until fifth grade and should not be introduced in fourth grade. Based on the call from their study, this study presents a large sample of students' multi-digit multiplication. This preliminary analysis includes data from fifth- and sixth-grade classrooms, with the primary goal of understanding which common algorithms children choose and the errors they typically make when using them. Therefore, this study examines fifth- and sixth-grade algorithm use, its success, and its associated errors.

Methods

Sample, Protocol & Data

Data were collected from 814 fifth- (n = 618) and sixth-grade (n = 196) students enrolled in 36 different teachers' classrooms in a Midwestern US state. Students self-identified primarily as White (75.4%), with others identifying as Black (11.1%), Asian (5.2%), Latine (2.1%), multiple ethnicities (6.5%), or as another ethnicity (0.5%). Additionally, 53.5% identified as female and 46.4% as male (0.1% as non-binary). As part of a larger study, students completed 10 whole number and decimal multiplication items aligned with their grade level. Certain whole number multiplication items for fifth (21×4; 19×16; 37×34; 86×49; 457×21; 2083×35; 789×382) and sixth grade (21×4; 19×16; 37×34; 86×49; 789×382) largely overlapped and are the focus of the current study. Participants' written work was scanned and then analyzed for the algorithm/strategy used, their responses, any errors employed in their work (i.e., arithmetic, multiplicative, partial products, or other), and whether they wrote expressions for partial products in their written work. We used a codebook from an earlier analysis of fourth-grade students' multiplication work (Egbedeyi & Kosko, 2025) to examine 308 of the 814 students' work. Both authors independently coded this subsample, with a Kappa of .97 ($p<.001$) indicating almost perfect agreement. Given this high level of agreement and following data reconciliation in the subsample, the remaining data were split evenly between the two authors for independent coding before proceeding to analysis.

Analysis & Results

We used descriptive statistics to examine students algorithm use, success and presence of any associated errors. Based on the analysis, the following primary algorithms were observed among both fifth- and sixth-graders: Repeated Addition, Doubling, the Box Model, the Partial Products Algorithm, Lattice, and the Standard Algorithm. We identified instances where students used multiple algorithms or employed other algorithms, with notes providing more specific details, such as the use of counting-by-1s.

Partial Products Algorithm | Box Method | Standard Algorithm

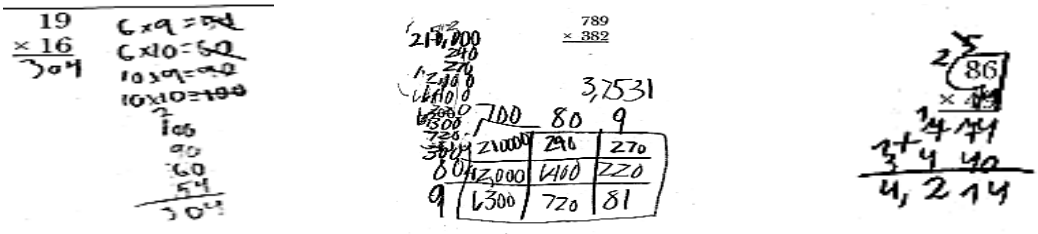


Figure 1: Examples of the Most Used Algorithms Among Fifth- and Sixth-Grade Students

Therefore, this analysis focused primarily on the Box Model, the Partial Product Algorithm, and the Standard Algorithm among both fifth- and sixth-grade students, as well as on the associated errors common to each algorithm. The full details of the analysis are given below

Table 1: Percentage of Usage and Correctness by the Most Common Algorithm Among Fifth- and Sixth-Grade Students

% of Fifth-and Sixth-Grade Students Using Algorithm			
	Box Model	Partial Products Algorithm	Standard Algorithm
Grade 5	19.7%	8.1%	72.2%
Grade 6	31.1%	3.6%	65.3%
% Correct on Task by Algorithm of Fifth-and-Sixth Grade Students			
	Box Model	Partial Products Algorithm	Standard Algorithm
Grade 5	60.9%	58.5%	61.7%
Grade 6	61.8%	66.7%	67.5%

Table 1 shows the percentage of students using each algorithm in the fifth and sixth grades. In both grades, the Standard Algorithm is the most used, while the Box Model is used by a smaller group of students, and the Partial Products Algorithm is rarely used at either level. A notable pattern is that use of the Box Model increases by about 11–12% from Grade 5 to Grade 6, while use of the Standard Algorithm and the Partial Products Algorithm both decline slightly across grades. Further analysis reveals that the Standard Algorithm, which most students use, produced the highest correctness in both grades, but only by a small margin. In fifth grade, it can be inferred that the correctness is almost the same across the three algorithms. Furthermore, in sixth grade, there is an increase in students' algorithmic correctness, with a noticeable increase in the Standard Algorithm and the Partial Products Algorithm, and a slight increase in the Box Model. Therefore, the statistical difference shows that in fifth grade, the Standard Algorithm exceeds the Box Model by only 0.8% and Partial Products by 3.2%, and in sixth grade, the Standard Algorithm is 5.7% above the Box Model and 0.8% above Partial Products; these remain modest gaps, indicating only a slight accuracy advantage for the Standard Algorithm while all three algorithms support broadly comparable levels of success.

We used descriptive statistics to compare the three major errors identified in the written work of fifth- and sixth-grade students across the Box Model, Partial Product Algorithm, and Standard Algorithm, as shown in Table 2. The analysis indicated three primary types of errors in the open coding process: arithmetic, multiplication, and partitioning. Arithmetic errors involved an error with addition or subtraction. Multiplication errors involved reporting an incorrect product in one or more steps when multiplying within the chosen algorithm, or incorrect placement of the place

value with the chosen algorithm. Partitioning errors including regrouping errors, involved miscoordination of partial products, for example, incorrectly partitioning a number using some of the partitioned multiplicands (e.g., 19×16 = 9×6 + 10×10), inability to multiply using the column value, or multiplying partitioned elements inappropriately.

Table 2: Percent of Error Types Made for an Algorithm Used (by Task) Among Fifth- and Sixth-Grade Students

	Fifth-Grade Student Algorithm Errors		
	Arithmetic Error	Multiplication Error	Partitioning Error
Box Model	17.8%	12.0%	5.6%
Partial Products	10.8%	10.8%	22.2%
Standard	5.6%	12.5%	18.4%
	Sixth-Grade Student Algorithm Errors		
	Arithmetic Error	Multiplication Error	Partitioning Error
Box Model	12.8%	17.4%	3.5%
Partial Products	6.1%	12.1%	21.2%
Standard	2.2%	12.6%	14.9%

Table 2 shows the errors comparison among the three primary algorithms used by both fifth- and sixth-grade students. The analysis shows that arithmetic, multiplication, and partitioning appeared across all the algorithms. However, this study specifically compares partitioning error among algorithms. The results indicate that partitioning errors are most frequent when students use the partial products algorithm in both grades. Likewise, the Standard algorithm shows notable partitioning errors, though fewer than the partial products, while the box model produces comparatively few partitioning.

Discussion

This study significantly contributes to expanding the existing body of knowledge on multi-digit multiplication. The results suggest that fifth- and sixth-grade students favor the standard algorithm for multi-digit multiplication among the remaining algorithms, though the box model receives more attention from sixth-graders than other algorithms. This result supports Egbedeyi and Kosko (2025), who found that the Standard Algorithm was the most frequently used strategy, followed by the Box Model and the Partial Products algorithm among fourth-grade students. Furthermore, the results showed that the Standard Algorithm is likely to produced more correct answers than the Box model and the Partial Products Algorithm; however, both the Box Model and the Partial Products Algorithm tended to yield correct answers when students were more exposed to them as alternative algorithms to the Standard algorithm (Thompson, 2008). In addition, the higher partitioning errors with Partial Products and Standard Algorithms, contrasted with fewer in the box model showed that many students might struggle to scaffold their understanding in applying the Partial Products Algorithm to derive partial products in multi-digit multiplication, especially those considered to be complex, while the Box/Area Models scaffold partitioning more intuitively (Anghileri, 2006; Egbedeyi & Kosko, 2025). While this study presents preliminary findings, the study therefore calls for students to have more exposure to the Box Model and Partial Products Algorithms in both the fifth- and sixth-grade classrooms because, despite their small popularity or usage among the students, they were able to produce almost a similar percentage of correctness level to the most commonly use algorithm (the Standard Algorithm) and likewise results of this study are highly significant in pointing toward future study to compare algorithms based on partial product tasks.

Acknowledgments

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FROM LINEAR INTUITIONS TO DIMENSIONAL REASONING: DEVELOPING NONLINEAR PROPORTIONAL THINKING THROUGH NANOSCALE STEM INVESTIGATIONS

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Proportional reasoning is foundational to mathematics and science learning, yet many middle school students rely on additive and procedural strategies that obscure relational understanding. This study investigates the development of proportional and covariational reasoning within a nanotechnology-focused integrated STEM summer program. Eleven middle school students engaged in ratio investigations, dimensional scaling, surface-area-to-volume (SA:V) modeling, and engineering design tasks grounded in nanoscale phenomena. Pre/post assessments showed significant gains in proportional reasoning (Wilcoxon $p < .05$, $r = .60$). Learning-trajectory analysis revealed progression from procedural fraction manipulation to multiplicative coordination and, for seven students, emerging nonlinear covariational reasoning in SA:V contexts.

Keywords: Integrated STEM, Learning Trajectories and Progressions, Rational Numbers and Proportional Reasoning, Middle School Education

Proportional reasoning is a cornerstone of middle grades mathematics and a foundational construct for scientific sensemaking, supporting reasoning about rates, densities, concentrations, similarity, and scale (Lamon, 2020; Lesh et al., 1988). Despite its centrality, proportional reasoning remains conceptually challenging for many students (Ayan & Isiksal Bostan, 2018). Research documents persistent reliance on additive comparisons, overgeneralized cross-multiplication, and procedural fraction manipulation that may yield correct answers while masking fragile relational understanding (Tourniaire & Pulos, 1985; Dougherty et al., 2016).

Conceptual proportional reasoning extends beyond computational proficiency to include multiplicative coordination, recognition of invariant scale factors, and interpretation of ratios as structured relationships (Lobato & Ellis, 2010). Students must transition from additive reasoning toward multiplicative coordination, a shift that often remains incomplete even when procedural accuracy is achieved (Vergnaud, 1983). These challenges are amplified in contexts involving dimensional growth and scale. Students frequently apply linear reasoning to area and volume situations despite the inherently nonlinear nature of surface-area-to-volume (SA:V) relationships (Tretter et al., 2006). Reasoning in such contexts requires covariational reasoning—the coordination of how quantities change together (Carlson et al., 2002). While covariational reasoning has been examined in algebraic contexts, its development within middle school STEM environments, particularly in relation to proportional reasoning and scale, remains underexamined.

Standards documents foreground ratio and proportional reasoning in mathematics and identify Scale, Proportion, and Quantity as a crosscutting scientific concept (Common Core State Standards Initiative [CCSSI], 2010; NGSS Lead States, 2013). Yet instruction across mathematics and science often remains fragmented, limiting students' opportunities to develop proportional reasoning in contexts where it is epistemically necessary for explanation (Lehrer & Schauble, 2015). Nanotechnology provides a particularly generative context, as reasoning at the nanoscale inherently involves ratios, dimensional scaling, and SA:V relationships that challenge

linear assumptions and foreground multiplicative and nonlinear structure (Ramsden, 1990). However, empirical research documenting how students' proportional reasoning develops in nanoscale STEM contexts remains limited. Grounded in a learning-trajectory perspective (Simon, 1995), this study addresses this gap by examining how proportional and covariational reasoning co-develop in a nanotechnology-focused STEM program.

Theoretical Framework

This study is grounded in a learning-trajectory perspective, which conceptualizes learning as movement toward a goal state through progressively sophisticated forms of reasoning supported by instruction (Simon, 1995). Within this framework, students' incomplete strategies are viewed as productive transitional reasoning rather than misconceptions. Research on proportional reasoning characterizes development as a progression from additive comparison to multiplicative coordination and invariant-ratio reasoning (Tourniaire & Pulos, 1985; Lamon, 2020). Procedural competence does not necessarily indicate relational understanding (Dougherty et al., 2016). From a learning-trajectory perspective, proportional reasoning is not a fixed endpoint but a construct that can be refined as students encounter contexts requiring more complex coordination of quantities.

Reasoning about dimensional growth introduces additional demands. Students frequently apply linear reasoning to SA:V relationships despite their nonlinear nature (Tretter et al., 2006). Reasoning in such contexts requires covariational reasoning (Carlson et al., 2002). In this study, nonlinear covariational reasoning is conceptualized as a refinement of proportional-reasoning trajectories rather than a transition to formal functional thinking. Students' reasoning remained grounded in ratio and multiplicative structure, even as they coordinated layered multiplicative effects across dimensions. Integrated STEM contexts, particularly nanotechnology phenomena involving SA:V relationships, served as epistemic drivers that foregrounded dimensional structure and supported this refinement (Moore et al., 2014).

Methods

This study employed a design-based research (DBR) approach to investigate the development of proportional and covariational reasoning in an authentic instructional context (Cobb et al., 2003). DBR was appropriate given the study's goal of identifying a realized learning trajectory rather than evaluating program effectiveness. Participants were 11 middle school students (ages 10–14) enrolled in a week-long, university-based nanotechnology STEM summer program that met daily for approximately 7.5 hours. The lead researcher served as the primary instructor, supported by two teaching assistants. Data sources included a 17-item pre- and post-proportional reasoning assessment scored using a 5-point analytic rubric; student artifacts; audio recordings of small-group discussions; and researcher field notes. Quantitative analysis examined pre/post differences using a Wilcoxon signed-rank test with effect size calculated using r. Qualitative analysis involved coding student reasoning as additive, procedural, multiplicative, relational, or nonlinear covariational. Interrater agreement was strong ($\kappa = .81$).

Instructional Sequence

Instructional activities were sequenced to support movement from additive toward multiplicative and nonlinear reasoning. The sequence began with a cornstarch–water ratio investigation that disrupted additive comparison by introducing ratio as a relational comparison. Ratio comparison and scaling tasks then supported the transition from procedural to relational proportional reasoning by maintaining mixture consistency across quantities using invariant scale

factors. Dimensional scaling tasks (length → area → volume) surfaced limitations of linear intuitions as students predicted effects of scaling objects. Surface-area-to-volume modeling with cubes foregrounded layered multiplicative structure and coordination of dimensional growth. Virtual nanoscale simulations connected abstract structure to scientific phenomena, and an engineering design challenge (Alka-Seltzer reaction rate) consolidated nonlinear reasoning through strategic application.

Results

Quantitative Growth

Students demonstrated significant growth in proportional and covariational reasoning. Median rubric scores increased from 3.1 to 4.2, indicating a statistically significant improvement ($Z = -2.80$, $p = .005$, $r = .60$). Gains were strongest on SA:V items, where reliance on linear assumptions decreased substantially.

The analysis of student work, discourse, and design revisions revealed a realized learning trajectory consisting of four phases (see Figure 1).

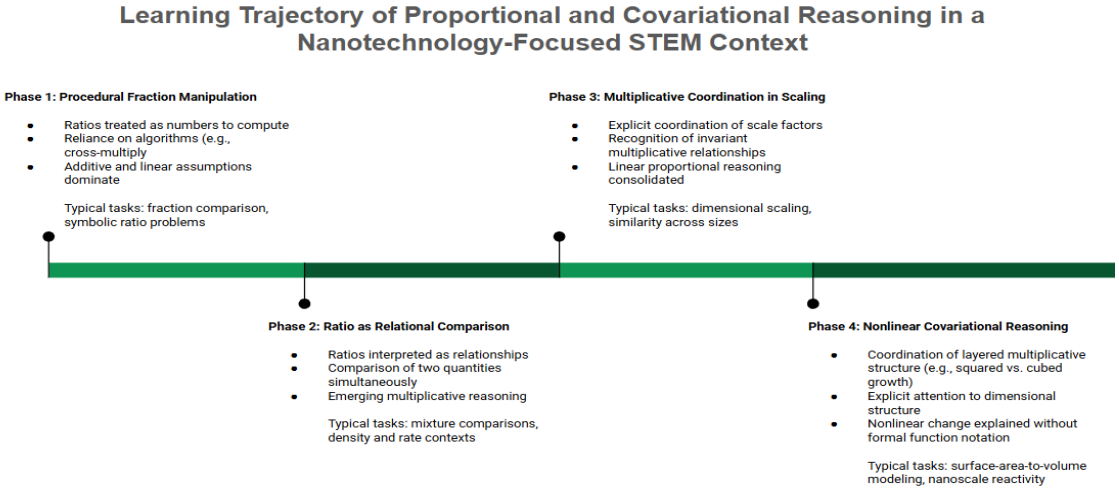


Figure 1: Realized Learning Trajectory of Proportional and Covariational Reasoning in Engineering Design Contexts

Realized Learning Trajectory

Analysis of student work, discourse, and design revisions revealed a learning trajectory consisting of four phases. In Phase 1 (Additive and Procedural Reasoning), students compared quantities additively or applied cross-multiplication without relational understanding. Phase 2 (Emerging Multiplicative Coordination) involved students beginning to recognize scale factors but applying them inconsistently across dimensions. In Phase 3 (Relational Proportional Reasoning), students coordinated quantities multiplicatively and identified invariant ratios. Phase 4 (Nonlinear Covariational Reasoning) involved students coordinating layered multiplicative effects across dimensions, recognizing that scaling effects depend on dimensionality.

Dimensional Revision

One analytic vignette illustrates this progression. During a surface-area-to-volume investigation, a student ("Maya") initially reasoned, "If the side doubles, everything doubles."

After calculating surface area ($SA = 24$) and volume ($V = 8$) for a two cm cube, she revised her explanation:

> "Surface area doubled twice, but volume doubled three times."

Maya then predicted cubic growth for larger cubes prior to calculation, demonstrating explicit coordination of dimensional structure and nonlinear covariation. Seven students demonstrated similar shifts, moving from linear scaling assumptions toward reasoning that accounted for layered multiplicative effects.

Engineering Design Application

All 11 students applied SA:V reasoning in an engineering design task involving Alka-Seltzer tablets. Students justified design decisions, such as crushing tablets to increase reaction rate, by coordinating surface area with reaction speed:

> "More surface touching the water means more reacting at once."

In this context, proportional reasoning functioned both strategically (to optimize performance) and explanatorily (to justify outcomes), indicating transfer of mathematical reasoning from modeling activities to applied engineering contexts.

Discussion

This study extends research on proportional reasoning by showing that middle school students' reasoning can develop beyond linear multiplicative coordination toward nonlinear covariational reasoning when dimensional structure is made epistemically central. Grounded in a learning-trajectory perspective (Simon, 1995), the findings position proportional reasoning not as a fixed endpoint but as a dynamic construct that can be refined through engagement with phenomena requiring coordination of scale, dimension, and rate of change.

Consistent with learning-trajectory theory, students' reasoning progressed through increasingly sophisticated forms as instructional activities disrupted additive and procedural strategies and supported relational coordination. Proportional reasoning developed in response to explanatory and design-oriented demands embedded in integrated STEM activities rather than through isolated symbolic practice. This finding aligns with prior research emphasizing that conceptual understanding is strengthened when mathematical structures are epistemically necessary for explanation (Lehrer & Schauble, 2015; Lobato & Ellis, 2010). Students' early reliance on additive and procedural strategies reflected productive transitional reasoning rather than conceptual failure (Simon, 1995).

The study further contributes by documenting the co-development of proportional reasoning and understanding of scale. Tasks involving dimensional scaling and SA:V relationships challenged students' linear intuitions and required attention to how quantities change across dimensions. Students came to recognize that proportional relationships are not invariant under dimensional transformation and that scaling effects depend on dimensionality. By foregrounding scale as a central organizing structure, instruction supported deeper relational understanding of proportionality (Lamon, 2020) and extends prior findings on persistent linear reasoning in area and volume contexts (Tretter et al., 2006).

Finally, the findings indicate that nonlinear covariational reasoning can emerge as a refinement of proportional-reasoning trajectories when dimensional structure is explicitly emphasized. Seven students coordinated layered multiplicative effects while remaining grounded in ratio-based reasoning. This extends covariational-reasoning research beyond algebraic contexts (Carlson et al., 2002) and challenges assumptions that nonlinear reasoning must be

deferred until secondary mathematics. Instead, nonlinear covariational reasoning appears as a natural developmental extension of proportional reasoning when learning trajectories foreground dimensional growth (Vergnaud, 1983; Lamon, 2020).

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STUDENTS' ENGAGEMENT WITH PARTIAL QUOTIENTS IN A BOX METHOD

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The study examined how two students (fourth & sixth grade) conveyed their multiplicative reasoning while solving multi-digit division problems using a Box Method. Both students participated in eye-tracking enabled clinical interviews. Transcripts were analyzed using Systemic Functional Linguistics (SFL) to examine how students interpreted and explained their own mathematical reasoning. Eye-tracking data and gestures were examined in concert with SFL to triangulate conveyed meaning. Findings highlight how differences in algorithm use and the participants' multiplicative reasoning affected their conveyed meaning of division.

Keywords: Elementary School Education, Number Concepts and Operations

Multidigit division can be difficult for students to understand the multiplicative meaning underlying the procedures (Hickendorff et al., 2019; Lampert, 1992; Silver et al., 1993). In particular, understanding the meaning of various digits within division algorithms has historically been found to be difficult for students (Hickendorff et al., 2019; Holland, 1942). Students may initially attempt to solve division tasks through additive reasoning such as via repeated subtraction or fair-sharing strategies (Neary & Nic Mhuri, 2021; Schulz, 2024; van Putten et al., 2005). A turning point in reasoning is marked by students' ability to partition dividends, and by consequence their quotients, by place value (Ambrose et al., 2003; Neary & Nic Mhuri, 2021; Schulz, 2024; Schulz & Leuders, 2018). This concept is known as partial quotients, and is a place value application of the Distributive Property to division (e.g., $177 \div 3 = 150 \div 3 + 27 \div 3$). Various scholars have advocated approaches to support students' learning partial quotients. These include: drawn groups-of models to write different place value components (Lampert, 1992); use of the Partial Quotients Algorithm, which is visually similar to long division, but where the partial quotients are explicitly written and accounted for (Losinski et al., 2021; Irons, 1975; Smith, 1950); and the Box Method for Division, which is an inversion of the same approach for multiplication (Egbedeyi & Kosko, 2025). Across these approaches is an effort to scaffold students' use of partial quotients.

Despite clear evidence of the importance of a conceptual understanding of partial quotients (Ambrose et al., 2003; Neary & Nic Mhuri, 2021; Schulz, 2024; Schulz & Leuders, 2018), and particular approaches meant to support this development (Lampert, 1992; Losinski et al., 2021; Smith, 1950), there is less understanding of the range of reasonings children engage with partial quotients. Multiplicative reasoning, from the perspective of Scheme Theory, emerges when children anticipate one level of units to construct two levels in activity. This first multiplicative concept (MC1) is characterized often through skip-counting, but may also be observed through repeated subtraction for division (Hackenberg, 2010; Kosko, 2019). Children at the second multiplicative concept (MC2) can anticipate two levels of units to construct three levels in activity. Gill et al. (2025) observed that such students can solve tasks like 8×33 by anticipating 33 as also 30 and 3 and then constructing 8×33 by adding 8×30 and 8×3. Yet, Gill et al. (2025) observed it is at the third multiplicative concept (MC3) when students can anticipate doing this with more than one number at a time (i.e., $24 \times 15 = 20 \times 10 + 20 \times 5 + 4 \times 10 + 4 \times 5$). The current study represents a preliminary analysis of students' division who operate at different levels of multiplicative reasoning (i.e., MC1 & MC3) with a goal of examining how these students engage

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

with partial quotients differently. Thus, the current study explores the following question: *How do students with different multiplicative reasoning convey meaning of division when using the Box Method?*

Methods

Participants & Protocol

Participants included two students as part of a larger study on fourth to sixth grade students' multiplication and division. Both analysis and data collection are ongoing at the time of this writing and involved a two-stage process. First, participating students completed a written assessment of their multiplicative reasoning and solving various multidigit division tasks. A subsample of students was purposefully selected from these responses for one-on-one eye-tracking enabled clinical interviews. Specifically, a target sample of 48 interviews is sought with individuals representing a range of multiplicative reasoning and written strategies for division. The current paper reports on the preliminary analysis of Lavada and Kit, who each used the Box Method for division. Lavada is a fourth-grade girl operating at MC1 and Kit is a sixth-grade boy operating at MC3. Both students wore Pupil Neon mobile eye-tracking glasses during the interviews and were tasked with solving each of the following tasks: 609÷3, 2400÷8, 708÷6, 179÷4. The first three tasks emphasize aspects of place value within division (Schulz, 2024) whereas the third task includes a remainder.

Analysis & Findings

Systemic Functional Linguistics (SFL) examines how grammar functions together for meaningful communication (Banks, 2019). Thus, elements of grammar may stem from language, different registers, gestures, and so forth (Jewitt et al., 2025). This analysis examines English grammar within the mathematics register where a register is a particular range of meanings and grammar associated with a specific activity (Halliday, 1975). For the purposes of this study, we examined reference chains to understand how students conveyed meaning about division. Reference chain analysis examines for cohesion of meaning by focusing on how nominal groups (nouns or sets of words that function as nouns) are referenced within and across a text. Meaning of nominal groups can be modified through transitive processes (i.e., verbs) and modality (Banks, 2019; Eggins, 2004). Modality conveys interpersonal meaning (Banks, 2019). Modality can be conveyed via modalization or modulation. Modalization can communicate a person's appraisal of the likelihood or frequency of propositions whereas modulation communicates appraisal of obligations (i.e., norms) or permissibility of propositions (Banks, 2019; Eggins, 2004). For the current analysis, we used transcribed interviews, as well as students video-recorded gestures and eye-gaze to examine their reference chains about division.

Lavada. For each division task, Lavada would draw a box with the same number of partitions as the divisor. She would then write numbers in each of the partitions and subtract the sum of these numbers from the dividend until reaching zero. For example, in solving 179÷4 she first wrote a 20 in each of the four partitions and subtracted 80 from 179. She then wrote a second 20 in each partition and so forth until she had subtracted all the dividend (see Figure 1). Lavada explained her division predominately through polar statements, conveying the truth value of her actions: "… we **do** 20, 40, 60, 80 … we **did** 179 subtracted by 80…" Interestingly, Lavada used modality in situations of uncertainty—particularly in instances when positing a number for one of her partitions in her Box Method. In the transcribed excerpt below, modality is italicized while transitive processes are in bold and nominal groups are underlined. Lavada hedges "you would do 20" while her finger points to and sweeps across the four 20s in Lavada's box method. Her eye-gaze also pauses at each 20, moving slightly ahead of her finger, while she makes this

statement (see Figure 1). Use of modals like *would*, *should have*, and *just* suggests a conveyed adherence to norms. In this case, we infer they are the norms of Lavada's box method where there are the same number of partitions as the divisor. Essentially, Lavada is conveying a skip-counting of four 20s two separate times. Her use of modals suggests these procedures adhere to her version of the box method.

Lavada: So, you *would* **do** 20, (sweeps first row of four 20s) // well you *should have* **done** 80, (circles all four 20s with finger as saying this) // but I *just* **broke** it down to 20. // So, we **do** 20, 40, 60, 80. // So, we **did** 179 subtracted by 80 equals 99. // And then I **did** 20 again, // which I **know is** 80. // I *just* **did** that. // So, 99 subtracted by 80 **is** 19. // And then I **did** four to get 16 // which **equals** three // and **there's not** three spaces.

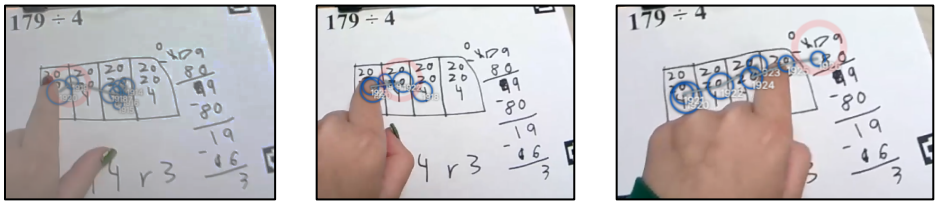


Figure 1: Segment (4 sec) from Interview for Lavada (MC1)

Analysis of Lavada's transcripts illustrates a conveyed goal of subtracting her partial quotients until reaching zero. This is evident in the excerpt above where material transitivity consistently conveys this pattern: "179 subtracted by 80… 99 subtracted by 80… I did four [fours] to get 16… which equals three." Moreover, across these tasks Lavada's box method focused on the value of the divisor instead of place value, which conveyed a groups-of meaning of division instead of a focus on partial quotients.

Kit. For each division task, Kit would quickly draw a box and place the divisor on the left side and write the digits by place value in the partitioned sections of his drawn box. He described this process clearly on the first task (609÷3) describing 609 as the dividend and 3 as the divisor. In describing his procedures, Kit would reference the digits, such as stating "six divided by three equals two." When the interviewer asked "is that actually a six?" Kit quickly said, "No, it's a 600" and the quotient there would be "200." Thus, Kit's reference chains consistently connected to the place-value meaning of the digits he acted upon. Most of Kit's statements were polar statements conveying a truth value related to his procedures: "then you **subtract** it… that **gives** you four… then you **carry** over the four." Use of modals appeared to be reserved primarily for contexts where the digits for one place value would have to be regrouped to another. For example, the excerpt below comes from Kit's explanation for solving 708÷6. As with Lavada's transcript, transitivity is in bold, key referents are underlined and modals are italicized. Within this excerpt, Kit initially uses *can* and *can't* to modulate statements about what is allowed to do with division in his box method. Later, he continues using such modals noting for the 700 that he "*can only* **divide** it by six one time," which again signals modulation within the context of regrouping quotients by place value.

Kit: So, you *can* **divide** seven by six one time. // So, you **put** the one above it, // but you *can't* **divide** it fully. // So, you **subtract** it // and then you **get** a one

Teacher: And the 1 stands for?

Kit: Oh, [1 **stands** for] 100. // And then you **carry** the one over to the 10 space // and since there**'s** zero there, // it *would* **equal** still as a 10 or whatever. // And then you **divide** it by six one time // it *can only* **divide** it by six one time // and then you **subtract** it // and that **gives** you four // And then you **carry** over the four // and the six divided by 48 **equals** eight.

Analysis of Kit's eye-gaze data confirms SFL analysis of his reference chains. Figure 2 illustrates an 8 second clip from the transcription above. As Kit explains dividing 700 by 600, his eye-gaze shifts to the tens column. As he states "and then you get a one" his eye-gaze centers on the 10 written in the 10s column. This action, coupled with his linguistic reference chain, suggests that Kit is conveying the "1" as a 10, but specifically a 10 in the tens place value. When prompted, Kit states that this 1—or 10 tens—also "stands for" 100. Throughout his explanation and observed actions in the interview, Kit reference place value, but uses modals when having to regroup digits in his quotients. This suggests Kit is appealing to his version of the box method, which emphasizes partitioning the dividend by place value.

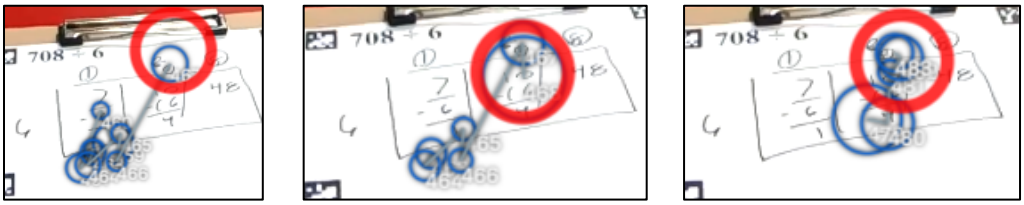


Figure 2: Segment (8 sec) from Interview for Kit (MC3)

Discussion & Conclusion

Various scholars have observed that partial products are important in students' multidigit division calculations (Hickendorff et al., 2019; Holland, 1942; Lampert, 1992; Silver et al., 1993). However, there is less research on students' reasoning with partial quotients. The purpose of this exploratory study was to examine how students conveyed multiplicative meaning when engaging with problems involving partial quotients. Findings indicated that Lavada and Kit used modals to convey an appeal to the norms of their algorithms, but these specific norms differed due to differences in the box methods used. Lavada's reference chains focused on repeated subtraction while her box method conveyed a groups-of approach to division (modified for place value). She hedged when placing quotients in each partition of her box method. By contrast, Kit's reference chains focused on the interplay of place value, and he hedged when having to regroup by place value. Important to note is that Lavada was operating at MC1 and was at the first half of fourth grade while Kit was operating at MC3 and was in the second half of sixth grade. Thus, these findings highlight key differences in students' reasonings with and exposure to multidigit division. For example, while Lavada did not explicitly reference place value in her explanations, she utilized it in her work. Thus, students at MC1 may be able to engage with place value in division but may convey different meanings than students at MC2 or MC3. Additional analysis is forthcoming, but findings provide important insights into differences in how students convey multiplicative meaning of their multidigit division algorithms.

Acknowledgments

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DEVELOPING COMPUTATIONAL FLUENCY THROUGH A VISUAL "HOW MANY" ROUTINE

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The How Many routine, in which an image of objects is used to elicit a variety of solutions, provides accessible, engaging computational practice. Yet, number sense routines are considered a "black hole" in mathematics education research, characterized by the lack of research directly studying the routine (Matney et al., 2020). This study examines the efficacy of How Many activities for promoting fluency development in elementary grades. A nine-lesson sequence of How Many tasks was taught in four second and third grade classrooms, and pre- and post-assessment interviews were conducted with 32 participants. Analysis of the interview responses via a multi-faceted fluency framework indicated statistically significant growth in fluency despite the brevity of the intervention.

Keywords: Number Concepts and Operations, Instructional Activities and Practices

Supporting all students to acquire computational fluency continues to be a challenge, as "many educators find that children, even in the upper grades, continue to draw tally marks and count by ones as their dominant solution approach in solving problems" (National Council of Teachers of Mathematices [NCTM], 2020, p. 82). Over the past 50 years, studies have shown that students who learn arithmetic via a focus on strategies as opposed to rote memorization consistently outperform their traditional counterparts (Thornton, 1978; Cook & Dossey, 1982; Henry & Brown, 2008; Brendefur et al., 2015). Yet many schools still rely on the rote practices of drill, worksheets, and timed testing, despite the overwhelming evidence that such practices do more harm than good (NCTM, 2020; Henry & Brown, 2008; Baroody, 2006).

Over the past two decades in an attempt to address this, number sense routines—brief, discourse-based routines focused on recognizing quantities and developing flexible mental computation strategies (e.g., addition, multiplication)—have gained prominence in elementary classrooms (MAISA, 2023). These routines reflect a shift to a contemporary definition of computational fluency as comprised of flexibility, accuracy, efficiency, and appropriate strategy use (National Research Council [NRC], 2001; NGA & CCSSO, 2010). Despite this, number sense routines have been characterized as a "black hole" in mathematics education research, meaning "an instructional practice for which there is a scarcity of blind-peer-reviewed research evidence supporting its efficacy, yet has attained critical gravity in the teaching field" (Matney et al., 2020, p. 247). Matney et al. (2020) explain that such "black holes" form when there is substantial tangential research that supports elements of the teaching practice, yet little-to-no research *directly* studying the efficacy of the practice itself. This study explored one such black hole: the How Many number sense routine.

The Multiplicity Lab (https://multiplicitylab.northwestern.edu/) at Northwestern University hosts a wealth of potential How Many activities (see Figure 1). The How Many routine consists of three components: (1) show an image to the class and ask, "How many do you

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see?," (2) provide children time to think, and (3) facilitate class discussion of as many different solutions as possible. This routine bears some resemblance to the number sense routine of number talks (Humphreys & Parker, 2015). While there is mathematical overlap between the concepts and practices that students might engage in through the How Many and Number Talks routines, one key difference is the entry point. In How Many, students are offered a *visual*, rather than a computational, mathematical prompt, and in most cases, multiple answers are defensible (e.g., total number of dice, total number of pips on the dice, identical sets of dice). For example, in the first image above, some children might count the dots, one by one, or add them in their visual order. Others might rely on their knowledge of combinations of 10 and doubles to group the 6 with a 4 to make 10, the two 5s to make 10, and then add the 8 from the two remaining 4s to reach 28. Still others may think multiplicatively, particularly about the three groups of 4. As students are exposed to the variety of strategies shared by their classmates, there are opportunities for them to develop flexibility, choose appropriate strategies, and move towards greater efficiency.

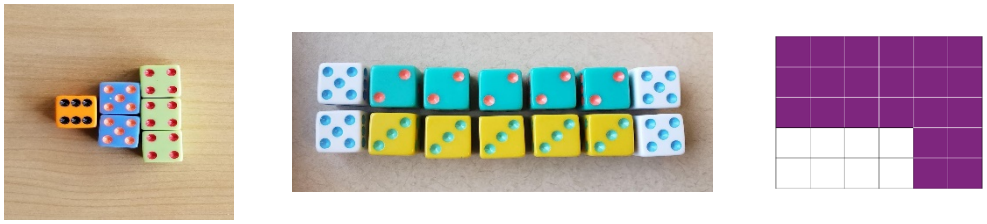


Figure 1: Three Different How Many Images

Defining procedural fluency in the complex way described above warranted new frameworks for fluency development. With respect to single-digit computation, a Three Phase basic fact fluency framework developed by Baroody and others (e.g., Baroody, 2006; Baroody et al., 2009; Baroody & Tiilikainen, 2003) is useful for assessing student progression as it distinguishes between deriving and knowing facts, a distinction that is often collapsed in other frameworks (e.g., Carpenter et al., 1993). According to Baroody's framework, students begin in Phase 1 with counting-based strategies (e.g., modeling and counting, mental counting, skip counting), then progress to Phase 2 where solutions are derived from known facts (e.g., $6 \times 8 = 5 \times 8 + 1 \times 8$). Phase 3, mastery, is demonstrated by highly efficient production of answers, either through recall or highly efficient strategy application. While the framework is critical for understanding fluency development with two addends or factors, it raises questions of how to assess fluency in the context of more complex tasks like those in How Many.

The purpose of our research was to explore the efficacy of How Many activities for promoting greater fluency development in elementary school children, and in doing so to develop analytic tools that build on earlier frameworks to allow us to assess fluency when many values are involved. In the current study, we ask: Do How Many activities promote fluency in elementary school students?

Methods

Setting, Participants, and Study Design

The study took place across four rural midwestern classrooms (two second grade and two third grade) and consisted of a week of pre-interviews with 32 participating students, three weeks of instruction, and a week of post-interviews. One researcher taught three lessons in each

classroom per week, for a total of 36 lessons (9 lessons/classroom x 4 classrooms). Each lesson lasted 30 minutes and utilized 1–2 images such as those shown in Figure 1, engaging students in whole class discussion as they shared different ways they might determine How Many.

Data Sources and Analysis

Data for this study include the pre- and post-assessments, which were video recorded. Our interview assessments, both pre and post, included the three images shown in Figure 1 as well as the computation problem 6 + 3 + 6 + 7. Children were presented with each task, asked "How many do you see?" and were provided with as many copies of the task as they wished to utilize in their responses. During the assessment, two research assistants took fieldnotes recording student strategies and solutions. These three sources, two sets of fieldnotes and the video records, were coordinated to analyze students' fluency before and after the How Many intervention.

By creating a list of indicators for the four components of fluency (NRC, 2001), we developed the fluency framework shown in abbreviated form in Figure 2 to measure participant growth and provide an indication of the potential efficacy of the How Many routine. This framework was developed iteratively by the research team and each task was scored by two coders; all coding differences were resolved through discussion with the entire research team. Each task on the interview could be scored up to four points on the framework; each task in which the student provided an accurate answer was given an additional point. Across all four tasks, each participant received a composite pre- and post-assessment score of up to 20 points. These composite scores as well as mean scores of individuals across the four tasks (up to five points) were tested for statistically significant growth using paired t-tests of the mean differences.

Level 1	Level 4
Efficiency: Relatively inefficient strategies that took a longer time than other available strategies. Flexibility: A single strategy is used throughout, without regard to the opportunities in the numbers. Appropriate Strategy Selection: Strategy did not take into account the particular numbers involved or the structures available.	Efficiency: Efficient strategies used exclusively throughout. The strategies coordinated maximize the structures available for efficiency. Flexibility: Students combined and moved between multiple strategies throughout the task. Appropriate Strategy Selection: Strategies fully accounted for features of the particular numbers involved or structures available.
Level 2	Level 3
Efficiency: Partial increase in efficiency, often at the beginning, but then primarily relied on inefficient strategies. Flexibility: Either a single strategy is used throughout that attended to one feature of the numbers, or students used such a strategy and then moved to an inefficient strategy for the remainder of the task. Appropriate Strategy Selection: Strategy took into account one feature of the particular numbers involved or the structures available.	Efficiency: Primarily relied on efficient strategies, but finished with or included a less efficient strategy for a portion of the task, OR used a large collection of strategies such that coordinating or moving between them was inefficient *(not maximizing use of structure)*. Flexibility: Students used more than one efficient strategy that attends to features in the numbers or their structure. Appropriate Strategy Selection: Strategy took into account more than one feature of the particular numbers involved or the structures available.

Figure 2: Abbreviated Fluency Scoring Framework

Findings

Our analysis found statistically significant growth in students' fluency from the pre- to post-assessments after the nine-lesson How Many intervention. The mean total score on the pre-assessments was 10.53 (out of 20; 5 points for 4 tasks), or a mean score per task of 2.63. On the post assessments, the mean total score was 13.63, or a mean score per task of 3.41. The pre- to post-assessment difference was highly statistically significant at the $\alpha = 0.05$ level ($p = 0.0008$). These differences were similar in both second and third grades.

Analyzing the shifts at the student level, we found that the number of students with a mean per task score lower than 3 shifted from pre- to post-intervention (see Table 1). In the pre-assessment, seven students had a mean score below 2 and another 12 students had a mean score of at least 2 but less than 3, for a total of 19 students with a mean fluency score per item below 3. In the post-assessment, the number of students in this group was more than cut in half; only one student had a mean per item score below 2 and eight scored in the next highest tier, for a total of nine students with mean scores below 3. Proportion testing indicated that this was a statistically significant shift at the $\alpha = 0.05$ level as well ($p = 0.0117$).

Table 1: Fluency Shifts Pre- to Post-Intervention

Mean Fluency Score (per task, out of 5)	Number of Students (Pre-Assessment)	Number of Students (Post-Assessment)
Score ≥ 4	2	8
$3 \leq$ Score < 4	11	15
$2 \leq$ Score < 3	12	8
Score < 2	7	1

Discussion

Our results indicate that the nine-lesson How Many intervention, despite its brevity, may have encouraged fluency development for the participating second and third graders. The duration of this intervention was bound by practical constraints at the end of a school year; our findings raise questions about what additional gains might be possible if this routine were truly routine, even once or twice a week, across a school year. Additional research is needed to explore this possibility, as well as to investigate the potential for supporting fluency development in both younger and older students. Furthermore, given the limited research on fluency frameworks involving multiple addends and factors, we posit the framework shared in Table 1 has scholarly significance as well. Indeed, we believe that these types of tasks may offer unique opportunities for developing appropriate strategy selection, flexibility, and efficiency, given the multiple entry points available when the number of addends stretches to six, ten, or more. We next seek to investigate the nature of the discourse and facilitation of the intervention itself to inform how others might seek to implement the How Many routine for future research and in practice.

Acknowledgements

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A DYNAMIC MEASUREMENT APPROACH TO MULTIPLICATIVE REASONING

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This study investigates how Dynamic Measurement for Area (DYME-A), an approach that engages students in exploring models of area as a dynamic sweep of a line segment over a distance, can support students' multiplicative reasoning. We report on a design experiment with one third-grade student to illustrate how experiences with generating and manipulating dynamic area models supported the student in constructing multiplication expressions as dimensionalized objects of base × height, that could be dynamically stretched, partitioned, and iterated, fostering flexible multiplicative reasoning.

Keywords: Cognition, Design Experiments, Number Concepts and Operations, Technology

Area models have been a key visual representation for several forms of multiplication, including whole numbers, fractions, and binomials (e.g., Kwon et al., 2019; Lischka & Stephens, 2020; Pratt, 2017). An area model is a visual representation of a rectangular area filled with a grid of rows and columns (e.g., Figure 1a) or split into unequal area parts (e.g., Figure 1b) (Kwon et al., 2019). For example, to find the product of 3×4, a student can draw a rectangle of three rows and four columns (Figure 1a) and count the total number of squares units.

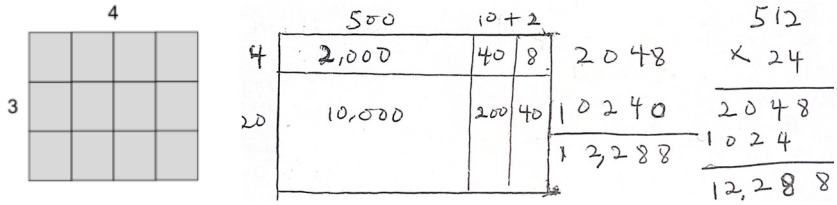


Figure 1: Area Models of Multiplication

Area models provide several benefits for multiplication including supporting students in making connections between quantities and operations (Kwon et al., 2019), developing mental images that help students reason abstractly (Englert & Sinicrope, 1994; Schultz, 1991), and reasoning about continuous quantities (Lischka & Stephens, 2020; Smith et al., 2013). Englert and Sinicrope also note that area model representations make the parts of partial product multiplication visible and support students to notice patterns relating the area models to the computations. Additionally, area models are particularly useful in visually representing real-world experiences such as "carpeting, fabrics, paint, wall coverings, and lawn and gardening products" (Schultz, 1991, p. 42). Lischka and Stephens (2020) argue that "developing an understanding of the area model in the elementary grades lays a foundation for understanding other concepts involving multiplication in later grade levels" (p. 187). These include ratios,

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

probability, even covariational reasoning and algebra in later years of schooling (e.g., Ellis et al., 2020; Fyfe et al., 2017; Lischka & Stephens, 2020; Lovin, 2020; Pratt, 2017).

Because of these benefits, the state learning standards (NGA, 2010; NJDoE, 2020) include area models in numerous mathematics standards for whole number multiplication and fraction operations and students are expected to be able to use them to represent, illustrate, explain, and justify their reasoning. Despite emphasis in the curriculum, the NAEP report (USDoE, 2022) shows that fourth-grade students struggle in both area measurement and multiplication. One reason for these struggles is that existing approaches lead students to focus on the number of parts rather than the area of the parts in an area model (Čadež & Kolar, 2018; Smith et al., 2013; Van de Walle et al., 2013). As a result of treating area as the count of discrete squares and not a multiplicative relationship, when area models are used for multi-digit partial product multiplication (Figure 1b) or the distributive property, students may "not be sure if numbers used for factors represent the lengths of sides of a given rectangle or the numbers of rows and columns in the model" (Kwon et al., 2019, p. 364). Therefore, students struggle with creating area models without grids for multiplying larger numbers (Figure 1b). Consequently, Kwon et al. (2019) call for more research to determine the essential ideas embedded in the use of area models for multiplication, as well as on the different kinds of area models and how students use them.

In this paper, we discuss how we reconceptualized area models through Dynamic Measurement for Area (DYME-A), an approach that engages students in exploring models of area as a dynamic sweep of a line segment over a distance (Panorkou, 2020a). Our conjecture was that this approach can engage students in dynamically generating and transforming area models and support them in developing more flexible multiplicative reasoning through a deeper connection with measurement. Specifically, we examined: (a) What kind of DYME-A model design can support students' multiplicative reasoning? and (b) What forms of multiplicative reasoning do students exhibit as they interact with DYME-A area models?

Area Models: What We Know and Pushing Forward

There are two distinct types of area models in the literature (Kwon et al., 2019): the area-to-area model and the lengths-to-area model. The area-to-area model starts with an area unit as an initial quantity and emphasizes its change through the one- or two-dimensional arrangement of area units in groups (Figure 2a). It is a model that is analogous to the traditional array model. In contrast, the lengths-to-area model conceptualizes an area unit as the product of two coordinated lengths of one unit each (Figure 2b) and generates area in terms of a specific length and width (i.e., factor × factor). While textbooks usually focus on the first type, the lengths-to-area model can support students to understand how lengths can generate and define an area.

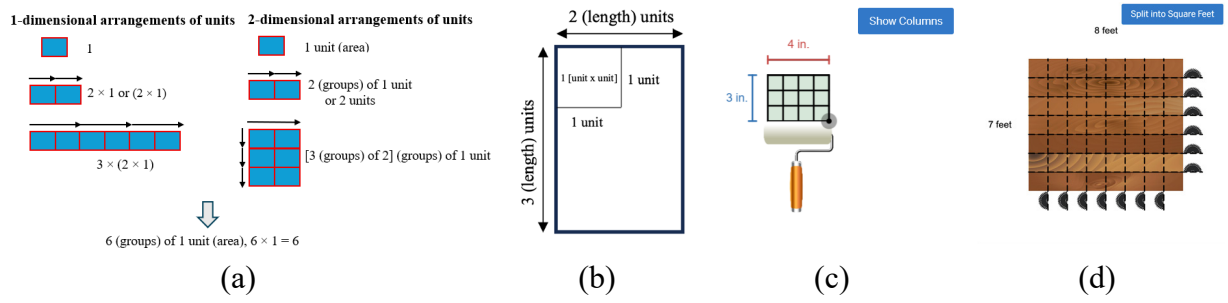


Figure 2: a) Area-to-Area and b) Lengths-to-Area Models Adapted from Kwon et al. (2019); c) Generating Area Models and d) Splitting Area Models into Square Units

In addition to the two models, we also considered the *containment* and *generation* perspectives of measurement (Panorkou, 2020b; Panorkou & Pratt, 2016) and how these may apply to how area models are drawn, manipulated, and conceived. The containment perspective entails conceptualizing an area model as a 2D space that contains other 2D (square units) and 1D parts (length, width), while the generation perspective entails conceptualizing that the area model can be generated by tracing a length over a distance, or by iterating other area units. In our study, we aimed to provide students with opportunities to experience both generation and containment perspectives. For instance, we had students construct area models by using draggable points to generate a specific length (e.g. 4 in.) then drag that length over various width measures and visualize how the area grows simultaneously (Figure 2c). To help students connect the 1D parts (length and width) with the 2D parts (square units) of the rectangle, we made use of features that allowed students to toggle between continuous area models (only showing the dimensions) and discrete area models with square units. We also had students dynamically cut rectangles into the contained square units given the dimensions of the rectangle (e.g., Figure 2d).

Theoretical Framework: Studying the Interplay Between Area and Multiplication

Over the years, research on students' measurement and multiplicative reasoning has developed along separate paths. A study that aimed to connect the two was the Elkonin-Davydov Curriculum (Davydov et al., 1999) and its US adaptation Measure-Up (Dougherty et al., 2004) that studied measurement with multiplication as a change of unit measures. The difference in the DYME-A approach is that students construct a unit of measure as a dimensionalized object (Thompson, 2000) that depends on the specific length and width. For instance, in Figure 2c, the construction of an area (e.g., using the length of 4 in. dragged by 1 in. to create an area of 4 square in.) can be perceived as a dimensionalized object that can be iterated by 3 in. to generate a total area of 12 square in. In constructing our framework, we focus on students' possible mental processes as they generate and act upon those dynamic area models. We hoped that this approach will give us insights into their early multiplicative reasoning that will be constructed and developed as they interact with these different models. Table 1 presents the mental processes that we considered while we were open for any others not reported on that list.

Table 1: Possible Mental Processes During Students' Area Models Explorations

Mental Process	Definition
Equipartitioning	Cognitive behaviors that have the goal of producing equal-sized parts of wholes (Confrey et al. 2009).
Re-assembling	Recombining equal parts to produce the original whole as "n times as many" or "n times as much" as a single part (Wilson et al., 2012)
Iterating	"Repeatedly instantiating a part of a unit in order to make a larger amount" (Hackenberg, 2010, p. 385)
Decomposing	Breaking a whole into non-equal parts (Sarama & Clements, 2009)
Composing	Recombining non-equal parts to produce the original whole (Sarama & Clements, 2009)
Stretching	Changing the whole by multiplicatively changing the base or the height by a factor greater than 1.
Compressing	Changing the whole by multiplicatively changing the base or the height by a factor between 0 and 1.

These processes illustrate an integration of the containment and generation perspectives. To elaborate, growing the base or the height to grow area presumes a conceptualization of area as both containing those objects and being generated by these objects. In a similar manner, equipartitioning an area into square units or re-assembling square units to produce area involves a conceptualization of area as containing square units and being generated by these square units.

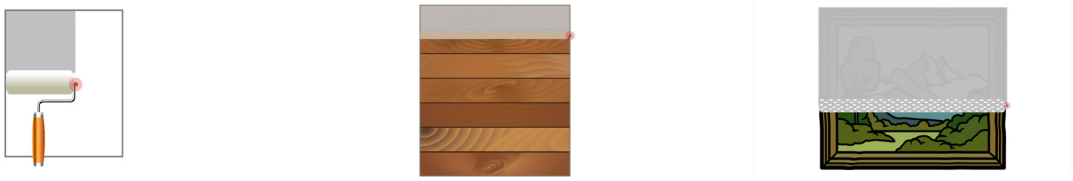
Methods

This paper reports on the findings from a design experiment (DE) (Cobb et al., 2003) with a third-grade student, Alexa, to initiate the discussion around the qualitative different mental processes and varied forms of multiplicative reasoning that are possible as students interact with the DYME-A tasks as well as the type of design (tasks, questioning) that might support these forms of reasoning. This DE is part of a larger design-based research study where we conducted a series of 10 DEs with pairs of third- and fourth-grade students aiming to study the interplay between design and students' reasoning. The particular DE presented here was part of our second iteration and lasted for 13 sessions. Each session ranged between 38–85 minutes.

The activities on area models and multiplication were designed using the Amplify (formerly Desmos) classroom platform due to its dynamic tools. Each session was video-recorded and student screen activity was recorded. All recordings of the sessions were multimodally transcribed (Chanier & Lamy, 2017) to capture verbal language, gestures, and student actions. First, we analyzed the transcriptions to identify episodes of student reasoning about area and multiplication based on our theoretical framework described previously. Each member of the research team coded the transcripts and then met to discuss the coding. In this paper, we report the second stage of analysis that included constructing a chronological account of student reasoning episodes to describe the relationship between their mental processes and their activity.

Findings

In terms of prior knowledge, Alexa was familiar with the term area, but she did not "really know what it means." She could also find the products of expressions such as 10×9 by using arrays and equal groups. She acknowledged the challenge with using arrays with big numbers, stating, "I won't have enough space" on the whiteboard to construct one. The initial tasks of DYME-A asked students to dynamically sweep a line segment over various distances and reason about the space generated using contexts such as painting, bubble wrap rolling, or carpet rolling (Figure 3). To create common language, we introduced the terms *base* (measure of one side), *height* (measure of a side that is perpendicular to the base), and *area* (measure of the surface generated).



(a) (b) (c)

Figure 3: Dynamic Sweep of a) a Paint Roller, b) a Carpet Roller, c) a Bubble Wrap

After dragging a roll of bubble wrap to cover a painting (Figure 3c), Alexa was asked how we could make the area bigger. She responded, "By making that height bigger, we can stretch," illustrating a dynamic conception of height as a quantity that can "*stretch*" and influences the area. We then prompted, "We cannot stretch the base?" and she acknowledged, "Yes, we can."

To support students to quantify the area, we then introduced the notion of square units by using a step-by-step animation (Figure 4) that illustrates how an area model for whole number multiplication can be generated, beginning with a single square unit (Figure 4a), then a row of five square units (Figure 4b), and rectangle of eight rows (Figure 4c) that dynamically forms a 40-unit rectangle. When Alexa was asked what she noticed about the animation, she stated, "The purple one is one unit, and the other one is five units because it is the base." She then continued, "The height got bigger, seven more," and "the total is 40 square units." Alexa's response shows that she constructed the 1-square-unit as an area unit, then quantified the one row based on those area units by looking at the measure of base. Then she quantified the large rectangle as 40 square units by considering the increase in the height and *iterating* the row seven times. Next, we asked Alexa to draw and label a 7×5-unit rectangle herself on a digital grid and find its area. She responded that to find the area of a rectangle, "we need to know how big is the base and the height," illustrating that she began connecting the base and height to the square units.

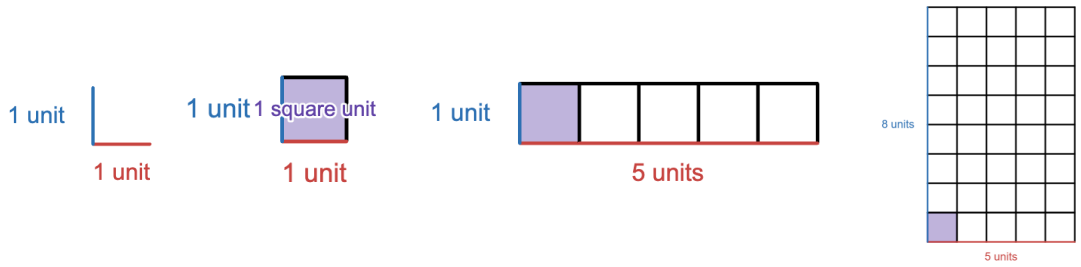


Figure 4: Construction of a) One Square Unit, b) Base Row of Units, and c) Entire Rectangle

To support the connection between lengths-to-area and area-to-area models further, we introduced a task using draggable paint rollers to generate rectangles. She dragged a paint roller to create a 4in. × 3 in. rectangle and stated, "This is four, this is three, so that is like, so that's 12," referring to the area (Figure 5a).

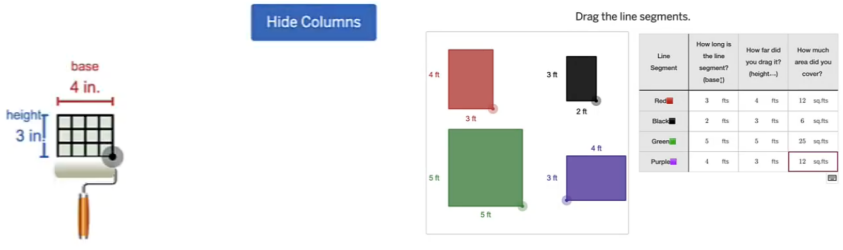


Figure 5: a) Hide/Show Columns Feature, and b) Dragging Line Segments

Next was an 8 in. × 3 in. rectangle and Alexa immediately stated, "So this one is 24," without generating the digital rectangle. When asked how she knew the 8 in. × 3 in. dimensions would result in a rectangle with an area of 24, she replied, "Because I can multiply the base and the height because it will equal to 24 since it's multiplication." We interpreted this as an early indication of her conception that area is determined by the product of base and height, connecting multiplication to the area model. Alexa then began working on dragging line segments to generate rectangles (Figure 5b). At this time, the "show columns" feature, which enabled students to see the square units, was no longer accessible and she was able to find the area of each rectangle. At the end of the activity, Alexa was asked, "How do we find area?" and Alexa stated, "By multiplying the base and the height," suggesting that the absence of square units in the design supported her shift between discrete and continuous models of area.

Our next goal was to support Alexa in recognizing the multiplicative coordination between base, height, and area. Alexa was asked to double the area of a 4 in. × 1 in. candy bar (Figure 6a). She first doubled the area of the rectangle by first *iterating* the area and connecting it to *stretching* the height. Alexa explained, "because if I add four more [square units], this [circles 1in height in her drawing] change into two [inches]." When asked to do it differently, she *stretched* the base. Then she was asked, "Based on what you did, how would you explain to someone how to double the area of the rectangle?" and she responded that she "can use the base or the height." She was then asked to triple a 4 in. × 1 in. rectangle and she *stretched* the height to create a 4 in. × 3 in. rectangle (Figure 6b). She then did it a second way by incrementally *stretching* the base to create an 8 in × 1 in. and then a 12 in. × 1 in. rectangle. She was then asked to triple the area again and stated that multiplying either the base or height by three would make the area "three times bigger." She was then prompted to reason about larger factors, such as "if I make the height ten times bigger, what will happen to the area?" She explained that when we make the base or height ten times bigger, the area will also become "ten times" bigger. Her reasoning shows that she connected the act of *stretching* with multiplication, recognizing that *stretching* either the base or height by a factor results in *stretching* the area by the same factor.

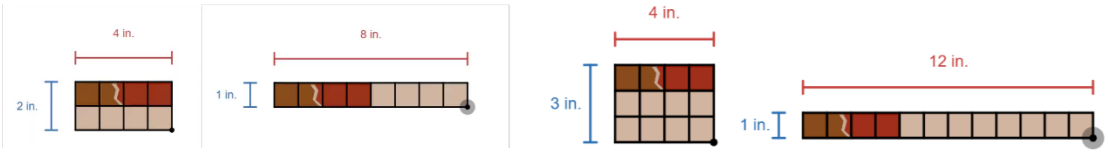
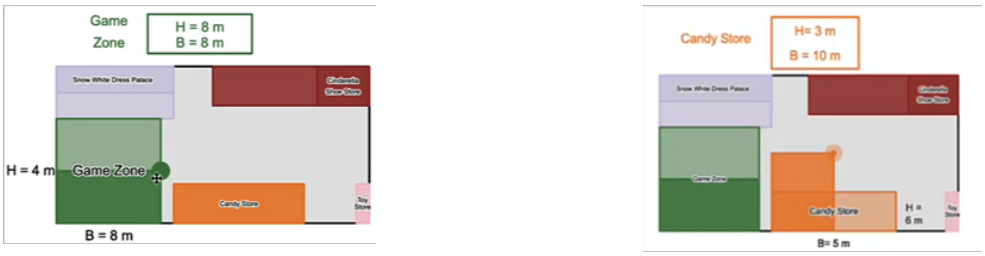


Figure 6: Alexa (a) Doubled and (b) Tripled the Height or the Base of a 4 sq. in. Candy Bar

The tasks that followed also focused on the multiplicative coordination of base, height, and area but illustrated change in continuous models where the square units were not visible. In one task, we asked Alexa to multiplicatively change the dimensions of various stores in a shopping mall and reason about the change in area. When we asked her to halve the height of the 8 m × 8 m game zone (colored green, Figure 7a), she *compressed* the area and stated that the height and area each "got two times smaller."

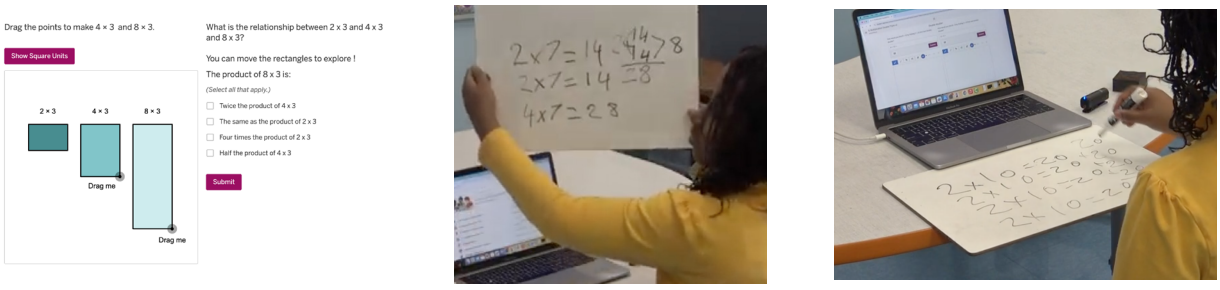


(a) (b)

Figure 7: a) Alexa Halved the Height, b) Alexa Doubled the Height and Halved the Base

Alexa was then asked to double the height and halve the base of a 10 m × 3 m candy store (colored orange, Figure 7b) and she replied, "It changed because it was like, first this [points at the original area], and it got longer [points at the new area], and the area and the base got smaller. But it still equals the same; it's still equal to the same area." Alexa's responses indicate that she was able to coordinate the change in length, base, and area flexibly, even when both dimensions changed, without the need of visible square units.

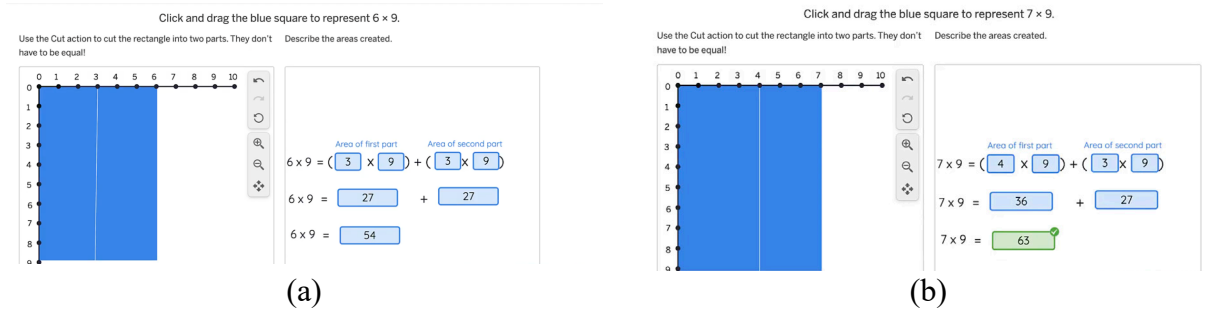
Subsequent tasks aimed to extend students' experiences with dynamic area models to multiplication by asking them to investigate relationships between factors that are multiples of two, with the goal to help them recognize that multiplying a quantity by four or eight is equivalent to doubling that quantity twice or three times, respectively. For instance, students were asked to drag the height of a 2×3 rectangle to generate 4×3 and 8×3 rectangles and explore questions such as, "What is the relationship between 2×3 and 4×3 (or 8×3)?" (Figure 8a). Students could also overlap the rectangles to compare them or toggle the square units. This series of tasks supported students in finding larger products by *equipartitioning* and *iterating*. For instance, when Alexa was asked to find 4×7, she stated, "I can use 2×7. So, I put 2×7, two times. It equal to the same number" (Figure 8b). We interpret Alexa's reasoning to show that she constructed 4×7 as a dimensionalized object that she split in half to a smaller dimensionalized object with a familiar product. She then doubled the 2×7 to *reassemble* the original. Similarly, when she was asked to find 8×10, she stated, "I can use 4×10 to do 8×10 since it will be easier." She continued, "I am going to use 2×10, four times, since it's double-double." (Figure 8c). Like before, we characterize her reasoning to illustrate that she constructed 8×10 as dimensional object that she *equipartitioned* in four smaller dimensional objects with a familiar product. She then *reassembled* the original by quadrupling the 2×10 part.



(a) (b) (c)

Figure 8: a) Relationship Between ×2, ×4, ×8; b) Alexa's 4×7 Work, c) Alexa's 8×10 Work

Likewise, Alexa was then asked to find products with factors ×6 and ×12 but she struggled to mentally coordinate the processes associated with *partitioning* and *iterating* these dimensionalized objects. To support her, we designed a partitioning task that she could actively manipulate from both generation and containment perspectives. We asked her to create rectangles using dragging (e.g. 6×9 in Figure 9a) and then use the Cut tool to split it into equal parts and determine each part's area. This task supported her exploration of how multiplying a quantity by 6 is equivalent to multiplying the same quantity by three and then doubling it. For instance, Alexa explained, "You can just use rectangles to find the answer, and you can just split a rectangle in half. Which is half of the area, you can use any area that can be split and after, you can use the area of the first part and the second part. You can add them up together to make it the product." The task allowed Alexa to equipartition the rectangle into smaller, more manageable dimensionalized objects.



(a) (b)

Figure 9: Cutting Rectangles Task for Finding (a) 6×9, and (b) 7×9

Subsequently, we introduced multiplication sentences with both factors odd (e.g., 7×9) to examine whether students could *decompose* dimensionalized objects. Alexa was able to *decompose* seven into three and four (Figure 9b) stating, "So what I did was cut the rectangle in four. And it doesn't equal to a half, because it's an odd number, but it doesn't have to be equal. So, the area of the first part is four times nine, in the second one is three times nine. And when I add the area of the first part it's 36 and the second part is 27." Alexa was able to *decompose* a bigger dimensionalized object into two smaller and more manageable dimensionalized objects and then *composed* them to find the total area. These *decomposition/composition* processes supported her to find more complex products later such as two-digit by one-digit, three-digit by one-digit, and two-digit by two-digit.

Concluding Remarks

Our tasks allowed students to work with area models as a combination of area-to-area and length-to-area models (Kwon et al., 2019) through the dynamic generation of both base and height, and square units. First, our findings show that the dynamic generation of area models allowed Alexa to see that the area of a rectangle depends on the base and the height. Then, Alexa demonstrated that the tasks supported her construction of the multiplicative relationship between base, height, and area as well as the connection to the square units used to measure area, illustrating the containment of space. Alexa also generated both 1D and 2D area models that supported her construction of the multiplicative coordination between base and height and area. These experiences provided Alexa with the opportunity to construct multiplication expressions (e.g., 8×7) as dimensionalized objects (Thompson, 2000) of base × height. These

dimensionalized object expressions modeled areas that could be stretched or compressed dynamically to generate larger or smaller areas. They can also be equipartitioned or decomposed to smaller manageable dimensionalized objects, which can be reassembled, iterated, or composed back to find the total area of the original. Shifting between discrete and continuous area models and between containment and generation (e.g., Panorkou, 2020b; Panorkou & Pratt, 2016) in the area model design could be a new perspective to support students' flexible multiplicative reasoning through measurement.

Acknowledgments

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THREE-DIMENSIONAL CONSTRUCTION THAT DEVELOPS SPATIAL VISUALIZATION IN YOUNG CHILDREN

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This study analyzed the spatial visualization abilities of 41 five-years children through their cube construction strategies and the mental manipulation of nets. Four types of cube construction strategies were observed. Mental manipulation of cube nets was difficult for children. The children were observed to tend to judge whether they could build a cube based on appearance alone. However, four children used gestures and took a long time to complete mental manipulation. Two of these four children created cubes using strategies that utilized the relative positions of the faces. For these two children, a correlation was considered to exist between cube construction and mental manipulation.

Keywords: Spatial Reasoning, Spatial Visualization, Mental Folding, Geometry

Introduction

Spatial reasoning is broadly defined as the ability to mentally represent, organize, and transform objects and their relations (Linn & Petersen, 1985; Newcombe & Shipley, 2015). Spatial reasoning plays an important role in predicting overall mathematics success with even greater predictive power than general mathematics scores (Farmer et al., 2013). Spatial skills are strongly related to success in Science, Technology, Engineering, and Mathematics (STEM) educational pursuits and career choice (Wai et al., 2009). However, this is a particularly underdeveloped area of current mathematics curricula, especially in Japan.

Theoretical framework

Spatial Reasoning that Operates on Transformations Between 2D and 3D Objects

Two object-based spatial skills that frequently appear in mathematics research are mental rotation and spatial visualization (Harris, 2023). Mental rotation skill is the ability to mentally rotate a two- or three-dimensional geometric figure quickly and accurately; and spatial visualization is associated with the ability to recognize complicated, multi-step manipulations of spatial presented information (Uttal et al., 2013). Spatial visualization has been singled out by cognitive scientist as a central metric of spatial reasoning (Jansen et al., 2013). In this article we discuss what we mean by spatial visualization, why constructing solids can foster spatial visualization in young children. Spatial visualization involves understanding and performing imagined movement of 2D and 3D objects (Uttal et al., 2013). Many studies have reported that educational interventions have improved spatial visualization in young children (e.g. Bruce & Hawes, 2015). Even at an early age, some children can mentally construct planes using spatial visualization and imagery. Furthermore, the children with lower levels of spatial visualization skills have improved through educational intervention (Uttal et al., 2013). However, the detailed reasons for this have not been reported, and there is no clear explanation for the relationship between spatial visualization, imagery, and intervention experiences in young children. Several spatial visualization tests have been studied to date (Davis, 2015). For example, both surface development tests (a 2D to 3D transformation task) and the paper folding tests (a mental folding task) may require dynamic spatial manipulation abilities and visuospatial working memory capacity. These tests require children to engage in dynamic thought processes related to a single

object. However, there is little detailed research on experiences and training to improve spatial visual abilities in young children. Currently, there is relatively little research focusing on spatial thinking and the environmental factors that influence its improvement.

This paper focuses on surface development tasks in young children. Creating three-dimensional objects from two-dimensional planes is a common activity in young children's play. Clements (2004) suggests that young children with extensive experience in creating three-dimensional objects have high spatial abilities. Creating three-dimensional objects is an enjoyable activity for young children. However, the way they approach it varies from person to person. We hypothesized that this experience might influence spatial visualization. Therefore, we decided to study the relationship between the strategies used when creating a cube from a two-dimensional plane and the strategies used when imagining folding the net of a cube. Based on the above, we address the following research questions.

- What types of strategies do young children use to construct a cube from six faces?
- What is the relationship between young children's cube-construction strategies and their mental manipulation of cube-nets?

First, we will study the process by which young children create cubes using six faces and define the process by which they understand how to create a three-dimensional cube from a two-dimensional plane. Next, we will study the process by which young children imagine a three-dimensional cube by mentally folding the net of a two-dimensional cube and identify the difficulties they face. Finally, we will study the relationship between the strategies of making a cube by hand and mental manipulation.

Method

A total of 41 five-year-old children participated. Each participating child (n=41) was individually interviewed by author in a quiet location at school. We observed and recorded how the children responded to the tasks. First, we placed a cube in front of the children, with squares of the same color facing each other. Next, we gave each child six squares (2-red, 2-yellow, 2-green). Then we told them, “Use these six squares to create the same cube as before.” Each child assembled a cube based on the color information. We observed, recorded, and categorized how each children assembled their cube. Second, we prepared a cube one top face missing and its net, and asked the children, “If you make this cube net, will it become the cube in front of you? Think about it in your mind and try to guess whether it will or won’t.” The children answered “Yes,” “No,” and “I don’t know.” The reason we used a net with one face missing was that the results of mental folding tests using six-faces and five-faces nets of the same shape were similar from first grade to sixth grade. The children worked on seven different nets (Figure 1).

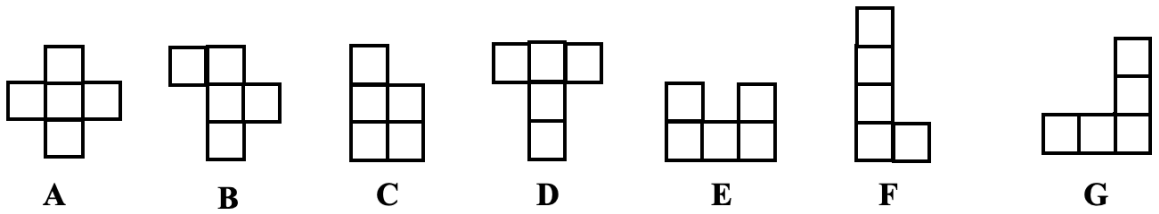


Figure 1: Seven Kinds of Cube Nets

The difficulty level of the thought processes for each type of cube net was analyzed based on the correct answer rate. In addition, the thought processes of students with high correct answer rates were observed and analyzed, their relationship to cube construction strategies examined.

Results

Two children were ultimately unable to assemble the cube. One child tried to make a cube by connecting six squares in straight lines to form a ring, while the other tried to make a rectangle by connecting two squares vertically and three squares horizontally, then folding that to make a cube, but both ultimately failed. Four strategies for constructing the cube were observed. First strategy to construct a cube was using three faces attached to one apex. Children who used this strategy connected three different colored faces and placed them on the sample cube, ensuring that the same colors corresponded. Eleven children used this strategy. Second strategy was to set faces of the same color on the bottom face. Nineteen children used this strategy. These children could construct a cube with the help of observation of sample cubes. In the third strategy, children first joined four side faces to form a cube, then attached two bases. In this case, children alternated between attaching two different colored faces and then bent them to see if faces of same color were facing each other. They already knew that when the cube was unfolded, the four sides of the cube would line up in a skip position. Four children used this strategy. Forth strategy is to construct a cube from cube net of cross shape. In this case, children had already known how to construct a cube from a cross-shaped cube net. Five children used this strategy, and these children already understand where the faces of a 3D cube fit onto the 2D cube net.

Next, we will discuss the characteristics of the cube net that make it easy and difficult for young children to mentally manipulate it. The average number of correct answers is 1.74 out of 7 questions. In the case of plane A, 95.1% of children said the correct answer. The next highest correct answer rate after plane A was plane E at 43.9%, and the lowest correct answer rate was plane G at 21.9%. Thus, most children could not imagine cube nets other than plane A. The incorrect answers demonstrate the difficulty in mentally manipulating the shape is their mind. Plane B can be constructed correctly, but 20 children answered “No.” After they answered, they tried to make this cube, but stopped when they raised one face. Because children had no experience moving two faces together. So, they couldn’t construct plane B in their mind.

Many of the children were not observed to be engaging in mental operation. However mental operation was observed in four children (Table 1). These four children were observed to approach the problem in a way that was clearly different from the other children. First, they do not answer immediately. They were observed thinking carefully while moving their hands. Ehrlich et al. (2006) suggested that gestures thus provide useful information about children’s spatial strategies.

Table 1: Answers of Four Children Observed Using Mental Operations

	A	B	C	D	E	F	G	Total of correct answer
Child A	✓	X	✓	✓	✓	✓	X	5
Child B	✓	X	✓	✓	✓	X	X	4
Child C	✓	✓	X	✓	✓	X	X	4
Child D	✓	X	✓	✓	✓	X	X	4

The average number of correct answers for 37 children, excluding the four children, was 1.7. Therefore, these four children got far more correct answers than the other children. Two of the four students looked at the sample cube and constructed a cube by placing faces of the same color opposite each other on the bottom. One of the four children first connected two pairs of same-colored faces in a skip position to complete a side and then added a top and bottom face to create a cube. Another one of the four children made a cube using the cube nets of cross. It was observed that these two children already knew how the faces of a 3D cube would be positioned when unfolded. A correlation was observed between the strategy of constructing a cube using the relative positions of the cube's faces and using mental rotation to create a cube in these two children.

Discussion

This study observed four strategies for constructing cubes. Young children relied on visual information, such as opposite faces being the same color or three faces meeting at a single vertex, to construct cubes. They also utilized knowledge such as alternating face colors and arranging faces in a cross shape to form a cube. Furthermore, it was found that it was extremely difficult for young children to mentally manipulate various shapes' nets. They did not appear to judge whether a net would form a cube based on its appearance, nor did they seem to engage in any mental manipulation. In particular, it was observed that children immediately recognized symmetrically arranged nets as cubes. Figure B (Figure 1) is a net that has a low success rate not only young children but also students from first to sixth grade of elementary school. It is thought that students pay more attention to this net than other because it required the mental operation of bending each face in a different direction, thus demanding more working memory. While many young children answered immediately, four children were observed taking their time, working with their hands and mentally considering the movement of faces in the net. One of these children knew that four squares arranged together form the four sides of a cube. Another student knew that unfolding a cross-shape cube net would results in a cube. It was observed that the spatial visualization of these four children were supported by knowledge gained through concrete manipulation of various three-dimensional objects. For children to be able to manipulate the movement of objects in their minds, they need to form the basis of images trough concrete experiences. Therefore, using concrete manipulation experiences and the use of images are essential for strengthening spatial visualization. Future research is expected to involve observing and coding the specific operations and mental processes of more students, conducting research

through actual learning experiences, and clarify what kinds of activities will foster students' spatial abilities.

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PROSPECTIVE TEACHERS' REASONING ABOUT MULTIPLICATION OF RATIONAL NUMBERS

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Keywords: Number Concepts and Operations, Preservice Teacher Education, Rational Numbers and Proportional Reasoning, Teacher Knowledge

Multiplication involving rational numbers has long posed ongoing conceptual challenges among teachers and students (Alghazo & Alghazo, 2017; Olanoff et al., 2014; Young & Zientek, 2011). Purnomo et al. (2022) noted that multiplication has sometimes been seen by learners to result in larger numbers, but this idea does not hold for multiplying a rational number between 0 and 1, as multiplication in this case results in a number smaller than the original value. Research has shown that in-service and prospective teachers (PTs) often find it challenging to create coherent meanings for such operations, and this indicates that this cognitive conflict extends beyond learners at the elementary grades (Depaepe et al., 2015; Kang, 2022; Son & Lee, 2016).

In this study, we examined PTs' reasoning about the multiplication of a positive number and a rational number between 0 and 1. We utilized Mathematical Knowledge for Teaching (MKT) as our framework (Ball et al., 2008) in this research. In addition, we grounded our work in literature on conceptual and procedural understanding of mathematical concepts to analyze PTs' understanding of rational number multiplication (Hiebert & Lefevre, 1986). Together, the MKT framing as well as this literature guided our interpretation of the findings related to PTs' understanding of multiplication of rational numbers. This literature further helped us identify some misconceptions that PTs had (Alghazo & Alghazo, 2017; Depaepe et al., 2015).

This study took place in a northeastern U.S. university where PTs were taking a mathematics course taught via problem solving. There were two sections with two instructors and four graduate mathematics education students who served as interns. Also, there were 30 PTs whose data were complete, and thus were used in reporting the findings of the study. In order to gain insight into how PTs conceptualize multiplying positive number and a rational number between 0 and 1, we used a qualitative technique using an interpretative paradigm. We further used three data collection tools. We used pre-unit and post-unit assessments that focused on PTs' thinking around multiplying a positive number by a rational number between 0 and 1. We prompted the PTs to describe whether the outcome was a smaller or larger value. A third data collection method we used were memos written by interns based on their observations and PT interactions.

The findings from the study indicated that most PTs were able to determine that multiplying a positive number by a rational number between 0 and 1 gives a result smaller than the positive

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number. However, the explanations for this varied among PTs as some of them demonstrated conceptual understanding and others relied more on procedures. At the beginning of the unit, some PTs had misconceptions; however, by the end most of the responses were void of erroneous explanations. Through this study we discuss particular junctures at which some misconceptions hindered conceptual understanding and procedural fluency. We further recommend that instruction should thoughtfully combine visual and contextual models, such as number lines, area models, and scaling scenarios, to assist future teachers as well as students in developing an understanding of the rationale behind the rules for multiplying rational numbers.

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GESTURE, INSCRIPTION & PROPORTIONAL REASONING: A PILOT STUDY

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Keywords: Rational Numbers and Proportional Reasoning, Assessment, Cognition

Objective & Theoretical Framework

This study investigated how students' gestures and inscriptions during proportional reasoning tasks might signal mathematical reasoning. The following question guided the pilot study: *Are specific types of gestures or inscriptions associated with more sophisticated proportional reasoning?*

Grounded and Embodied Learning (Nathan, 2022) encompasses the many embodied and situated ways students build and demonstrate mathematical understanding. This suggests assessment should attend to gestural and representational activity as meaningful evidence of reasoning. The long-term goal is to develop empirical foundations for technology-mediated, embodied assessment systems capable of detecting mathematical reasoning through gesture and inscriptions—moving beyond static, decontextualized assessment formats.

Method

Sixty-five grade 6 students were video recorded as they completed between 1–4 proportional reasoning problems and explained their thinking. These videos were coded for gesture, inscriptions, correctness, and proportional reasoning sophistication. Analyses focused on frequencies, proportions, and chi-square tests of association, interpreted as exploratory given the nested data structure.

Results & Implications

Gesture was present in 74% of attempts and more common during explaining (70%) than solving (22%). Gestured attempts were more likely to involve proportional reasoning than those without (71% vs. 46%; $\chi^2(1) = 11.37$, $p < .001$, $\varphi = .22$). Inscriptions were present in 41% of attempts and more likely to accompany proportional reasoning (74.5% vs. 57.3%), yet slightly more common in incorrect attempts (37.8%) than correct ones (34.7%), suggesting no straightforward relationship between inscription use and success. Creating number lines or ratio tables was associated with the highest proportional reasoning rates (85% and 77%), while annotating a given model showed the lowest (25%).

These exploratory findings suggest that gestures and inscriptions carry diagnostic information about proportional reasoning that correctness alone does not reveal. The pattern of inscription use occurring more often with proportional reasoning but not necessarily with correct answers warrants further investigation. This pilot study points toward explanation-phase gesture and student-generated representations as potentially meaningful signals for assessment purposes. Future work will examine whether these patterns hold across larger and more diverse samples, and what they mean for the design of technology-mediated assessment systems capable of detecting reasoning through gesture and inscription in real time.

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WHEN "COUNTING BY TENS" ISN'T "COUNTING BY TENS"

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Keywords: Elementary School Education, Number Concepts and Operations

Place value is an important concept that children begin to develop early in elementary school (NGAC & CCSSO, 2010). From a Scheme Theory perspective, students first internalize number after constructing the Initial Number Sequence (INS), which involves the ability to count-on to other numbers by 1s (Steffe, 2025). When students develop the Tacitly Nested Number Sequence (TNS), they can anticipate one level of units to operate on, including skip-counting by 10s (Osana et al., 2023; Steffe, 1992). Thus, a child at TNS can simultaneously conceive of a 10 as one unit and also as ten 1s units (Silveira, 2000). Yet, practitioner literature includes examples of counting by 10s that are not considered as skip-counting (Battista, 2012; Wright et al., 2011). Such instances often occur in first grade (Hiebert & Wearne, 1992). Thus, the purpose of this study is to examine one first-grade child's construction of number in the context of place value.

A six-year-old, first grade student participated in this study. His performance was consistent with INS, as demonstrated by his ability to count onto by ones on Task 2 (see Figure 1). Across tasks, he relied primarily on these counting-on strategies, progressing sequentially from one tick mark to the next. Confirmatory evidence in the form of eye-gaze movement and gesture reinforced these observations. In several instances, he appeared to count by tens based on his verbal references to decade numbers (see Task 8 in Figure 1). However, closer analysis revealed disconfirmatory evidence suggesting that he treated each position as a single countable unit while labeling place value positions with decade names, rather than coordinating tens (or hundreds) as composite units. For example, when asked to interpret quantities composed of tens and ones, he stated that 6 tens and 30 ones equals 36, and 2 tens and 4 ones equals 60. These responses indicate that although he used decade terminology, he did not coordinate tens and ones according to place value structure. Thus, these findings illustrate that apparent counting by tens may occur without the construction of composite units, highlighting that verbal decade references alone are insufficient evidence of TNS development.

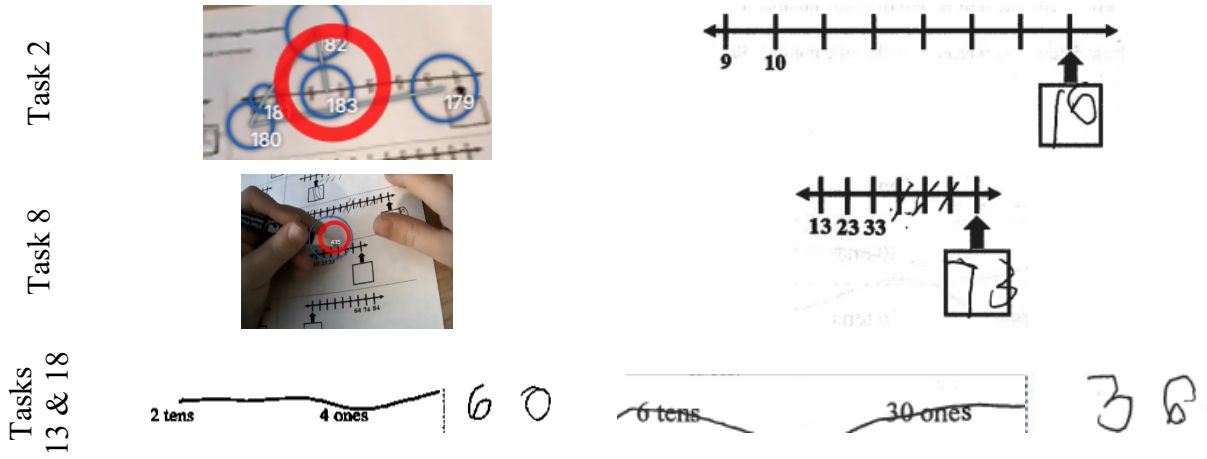


Figure 1: INS-Level Counting and Eye-Gaze Coordination During Number Line Tasks

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A STUDENT'S CONVEYED REASONING OF TWO VS FOUR PARTIAL PRODUCTS

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Keywords: Elementary School Education, Number Concepts and Operations

Partial products involves partitioning factors in multiplication by place value (Ambrose et al., 2003; Baccaglini-Frank et al., 2023; Hinton & Flores, 2024). They are the basis for algorithms of multidigit multiplication (Ambrose et al. 2003). There is preliminary evidence in the literature about the reasoning required for engaging in partial products (Gill et al., 2025), but more scholarship is needed. This study reports on an analysis of one student's conveyed meaning of multiplication when engaging with two and four partial products.

Systemic Functional Linguistics (SFL) was used to examine how one fifth-grade student, Hunt, conveyed meaning about partial products in different multiplication tasks. SFL analyzes grammar in text for how ideational meanings are conveyed by speakers (Banks, 2009). In this study, we examined transitive processes (verbs) and modality (hedging) in Hunt's transcribed text alongside his written work and physical actions (i.e., gestures). Hunt solved various tasks (14 × 2, 8 × 42, 43 × 38), but we focus on his explanation for 43 × 8 (see Figure 1). Across all tasks, Hunt engaged in partial products effectively. When solving 43 × 38, Hunt partitioned 38 into 30 and 8 two times and drew a line between both sets. He then partitioned 43 into two 40s and two 3s, placing them in both halves for the partitioned pairs of 38, respectively. Examining his transcribed interview, Hunt consistently referenced these partial products, but in pairs: "30 and eight … two 40s … the threes." Alongside his linguistic pairing of two partial products at a time, he used a series of transitive processes to convey his partitioning: "If you **break** …, then you **break** up … and you **put** …" Recall that Hunt drew a line between the first and the last two equations, thus conveying two pairs of partial products. Moreover, in explaining his procedures, he noted that when he had solved 8 × 42 that it "is a one-digit times a two digit," but that 43 × 38 was different because "you *have to* **do** the same exact thing [twice]." Both in his written work and spoken discourse, Hunt consistently conveyed his operating on four partial products as two sets of two partial products. Yet, Hunt lacked confidence in his understanding of the approach stating "I **don't know** how the strategy works; I *just* **learned** it." Despite this conveyed uncertainty, Hunt's description of four partial products as a pair of two partial products is appropriate. It may be indicative of a key transition between students' understanding two and four partial products.

So, I **don't know** how the strategy works, // I just **learned** it. //
If you **break** up the 38 and the 43, // then you **break** up the 30 //
and you **put** 30 and 8 // and then [**put**] 30 and 8 down here. //
And then you **put** two 40s on one of this half //
and then [**put**] on the other half // you **put** the threes
(motions to written 30×3. And 8×3)…

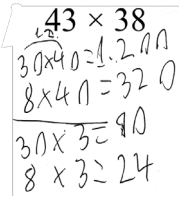


Figure 1: Excerpt from Interview (left) with Scanned Work (right) for Hunt (MC3)

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POLICY, INSTRUCTIONAL LEADERSHIP, AND TEACHER EDUCATORS

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MATHEMATICS EDUCATION DOCTORATES' WORK AND PREPARATION

EL TRABAJO Y LA PREPARACIÓN DE LOS DOCTORADOS EN EDUCACIÓN MATEMÁTICA

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Mathematics education doctorates perform various tasks and play versatile roles. However, the field lacks consensus on what doctoral programs prepare future mathematics education doctorates for, or what preparation they need to perform their jobs. In this paper, we focus on the work and doctoral preparation of mathematics education doctorates, using interviews with 35 recent graduates. The results revealed that the work of mathematics education doctorates is far more complex than the traditional research/teaching dichotomy suggests. Doctoral programs need to be flexible to meet individual needs/goals, given each student's background. Some recommended experiences in doctoral programs include interdisciplinary collaborations, professional networking, and policy work. Gaps between the work and preparation for mathematics education doctorates are highlighted, and recommendations are provided.

Keywords: Doctoral Education, Doctoral Preparation and Training, Work of Mathematics Education Doctorates, Doctoral Competencies

Three National Conferences on Doctoral Programs in Mathematics Education in 1999, 2007, and 2022 focused on the competencies required for doctorates in mathematics education and on their preparation in doctoral programs. After each of the first two conferences, an edited volume was published to summarize the discussions and capture various ideas about doctoral preparation and competencies (Reys & Kilpatrick, 2001; Reys & Dossey, 2008). The Association of Mathematics Teacher Educators (2003) also published *Principles to Guide the Design and Implementation of Doctoral Programs in Mathematics Education*, listing and elaborating on the required competencies, such as mathematics and research. Since the first conference, several studies have been conducted on doctoral preparation, primarily using survey data to profile current doctoral preparation across programs and to call for further research (e.g., Reys & Reys, 2016; Reys et al., 2019).

Our work on mathematics education doctorates' competencies and doctoral preparation consists of a survey study with 218 recent graduates and a follow-up study in which we interviewed 35 of the participants about the work they performed in their current positions and the preparation they received in doctoral programs. The goal was to unpack and elaborate on the competencies needed for mathematics education doctorates, which we posit informs doctoral programs to better prepare them for their work and roles. We report the competencies recommended for mathematics education doctorates elsewhere. In this paper, we focus on the following two research questions and the related data analyzed.

1. What roles do mathematics education doctorates perform?
2. How well did the doctoral programs prepare them for their roles?

Framework

Mathematics education doctorates presumably perform various tasks and play versatile roles. However, the field lacks consensus on what doctoral programs prepare future mathematics education doctorates for, or what they need to learn or prepare to perform their jobs (Chinen & Din, 2024; Reys et al., in press). To answer these questions, we need to examine the interplay among mathematics education doctorates' work, the competencies required to perform that work, and the doctoral preparation needed to develop those competencies. Our work is grounded in this interplay among the three (see Figure 1).

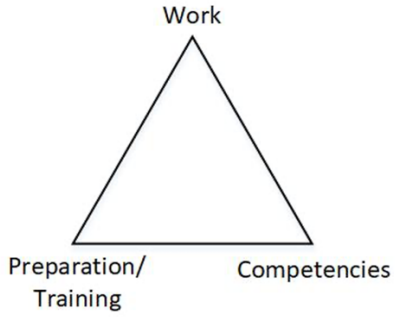


Figure 1: Interplay among Mathematics Education Doctorates' Work, Competencies, and Preparation/Training

Drawing on the definitions of Durette et al. (2016), we view competencies as attributes and resources that enable competent action, including knowledge, skills, and dispositions. Besides the core knowledge and skills in a specific field, the doctorates need general but critical competencies, such as communication, the ability to collaborate and work in a team, and the capacity for adaptation (Durette et al., 2016). Examining the nature and scope of the work and the roles that mathematics education doctorates perform is pivotal for determining the competencies required and for better designing doctoral programs. Hiebert et al. (2008) argued, "Answering questions about essential competencies requires describing clearly the work of mathematics educators. It makes no sense to design doctoral programs that are not aligned with the kind of work graduates actually do" (p. 246). The work that mathematics education doctorates do evolves; so do the competencies needed, and the preparation and training must evolve as well. Continued training is needed after graduation, as in any other job. Doctoral programs need to provide core experiences and training to support this evolution and the growth of mathematics education doctorates throughout their careers. This study examines the work experiences of recent graduates and reflects on current doctoral preparation.

Literature Background

The literature on doctoral programs and doctoral preparation, and on the competencies (including knowledge and skills) required for mathematics education doctorates, is of two kinds: (1) positions and recommendations based on professional conversations, authors' perspectives, and related literature; and (2) empirical studies that relied on surveys. A body of literature, including the reports and articles in the edited volumes after the National Doctoral Conferences (e.g., Ferrini-Mundy, 2008; Middleton & Dougherty, 2008; Zbiek & Hirsch, 2008), discusses and recommends what mathematics education doctorates need to learn and experience during doctoral training (e.g., Kilpatrick & Spangler, 2016). Typically, this line of literature lists

categories of core knowledge and skills and elaborates on them, discussing what preparation/training can help doctoral students acquire those competencies.

Chinen & Din (2024) reviewed and summarized the literature (articles or reports) on the knowledge and skills doctoral students need to learn through doctoral programs, from the perspective of educating future mathematics teacher educators. They found only a limited number of empirical studies on doctoral preparation, whereas other literature was mostly based on authors' perspectives. Existing empirical studies relied heavily on surveys to investigate aspects of doctoral preparations in mathematics education. In particular, Reys and his colleagues profiled doctoral programs using various aspects, such as the number of doctoral graduates and the number and types of courses (e.g., Reys & Reys, 2016), as well as special niches in doctoral programs recognized by peer faculty (Reys et al., 2021). Although surveys are useful tools for gathering large datasets and condensing the information efficiently, the method is limited in capturing the depth and specifics of the information. We also used a survey to gather the input from recent graduates. As we analyzed the survey data, we confirmed the need to follow up on their responses (Kim et al., 2024). One study collected data through a combination of a survey and interviews with a subset of participants. Glasgow (2000), in addition to a survey, collected data through interviews with 12 respondents about their doctoral preparation and current work. His criterion for selecting interviewees with a strong scholarly record (publications, presentations, etc.) was to hear from the "successful" doctoral recipients. In our study, we aimed to gather diverse voices from across positions.

Both types of literature described above focused on doctoral preparation for academia mostly. Centering on the role of mathematics education doctorates in higher education, the literature highlighted typical recommendations, such as research and teaching (e.g., Chinen & Din, 2024; Kilpatrick & Spangler, 2016; Shih et al., 2020). Glasgow (2000) gathered data from K-12 position holders and the commercial sector, as well as from faculty in higher education, yet his view of "successful" doctorates was limited to scholarly activities. As mathematics education doctorates play versatile roles, this study aimed to examine "the nature of the professional practice" of various groups of mathematics education doctorates.

After more than two decades of discussions and research on doctoral preparation in mathematics education, Reys et al. (in press) continue to report a lack of research on doctoral preparation. They call for descriptive research and case studies on doctoral programs (e.g., specific elements of doctoral preparation). This study, drawing on the voices of 35 recent graduates, describes their doctoral preparation, including the strengths and weaknesses of their doctoral studies and experiences. Reys et al. (in press) also suggest we examine what other fields do in doctoral preparation (e.g., science education). Durette et al. (2016) highlight a shift in doctoral education from a doctoral degree as a product to a degree as "a process, that is, a training providing the necessary competencies" (p. 1355) to become a competent worker. This study examines the recent graduates' work and doctoral preparation/training, which can illuminate the competencies needed and the continued training in their careers.

Methods

This study is part of the ongoing work on competencies that mathematics education doctorates need to perform their work, and the doctoral training needed for those competencies. This study drew on data from that project to answer the research questions.

Data Collection and Participants

The research team distributed a survey to mathematics education doctorates who graduated in the past 10 years to gather data on their work, doctoral preparation, and how they apply their

knowledge and skills in their current positions. As part of the survey, respondents were asked whether they would be willing to participate in an interview to elaborate on their responses. Out of 218 survey respondents, 72 volunteered to participate in an interview. Among the volunteers, 35 completed an interview either in a focus group (e.g., post-doc group) or individually during the summer of 2024. Focus group interviews were intended to capture both consensus and distinctions within each group, rather than interviewing everyone individually. Focus group interviews help capture detailed information and insights from the participants at a deeper level as they listen to others and articulate their thoughts (Eliot, 2005). Despite the benefits of the focus group interviews, we interviewed seven participants individually due to scheduling conflicts. The interview participants held positions across a range of professional contexts, including tenure-track, non–tenure-track, postdoctoral, and K-12 (see Table 1), thereby meeting our goal of hearing diverse voices. The interview questions focused on the participants' work, the competencies they used, and their doctoral preparation and experiences. We developed an interview protocol using guidelines for conducting focus group interviews (Eliot, 2005; Krueger & Casey, 2002) and were trained prior to the interviews. The first four authors conducted 14 interviews virtually (seven focus group interviews, each group consisting of two to six participants, and seven individual interviews), lasting 1 to 1 2/3 hours each.

Table 1: 35 Interview Participants

Position	# of participants	Percentage
Tenured	12	34.3%
Pre-tenured	7	20.0%
Non-tenure track	7	20.0%
Post-doc	4	11.4%
K-12 education	4	11.4%
Research Scientist	1	2.9%

Data Analysis

Following the interviews, the interviewers checked the accuracy of the AI-generated transcripts and cleaned them while watching the interview recordings. Each interview transcript was then converted to a tabular form and categorized by participants' professional positions (e.g., tenure-track) for analysis. The research team then color-coded and summarized the participants' responses in meaningful chunks. First, each team member independently coded the data (e.g., comments on the mathematics education doctorates' work and roles played in purple, and comments on their doctoral preparation in green) and summarized the main idea in each meaningful chunk. Afterward, the team met to review and ensure consistency among the coders, and to discuss the main themes from individual chunks and across chunks. Any discrepancies in coding and main themes were discussed until consensus was reached. Once gaining confidence in coding consistency after four interviews that we all coded and discussed, we coded the remaining interviews in pairs. After coding each interview, we summarized the main themes and the unique yet significant aspects of participants' responses in the categories of their work, preparation, and competencies in use from each interview.

Finally, we condensed the themes from each interview into the main categories in a large spreadsheet to notice common themes in the mathematics education doctorates' work, competencies, and doctoral preparation across all the interviews. Connections among the themes

across the 14 interviews were examined to identify significant emerging patterns. Additionally, comparisons were made across different professional groups (e.g., postdoctoral, tenure-track) to examine consistency and variation in themes and patterns. This study employed a purely qualitative approach, focusing on the participants' perspectives and experiences. For triangulation, the findings were compared across participants' reported competencies, doctoral preparation, and professional work. The themes of the mathematics education doctorates' work and doctoral preparation and their relationships are elaborated on in the following sections.

The Work Mathematics Education Doctorates Do

Doctoral preparation in mathematics education is often framed around specific professional roles, such as teaching, research, and service. Yet focus group interviews with mathematics education doctorates revealed that their lived professional work is far more complex. The participants represented a wide range of professional positions, including tenure-track and non-tenure track faculty positions, postdoctoral appointments, and K-12-based roles, reflecting the diverse institutional contexts in which math education doctorates work. The data show that they engage in highly diversified, context-dependent work that we have clustered into four interrelated domains: teaching, research, service, and integrated work.

Across roles, teaching emerged as a core responsibility. However, teaching expectations varied widely, with reported loads ranging from one course per year to 2-2 and 4-4 configurations, and with teaching ranging from a minor role to a primary responsibility. Participants described teaching preservice teachers, graduate students, in-service educators across elementary, middle, and secondary fields, and K-12 students. They reported teaching mathematics content courses at the undergraduate and graduate level, as well as methods and pedagogy courses in general education, math education, science education, STEM education, and research methods. However, teaching was not limited to course delivery and often involved substantial work with clinical supervision and mentoring, instructional coaching, course design and redesign, curriculum alignment, coordinating courses, supervising TAs, advising graduate students, and serving on doctoral committees.

Research expectations varied widely, ranging from clearly defined annual publication targets to roles where research was encouraged but not required. Some participants experienced formal pressure to publish and secure funding, particularly in postdoctoral and tenure-track positions, while others engaged in research more informally in clinical or teaching-focused roles. Research activities included study design, data collection and analysis, collaborative writing, grant development, and project management. Participants described research-related work that extended beyond traditional scholarship to include educator training, curriculum work, research literacy, technology-mediated inquiry, graduate mentoring, and professional development facilitation. Respondents in K-12 positions reported they collected and analyzed data to inform policy decisions rather than publishing research findings. Many participants described conducting research outside of formal workload allocations, highlighting tensions between research expectations and time available. As a result, research work was shaped as much by institutional context as by scholarly interest.

Service and leadership constituted a third major area of work and were described as both extensive and largely invisible. Participants reported engaging in departmental service, program coordination, accreditation and assessment efforts, mentoring students and colleagues, and acting as a liaison between departments, colleges, and external partners. This work required significant relational and organizational expertise but was rarely acknowledged as specialized work of doctorates.

Participants described "other work" as a practice-based and service-oriented activity that connected research, teaching, leadership, and policy. Mathematics education doctorates consistently described work involving interaction with a variety of stakeholders, including extensive engagement with K-12 schools and districts, state and accreditation agencies, and university communities, through leadership, service, research, curriculum work, and educator support. This included translating research for practitioners and adapting disciplinary knowledge for varied audiences. These responsibilities often operated outside formal workload categories but were central to mathematics education doctorates' roles.

Doctoral Program Preparation and Work

The doctoral preparation that the participants had varied within and across institutions. Aspects of their preparation that were discussed included general program descriptions and specifics related to the suggested competencies for mathematics education doctoral students. These included, but were not limited to, DEI, technology, teaching and research experiences, professionalism, networking, and collaborations.

Doctoral programs vary; however, at their core, they focus on math, research, and education. Some institutions place equal emphasis on all three, whereas others emphasize one or two areas, expecting training in the other core components to have been completed during Master's programs (e.g., math courses). Experiences within programs typically include coursework, practical experiences, often teaching and/or research opportunities, and integration into the field through collaborations with faculty and attendance at conferences. Many felt that their programs largely prepared them for positions at research institutions or in academia. Additionally, doctoral students needed to be self-sufficient and independent in their own learning, as well as proactive in taking initiatives to better themselves as students and scholars by pursuing additional coursework to become more versed in topics impacting education.

Teaching and Research Experiences

Teaching and research experiences varied and largely came in the form of assistantships or apprenticeships during one's doctoral program. Teaching included GTA or GA positions that involved being a graduate assistant or teacher of record for undergraduate courses, supervising student teachers, a university field instructor, or mentoring a course with a faculty member for a semester. Several participants noted that their teaching experience(s) gained in doctoral programs allowed them to use their teaching abilities in their current positions, even when it involves skills like teaching writing, which they may not have done before.

Research experiences also varied, with some noting that their only research experience consisted of readings and discussions about research in courses. Others noted that they were a research assistant on a grant and/or had opportunities to work closely with their major advisor or other faculty members on a research project. Those involved in these experiences were also more likely to have opportunities to travel to and present at conferences, to be involved in developing grant proposals, to write articles for publication, and to gain insight into the IRB process.

Teaching and research were often siloed within a program, with no connections between the two. Several interview participants noted a noticeable mismatch between the instructor's teaching methods and their research, highlighting gaps in the alignment of teaching practices and academic expertise. Additionally, several participants noted that their programs had a strong emphasis on teaching rather than research. Those in programs with a strong research focus reported that their research experiences were more focused on educational research than on teacher education research. Many expressed a lack of confidence in their research abilities, as the program primarily prepared them for teaching roles. At the same time, one participant noted that

their current job is teaching-focused and that their career goal is to teach future educators, so they felt the program's limited research preparation was not an issue for them. Teaching and research experiences were valuable to all participants; however, not all the experiences they received necessarily adequately prepared them for their current role.

Preparation for Current Work

Several participants noted varying degrees of connection between their doctoral preparation and their current work. Some felt their research training, although not extensive, was helpful in their current roles, particularly when assisting faculty and conducting research. Knowledge of policy and IRB procedures, as well as experience with policy during their doctoral training, provided confidence in adapting to potential future changes. Those with teacher education apprenticeships and teaching assistantships found their training valuable in their current positions. However, some felt their doctoral programs did not adequately prepare them for certain aspects of their careers, which are discussed in more detail in the next section. One participant mentioned that their program did not focus on the specialized field of teaching math for social justice, which they had to define and expand on their own as a graduate student.

Gaps and Strengths

Mathematics education doctorates varied in their overall satisfaction with their doctoral programs. Despite varying program experiences and pursuing a wide range of post-graduation career paths, participants shared common perceived gaps and recommendations for mathematics education doctoral programs. While the gaps stemmed from unmet needs and expectations, recommendations arose from positive experiences that participants wanted future doctoral students to share.

Doctoral Preparation Gaps

The most significant concern expressed by the participants was a lack of preparation for the academic job market. Participants reported feeling underprepared in several key areas, including research, scholarly writing, and professional networking. Research opportunities were described as uneven across doctoral training, and some participants noted limited integration between quantitative methods and mathematics education research. Participants highlighted the lack of guidance in research originality and emphasized the need for more authentic writing experiences. For example, one participant from the tenure-track group noted, “Something I didn’t get in my doctoral training that I wish I had a little more of was just experience writing grants. My professors had grants and I worked on them … but I never got experience writing them.” Overall, participants expressed a desire for greater exposure to writing and publication processes before entering the job market.

Although insufficient preparation in research and writing has been previously documented among postdoctoral scholars, our findings also revealed another significant area of concern: professional networking. Multiple participants cited feeling unprepared to engage with the broader mathematics education community and described challenges in building professional connections. Some also experienced a sense of disconnection within their own programs, citing insufficient mentorship and limited faculty support.

Doctoral Preparation Strengths

Overall, the participants emphasized the benefits of tailored advising, authentic teaching, and diverse professional experiences during their doctoral preparation. Since doctoral students come from diverse backgrounds, the participants praised programs that personalized content to fit students’ individual needs and goals. Many also advocated for opportunities to teach or supervise courses, with one participant stating, “If you’re going to leave a [doctoral] program and take a

faculty position, you need to know how to teach at the college level … it's just essential." Participants also stressed the importance of teaching doctoral students how to provide feedback to teachers and prepare doctoral students "to be part of the larger mathematics education community." Several recommended that doctoral students gain professional experience through service and leadership opportunities, including peer-reviewing for journals, attending conferences, writing grants, and supervising classroom teachers.

Participants also advocated prioritizing specific content areas within required coursework, such as diversity, equity, and inclusion (DEI), assessment, and curriculum. Several participants noted that, while their programs offered these topics as optional courses, they proved instrumental for their future work and should be required for doctoral preparation. One participant also suggested that instead of addressing DEI through an isolated course, it should be woven through all courses within the program.

Discussion

This research showcases that doctoral work in math education is more complex than traditional, single-role models or linear career trajectories suggest. Doctoral work often involves integrating research, teaching, service, and leadership across numerous contexts. Beyond teaching and research, mathematics education doctorates engage in curriculum work, leadership, mentoring and supervision, accreditation and policy work, grant and project management, translating research to practice, interdisciplinary collaboration, and networking. All of this requires preparation that centers on boundary-crossing, relational, and networking expertise, professionalism, and attention to diversity, equity, and social justice alongside traditional research and teaching competencies.

The data revealed a meaningful distinction between research producers and research consumers, with many roles requiring not only the generation of scholarship but also the interpretation and translation of research for different audiences. Also, teaching experience needs to encompass curriculum development, course design, and preparing others (e.g., TAs) to teach. Professionalism, networking, and interdisciplinary collaboration emerged as critical competencies, yet the participants described these areas receiving little or no attention in doctoral preparation.

This study is limited by its reliance on self-reported experiences of 35 recent U.S. graduates, and its findings may not generalize to all contexts. However, by centering the diverse professional roles of math education doctorates, this study provides insight into the evolving nature of math education doctoral work. Preparing future doctorates requires moving past a traditional teaching-research-service model toward models that recognize the integrated, relational nature and boundary-spanning roles embedded in mathematics education doctoral work. Alignment between doctoral preparation and current work differed across professional contexts. Given the diversity of career trajectories among our participants, the findings suggest that a single model of doctoral preparation may not be adequate to address the range of roles that math education doctorates assume.

Lastly, considering the high attendance of graduate students, doctoral advisors, and mentors at PME-NA, the topic of this study is highly relevant to the audience and informative for both students and faculty in mathematics education doctoral programs.

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CONSTRAINTS AS A TOOL TO DEVELOP CURRICULAR DECISION-MAKING IN MATHEMATICS METHODS INSTRUCTION

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Existing research on mathematics teacher educators as curriculum designers has identified factors that influence design, principles to guide design activities, and professional pedagogical structures used in design. In this paper, we report our use of self-study to improve our curricular design activities drawing from primary sources including designed activities, weekly instructor reflections, and analytic dialogues to develop awarenesses of curricular design informed by our commitment to relational constructivist teaching. Findings explain the emergence of constraints as a design consideration mathematics teacher educators use to surface evidence of preservice teachers' curricular decisions. We argue that mathematics teacher educator imposition of intentional constraints in designing instructional activities provides opportunities to observe and engage preservice teachers in discussions of their curricular decision-making.

Keywords: Teacher Educators, Curriculum, Instructional Activities and Practices, Pre-Service Teacher Education

Problem Statement

A growing field of research in mathematics education focuses on learning and growth of mathematics teacher educators (MTEs). Within the complex work of teaching preservice teachers (PTs) to teach mathematics (Li & Castro Superfine, 2018), MTEs' research of their own practices should have a goal of "self-understanding and professional development" (Chapman, 2008, p. 132) as well as contributing knowledge for other MTEs. Within the specific practice of curriculum design, MTEs design learning activities (Ackermann, 2012; Li & Castro Superfine, 2018) to support PTs as curricular agents (Felton & Koestler, 2015). Research on MTEs' design of curriculum has focused on PTs' reading (Drake et al., 2014) and use of curricular resources (Zaslavsky & Sullivan, 2011), with emphasis on principles (Saclarides et al., 2022) and factors impacting design, however insight into MTEs' work as curriculum designers is limited (Li & Castro Superfine, 2018).

As MTEs in US universities, we (the authors) faced challenges supporting PTs to develop curricular decision-making skills. We are relational (Kitchen, 2005) constructivist (Steffe & D'Ambrosio, 1995) MTEs who engage in inquiry into our practice through "reciprocal" critical friendship (Guilfoyle et al., 2007; Stolle et al., 2019, p. 23). In Fall 2023, we began to explore our MTE curriculum design practices for mathematics methods courses as an extension of our prior research and in response to legislative shifts related to teachers' curriculum use. We focused our inquiry on the question: How do MTEs design tasks to support PTs' development of curricular decision-making? Through our self-study inquiry (LaBoskey, 2007), we learned to use constraints (Coles, 2020) as a pedagogical tool which led to PTs' discussions of their curricular decisions and personal experiences and ideas informing their choices. We argue imposing intentional constraints on MTE designed tasks provides opportunities to both observe and bring into discussion PTs' curricular decisions, thus developing PTs' curricular decision-making.

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Background: Literature and Theoretical Framing

We, Alyson and Signe, practice relational (Kitchen, 2005) constructivist (Steffe & D'Ambrosio, 1995) teacher education in our mathematics methods courses. With this perspective, we strive to know our PTs' problems, convey respect and empathy, and grow in relationships (Kitchen, 2005) as we use their ideas and experiences to develop PTs' teaching practices. Ackermann (2012) described:

> Constructivist teachers need to support their students' initiatives by creating a "learning culture" in which the active contributions of every participant can become a valuable piece within a broader learning experience. They need to help the group of learners as a whole become its own critic, and learn when and how to overcome some perceived limitations. (p. 342)

Learning takes place as PTs engage in central activities of the teaching community, such as lesson planning, and are supported to contribute to their own growth (Ackermann, 2012).

To create a constructivist learning environment, we flow between facilitating learning and researching MTE knowledge (Zaslavsky, 2008) as we develop tasks with potential to support PTs to hone their teaching practices. The cycle of facilitating learning and researching MTE practice centered around MTE tasks creates possibilities for pedagogies of practice that push beyond representations, decompositions, and approximations of practice (Grossman et al., 2009) to the creation of pedagogical tools (e.g., discussion) that make it possible for PTs to examine their own curricular decision-making.

Li and Castro Superfine (2018) defined *design* as "the process in which decisions are made about the content, curriculum materials, instructional strategies, and assessments used to support learning in pursuit of instructional goals" (p. 180). MTEs as designers take up knowledge of the field and knowledge of learners to design tasks with goals of addressing themes important for the preparation of mathematics teachers, such as developing adaptability, selecting appropriate tools for teaching, and overcoming barriers to student learning (Li & Castro Superfine, 2018; Zaslavsky, 2008). As constructivist designers, we focus on designing activities that support PTs to build on their own experiences as they learn (Ackermann, 2012).

The flow between facilitating learning and conducting practitioner-based research (Borko et al., 2007; Zaslavksy, 2008) that occurs as designed tasks are used and redesigned in iterative cycles supports MTE's learning and the refinement of designed tasks (Zaslavsky, 2008). Such refinements include imposing constraints, which we define as boundaries on PTs' activities with the goal of focusing pedagogical inquiry. Through such iterations, PTs' responses make visible how they navigate intentional and unintentional constraints in their interpretations of the tasks. Constraints both bound the tasks and create space for PTs to express their ideas (Ackermann, 2007; Coles, 2020). In this paper we extend Li and Castro Superfine's (2018) account of MTEs as designers to provide an exemplar of MTEs coming to know and use constraints in supporting PTs to develop as curriculum decision-makers.

Methods

For over 12 years, we have met remotely each week in critical friend conversations (Schuck & Russell, 2005) to share our teaching practices, discuss task design, and reflect on our MTE practices. Adopting LaBoskey's (2007) conceptualization of self-study as self-initiated, improvement-aimed, interactive, and involving validation via trustworthiness, we approach self-study as open, collaborative, and involving reframing (Samaras & Freese, 2009). We engage in perspective taking cycles through design, teaching, and weekly analytical dialogues (Placier et

al., 2005) where we analyze alignment between our activity designs and relational constructivist principles (The Arizona Group, 1996).

For this study, our primary data sources included designed activities particular to developing PTs' curricular decision-making, weekly instructor reflections, and analytic dialogues from 2023, 2024, and 2025. Secondary data sources included PTs' work samples, recordings of class conversations, and conference proposals and proceedings encapsulating our curriculum design processes. We engaged in dialogic analysis, with the goal of becoming conscious of our design decisions and improving our design processes aimed at supporting PTs' curricular decision-making. Dialogic analysis revealed the role of constraints as pivotal in our curricular design. To support the dialogic analysis (Placier et al., 2005), we also reviewed course documents from each year to note changes in our designed activities and instructor journals and meeting recordings to note discussion of constraints applied to tasks. Findings are generated from both authors' practices and data, but we report only from Alyson's practice here due to space restrictions.

We are mathematics methods instructors at two different U.S. universities. Both of us are experienced MTEs with backgrounds in teaching mathematics prior to transitioning to teacher preparation. Alyson teaches methods courses for secondary PTs and Signe teaches methods courses for elementary PTs. Alyson and Signe are connected by their commitment to relational constructivist teaching (Kitchen, 2005; Steffe & D'Ambrosio, 1995).

Alyson's practice, from which we draw the data for this report, takes place in a UTeach replicate program (Hughes et al., 2020) in which PTs matriculate as a cohort and are in classrooms beginning in their first semester in the major. Alyson's methods course typically occurs in the semester prior to student teaching. Though the course does not have a field placement component directly connected to it, PTs have a field placement for another course during the same semester. This is the second of two mathematics methods courses in the program. The focus of this course is on unit planning (i.e., connected lessons encompassing more than two days of instruction) and the Mathematics Teaching Practices (National Council of Teachers of Mathematics [NCTM], 2014).

Findings

To share our findings, we describe Alyson's layers of 2023 methods course activity (Kastberg et al., 2024) related to curricular decision-making. We then share the evolution of our curricular design to support PTs' curricular decision-making in 2024 and 2025 to highlight the emergence of constraints as a productive design element.

Contextual Grounding: Layers of MTEs' Designed Activities

In 2023, as we concluded prior inquiry into our discussion practices (Kastberg et al., 2024), we were drawn to emerging questions about PTs' curricular decision-making amid changing local contexts. Legislation impacting teachers' curricular agency was enacted in Alyson's state and her partner school district adopted new curriculum. As constructivist MTEs, we started by engaging PTs in sharing their thinking. We wanted to understand what our PTs were bringing with them to this aspect of practice. Alyson asked PTs to create a concept map showing how they thought about lesson planning. PTs drew maps connecting constructs such as standards, curricular resources, technology, and scope and sequence documents to questions such as *What do students know?* and *How is the mathematics sequenced?* (PT submitted work, 10/10/2023). Using this and other reflective prompts, Alyson formed models of PTs' curricular decision-making, acknowledging differences among the authorities PTs leveraged in their planning. For example, some PTs gave authority to textbook authors to set the order of presentation of concepts whereas other PTs took internal authority to rearrange concepts or change the

facilitation of activities. Though our attention was shifting to making sense of PTs' models of curricular decision-making, our activity design lived within existing structures for courses and thus we did not alter major components of the course.

The existing structure of Alyson's course included two major activities: writing unit plans and rehearsals of teaching. PTs wrote two unit plans (one for three days of instruction and one for five days of instruction). PTs were provided with curriculum materials and other virtual resources for an assigned high school content goal and were required to produce lesson plans for consecutive days of instruction including all materials a teacher would need to teach the lessons. The PTs also engaged in three rehearsals of teaching. For each rehearsal, a PT prepared and taught a ten-minute portion of a lesson to classmates as learners for the lesson. Each rehearsal was video recorded and then watched by the whole class to support collaborative debriefing regarding the teaching decisions enacted in the rehearsal. These activities spanned the semester (see Table 1). In addition to these major activities, there were readings and shorter assignments that supported PTs to develop understanding of the Mathematics Teaching Practices (NCTM, 2014) and reflect on their own teaching practices as enacted in class assignments. Throughout all tasks, PTs engaged in self and peer feedback cycles and received feedback from Alyson.

Table 1: Layers of Instructional Activities for Curricular Decision-Making Development

Week	Description of Task
Week 1	PTs map the content taught in high school mathematics and explore high school standards
Week 2	Unit plan 1 topic assigned; introduction to the Mathematics Teaching Practices (MTP) (NCTM, 2014)
Week 4	Introduction to curriculum resources (PTs are provided with the teacher's edition for unit plan 1)
Week 6	Rehearsal 1 with video and debriefing (launch a provided task)
Week 7	Final draft of unit plan 1 submitted
Week 8	Unit Plan 2 topic assigned (5 day plan)
Week 8	Rehearsal 2 with video and debriefing (lesson launch using Unit Plan 2 provided curricular resources)
Week 11	Rehearsal 3 with video and debriefing (discussion following a technology enhanced lesson)
Week 14	Final draft of unit plan 2 submitted

Understanding Our Goals for PTs' Curricular Decision-making

The first mention of "constraint" in Alyson's meeting notes, "What are the constraints I impose?" (Meeting Notes, 1/8/2024), arose from a dialogue about centralization of curriculum as we wondered about our goals for PTs:

> Alyson: My goal is that as my PTs build lessons, they make decisions based on what is going to help their learners make sense of mathematics. And that they are reaching for different documents to make that happen. The document might be the textbook or internet resources or ideas they have about how they understand a concept. I want them to feel they have the agency to do that. … But the current legislative shifts are restricting the choices they have. (Critical Conversations, 1/8/2024)

We continued to discuss constraints PTs would experience in their future careers, such as limits to curricular materials, imposed scope and sequence for courses, and requirements to teach scripted lessons with fidelity. In our conversation, we acknowledged constraints we imposed. Alyson dictated the number of days unit plans were to span, assigned mathematical content as the focus of the plans, and provided descriptions of students in class contexts for which the plans were to be written. Rehearsals included constraints for PTs including tasks, required materials, and assigned standards. Constraints were present in our design, but at this time we only considered them to be ways to bound the tasks we assigned to PTs. Further, we were concerned with PTs' agency as curricular-decision makers and experienced a tension between constraining designed tasks and impacting PTs' autonomy (Critical Conversations, 1/15/2024)

Possibilities Offered by Constraints on Tasks

In 2024, supplied with ideas of how PTs' might approach curricular decision-making through the models we had constructed in 2023, we delved further into improving our design practices. Signe said, "We know how we are designing activities on some level" (Critical Conversation, 8/14/2024) however, we struggled to explicitly describe our design practices. Clearly articulating our curricular decision-making practice would support us to improve this aspect of our practice.

As we planned for the Fall 2024 semester, we asked questions during our weekly meetings like "How do we support our students to be wonderers about teaching?" and "How do we make decisions about constraints on tasks?" (Meeting Notes, 8/14/2024). In our August 14 meeting, amid final planning for courses to begin the next week, we shared planned questions for PTs about curricular decision-making, our goals for PTs' discussions of curricular decision-making and explored constraints we were imposing on activities. Alyson's Fall 2024 course would have three PTs enrolled: Carrie, Denny, and Whitney.

As a result of our dialogic conversations, Alyson adjusted constraints on rehearsals in her course. For the 2024 course, Alyson changed rehearsal requirements so that rehearsals were embedded in the unit plans PTs were assigned to write in the course. Other structures of rehearsals remained the same as in 2023. Alyson posited that rehearsals derived from plans PTs had written could produce more productive debriefing discussions with her PTs. For the first rehearsal, PTs were now required to launch a lesson from their first plan using provided curriculum materials. For the second rehearsal, PTs rehearsed presenting a strategy to support language learners drawn from their second unit plan. The third rehearsal did not change from the 2023 course and still required PTs to engage the class in discussion following the implementation of a mathematics action technology task.

Following the conclusion of the first round of rehearsals, Alyson provided example strategies PTs could use to support language learners in mathematics class including Frayer diagrams, color coding techniques, discussions about language use, use of examples and non-examples, and others. After PTs engaged as learners in short mathematics lessons including these strategies, Alyson displayed tasks PTs had launched for their first rehearsal (Figure 1, Class Files 10/10/24) and asked PTs how they might modify their tasks to include language learner supports.

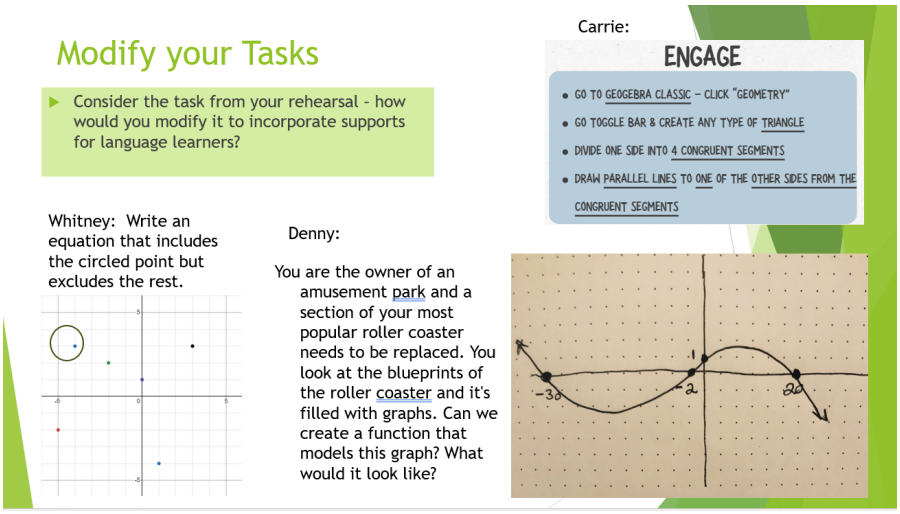


Figure 1: Slide of PTs Rehearsal 1 Tasks for Modification for Language Supports

After a few minutes of think time, Alyson asked PTs to discuss their curricular decision-making. Denny, whose task included a paragraph explaining a rollercoaster as a polynomial function, suggested removing the rollercoaster context from the task completely so language learners would not have to read the explanation. The other PTs continued:

Carrie: You could play a 30-second clip of a rollercoaster and say, "There is a function for this. What do we think it looks like?" That way you could keep the context, which might help them. I would also just simplify the language—they don't need all the information you included.

Whitney: You don't have to remove the context. Try displaying a rollercoaster [track] on your whiteboard and drawing the graph over it.

Discussion continued with Carrie suggesting she could create a video or set of image driven instructions to replace written directions for her task (see Figure 1). To which Whitney added, "you could use screen capture and get rid of the problematic words completely," encouraging Carrie to find a way to reduce the language difficulty for language learners (Class recording, 10/10/2024). The discussion continued with PTs offering each other a variety of adaptations with little input from Alyson.

In later critical conversations, Alyson would hold up this class discussion as an example of what was possible for PTs' discussions about curricular decision-making. However, during this semester, Alyson's concerns were focused on constraints imposed by PTs' mathematical content knowledge. She wrote in her meeting notes, "My students' lack of desire to work with the mathematics imposes a curricular constraint that is internal" (Meeting Notes, 11/18/2024).

At the conclusion of the 2024 Fall semester, Alyson and Signe engaged in dialogic analysis of activities in their courses and recognized constraints emerging as a tool that could be used to support development of PTs' curricular decision-making. Critical conversations included ways context constrained our decisions (Meeting Notes 2/24/2025), and wonderings about how PTs could make sense of self within our intentional constraints (Meeting Notes, 6/3/2025). In Alyson's 2024 course, she wrestled with constraints imposed by PTs content knowledge and constraints that supported productive discussions, helping us to see constraints could be helpful

and good (i.e., language learner constraint) or unproductive (i.e., PTs' content knowledge; Meeting Notes, 2/24/2025).

Successes and Challenges with Imposing Intentional Constraints

Reflecting on Fall 2024 led Alyson and Signe to shift their conversations regarding constraints on curricular design. Critical conversations in preparation for the start of the Fall 2025 semester included using common reflection questions for PTs as well as considerations for how constraints might be included in the planned tasks for courses. Alyson's class included five PTs in Fall 2025 and Signe's course maintained its larger population between 15 and 20 PTs. To accommodate her larger classes, Signe pushed for thinking about whole class constraints that might be implemented to develop PTs' curricular decision-making, explaining:

> The central takeaway for us at that time [2024] was trying to understand the different constraints that the preservice teachers were operating under with the different knowledge structures that they had. So we had evidence of their curricular reasoning, but then we hadn't really incorporated the idea of the constraints and how those might potentially affect that reasoning. We talk a lot about individual meetings [with PTs] and how those take a role. And I think when we were discussing this, we were talking about the fact that each one [PT] is different, and that makes it hard to have an instructional activity design that's going to capitalize on those differences or address them. (Critical Conversation, 9/29/2025)

Alyson reframed how she thought about the design of activities aimed at developing PTs' curricular decision-making. Particularly, Alyson thought about how adjusting constraints might encourage reflection and discussion about curricular decision-making.

To describe the second rehearsal for Fall 2025, Alyson explained that PTs must incorporate a strategy to support language learners in their class for their second unit plan. She described this requirement to the class as an intentional constraint that was authentic to what they may experience in schools (Class recording, 10/7/2025). During the second rehearsal, Matthew, the last PT to present his rehearsal, used a Frayer Diagram as a tool for developing understanding of exponential function graphs. He began his rehearsal by asking classmates to record the pieces they felt were important to the definition of an exponential function on a Frayer Diagram he provided. Matthew drew out classmates' ideas, recording them on the board and easily discussing different notations classmates had chosen. At the end of his rehearsal, Matthew asked if a negative value for b in $f(x) = a \cdot b^x$ would be possible. His explanation faltered. During the whole class debriefing conversation the PTs pointed to their lack of understanding related to the restrictions on b in the definition of exponential functions (Class Recording, 11/04/2025). The whole class worked through understanding why b must be positive. Then Matthew had an opportunity to meet with Alyson individually to further refine his understanding of the definition of exponential functions and improved his written lesson plans based on this learning. Alyson's intentional constraint to use language supports led to open discussion about mathematical knowledge and its role in curricular decision-making.

However, not all constraints led to success. Alyson continued her practice with her third rehearsal to require PTs to hold a whole-class discussion following the implementation of a mathematics action technology task. PTs engaged in each other's technology tasks prior to the start of the third rehearsal. Then rehearsals were conducted in which each PT led a discussion to summarize the concepts learned in the technology task. Throughout the preparation for and enactment of these rehearsals, Alyson was disappointed in PTs' lack of interest in holding a whole-class discussion and puzzled about why PTs seemed resistant to the activity. PTs

expressed frustration and said they would not necessarily hold a whole-class discussion following their activities (Class recording 11/18/2025). In her debrief, PT Kourtney reflected on her rehearsal saying:

> I didn't love this rehearsal, but I guess I could have asked more guiding questions to make it a more student-centered discussion. But the way I have this structured is that we are going to do four slides and then stop and talk about it. I guess I could have had a guided notes summary page to summarize after the fact. (Class recording, 11/20/2025)

Alyson's imagined class structure for implementing technology rich tasks with mathematics learners included a whole-class discussion of the concepts following the activity. However, this constraint for her task design was not aligned with PTs' visions for how using a technology task would take place in their classes. In this debrief, PTs critiqued the activity rather than discussing their curricular decision-making.

Alyson and Signe continued to meet weekly throughout Fall 2025 and collaboratively planned constraining activities to experiment with intentional constraints as supports for discussions of curricular decision-making among PTs. Signe experimented with changing the time constraints for teaching and inspired Alyson to plan an activity she called "What if it snows?". At the end of the semester, following completion of their second unit plan which encompassed five days of instruction for 90-minute class periods, Alyson asked PTs to edit their unit plans to remove 1 ½ days of instruction (imagining schools were closed due to a winter weather event). After making decisions, Alyson engaged PTs in discussion:

Alyson:	Talk to each other—How did you make the decisions you were making?
Fiona:	So, I just slid some things together. I cut out a couple of think-pair-shares and an entrance ticket. I also got rid of something I didn't really like. I cut out the walk-through of a poster presentation.
Deborah:	I did something like that too. I looked for things that were connected together. Like you can do hyperbola and ellipses together pretty easily. With a little less practice and putting ideas together it could work.
Reanne:	So rather than removing the sense-making, it sounds like we put connected ideas together and moved some homework to outside of class time. I did that too.
Kourtney:	Me too. Some of the days of practice could be grouped together differently. (Class recording, 12/2/2025)

The discussion continued as PTs built on each other's ideas and compared their reasons for the curricular decisions they made. Alyson noted it a success that PTs were discussing curricular decisions as a community with little input from her, though sometimes finding agreement where it did not fit, as in Deborah's connection to Fiona's statement.

Discussion and Conclusion

We set out to investigate how MTEs engage in curricular design to support PTs' curricular decision-making. Our account provides an exemplar of how MTEs engage in design, adding to research literature on MTEs' design practices (e.g., Li & Castro Superfine, 2018; Zaslavsky, 2008). The key finding from our study was MTEs' intentional use of constraints to productively support PTs' development of curricular decision-making through discussion. Constraints ranging from assigning PTs particular standards to identifying learner needs for PTs to address had always been part of our curricular design. However, cycling between facilitator and researcher

(Zaslavsky, 2008) supported reflection on incidental constraints (format and bounding) and intentional constraints (technology use and temporal) as design elements. This reflection shifted our attention to the content of PTs' reasoning regarding their curricular decisions. We saw something different about our PTs' curricular decisions than we were able to see before. Through discussions, we gained insight into the rationales PTs provided for their decisions.

We began to see constraints as offering possibilities to learn about PTs' curricular decision-making through class discussion. These possibilities extended the learning culture in our classrooms through the intentional use of constraints to create the basis for discussions with active contributions from every PT (Ackermann, 2012). As Coles (2020) stated, "A boundary is offered, in each case, in the belief that the constraint will occasion freedoms and learning in future communications" (p. 20). Originally, constraints were a way to bound activities so PTs would produce similar products. Now we view boundaries as making space for growth, as in Matthew's experience with exponential function definitions, and creating possibilities for discussions about PTs' curricular decision-making.

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FROM FELLOWSHIP OPPORTUNITIES TO LEADERSHIP SPILLOVER: APLUS IN MATH FELLOWS' CASE JOURNEYS

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This brief report examines how fellows in the APLUS (Alabama Practitioner Leaders in Underserved Schools) in Math Master Teacher Fellowship used program opportunities and translated learning into leadership beyond their classrooms. Using maximum-variation case selection, we analyze three fellows with contrasting preparation pathways and initial professional capital. Data include interviews and artifacts from Teacher-Led Community (TLC) planning and facilitation, preservice teacher mentoring, and a schoolwide student initiative. Within-case journey maps and cross-case thematic analysis trace opportunity uptake and spillover outcomes. Preliminary findings indicate three pathways: (a) iterative district-facing PD design leadership grounded in TLC data use, (b) mentoring-centered leadership evidenced in novice teacher growth, and (c) schoolwide initiative leadership extending across disciplines.

Keywords: Teacher Leadership, Professional Development, Identity/Agency, Systemic Change

Mathematics teacher leadership in underserved schools demands more than improving individual classroom practice; it requires sustained professional development that positions teachers to influence colleagues and contribute to school and district improvement. In APLUS in Math, we designed a five-year, multi-phase curriculum model that integrates graduate coursework, National Board Certification preparation in collaborative professional learning communities, and leadership project development to cultivate secondary mathematics teachers as instructional experts and emerging change agents (Zelkowski et al., 2025a). APLUS was organized around goals targeting instructional expertise, content knowledge, leadership enactment, clinical mentoring capacity, and dissemination through professional venues Prior analyses of the program's first phases documented alignment among evidence sources from the classroom observations of Mathematics Teaching Practices and Student Mathematical Practice engagement (Gleason et al., 2017), National Board component outcomes, and suggested growth in instructional practice and leadership readiness (Zelkowski et al., 2025b). However, outcomes and readiness indicators alone do not explain how different participants take up fellowship opportunities or how learning extends beyond the classroom in distinct ways. Accordingly, this brief research report uses a maximum-variation multiple-case design to examine (1) which aspects of the APLUS project participants identify as most consequential for instructional growth and student learning and (2) what leadership enactments, if any, extend beyond participants' classrooms (e.g., peers, school structures, district professional learning), and how are those enactments described and evidenced.

Perspective & Theoretical Framing

This study adopts the perspective that teacher leadership development that sustains instructional improvement over time is best understood as participation in professionally

organized communities wherein the teacher has an increasing central role over time. APLUS was designed as a structured three-phase curriculum that engages teachers in graduate coursework, collaborative National Board Certification preparation, and leadership project development, with the explicit intention of cultivating professional identity, instructional expertise, and leadership capacity (Zelkowski et al., 2025a). The curriculum is grounded in Wenger's (1998) Communities of Practice and the NBPTS Five Core Propositions (National Board for Professional Teaching Standards [NBPTS], 2016), framing Professional Learning Communities (PLCs), peer mentoring, and shared inquiry as mechanisms through which teachers develop identity and agency alongside increasing responsible teacher leadership.

Within this framing, the fellowship operates as an opportunity structure with a set of designed opportunities aligned to program goals that participants may take up (e.g., collaborative PD task design, facilitation routines, mentoring of novices and preservice candidates, and dissemination through professional venues). We distinguish between program opportunities (structured components of APLUS), program engagement (participants enacted participation in components), and leadership spillover (leadership activity that extends beyond the participant's classroom and beyond routine program participation, evidenced through initiatives, roles or practices taken up in peer, school, or district contexts) rather than changes in beliefs or intentions alone. This focus is consistent with prior reporting that positions APLUS as a multi-phase professional development model intended to prepare teachers for schoolwide or districtwide influence through the combined use of graduate coursework, National Board preparation, and leadership projects (Zelkowski et al., 2025b).

Methods

This study uses a maximum-variation multiple-case design (Patton, 2014) to examine how APLUS fellowship opportunities translated learning into leadership influence beyond the classroom. Three fellows were purposefully selected to represent contrasting entry pathways and professional capital and differing loci of leadership enactment (Figure 1): Teacher A (high initial preparation; district-facing TLC design leadership), Teacher B (limited early preparation opportunities; mentoring-centered leadership with TLC support co-leadership), and Teacher C (elementary-to-middle-grades pathway; schoolwide initiative leadership alongside TLC co-lead).

Data sources included semi-structured interviews focused on perceived high-value fellowship opportunities, uptake episodes, and leadership enactments, as well as artifacts aligned to those enactments (Creswell, 2018). Artifacts included TLC planning materials (agendas, PD task drafts, participant feedback/data instruments), masked mentoring/coaching documentation related to preservice teacher candidates, and initiative-related artifacts connected to Teacher C's implementation role. Participation records were used to corroborate leadership responsibilities.

The three pathways were developed iteratively: we first identified salient enactment episodes within each case, then compared cases by locus of influence (peer/school/district), mechanism of leadership (PD design/mentoring/initiative leadership), and evidence of spillover. Thus, the pathways are provisional analytic claims from this early case analysis rather than a complete typology. Credibility was supported through interview-artifact triangulation and analytic notes documenting how case-level evidence informed cross-case assertions.

Findings

Figure 1 summarizes three contrasting fellow trajectories illustrating how the same fellowship opportunity structures can produce different forms of leadership spillover beyond the classroom. In response to RQ1, fellows identified opportunities as consequential when they

connected individual instructional growth to shared professional work; in response to RQ2, spillover was evidenced through leadership enactments in district PD design, mentoring of novices, and schoolwide initiative leadership. Across cases, spillover was not a generic "outcome" of participation; it was enacted through identifiable mechanisms in forms of PD design routines, mentoring/coaching practice, and schoolwide initiative leadership that were shaped by entry conditions and organizational role access.

Case	Entry conditions	APLUS opportunity structures Goals 1–5	Uptake episodes (illustrative)	Spillover locus / form
Teacher A (HS)	Rigorous preparation; high math capital	**G1** Instructional refinement **G2** Content leadership/legitimacy **G3** Leadership roles **G4** Clinical mentoring network **G5** Dissemination/credentialing	Leads TLC planning cycles; co-designs PD tasks; directs TLC data collection/feedback; sustains candidate supervision	**District:** PD design leadership; stewards' iterative improvement; builds other leaders' capacity **Peer/Clinical:** mentoring pipeline
Teacher B (HS)	Limited early PD; fellowship as access	**G1** Instructional learning via cohort **G2** Content learning/structured reflection **G3** Scaffolded leadership participation **G4** Candidate mentoring entry points **G5** Credentialing/ recognition pathways	Co-leads TLC planning (support role); enacts focused mentoring/coaching; expands mentoring responsibilities	**Peer/Clinical:** mentoring-centered leadership; novice growth episode as marker **District:** support-oriented TLC co-lead
Teacher C (MG)	Elementary → middle grades endorsement; rapid uptake	**G1** Instruction linked to initiative **G2** Endorsement/content pathways **G3** Schoolwide leadership roles **G4** Candidate mentoring entry points **G5** Peer learning dissemination/support	Co-leads TLC planning (hybrid); leads cross-disciplinary Whole Child work; facilitates/supports peer learning; strong mentor of candidates	**School:** initiative leadership across disciplines **Peer/District:** TLC co-lead + facilitation

Figure 1: Cross-case Journey Map Comparing Entry Conditions, Program Opportunities, Uptake Episodes, and Leadership Spillover Across Three Provisional Pathways

Teacher A's pathway foregrounds spillover enacted through collective PD design and iterative improvement. Teacher A assumed primary responsibility for coordinating TLC planning, setting agendas, facilitating co-design of PD tasks, and organizing collection and use of participant feedback to inform subsequent sessions. This trajectory (Figure 1) illustrates how leadership opportunity structures (Goals 3 and 5) were taken up as recurring design and data-use routines that supported district-facing instructional leadership while building other teacher leaders' facilitation capacity.

Teacher B's trajectory illustrates how fellowship participation can function as access to a rigorous professional community and legitimacy for leadership enactment. While co-leading TLC planning primarily in a support role, Teacher B enacted leadership most visibly through mentoring and coaching preservice teacher candidates. The salient enactment marker was the candidate's rapid growth early during student teaching, attributed to structured coaching and feedback routines. Figure 1 characterizes this pathway as mentoring-centered spillover, with TLC participation serving as an on-ramp for leadership participation and identity development.

Teacher C's pathway highlights spillover enacted at the school level, extending across disciplines. In addition to co-leading TLC planning in a hybrid lead/support role, Teacher C served as the school leader for *Whole Child* implementation engaging teachers beyond

mathematics. Figure 1 captures a trajectory in which organizational openings for schoolwide improvement supported translation of fellowship participation into cross-disciplinary leadership.

Cross-case comparison showed that the cases differed not only by participant background but by the locus and mechanism of leadership: Teacher A's influence centered on district-facing PD design and data-use routines, Teacher B's on mentoring/coaching routines, and Teacher C's on schoolwide initiative leadership. Case journey maps (see Figure 1) show how fellows' entry conditions and participation in structured APLUS opportunities are translated into distinct forms of leadership spillover across peer, school, and district contexts. Across cases, the fellows' trajectories can be characterized as three distinct but convergent leadership pathways. Teacher A exemplifies an *iterative PD design leadership* pathway in which leadership spillover is enacted through organizing and steering TLC planning cycles, coordinating collective task design, and using participant data to refine subsequent professional learning. Teacher B illustrates a *mentoring-centered leadership* pathway in which the fellowship functions as an access point to rigorous professional community and legitimacy, with spillover enacted most visibly through coaching and supporting novice development alongside a supportive TLC co-leadership role. Teacher C represents a *schoolwide initiative leadership* pathway in which spillover is enacted through cross-disciplinary improvement work, leveraging organizational openings to lead *Whole Child* implementation while also contributing to TLC planning and facilitation. Taking together, these pathways suggest that APLUS supports multiple routes from opportunity uptake to beyond-classroom influence, with the form of spillover shaped by entry conditions, role access, and local educational supports.

Conclusions and Implications

This brief report extends prior APLUS work by illustrating how a common, multi-phase fellowship design can yield multiple pathways from professional learning to instructional leadership. Across three contrasting cases, leadership spillover was enacted through (a) iterative district-facing PD design routines, (b) mentoring-centered coaching of novices, and (c) schoolwide cross-disciplinary initiative leadership. Rather than treating "teacher leadership" as a single outcome, these cases suggest that leadership development is better understood as opportunity uptake within structured communities (e.g., TLC) that affords legitimacy, role access, and tools for influence beyond the classroom.

For APLUS design, the findings point to several implications. First, TLC structures that emphasize collective task design and iterative improvement informed by participant data appear to function as a high-leverage mechanism for district-facing spillover and capacity building among teacher leaders. Second, mentoring-centered pathways warrant intentional scaffolds—such as coaching routines, protected time, and explicit mentoring roles—to translate instructional growth into novice support and induction capacity. Third, schoolwide leadership pathways depend on organizational openings; engaging administrators and clarifying role authority may increase the likelihood that fellows can lead cross-disciplinary improvement work while sustaining mathematics-specific instructional leadership.

Future research in the larger APLUS study will examine the prevalence and stability of these pathways across the full cohort and across phases of participation, including how entry conditions (e.g., preparation history, prior leadership) and contextual supports (e.g., administrator support, formal roles) mediate spillover. Methodologically, the journey-map approach used here can inform broader analyses by specifying measurable indicators of uptake and spillover (e.g., TLC roles, mentoring enactments, schoolwide initiative artifacts), supporting an account of how teacher leadership develops in sustained fellowship models.

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PROFESSIONAL EXPERIENCES RECOMMENDED FOR THE PREPARATION OF MATHEMATICS EDUCATION DOCTORATES

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The knowledge and competencies necessary for mathematics education doctoral preparation have received growing attention over the past two decades. Recently, Kim et al. (2024) administered a survey examining the knowledge, skills, and competencies necessary for doctorates in mathematics education as they embarked upon varying career paths. In this report, we examine the interview data of 35 recently graduated mathematics education doctorates who participated in the survey. The findings expand on previous work by revealing additional recommendations for mathematics education doctoral preparation. Specifically, participants highlight the importance of opportunities for professionalism and networking and share experiences from their own doctoral programs. These findings have implications for faculty and university administrators seeking to strengthen their mathematics education doctoral programs.

Keywords: Doctoral Education, Doctoral Preparation and Training, Competencies Required for Mathematics Education Doctorates, Professionalism, Networking

Over the past two decades, research investigating the experiences of graduates from doctoral programs, specifically within the field of mathematics education, has grown significantly. The first National Conference on Doctoral Programs convened in 1999 and sparked conversations about the components necessary within doctoral programs to promote the success of graduates in their future careers. Following the conference, the Association of Mathematics Teacher Educators (AMTE) published the *Principles to Guide the Design and Implementation of Doctoral Programs in Mathematics Education* (2003). This document outlined the essential knowledge necessary for doctorates in mathematics education and provided recommendations for the development of high-quality doctoral programs.

The second National Conference on Doctoral Programs was held in 2007. After almost a decade, scholars continued to address doctoral student preparation and possible program improvements. Reys and Dossey (2008) contributed to this effort by publishing a monograph detailing the core components necessary for mathematics education doctoral preparation. In this monograph, the authors elaborated on the core knowledge necessary for mathematics education doctorates by identifying specific components (e.g., curriculum). Ultimately, Reys & Dossey's (2008) work encouraged others to explicitly define the components necessary for mathematics education doctorates, rather than providing broad overarching recommendations.

The third and most recent National Conference on Doctoral Programs took place in 2022. Following the third conference, the question remained: How can programs ensure that mathematics education doctorates are successful after graduation when there is such a vast spectrum of career paths they might pursue? Kim et al. (2024) sought to address this concern by identifying the scope of work that mathematics education doctorates engage in. Additionally, the authors wanted to know how mathematics education doctorates use their doctoral program

experiences in their current roles (post-graduation). To understand and document these experiences, Kim et al. created a survey to explore the knowledge, skills, and competencies necessary for doctorates in mathematics education.

Core Competencies for Mathematics Education Doctorates

Although early literature examining the necessary components for strong mathematics education doctoral preparation focused primarily on mathematics, research, and teaching, additional competencies began to emerge as research progressed. Beginning in 2002, Reys & Reys (2002) developed a list of essential knowledge that mathematics education doctoral programs should include to promote the success of their graduates. They recommended that programs place an emphasis on the following areas: (1) mathematics content, (2) research, (3) social, historical, political, and economic context of education, (4) learning, (5) teaching and teacher education, (6) technology, (7) curriculum, and (8) assessment (Reys, 2002).

Shortly after the second National Conference of Doctoral Programs was held, Ferrini-Mundy (2008) highlighted the components that she believed to be at the core of mathematics education doctoral programs, expanding the previous work of Reys (2002). These components included mathematics, learning, and research. In addition to these foundational core components, she recommended that "core plus" components be recognized as well. Her findings suggested the inclusion of aspects such as: (1) resources for learning, (2) contexts, (3) diversity and equity, (4) ethics, (5) teacher education, (6) teaching, (7) policy, (8) technology, and (9) international issues (Ferrini-Mundy, 2008). Almost a decade later, Kirkpatrick & Spangler (2016) expanded the list of recommendations further by highlighting the need for teaching experience prior to entering a doctoral program. They emphasized the need for mathematics education doctorates to have experience teaching pure mathematics (K-12 or undergraduate level) and experience teaching mathematics education. In addition to the proposed teaching experiences, Kirkpatrick and Spangler (2016) also advocated for mathematics education doctorates to have access to a wide variety of research experiences during program enrollment.

Although each of these reports contributed to the literature examining the necessary components for mathematics education doctoral preparation, they lacked empirical data, each resulting from either a task force (Reys, 2002) or general noticing within an individual doctoral program (Ferrini-Mundy, 2008; Kirkpatrick & Spangler, 2016). To address this gap, Kim et al. (2024) compiled the collection of recommended competencies into a single list seeking to gather information about each competency that would inform future mathematics education doctoral programs. The preliminary list consisted of 12 core competencies: (1) mathematics, (2) curriculum, (3) research, (4) diversity and equity, (5) policy, (6) standards, (7) teacher education, (8) technology, (9) learner development, (10) assessment, (11) history of mathematics education, and (12) interdisciplinary connections.

Ongoing Study

Despite ongoing discourse related to doctoral student preparation, the question remained: What core components should mathematics education doctoral programs attend to in order to best prepare their doctoral students for success after graduation? In response to this question, Kim et al. (2024) conducted an empirical research study that developed and administered a survey designed for recent graduates of mathematics education doctoral programs. The survey sought to gather information about recent mathematics education doctorates regarding the responsibilities of their current role, experiences prior to their doctoral program, experiences during their doctoral program, and the impact that these experiences had on their current role.

Additionally, the survey gathered input regarding the use of Kim et al.'s list of core competencies. The list was based on existing literature and additional important areas related to the mathematics education doctorates' work, such as assessment.

The survey was administered through Qualtrics, and participants were notified of its opening through email listservs, personal connections, and social media. Survey items were presented in both multiple-choice and open-ended formats. The survey was left open for one month, and at the time of closing, 309 individuals responded. The authors define mathematics education doctorates as "doctoral graduates from a mathematics department or a department/school/college of education, with an emphasis on mathematics education" (Kim et al., 2024). Additionally, participants were only included in the study if their current role was within the area of mathematics education. Out of the 309 survey respondents, 218 completed the most significant portions of the survey (> 85%), and their responses were analyzed. Although all participants were recent graduates of mathematics education doctoral programs, their current roles within mathematics education varied significantly.

Following the survey study, interviews were conducted with 35 of the survey participants to further explore the competencies needed for the work of mathematics education doctorates. The full results on the competencies are reported elsewhere; in this report, we focus on professionalism, which we found significant aspect of the competencies that mathematics education doctorates need to develop. This report explores a subset of the interview data collected during this study. Specifically, it examines the responses of mathematics education doctorates who discussed professional experiences related to professionalism and networking. Professionalism and networking both encompass experiences required to navigate the landscape of mathematics education successfully. However, professionalism opportunities are oriented inward, toward an individual's capacity to function effectively within the profession. While networking is framed through an individual's outward relationship to the mathematics education community. This report seeks to answer the following research question: What professional experiences are most needed for mathematics education doctorates after graduation?

Methods

Data Collection and Participants

The data presented in this report was gathered through an extension of the survey administered by Kim et al. (2024). In addition to completing the survey, respondents were asked if they would be willing to participate in an interview with one of the members of the research team to elaborate on their responses. Out of the 218 respondents, 72 mathematics education doctorates agreed to participate in a follow-up interview. Ultimately, only 35 respondents were able to participate in a follow-up interview. The data discussed in this report reflects the responses of these 35 participants. Each participant was placed in an interview group based largely on their responses to the survey and their self-reported current position (post-doc, tenured faculty, pre-tenured, non-tenure track, post-doc, K-12 educator, research scientist). In total, 14 interviews were conducted either through focus groups of one to five participants or individual interviews. As seen in Table 1, participant positions varied greatly, with tenured faculty representing the largest subset (34.3%).

Table 1: Current Positions of Mathematics Education Interview Participants

	Position					
	Tenured	Pre-Tenured	Non-Tenure Track	Post-Doc	K-12 Education	Research Scientist
Participants	12	7	7	4	4	1
Percentage	34.3%	20%	20%	11.4%	11.4%	2.9%

Data Analysis

Following each interview, the responses were transcribed and the first phase of coding began. To organize and analyze participant responses, our research team developed a series of codes reflecting the trends present within the interviews through open coding. From these trends, five categories emerged: (1) current role or work they do, (2) doctoral program experiences, (3) experiences before their doctoral program, (4) how they use certain skills or knowledge to do their job, and (5) recommendations for future doctoral programs. Each category, seen in Figure 1, was assigned a color for coding to distinguish it within the transcripts. If responses did not fit any of the categories, but appeared to embody a significant trend, they were highlighted in grey.

Color	Categorization	Description
Purple	Current Role/Work They Do	Aspects of their current position: - Job title - Main duties & expectations - Research focus
Green	Doctoral Program Experiences	Experiences within their doctoral program: - Teaching experience - Research experience - Other
Blue	Experiences Before Doctoral Program	Experiences prior to their doctoral program: - Undergraduate experience - Graduate experiences (Master's degree) - Previous career experiences
Yellow	How They Use Certain Skills or Knowledge to do Their Job	Relevance of the core competencies in their current position: - Each core competency - Any additional relevant components
Pink	Recommendations for Future Doctoral Programs	Suggestions for future doctoral programs based on their doctoral program experience: - Helpful experiences included in their program that they would recommend - Experiences that would have been helpful, but were not included within their program

Figure 1: Interview Data Color Coding Sequence

After all codes were established, the research team analyzed several transcripts together to establish a consistent coding scheme. Members would analyze transcripts individually before meeting as a research team to establish reliability within coding assignment. The team discussed all discrepancies in coding to determine specific descriptions of each category in an effort to limit any overlap between codes. Upon establishing descriptions of each code, the team divided into pairs to complete the analysis of the remaining interviews. Each pair was assigned two to three interviews to analyze, coding individually before meeting together to establish reliability in coding.

Following the analysis of each transcribed interview, all coded data was compiled and displayed in a table. The team then identified sub-categories within the broader categories, seen

in Figure 2, and cataloged all responses accordingly. As this process continued, the team decided to merge the category recommendations for doctoral programs (previously pink) with the reported doctoral program experiences (green). Additionally, sub-sets of uncategorized significant trends (grey) began to emerge and were included to represent important responses from participants that did not fall under any previously established category.

Current Role/Work They Do	
- Job Title - Main Duties & Expectations - Research Focus - Research Related Work - Teacher Related Work	- Other Work (Professional Development, School Boards, etc.) - Other Significant Components (Job Satisfaction, Job Fit, etc.)
Doctoral Program Experiences	
- General Program Information - Equity & Diversity Experiences - Technology Experiences - Coursework - Teaching Experiences	- Research Experiences - Comprehensive Exams - *Recommendations for Doctoral Preparation* - Other Significant Components (Professionalism, Networking, etc.)
Experiences Before Doctoral Program	
- Undergraduate Experiences - Master's Program Experiences	- Previous Career - Other Significant Components (Standards, etc.)
How They Use Certain Skills or Knowledge to do Their Job	
- Mathematics - Curriculum - Research - Diversity/Equity - Policy - Standards - Teacher Education	- Technology - Learner & Learner Development - Assessment - History of Mathematics Education - Interdisciplinary Connections - Other Significant Component Recommendations
Additional Important Factors	
- Effects of Educational Political Climate - Effects of Covid-19 Pandemic	- Lack of awareness of possible career paths (outside of higher education)

Figure 2: Interview Data Code Sub-Categorization

Findings

The findings in this report highlight the experiences of mathematics education doctorates as they describe the importance of professionalism and networking opportunities during their doctoral programs. Professionalism experiences emphasized the development of skills needed to perform one's professional role after entering the field. While networking opportunities described the connections individuals had with others both inside and outside their institutions.

Professionalism

As we coded the participants' responses on their doctoral experiences, the topic of professionalism was a recurring theme. Out of 35 participants, 10 explicitly discussed professionalism experiences during their doctoral program. This report defines professionalism as the competencies or skills required to navigate the landscape of mathematics education

successfully. Although less than a third of participants spoke about their professionalism experiences, those who did stressed how valuable these experiences were. One participant praised her "preparing future faculty program" where doctoral students "did all those professionalism things, and a whole bunch of people all got in a room. They're all looking for jobs. We had all those [professionalism] conversations … it was like an extra thing, but so essential, just so essential." For mathematics education doctorates, there are several aspects of professionalism that do not become apparent until they enter the workforce. After using a constant comparative analysis to examine the data, three themes emerged from participants' discussions of professionalism: (1) seeking multiple perspectives, (2) assistantships and apprenticeships, and (3) academic assessment. Excerpts from participants, seen in Figure 3, represent both positive (unshaded) and negative (shaded gray) professionalism experiences. (Please note that the names in Figure 3 here and Figure 4 later are pseudonyms.)

Sub-Category	Example 1	Example 2
Seeking Multiple Perspectives	"There was a preparing future faculty program that was offered at my grad institution that basically did all those professionalism things, and a whole bunch of people all got in a room. They're all looking for jobs. We had all those conversations. There were panels – someone who just knew the structures of universities and how they work, because that's not always a conversation or something that you learn about. And just hearing perspectives from deans writing cover letters and all of that stuff, that was all in a summer course. So it was like an extra thing that I did but so essential, just so essential." - **Claire (Postdoctoral Scholar)**	"I wouldn't have that knowledge necessarily had I not done graduate school. You know through graduate school I got to talk to Matt Larson a few, quite a few times…and that was when he was starting to take leadership roles at NCTM. And he was bringing to these courses that I was teaching and conversations I was getting to have with him, some really interesting perspectives" - **Leslie (Education Department Chair)**
Assistantships & Apprenticeships	"I want to share something that I felt like I have seen as being tremendously helpful for my graduate training related to teacher education. Before I even had a teaching assistantship, I had a semester to apprentice with a tenured professor who had designed the methods course…That was tremendously helpful, for then my own teaching and being able to also make sense of future classes. Instead of 'well, this is the way that I did it, so this is the best way,' I had a model of what that looks like and experienced someone else there to watch what I was doing and give feedback. That I feel like has really strengthened what is now a very core part of what I'm expected to do in my current role." - **Alan (Assistant Professor)**	"I did a little bit because I asked for later in my Ph.D. program more of a research assistantship. But I felt like coming out of the program, I didn't feel like I was prepared for or would be ready to take on a tenure track position" - **Amy (Assistant Professor)**
Academic Assessment	"I never would have thought about it in grad school that the assessment of my colleagues and my peers was something that I would need to be thinking about in my job as a faculty member. But it is a huge part of my job at this point. Whether it's my staff, my lectures, or my faculty, I am assessing a lot of their work, their teaching, their service and their scholarship with my students." - **Leslie (Education Department Chair)**	"I guess the one thing I did not really do or don't have as much experience in is assessing courses and programs for teachers. I've…I don't think I've ever done that. Obviously, I plan and teach courses and I've designed courses, but I've never assessed." - **Nicole (Assistant Professor)**

Figure 3: Professionalism Experiences of Mathematics Education Doctorates

Most participants who discussed seeking multiple perspectives during their doctoral program said that it was not only a positive experience, but one that helped them succeed in their future role. However, the opinions about graduate apprenticeships/assistantships were mixed. Assistantships were generally cited to prepare mathematics education doctorates for teaching but fell short in preparing them for research. Finally, participants unanimously expressed negative feelings regarding their experience with academic assessment. Whether in reference to assessing peers or evaluating courses, participants did not feel that their doctoral program prepared them for this task.

Networking

As participants discussed the experiences they had in their doctoral programs, they also commented frequently on their networking experiences. Out of the 35 participants, only 13 discussed networking experiences (good or bad) during their doctoral program. When asked what she would recommend doctoral programs to focus on, one participant suggested "Maybe professional networking … because it's really what you're doing when you step outside of your university. You're getting involved in other professional organizations. You're learning … you're investing in your field, and you're investing in your profession." Several mathematics education doctorates echoed this sentiment and emphasized the importance of their networking experiences. After using a constant comparative analysis to examine the data, three themes emerged from participants' discussions of networking: (1) student and faculty interactions, (2) proactive process, and (3) conference exposure. Excerpts from participants, seen in Figure 4, represent both positive (unshaded) and negative (shaded gray) networking experiences.

Sub-Category	Example 1	Example 2
Student & Faculty Interactions	"A lot of programs have some sort of… We called it Friday Seminar. But just a time when all of the students and faculty can get together. Maybe it's a semi-structured time, but it's an important time." - **Josh (Research Scientist)**	"And [mentoring] is also something that I was supported to focus on at my doctoral institution, and had great mentors who centered that and their work as well." - **Francis (Associate Professor)**
Proactive Process	"I had to reach out kind of regularly to a handful of professors that I like to work with, and say, 'Hey, what do you have coming up? I'd love to work with you on something' or 'I heard you have something coming up. Do you have space for a GA?' So that's how it worked. It was really like it had to be self-initiated." - **Stephanie (Assistant Clinical Professor)**	"You don't know what you don't know, so I think you have to have somebody to kind of say, 'Hey, like check out these spaces and check out these things.'" - **Amy (Assistant Clinical Professor)**
Conference Exposure	"I didn't really know for many years what conferences to go to because no one had said 'These are math ed conferences' or 'This is what you should be interested in'…I think if I were to mentor doctoral students…I would tell them 'Here are the major conferences you should be interested in [and] this is when the proposals are generally due.' - **Blair (Assistant Professor)**	"You know, grad programs [providing] funding for going to AMTE or NCTM…Maybe that is part of the requirements of [doctoral students], that they're going. Maybe [they're] presenting, but maybe just learning what it's all about. Because if you've never been to one, I don't think you have any idea what to look for." - **Leslie (Education Department Chair)**

Figure 4: Networking Experiences of Mathematics Education Doctorates

Most participants who discussed their interactions with faculty during their doctoral program highlighted both their importance and positive impact on their career. Additionally, while some viewed self-initiated discovery as a positive experience, others critiqued the hands-off nature of their program, with one participant stating, "You don't know what you don't know." Unfortunately, a common negative experience that emerged throughout the discussions of networking was the lack of conference exposure for mathematics education doctorates. While some commented on their lack of knowledge surrounding conferences during their doctoral program, others highlighted the financial burden that conference attendance can have on a doctoral student.

Discussion

Although the literature examining core competencies necessary to promote success among mathematics education doctorates is growing (Hiebert et al., 2008; Reys & Reys, 2016; Shih et al., 2020; Yow et al., 2016), there is a need to continue these discussions through empirical research. The structure of mathematics education doctoral programs has direct implications for

the success of graduates who go on to serve in various roles, including K-12 teaching, curriculum development, research, and higher education.

While discussions of competencies recommended for mathematics education doctorates have largely centered on domains of knowledge, less attention has been given to professional experiences as a critical competency. Since aspects of doctoral preparation, such as professionalism and networking, are discussed less frequently than other components they are often overlooked. However, these "invisible components" present the greatest challenges for mathematics education doctorates upon entering their professional roles. Thus, including professional experiences within the core competencies is essential to better prepare mathematics education doctoral students for the full range of demands they will face in their professional careers.

It would be unrealistic to assume that doctoral programs can prepare their graduates for every possible career path available to them. However, by documenting the most significant core components that contribute to mathematics education doctorates' success, research can inform future doctoral programs and provide recommendations that encourage future faculty to thrive. By taking simple steps, such as encouraging conference attendance through graduate funding or establishing faculty mentorship through graduate assistantships, doctoral programs can promote professionalism and networking experiences and better prepare their graduates for varying career paths. Future research should continue to explore the experiences of mathematics education doctorates as they transition into their professional roles.

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REASONING ABOUT COLLABORATIVE DESIGN FOR SYSTEM-LEVEL LEARNING IN RESEARCH-PRACTICE PARTNERSHIPS

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This paper examines how multi-tiered research–practice partnerships reason about organizing collaborative design with education systems in efforts to support system-level learning. Drawing on three distinct design situations from a long-standing statewide mathematics RPP, we analyze reflective accounts of design decisions elicited through a structured analytic tool. We identify recurring forms of reasoning about the purposes of design, interpretations of system conditions, configurations of expertise and resources, enactments of design activity, and the tradeoffs these judgments entail. From these patterns, we theorize a methodology for collaborative design with education systems that can inform how other research–practice partnerships reason about organizing design under shifting system conditions.

Keywords: Systemic Change, Research Methods, Design Experiments

Meaningful and sustained instructional improvement at scale in mathematics remains a persistent policy, practice, and research problem. In the U.S., instructional improvement is pursued within education systems where authority and responsibility for change are distributed across multiple levels and organizations, resources and expertise are unevenly allocated across contexts, and sociopolitical dynamics shape which forms of instructional change are prioritized, supported, or contested (Polikoff, 2021). As a result, efforts to improve teaching and learning must extend beyond instructional quality in individual classrooms to contend with variation in local capacity, organizational structures, policy pressures, and political dynamics (National Academies of Sciences, Engineering, and Medicine [NASEM], 2025). Mathematics education research has largely approached instructional improvement through research on teaching and learning and through the development of curricula, instructional tools, and professional learning programs. While this scholarship has generated important insights, it offers limited guidance for efforts aimed at instructional improvement when the unit of change is an education system rather than a bounded setting (i.e., school, district). These realities cast the central challenge of improving mathematics instruction as a problem of organizing and coordinating research and design efforts across roles, organizations, and time under changing system conditions.

Multi-tiered research–practice partnerships (RPPs) (Farrell et al., 2021) have increasingly emerged as key actors in this space (McMahon et al., 2022; NASEM, 2025). Positioned across research institutions, state agencies, districts, schools, and professional organizations, RPPs are uniquely situated to convene diverse expertise, navigate the contours and constraints of education systems, and coordinate research and design efforts that span roles, organizations, and time. For long-term RPPs, this coordination often extends beyond the design of specific resources or interventions to include an intentional organization of relationships, routines, and conditions that support instructional improvement (Penuel, 2019). Yet despite a growing body of scholarship documenting the effects of interventions developed by RPPs and examining

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partnership dynamics (Coburn & Penuel, 2016; Farrell et al., 2021), there remains relatively little insight into how partnerships reason about organizing collaborative design in response to system conditions. For example, partnerships may elect to address externally imposed problems such as policy mandates that require rapid coordination across levels of the system. At other times, they may engage in negotiated and jointly defined problems that emerge from sustained deliberation and shared commitment. Beyond contextual, such differences call for distinct ways of convening expertise and organizing participation for design activity.

In this paper, we examine how multi-tiered RPPs reason about organizing collaborative design with education systems in efforts to support system-level learning. Our central claim is that, in RPPs focused on system-level learning and instructional improvement, collaborative design is carried out through recurring forms of methodological reasoning about how design activity is organized and sustained under changing system conditions. This study addresses the following research question: *How do multi-tiered research–practice partnerships reason about organizing collaborative design under differing system conditions in efforts to support system-level learning and instructional improvement?* To address this question, we analyze three distinct design situations from a long-standing statewide RPP in order to elicit reflective accounts of how the partnership approached design under differing system conditions. Through this analysis, we theorize a methodology for collaborative design with education systems that specifies the forms of reasoning and judgment through which partnerships organize system-level instructional improvement.

Theoretical Perspective

Our aim in this study is to theorize a methodology for collaborative design with education systems. By methodology, we refer to a set of procedures and the accompanying rationale that guides how they are selected and used. Methodology thus concerns not only activity but also the logics, commitments, and warrants that shape judgments in pursuit of particular goals (Guba & Lincoln, 1994; Kelly, 2009). In contrast to research methodologies, Edelson (2002) characterizes design methodologies as prescriptive theories that are developed through sustained engagement in design. Design methodologies offer guidance for organizing activity when encountering new situations that share salient features and goals with prior design efforts by foregrounding judgments about how roles, expertise, and structures are configured. In situations characterized by multiple and often competing goals, this form of reasoning is less about optimizing toward a single solution than about navigating tensions and balancing incommensurate aims (Tatar, 2007).

By collaborative design, we refer to a range of participatory approaches to educational design that involve researchers and practitioners working together to address practical problems, along with a shared orientation toward joint inquiry, learning, and authority in design. Such approaches are often referred to as co-design (e.g., Penuel et al., 2007) or participatory design (e.g., Bjögvinsson et al., 2012) and typically described as processes and techniques for engaging practitioners in the development of educational innovations. In this paper, we adopt a broader interpretation of collaborative design that extends beyond particular methods or design processes to encompass the commitments, orientations, and judgments through which joint work is organized, including how roles and expertise are configured, how participation is structured, and how design activity is adapted to shifting conditions, goals, and constraints.

This framing of collaborative design foregrounds the forms of reasoning involved in organizing design activity and is especially relevant in multi-tiered RPPs, where design unfolds across roles, organizations, and levels of the education system. In such partnerships, design often extends beyond the development of specific instructional resources or interventions to include

the intentional organization of relationships, routines, and conditions that support instructional improvement over time. This form of long-term, system-oriented activity has been described as infrastructuring, a practice oriented toward shaping the social, organizational, and material conditions that enable the implementation, adaptation, and sustainability of educational innovations (Penuel, 2019). From this perspective, collaborative design in RPPs is both a means of producing artifacts and a strategy for organizing research and design with education systems under changing conditions. This view motivates theorizing a methodology for collaborative design with education systems that accounts for how partnerships reason about organizing design activity, configuring expertise, and navigating tradeoffs as they encounter new but related system-level design situations.

Methods

Context: The Partnership and Design Situations

To develop the methodology presented in this paper, we examined the work of the North Carolina Collaborative for Mathematics Learning, a long-standing, statewide research–practice partnership focused on expanding equitable professional learning opportunities and supporting system-level learning and instructional improvement in mathematics education across North Carolina. Since 2016, the partnership has convened university-based researchers, state and local education agency leaders, school-based instructional leaders, and classroom teachers to collaboratively design professional learning resources and structures that support systemic coherence through a shared instructional vision. These activities unfolded across multiple initiatives, each organized under distinct system conditions, including differences in the broader sociopolitical landscape of the system, the nature of the problem being addressed, the urgency with which action was required, the locations of authority and expertise involved, and the scope of influence the initiatives were intended to have. Drawing on this variation, we selected three initiatives that illustrate meaningful differences in these system conditions. For analytic purposes, we considered each initiative as a distinct design situation that enabled our examination of how collaborative design was reasoned about under contrasting conditions.

Initial design situation. The first design situation occurred in 2016–2017 following North Carolina's adoption and immediate implementation of new high school mathematics standards, a decision that left many districts with little time to prepare. The problem space was initially surfaced by district and state leaders, shaped by political tensions surrounding the state's prior adoption of the Common Core State Standards for Mathematics, and constrained by districts' capacity to respond. Under these conditions, the partnership faced significant pressure to act quickly, with limited opportunity for broad participation or extended deliberation. Given these constraints, it leveraged existing relationships and organized collaborative design around a small group of university-based researchers and trusted district and state partners to develop mathematics professional learning and implementation resources intended for statewide use and broad distribution (see https://www.nc2ml.org/high-school-resources/hs-math-standards). Design decisions were heavily influenced by the compressed timeline, the need to navigate politically charged policy contexts while safeguarding participating organizations, and the need to address multiple levels of the state education system simultaneously.

Second design situation. The second design situation occurred between 2018 and 2020, following North Carolina's adoption of new K–8 and fourth-level high school mathematics standards under a planned yearlong implementation timeline. The problem space centered on the widespread use of locally developed pacing guides that varied substantially in quality, coherence, and alignment with research on student learning. These guides were often created under resource

constraints, used prescriptively, and widely viewed as barriers to supporting high-quality instruction in classrooms. Under these conditions, the partnership faced a challenge of supporting statewide coherence while respecting meaningful differences across local contexts and capacities. In response to this context, the partnership developed 15 grade-level and course-specific curricular and instructional guidance resources that clustered and sequenced the new standards, explained the underlying mathematics and research on student learning, and suggested pedagogical approaches for teaching them in diverse settings (see https://www.nc2ml.org /k-5-resources/#frameworks). It expanded participation to include design teams composed of teachers, instructional coaches, district leaders, and state education agency staff from across the state. Design decisions were shaped by the need to balance inclusion with coherence, position expertise across multiple roles, and support adaptation in local contexts.

Third design situation. The third design situation occurred between 2022 and 2025 with the launch of the partnership's VISIONS Project, a multi-year effort oriented toward system-level instructional coherence. Rather than responding to policy adoption or implementation timelines, the partnership focused on developing a shared vision of high-quality and equitable mathematics instruction as a means of supporting instructional improvement (NC^2ML, 2025). This focus emerged from instructional challenges identified by more than 280 mathematics teachers and leaders during regional listening sessions, partnership leaders' reflections on persistent variation in instructional visions across the state, and the broader post-pandemic context. In this situation, the partnership faced a challenge of engaging the education system around a research-based vision of high-quality, equitable mathematics instruction without the unifying pressure of a policy deadline. In response, the partnership organized collaborative design through multi-year, heavily facilitated grade-band design teams coordinated by a steering committee composed of educators representing diverse roles, regions, and perspectives. Design decisions required ongoing reasoning about how to support inclusive participation, manage complexity across grade bands, and sustain engagement over time while maintaining a coherent system-level orientation.

Together, these three design situations provided contrasting contexts for examining how the partnership reasoned about organizing collaborative design in support of system-level learning. Although each situation involved distinct problems, timelines, and configurations of participants, all required the partnership to make judgments about how its activities should be organized given who was involved, what forms of expertise were needed, and what the effort was intended to accomplish given the constraints and affordances of each situation. Treating these situations as analytically distinct enabled us to examine recurring forms of reasoning about how collaborative design was organized across contrasting situations.

Data Sources and Analysis

To support a systematic examination of these situations, we developed an analytic tool referred to as the BAT. The BAT was designed to elicit reflective accounts of how the partnership configured its activities and reasoned about collaborative design under particular system conditions. It was informed by Edelson's (2002) conception of design methodologies as prescriptive theories that guide decisions about how design activity is organized, what forms of expertise are required, how responsibilities are distributed, and how design processes are enacted. The BAT operationalized this perspective through 11 questions focused on how design problems were defined within particular contexts, how expertise and participation were positioned, and how activity was organized and sustained. Using a common set of questions across multiple design situations, the BAT elicited the judgments, rationales, and tradeoffs

through which the partnership reasoned about collaborative design. Each respondent completed the BAT in a single retrospective session and treated each situation as a distinct case.

The primary data source for this analysis consisted of completed BAT responses from five core members of the partnership who were centrally involved in organizing and sustaining the partnership's efforts across the three design situations. These respondents included partnership researchers and design leaders who had direct responsibility for configuring design activity, facilitating collaboration across roles and organizations, and navigating system-level constraints. In addition to the BAT responses, we used partnership records to support interpretation and ensure accuracy when constructing preliminary accounts of each design situation.

Our analysis proceeded in three stages. First, we constructed analytic accounts for each design situation by synthesizing BAT responses and partnership records with attention to how collaborative design was organized and sustained under the conditions characterizing each situation. These accounts were then circulated among 15 additional designers who had participated in each initiative for scrutiny, correction, and elaboration. Second, we compared these accounts across design situations to identify recurring patterns of judgment that shaped how the partnership organized its work. Third, we synthesized these comparisons to develop higher-level categories that characterize recurring forms of reasoning about collaborative design across system conditions. Throughout this process, we iteratively refined our interpretations through discussion and feedback from designers involved in the initiatives.

Results

Across the three situations examined in this study, our analysis revealed recurring forms of reasoning through which the partnership organized, enacted, and sustained collaborative design in response to shifting system conditions, timelines, and goals for system-level learning. These forms of reasoning shaped what was designed and how the activity itself was configured, including who participated, how expertise and responsibilities were positioned, how work was coordinated across roles and organizations, and which tradeoffs were accepted at particular moments. Examining how these forms of reasoning operated across distinct design situations enabled us to move from describing situated decisions to articulating an empirically derived design methodology that delineates a space for organizing and sustaining collaborative design over time with education systems.

Figure 1 synthesizes the methodology by illustrating how a partnership's orientation toward an education system and the purposes of collaborative design shapes its interpretation of system conditions, configuration of resources and expertise to support design activity, organization of facilitation and participation, and engagement with recurring tradeoffs as conditions and goals evolve. In this way, Design Orientation, Design Conditions, Design Structures, Design Practices, and Design Considerations function as interrelated components that together define a relational space for methodological reasoning across time and initiatives. Each component is elaborated through a set of dimensions that specify ranges along which partnerships reason about how collaborative design should be organized under differing system conditions. In what follows, we elaborate each component by clarifying the forms of reasoning it foregrounds and how it informs decisions about organizing collaborative design for system-level learning.

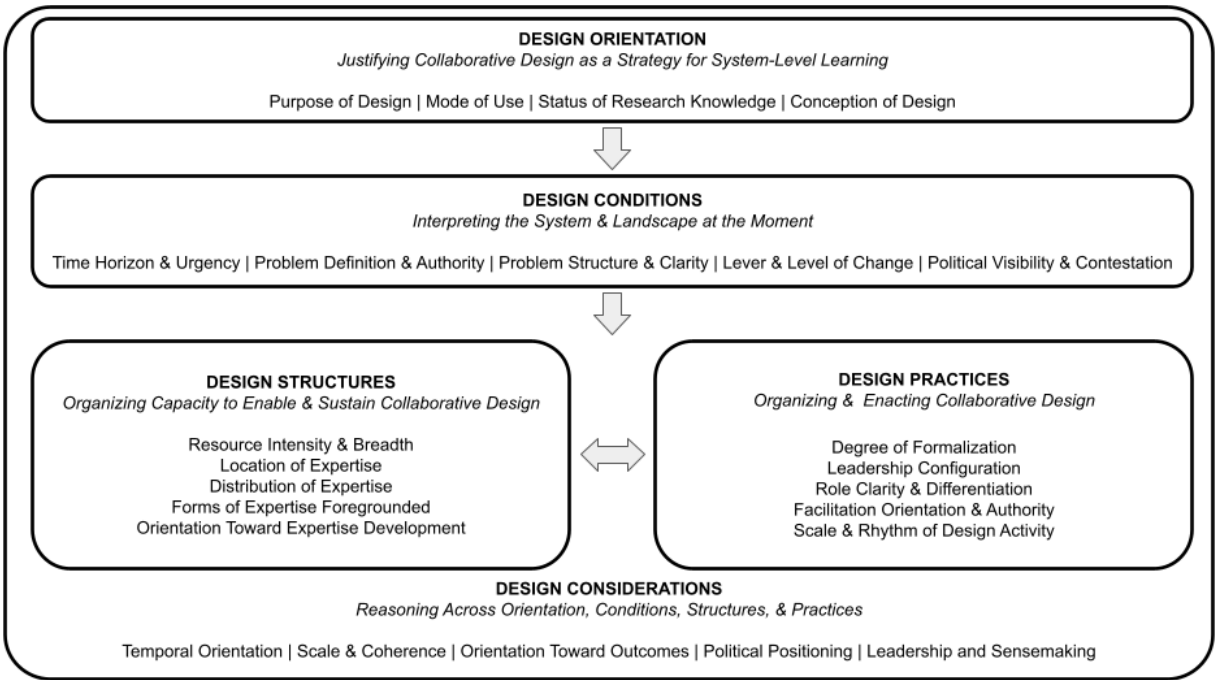


Figure 1: Methodology for Collaborative Design with Education Systems

Design Orientation describes how a partnership reasons about the purposes of collaborative design and its role in supporting system-level learning. It foregrounds a partnership's commitments that shape the kinds of learning that are prioritized (e.g., individual practice, collective sensemaking, or organizational capacity), how progress is understood in relation to system-level goals, and which tradeoffs are viewed as acceptable when organizing design activity. Orientation is elaborated through four dimensions, shown in Figure 1, that define a space for reasoning about why collaborative design is an appropriate response to system-level challenges and how it should be oriented under differing conditions. For example, one dimension concerns the *purpose of design* and captures how partnerships balance an emphasis on producing resources with using design activity as a mechanism for capacity-building, alignment, and coordination across roles and organizations. Where a partnership positions itself across these dimensions informs how system conditions are interpreted and, in turn, how structures and practices are configured and considerations are weighed.

Design Conditions characterize how a partnership makes sense of the education system and broader landscape at a given moment when organizing collaborative design. Rather than articulating commitments or aims, the component highlights features of the system that shape what forms of activity are feasible, constrained, and consequential, including temporal pressures, political dynamics, authority structures, and the nature of the problem being addressed. Conditions are developed through five dimensions that assist partnerships in interpreting the demands and constraints of a given system moment. For example, one dimension concerns *time horizon and urgency* and captures how partnerships reason about the extent to which a situation demands rapid response versus sustained deliberation and sensemaking over time. How a partnership reads system conditions influences judgments about who should be involved and in what roles, how design activity should be coordinated across actors and organizations, and how

quickly design activity should move. In this way, conditions shape how structures and practices are selected and enacted in response to the system.

Design Structures concern how a partnership organizes capacity to make collaborative design with an education system possible and sustainable. This component emphasizes organizational arrangements that shape who can participate and in what roles, how activities are resourced and coordinated, how responsibility for decisions is distributed, and how expertise is positioned and developed. Structures are operationalized through five dimensions that describe ways of assembling capacity to support design activities under particular system conditions. For example, one dimension concerns *orientation toward expertise development* and distinguishes between approaches that treat expertise as a prerequisite for participation and those that treat it as an outcome of participation itself. Where a partnership positions itself along these dimensions influences which forms of activity become feasible, how participation is sustained, and how capacity develops over time.

Design Practices attend to how design activity unfolds over time. This component foregrounds reasoning about how partnerships facilitate and coordinate design activity in practice, including how responsibility is exercised, how participation is organized, and how learning across roles and organizations is supported as conditions and capacity shift. Practices are described through five dimensions that define a space for reasoning about how collaborative design is enacted under particular system conditions and structural arrangements. For example, one dimension concerns *facilitation orientation* and captures how partnerships balance facilitation aimed at coordination and efficiency with facilitation oriented toward collective sensemaking, alignment, and capacity-building. Where a partnership positions itself along these dimensions shapes how design activity unfolds, how participants experience collaboration, and how shared learning and ownership develop over time.

Design Considerations capture persistent tensions that partnerships must reason through when organizing collaborative design under particular system conditions. Rather than describing features of the system or design activity, this component highlights how partnerships weigh competing aims and make tradeoffs across orientation, conditions, structures, and practices. Considerations are elaborated through a set of five tensions that surface persistent dilemmas when designing with education systems and define a space for reasoning about what is responsible, strategic, and sustainable over time. For example, one consideration concerns *scale and coherence* and captures how partnerships reason about supporting system-level alignment while allowing for local interpretation and adaptation. How a partnership engages these considerations informs judgments about how collaborative design is scoped, who is involved and in what roles, how explicit or flexible designs should be, and how alignment and adaptation are balanced under particular system conditions.

Discussion

This study examined how a multi-tiered research-practice partnership organized collaborative design under differing system conditions in efforts to support system-level learning and instructional improvement. Across three design situations, we found that the partnership's approach to collaborative design was shaped by recurring forms of reasoning about how design activity could support system-level learning, how system conditions were interpreted, and how expertise, resources, and participation were configured in response to shifting constraints and possibilities. Together, these forms of reasoning constitute an empirically derived methodology that characterizes how partnerships organize collaborative design in ways that remain responsive to changing system conditions.

This methodology extends existing scholarship on research-practice partnerships by focusing attention on the methodological reasoning partnerships use to organize and sustain collaborative design with education systems. Prior RPP research has documented internal partnership dynamics associated with effective collaboration, including trust, mutualism, and communication routines (Coburn & Penuel, 2016). This study builds on that scholarship by specifying how partnerships make and revise judgments when organizing design as conditions, timelines, and priorities change. In doing so, the methodology complements existing RPP frameworks by foregrounding the tradeoffs and strategic decisions involved in coordinating design work with education systems.

Our findings have implications for how system-level learning and instructional improvement is understood and pursued. In U.S. education systems, governance, expertise, and resources are distributed across multiple levels, organizations, and actors, and improvement therefore unfolds differently across contexts (NASEM, 2025). The methodology presented here highlights how partnerships can reason deliberately about these realities when organizing collaborative design. By attending to orientation, conditions, structures, practices, and considerations as interrelated components, partnerships can use the methodology as a planning and reflective tool to organize collaborative design in ways that remain responsive to shifting system demands while sustaining learning, coordination, and long-term viability.

The methodology developed here draws on a single long-standing statewide partnership and a bounded set of design situations and should be considered with that scope in mind. Our aim was not to offer a generalizable model, but to develop an analytic and practical resource for reasoning about collaborative design with education systems whose value depends on examination, adaptation, and refinement through use across contexts. Future research might examine how similar forms of methodological reasoning emerge in other partnerships or policy contexts, how these forms evolve as partnerships mature or confront new system conditions, and how this methodology is taken up, adapted, and revised as a planning and reflective tool. Such efforts would further develop an empirically grounded theory of collaborative design as a methodological practice for system-level educational improvement.

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MATHEMATICAL IDENTITY MAKING, SELF-STUDY THROUGH DUOETHNOGRAPHY, AND TEACHING GEOTECHNICAL ENGINEERING AS A FIELD OF STUDY

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Keywords: Mathematical Identity, ADAS, Authorial Voice, Self-Study, Duoethnography, SOAR

Mathematical identity has been extensively researched in contexts of mathematics education. The bulk of this focus, however, was mostly conducted in K-12 settings (Heyd-Metzuyanim & Graven, 2024; Sfard, 2019). While few studies have focused on mathematical identity in university level settings, none, as far as we know, focused on university instructors' mathematical identity experiences. We problematize this gap because this missing piece precludes a view onto the relationship between mathematical identity making and pedagogical approaches in settings of higher education. The study is conducted as part of an interdisciplinary collaboration between two university instructors. We import a four-part framework of mathematical identity making to capture, surface, and discern connections with pedagogical approaches and instructional practices.

We use the ADAS framework (Fellus, 2019), to carve the landscape of one's *Autobiographical* (experiences with learning mathematics), *Discoursal* (how one is spoken to and about as a learner of mathematics), *Authorial* (opportunities one gets to develop ownership, autonomy, and confidence in mathematics), and *Socioculturally* available narratives about mathematics. This approach is aimed at addressing what Graven and Heyd-Metzuyanim (2019) identified as "the implicit unification of different types of identity narratives; and … the neglect … in the field to differentiate between … various identity stories of different authors" (p. 364). The ADAS framework allows us to dive deeply into different types of identity narratives so that we can differentiate between various identity stories and critically reflect on their connections with instructional practices.

We engage in a two-pronged methodology that involves self-study (Kitchen, 2009) and duoethnography (Norris & Sawyer, 2017) to embark on a journey of critical reflection of and awareness to connections between mathematical identity and instructional practices. The data comprise emails between the authors, more than a dozen scheduled recorded dialogic online conversations, and a growing record of the first author's experiences in mathematics and the second author's feedback and follow-up questions. The data were analyzed as sites of inquiry and theory-informed reflections through the second author's role of critical friend.

Several insights pertain to how one's experiences with learning school mathematics project onto one's instructional practices. We showcase how opportunities the first author experienced in taking ownership over mathematical ideas (the second A in the ADAS framework) while strongly associated with the other identity-related experiences (i.e., autobiographical, discoursal, and socioculturally available narratives about mathematics) have also been the catalyst for self-regulation, fluency, and flexibility. We thus offer an identity-making pedagogical approach that pushes to the front opportunities for learners to develop mathematical voices through self-regulation, ownership, autonomy, and research (SOAR).

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ROOTED IN PLACE: EXPLORING AFFORDANCES AND CONSTRAINTS OF RURAL MATHEMATICS TEACHER LEADERSHIP

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Keywords: Instructional Leadership, Rural School Contexts

Rural schools constitute approximately one-third of public schools in the United States (Showalter et al., 2017) and enroll a student population that is becoming increasingly diverse (Lavalley, 2018; Johnson et al., 2014; Thier et al., 2021). Due to an uneven distribution of economic, social, and political capital, including a lack of specific attention in policy and research, rural schools typically face acute inequities in meeting the educational needs of these student populations (Hartman et al., 2022; Johnson & Howley, 2015). However, rural school contexts also have characteristics of cultural richness, self-reliance, and deep connection to community and place, which all serve as assets to leverage toward educational improvement (Gallo, 2020; McHenry-Sorber et al., 2023).

Rural schools and districts face specific challenges in teacher recruitment and retention, especially in STEM fields, which further exacerbate issues of teaching and learning (Carver-Thomas & Darling-Hammond, 2017; Goldhaber et al., 2021; Ingersoll & Tran, 2023). Opportunities for teacher leadership, and systems that support and leverage teacher leadership, are an effective means for improving teaching and learning in schools and increasing teacher retention (Lumpkin et al., 2014; Shen et al., 2020). Supporting mathematics teacher leadership in rural schools is shaped—through both affordances and constraints—by the place-based, cultural, and geographic contexts of these settings. While a small subset of the literature on mathematics teacher leadership has focused particularly on rural contexts (e.g., Bolyard et al., 2025; Lotter et al., 2020; Nelson, 2022), the place-specific variation across rural contexts necessitates intentional efforts to further understand the development of mathematics teacher leaders and the cultivation of systems that support and leverage mathematics teacher leadership.

We present initial findings from collaboration across two different research studies, one in Idaho and one in West Virginia, that engaged in developing mathematics teacher leaders and supporting their work in school and district contexts. The intent of both studies was to better understand the interplay between aspects of the rural context and teacher leadership, with the aim to articulate affordances and constraints of the journey of becoming and being a teacher leader in a rural space. We intentionally focus on rural contexts across two states in different regions of the United States to highlight the variance of affordances and constraints related to rural teacher leadership, despite the unifying characteristics of rurality.

Findings across both projects highlight shared themes, such as the role of professional networks in validating teacher leader identity and the tension between teacher agency and district-level structures. Notable differences also help further elaborate the nuances of teacher leadership, including how geographic isolation in Idaho and regional cultural norms in Appalachian West Virginia distinctly shape teachers' experiences of and pathways into leadership. This poster represents the beginning of a research collaboration aimed at developing a more nuanced understanding of mathematics teacher leadership in rural contexts, with the goal of building a framework that informs the design of programs and systems that more effectively support and leverage rural mathematics teacher leadership across diverse rural contexts.

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UNDERSTANDING THE MATH LEARNING PARTNER ROLE IN ONTARIO'S DETRACKED GRADE 9 MATHEMATICS CLASSROOMS

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Keywords: Detracking, Instructional Leadership, High School Education, Curriculum

Purpose and Methods

In Ontario, Canada, mathematics performance has prompted renewed attention to classroom instruction and implementation of supports for teachers. A directive from the Ministry of Education (2024) has increased funding and classroom supports for roles such as board math leads and central math support educators (2016, 2024). While titles vary, the function is clear: to support teachers through classroom-embedded professional learning (Ellington et al., 2017; McGraw et al., 2025; Walkowiak, 2016). This study examines how MLPs and Math Coordinator perceive and enact their roles within a large urban board. Research questions include:

1. How does the school board define the Math Learning Partner role? How does this align with or differ from the MLPs perspective of their own role?
2. What training and professional learning do MLPs experience? How do these structures prepare them to impact teacher practice and student learning beyond their direct, in-person support?

MLPs are expected to work directly in classrooms through co planning and co-teaching, positioning the role as classroom embedded and partnership oriented (Saclarides, 2022). Research highlights the importance of infrastructure, centralized leadership, time for collaboration and clarity in role expectations (Patel et al., 2021; Walkowiak, 2016), as well as the role of principals in supporting instructional partnerships (West, 2017).

This qualitative study explored experiences of 5 MLPs and one Math Coordinator working in destreamed Grade 9 mathematics. Semi structured interviews focused on perceptions of the role, supports, training, challenges and indicators of success. Data was coded inductively.

Results and Conclusion

Participants described clear expectations related to improving mathematics achievement, particularly aligned with EQAO performance and data use. MLPs were expected to support High Impact Instructional Practices, ensure curriculum fidelity and design professional learning. At the same time, participants reported mixed messaging across levels of leadership and a strong sense of accountability. MLPs navigate a mix of instructional, relational and organizational demands. The role extends beyond pedagogy to include navigating school contexts, building relationships and responding to local needs. Time limitations, systemic barriers and the instability of funding were identified as ongoing challenges. Training includes formal learning, collaboration and shadowing, grounded in current research and supported through team structures. This study highlights the importance of principal partnership, consistent messaging and time for in class collaboration. The findings show that the role is shaped by formal

expectations and relational/organizational demands, contributing insights into how board level teacher educator roles can be supported in detracked mathematics contexts.

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COMPUTATIONAL DEGRADATION OF OVERLY-BUREUACRATIZED MATHEMATICS EDUCATION COMPLEX SYSTEMS

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Keywords: Policy, Professional Development, Standards, Systemic Change

The purpose of this research is to better understand how the growing educational bureaucracy of the United States (U.S.) affects mathematics education reform progress. This research extends previous research (Ricks, 2025) describing how initially useful educational bureaucracy (promoting equality and efficiency) stifles reform potential by excessive reductionistic fixation on trivializing metrics and alignment compliance. The original standards-based reform movement has taken a potentially dangerous bureaucratic turn. This new era of reform-by-bureaucracy needs clarifying research about its effects. This study seeks to develop explanatory theory about this phenomenon and to suggest future action for praxis. I followed the four cyclic stages of Glaser's (1965) constant comparative method—comparing, integrating, delimiting, and writing—to perform a meta-analysis across empirical and theoretical material from eight prior bureaucracy-based research studies. I utilized an integrated theoretical framework viewing overlapping, nested mathematics education systems—such as mathematics classes, departments, teachers' learning communities, schools/districts, states' departments of education (some with numeracy divisions), special interest groups, professional organizations, etc.—as *computational* complex systems. All complex systems *compute* as part of their systemic maintenance—e.g., self-emergent agentic rule-bound behavior enfolding through feedback loops determining actual morphological structures—and hence, enact systemic-specific forms of mathematics. The quality of computation is directly linked to the health of the system—i.e., its dynamic adaptability, creative integration of diversity, problem-solving capacity, etc. This perspective illuminates how organizational hierarchies and algorithmic protocols mandated by educational bureaucracies affect the information-processing structures of mathematics education complex systems. The results of this study indicate that overly-bureaucratized mathematics education systems often suffer from procedural limitations to the necessary phase space of morphological configurations needed for reform. I adapt Stein et al.'s (2009) student-based *Mathematical Task Framework* to the super-systemic level, and show how bureaucratized mathematics education systems' behavior is mostly low-level *memorization or procedures-without-connections*. Bureaucratic mandates like forced digital platformization, scripted curricula, value-added teacher accountability, high-stakes disciplinary pressures upon schools/districts, etc. all contribute to constrained computational capacities. This research suggests ways to leverage educational bureaucracy to encourage higher-level systemic computation like *procedures-with-connections* and possibly *doing mathematics* (e.g., administrative supports for teacher professional learning communities like lesson study) to encourage better super-systemic mathematization.

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PRE-CALCULUS, CALCULUS, AND HIGHER MATHEMATICS

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

TEXTBOOK CONCEPTIONS AND STUDENT RESPONSES ABOUT PIVOTS IN LINEAR ALGEBRA

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This study investigates how students reason about pivots in linear systems, considering both potential conceptions in a linear algebra textbook and students' responses to open-ended questions addressing the notion. Using Balacheff's cK¢ model, we identified problems, operations, representations, and control structures embedded in the textbook explanations. We then analyzed 68 student responses, identifying which control structures were evident. Findings indicate that students frequently mobilized the Zero Row control structure promoted by the textbook, but also activated control structures related to the number of solutions. Some responses lacked the full constraints of the reduced row echelon form. This suggests that students selectively engage with textbook structures and mobilize strategies from related contexts, offering insights that can inform teaching strategies and the design of instructional materials.

Keywords: Undergraduate Education, Algebra and Algebraic Thinking, Integrated STEM / STEAM, Curriculum

This study is situated within PROTEUS (PreTeXt-Runestone: Open Textbooks for Engaging Undergraduates in STEM), a large-scale project aimed at developing interactive questions for open-source textbooks in calculus and linear algebra using the Runestone Academy platform (Mesa et al., 2023). The textbooks and the platform are freely available, allowing instructors to adopt these materials with their students. The questions developed within PROTEUS include mainly open-response items, fill-in-the-blank questions, drag-and-drop activities, matching tasks, and Parsons problems.

A key challenge in this context concerns the interpretation of students' responses to open questions. Unlike automatically graded question types, open responses require instructors to examine each answer individually in order to infer students' thinking and, when desired, provide personalized feedback. This makes understanding patterns in student reasoning particularly important for both instructional and research purposes.

Within this setting, students may rely on the textbook, and potentially other resources, when responding to open questions before formal instruction on the topic. Motivated by this, we focused on tasks related to pivots to identify from how the concept is presented in the textbook the possible conceptions that can emerge and then contrasting student responses to those possible conceptions.

Theoretical Framework

In this study, Balacheff's cK¢ model (Balacheff & Gaudin, 2009) was chosen because it supports the formulation of possible students' reasoning through the identification of control

structures evident in responses to specific tasks. In this model, knowledge refers to the system of constructs that are institutionalized by a social body, such as the mathematical community, while knowing refers to the personal constructs developed by an individual. A conception is defined by the interaction between a subject's knowing and a mathematical milieu in which a mathematical notion is needed.

Instead of assuming the subject's internal mental processes, the model utilizes a quadruplet connecting problems (P), operators (O), representation systems (R), and control structures (Σ) to provide a model for a student's conceptions about a mathematical notion. By applying this model to textbook analysis, prior work has identified the "potential conceptions" that are intended by textbook authors by examining how expository text and examples can guide readers through specific tasks (Mesa, 2010; Gerami et al., 2024).

A major focus in our study is the control structure, which contains the criteria that is used to decide what actions to take to solve the problem, to determine when an answer is reached, and to establish the correctness or plausibility of that answer. When analyzing student responses, the model allows researchers to treat students' justifications as empirical evidence of their internal control structures (Gerami et al., 2024). Gerami and colleagues found that students often employ a wider and more varied range of control structures than those explicitly presented in their textbooks, which suggests that their "knowing" is shaped by a more complex interaction between instructional design and personal experience (see also Deng et al., in press).

Specifically, we examine student responses to three open questions embedded in the textbook to answer the following research questions (RQ):

- What conceptions of pivots can be inferred from the content related to the pivots in an interactive undergraduate linear algebra textbook?
- What control structures are evident in student responses to open questions about pivots?

In this study, we use the cK¢ model to analyze student responses collected via an interactive textbook, focusing on the *Understanding Linear Algebra* textbook (ULA; Austin, 2025) housed in the Runestone learning academics platform. ULA is an open source and interactive resource, it provides a unique milieu in which students encounter interactive reading questions after the expository text and examples; Runestone captures all the responses students type, making them available for analysis.

We analyzed responses to determine the specific control structures that were called for in answering the questions, and contrasted them with the possible conceptions the textbook would have promoted. The contrast allows a differentiation between knowledge (as evidenced in the textbook) to knowings, from students' responses. We can then determine if the students rely on the control structures demonstrated in the textbook, or if they develop alternative control structures to reason and verify their answers.

Analyzing the ULA textbook is important because its interactive format may make available a wider range of mathematical conceptions to students than traditional printed texts. This digital environment can promote the establishment of multiple connections, between representations, procedures, and meanings (Gueudet et al., 2017).

In this sense, the ULA textbook not only presents content, but also structures how students may interpret and justify mathematical situations (Love & Pimm, 1996). Thus the cK¢ model provides a suitable analytical lens to examine these conceptions as they are embedded in the textbook content and to relate them to the control structures evident in students' responses to open-ended questions.

Methodology

We analyzed Section 1.4 of the Understanding Linear Algebra textbook, entitled "Pivots and Their Influence on Solution Spaces." This section presents a definition of pivot positions, six questions in a Preview Activity, an explanation of the existence of solutions, and a discussion of solution uniqueness. It concludes with a summary containing five open-ended checkpoint questions, followed by a final list of exercises.

Three researchers independently analyzed this section using cK¢ model, based on the two fundamental questions the author proposes: the existence and uniqueness of a system linear solution. The unit of analysis was each worked example in the section that addressed these topics, examining how existence and uniqueness were operationalized.

By examining the problem formulations, the representations mobilized, the expected operations, and the associated control structures, we inferred the mathematical conceptions that the textbook makes available for addressing the proposed problems to address the first RQ. Any disagreements were minor and were resolved through consensus among all authors. A table (Table 1) was constructed to systematize this analysis.

While the first part of the study focuses on the conceptions of pivots made available by the textbook, the second part examines how students mobilize control structures when responding to open questions about pivots.

For that, we focused on the first three PROTEUS questions (parts a, b, and c) from a single activity in the section (runestone.academy/ns/books/published/PROTEUS_Year2_ULA_Template/sec-pivots.html) which share the same opening prompt (Figure 1):

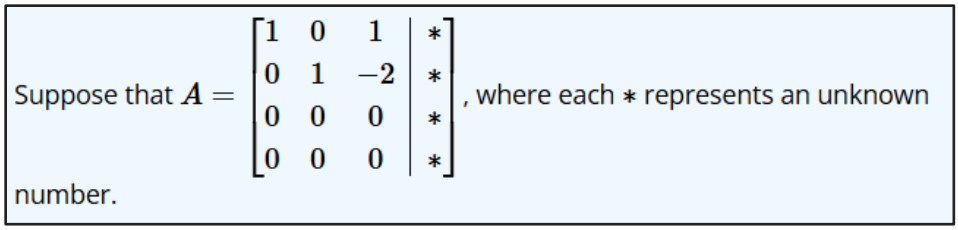


Figure 1: Opening Part of the Prompt

a) If the linear system represented by A is consistent and A is in reduced row echelon form, can we determine any of the unknown numbers? If so, which ones? Explain your thinking.
b) If the linear system represented by A is inconsistent and A is in reduced row echelon form, can we determine any of the unknown numbers? If so, which ones? Explain your thinking.
c) If the linear system represented by A is consistent and A is in reduced row echelon form, can you guarantee that there is a single solution or that there are infinitely many solutions? Explain your thinking.

For space reasons, this paper focuses on the first two questions of the activity. We had access to 82 student responses to these two open-ended questions, collected from courses taught in Fall 2025 by three instructors at different U.S. universities. Responses that did not include an explanation or for which students' reasoning could not be inferred were excluded from the analysis: three from the first question and eleven from the second. The higher number of

exclusions for the second question was related to the question itself, which stated that the matrix was inconsistent; some students therefore appeared to conclude that there was no possible value for the asterisks and did not provide further justification. As a result, the final dataset consisted of 68 student responses.

Based on the textbook analysis and the focus of questions (a) and (b), we identified four control structures relevant to determining the consistency or inconsistency of a linear system. These control structures, their definitions, and illustrative examples from student responses are presented in Table 1.

Table 1: Control Structures Identified and Illustrative Examples from Students' Responses

Name of Control Structure	Definition	Example Response (question, student ID)
Pivot position (CS1_1)	Identify pivot positions in a RREF matrix and determine the corresponding zero and nonzero entries in the same column.	[...] This pivot point must be in R3, because it can be the only nonzero value in its column, and if it were in R4, R3 would contain all zeros, which would not meet the standards for RREF. (Part B, 212I-001-06)
Rightmost column (CS1_2)	Check if the rightmost column has a pivot to conclude consistency/inconsistency.	Either row 3 or 4 would have a number other than 0 in the rightmost column. There would be no solutions. (Part B, 212I-002-04).
Equations (CS2_2)	Translate matrix rows into equations and use their solvability to determine consistency or inconsistency.	If consistent, then the forth column in R3 and R4 have to be 0 for that statement (0 = 0) to be true. I think we'll require more information to determine the other two asterisks. (Part A, 212I-003-02)
Zero row (CS3_2)	Examine zero rows in the RREF matrix to determine consistency based on whether the last entry is 0 or 1.	The 0 rows have to be 0. (Part A, 212I-003-15)

In addition to these four control structures, our broader analysis identified five further control structures related to reasoning about the number of solutions of a linear system (uniqueness versus infinitely many solutions). In the full coding scheme, these control structures were distinguished as CS1_3 through CS5_3.

However, because questions a and b do not address the number of solutions, and instead focus exclusively on consistency and inconsistency, these additional control structures are not examined in detail in this paper. Since they all refer to the same underlying problem, reasoning about solution multiplicity, they were grouped and labeled generically as CS*_3.

The first three authors independently coded all 68 student responses for evidence of the nine identified control structures. Multiple control structures could be assigned to a single response. This analytical process provided the basis for examining how students' reasoning aligned with,

diverged from, or partially reflected the control structures made available in the textbook explanations.

Findings

To address our first RQ related to conceptions of pivots, we examined the textbook explanations regarding pivot positions and the consistency of linear systems. Using the framework, we identified four conceptions embedded in the textbook, three of which are illustrated in Table 2. These conceptions describe the ways in which the textbook guides students to reason about pivot positions and the consistency of systems through specific representations, operations, and control structures.

The first conception corresponds to the Pivot Position control structure (CS1_1), formalized as the following quartet:

P: Determining pivot positions in a matrix.

R: Matrix representation in reduced row echelon form (RREF), with pivot entries visually emphasized.

O: Identify the leftmost nonzero entry in each row of the RREF matrix.

Σ: If row i contains a leading 1 in column j, then (i, j) is a pivot position.

Table 2: Conceptions Presented in the Pivot Section of the Textbook

Elements of the Conception	Conception 1	Conception 2	Conception 3
Problem 2	*Determine whether the linear system is consistent or inconsistent.*		
Representations	Matrix in $R^n \times R^n$ focusing on pivot positions.	Algebraic notation for systems of linear equations.	Augmented matrix focusing on zero rows.
Operations	Find the RREF matrix, find the pivot position.	Find RREF of the matrix, put in terms of variables (x, y, z), and calculate those equations.	Find the RREF matrix, check the 0-row in the coefficient matrix.
Control Structure	There is a pivot position in the rightmost column of the corresponding augmented matrix → conclude inconsistency. There is not a pivot position in the rightmost column of the corresponding augmented matrix → conclude consistency.	If it is possible to relate a row with an equation in the form $0x + 0y + 0z = 1$ → conclude inconsistency. If all the rows related to equations have solutions, like $0x + 0y + 0z = 0$ → conclude consistency.	If there is a 0-row in the RREF matrix in this format $(0 \cdots 0\ 1)$ → conclude inconsistency. If all the 0-rows in the RREF matrix are $(0 \cdots 0\ 0)$ → conclude consistency.

The other three conceptions, illustrated in Table 2, correspond to the control structures Rightmost Column (CS1_2), Equations (CS2_2), and Zero Row (CS3_2). Each describes how students can determine consistency or inconsistency of a system by checking the rightmost column, translating rows into equations, or examining zero rows, respectively.

To address our second RQ, related to the students' control structures, three of the authors independently coded student responses to locate evidence for any of the nine control structures identified across the questions. Multiple codes could be assigned to a single response.

This analytical process provided the basis for examining how students' reasoning aligned with, diverged from, or partially reflected the control structures embedded in the textbook explanations. Across 68 responses, we identified a total of 89 instances of control structures: 52 responses contained a single control structure, 12 contained two, 3 contained three, and 1 contained four.

For instance, one student response for part a illustrates how multiple control structures could be applied in a single answer: "Yes, we can determine a few of the unknown numbers - For the system of A to be consistent the right-hand side of Columns 3 and 4 must be 0 (Rightmost column - CS1_2). The Column 1 and 2 give pivot variables so they can give out numbers (Pivot position - CS1_1)." The frequencies of each code are presented in Table 3.

Table 3: Frequency of Control Structures in the Dataset

Code	Frequency
Pivot position (CS1_1)	8
Rightmost column (CS1_2)	9
Equations (CS2_2)	8
Zero row (CS3_2)	40
Other control structures (CS*_3)	24
Total	89

Looking at Table 3, the most striking observation is that Zero row (CS3_2) was used far more frequently than any other control structure, accounting for over half of the 68 responses. This emphasis mirrors the textbook, where a 0-row is highlighted to differentiate between rows ending in 1 or 0, corresponding to inconsistent or consistent solutions.

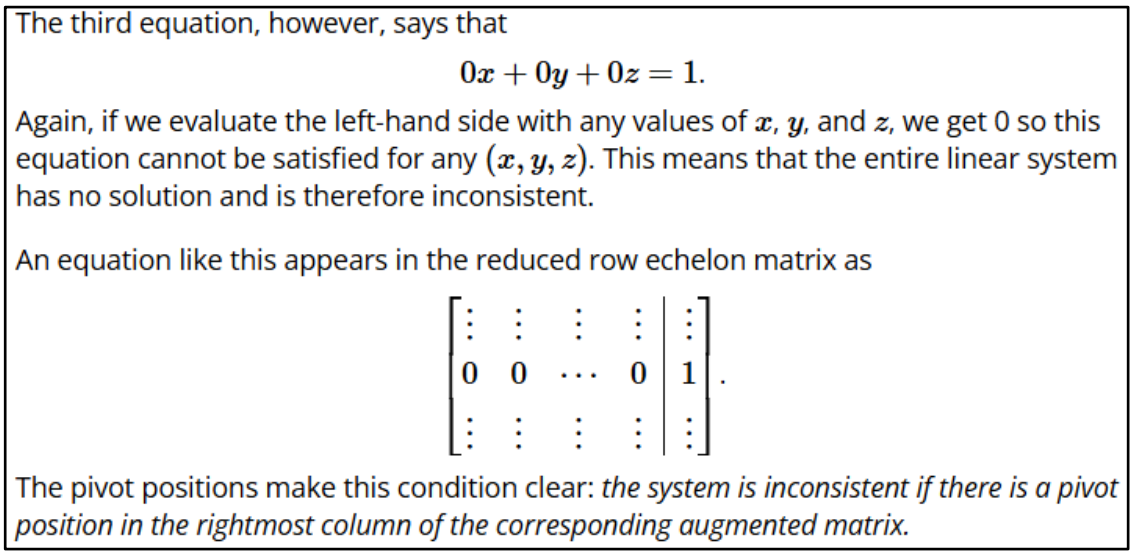
This visual highlighted example in the textbook may have influenced student responses (Love & Pimm, 1996). Many student responses focused on this 0-row control structure, describing the values in the last column of Rows 3 and 4 without explicitly connecting these entries to the pivot positions in the RREF matrix.

For example, some responses suggested that the unknown numbers in Rows 3 and 4 could be arbitrary to indicate inconsistency, while the top entries were left unspecified: "We can determine that the unknown numbers in Rows 3 and 4 can be non-zero, since if they are, the linear system cannot be consistent. Apart from that, the unknowns in Rows 1 and 2 can be anything, and it will still be inconsistent" (using Zero row as the control structure).

Considering the pivot positions associated with the RREF matrix would provide additional structure for interpreting the last column, indicating that the first two entries, and the last one should be 0 and that only the Row 3 entry should be 1. In this sense, the responses illustrate reasoning about inconsistency that does not yet fully integrate the RREF structure.

In the textbook, the 0-row ending with a 1 is visually emphasized, while the other entries are represented with dots (Figure 2). These visual cues appear to guide student attention to the last column, highlighting the connection between row structure and consistency, while leaving the other entries open to interpretation. This alignment with Balacheff and Gaudin (2009) reflects how student response patterns can emerge from the textbook's structure, reinforcing the idea that a conception is characterized as a situated knowing.

In addition, the translation of the 0-row ending with 1 into the equation $0x + 0y + 0z = 1$ emphasizes the absence of a solution. This explicit representation may have contributed to reasoning that any nonzero number in that position could generate inconsistency, without necessarily considering how the pivot positions in the RREF matrix constrain the other entries.


The third equation, however, says that

$$0x + 0y + 0z = 1.$$

Again, if we evaluate the left-hand side with any values of x, y, and z, we get 0 so this equation cannot be satisfied for any (x, y, z). This means that the entire linear system has no solution and is therefore inconsistent.

An equation like this appears in the reduced row echelon matrix as

$$\left[\begin{array}{cccc|c} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right].$$

The pivot positions make this condition clear: *the system is inconsistent if there is a pivot position in the rightmost column of the corresponding augmented matrix.*

Figure 2: Textbook Excerpt Illustrating Control Structures for Linear System Solutions

Another important observation from Table 3 is the high number of responses using CS*_3. Many students applied reasoning associated with the number of solutions of a linear system (CS*_3) when responding to questions about consistency (Parts A and B).

For example, to part a when the matrix is consistency, some stated, "We can't determine the value of a free variable" or "No, because there is a free variable," while when the matrix is inconsistency (Part B), one wrote: "We will not be able to, because if the linear system is inconsistent, there will be no solution."

These responses indicate that students were reasoning from the perspective of the system's solutions rather than focusing on the values of the entries in the rightmost column (the asterisks). In other words, students often mobilized control structures related to the number of solutions to justify what could or could not be inferred about individual entries, showing how these structures can be activated across related but distinct problem contexts.

Discussion and Conclusions

This study examined how pivot positions are presented in an interactive linear algebra textbook and how students activate related control structures when responding to open-ended questions. Regarding our first RQ related to conceptions of pivots, the textbook conveys several conceptions of pivots, emphasizing the identification of pivot positions, the connection between

rows and equations, the role of zero rows, and the significance of the rightmost column in determining consistency. These conceptions are embedded in visual and symbolic representations, which structure the potential ways students may interpret and reason about linear systems (Mesa & Goldstein, 2017).

In addressing the second RQ related to students' control structures, we observed that they activated a range of such structures in their responses. Most responses reflected reasoning about zero rows, and number of solutions. A particularly interesting pattern was that, when asked to reason about consistency, some students mobilized control structures more closely associated with determining the number of solutions. This suggests that students may treat consistency and number of solutions as closely connected, or even interchangeable, even though they involve different levels of classification. While these control structures are mathematically relevant, they were not always directly aligned with the question being asked. This demonstrates how student reasoning can selectively engage structures promoted in the textbook while also suggesting areas where additional scaffolding might be beneficial.

At a broader level, this analysis highlights the alignment and divergence between the conceptions embedded in the textbook and the control structures evident in student responses. This comparison is grounded in the understanding of the textbook as an instrument for acquiring mathematical knowledge (Rezat, 2006). While the textbook provides explicit cues for identifying pivots, student interpretations do not always fully exploit these available frameworks. These insights suggest that instructional support could be enhanced through exercises that explicitly bridge the relationships between pivots, zero rows, and the reduced row echelon form.

While this analysis provides valuable observations, the number of responses analyzed here was relatively small (68 in total), which may limit the generalizability of the findings. However, we anticipate receiving approximately 300 additional responses, which could provide a more comprehensive understanding of the control structures students employ when reasoning about pivots.

Future work should extend this analysis to a larger dataset to examine how control structures are used across a broader sample and to refine our understanding of student reasoning patterns in the context of pivot positions. Such work can also inform the development of teaching interventions and learning materials, providing guidance both for instructors teaching pivots and for textbook authors designing instructional content.

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WHICH STANDARDS MATTER MOST? A QUANTITATIVE ANALYSIS OF CONTENT RETENTION AND GRADES IN COLLEGE ALGEBRA

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This paper describes a quantitative analysis of students' retention of content knowledge and the predictability of their final grade based on their mastery of content in a standards-based College Algebra course. The sample consisted of semester and final exam mastery data, along with final grades, for 39 students enrolled in a single section of College Algebra. Our analysis consisted of a matched-pairs analysis of students' semester and final exam mastery scores, along with various statistical diagnostics and measures, such as Goodman-Kruskal gamma and Ridge Ordinal Regression, to identify central standards with significant predictive ability for final grade. Our results identify four standards and two themes that exhibit significant predictive power for students' final grades, which we connect to the literature on students' thinking on these topics.

Keywords: Precalculus, Standards, Assessment, Data Analysis and Statistics

Alternative grading (AG) practices (e.g., standards-based grading, specification grading, ungrading) are commonly used in K-12 mathematics settings in the United States and are increasingly used in postsecondary classrooms (Lewis, 2022). Researchers have shown that AG implementation can yield positive affective outcomes for students, such as reduced test anxiety (Harsy & Hoofnagle, 2020; Henrikson et al., 2020) and a re-envisioning of office-hours conversations and assessments as learning opportunities (Collins et al., 2019; Zimmerman, 2020). Various researchers have also shown that AG students perform no worse than their peers in traditionally graded classes on cumulative exams (Harsy & Hoofnagle, 2020; Morris et al., 2024) or standardized tests (Pollio & Hochbein, 2015; Welsh et al., 2013). At the College Algebra level, there is a litany of research categorizing students' thinking about using quantitative reasoning to model real-world phenomena with functions (Pollio & Hochbein, 2015; Welsh et al., 2013), logarithms (Kuper & Carlson, 2020; Weber, 2002), and exponential functions (Ellis et al., 2016). However, there is very little research relating students' performance on individual standards to their success in College Algebra (Eckman et al., in press), retention of knowledge on a cumulative final exam, or the ability of these data to predict final course grade. The purpose of this paper is to present a quantitative analysis of the relationship between students' final grade performance in one section of a coordinated AG college algebra course, their mastery of individual standards during the semester, and retention of content knowledge on a cumulative final exam. Our discussion will be framed around the following research questions (RQ):

1. Which College Algebra standards showed the greatest retention differences from semester to final exam?
2. What is the independent effect of each standard on final grade after controlling for possible statistical interactions between standards?

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Theoretical Background and Course Structure

The AG College Algebra course (the equivalent of a U.S. high school Algebra II course) that we describe in this paper was constructed using the Four Pillars of Alternative Grading (Clark & Talbert, 2023) and Collins et al.'s (2019) characteristics of Mastery-based Testing (MBT). In this section, we describe the structure of our course using the Four Pillars.

Pillar 1: Clearly Defined Standards

To our knowledge, there is no widely accepted definition among AG proponents for the word "standard." For instance, Clark and Talbert (2023) leave decisions about what content to cover and the determination of mastery benchmarks to the instructor's discretion, and Collins et al. (2019) state that standards should be clearly communicated on study guides and assessments.

To construct the original 18 standards for College Algebra, we evaluated a common final exam using the principles of open and axial coding from grounded theory (Strauss & Corbin, 1998). The process included creating lists of content skills students needed to answer exam questions (open coding) and grouping these skills into topics (axial coding). The 18 standards, included in Table 1 below, addressed four themes: general properties of functions and their graphs (8 F standards), exponential and logarithmic functions (5 EL standards), polynomial and rational functions (4 PR standards), and polynomial theory (1 PT standard). We consider our list of standards to be a work in progress and continually evaluate the need for wording updates, combinations, or new types of standards (e.g., synthesis standards).

Table 1: College Algebra Standards in the Approximate Order of Instruction

Code	Standard Summary
F1	Distinguish and interpret functions and non-functions in various forms.
F2	Recognize parent or toolkit functions in graphical or symbolic form.
F3	Given a function in symbolic, tabular, or verbal form, identify its domain and range.
F4	Identify features of a function graph and describe intervals over which these occur.
F5	Find the inverse of a one-to-one function and interpret it in context.
F6	Identify or compute the average rate of change or difference quotient of a function.
F7	Algebraically or graphically transform a function through scaling or rigid motions.
F8	Perform function arithmetic or composition. Identify the domain of the new function.
EL1	Graph an exponential function and identify algebraic or graphical parts of the function.
EL2	Graph a logarithmic function and identify algebraic or graphical parts of the function.
EL3	Convert between exponential and logarithmic form. Use logarithm properties.
EL4	Solve exponential and logarithmic equations algebraically and provide exact answers.
EL5	Model and solve applications of exponential functions.
PR1	Identify and graph linear, quadratic, and piecewise functions.
PR2	Model and solve applications of linear and quadratic functions.
PR3	Graph or write rules for polynomials using roots, multiplicities, and end behavior.
PR4	Construct graphs of rational functions based on their properties.
PT	Determine the complex zeros of a polynomial and write it in factored form.

Pillar 2: Helpful Feedback

AG College Algebra instructors provide weekly assessment and reassessment opportunities and give specific, targeted feedback to student work on these items. Collins et al. (2019) stated

that the purpose of such feedback is not to penalize students but to help them progress toward mastery of the course material by highlighting areas that require further study.

Pillar 3: Marks Indicate Progress

Instructors use a binary system to provide marks on students' assessments and reassessments: *pass* and *not pass yet* (Clark & Talbert, 2023). Instructors base *pass* marks on conceptual mastery of a topic and utilize *not pass yet* marks to indicate the need for additional reflection on course material, rather than penalizing (often computational) mistakes (Collins et al., 2019).

The number of standards a student masters (passes twice) throughout the semester constitutes the primary measure of final course grade. Table 2 presents the letter grade initially assigned to students before other modifiers, such as homework and final exam performance, are applied. Consequently, students' final letter grade in the course indicates the amount of AG College Algebra content for which they have exhibited proficiency. When we make claims in the results about individual standards and students' retention (RQ1) or final-grade predictability (RQ2), we do so to highlight important topics in a College Algebra class.

Table 2: Letter Grade Scale for Standards Mastered

Grade	A	B	C	D	F
Standards Mastered	17–18	14–16	11–13	9–10	0–8

Pillar 4: Reattempts Without Penalty

A major component of our AG College Algebra class is that students can reassess, outside of class, standards they have previously assessed twice in class but have not yet mastered. Clark and Talbert (2023) call this process *reattempts without penalty*. At our institution, students complete a form to schedule reassessments in a dedicated reassessment room, which student proctors monitor. In this report, we compare students' semester mastery status after (re)assessments with their performance on the same standards on the final exam (RQ1).

Methodology

Data Collection and Final Exam Makeup

Sample description. The students in this sample were enrolled in a single section of College Algebra at a mid-sized public university in the Pacific Northwest during the Fall 2025 semester. The first author served as the primary instructor of this course and was the only College Algebra instructor to offer a cumulative final exam. We collected three types of instructor-provided data: students' semester standard mastery scores based on (re)assessment grades, students' performance on each standard on the final exam, and students' course grades.

Description of assessments, reassessments, and final exam. The course assessments and reassessments were single-page quizzes containing 1–3 questions per standard. In contrast, the final exam was a nine-page assessment in which the standards were clearly labeled, and each question corresponded to a different standard. Unlike regular assessments, the final exam used three evaluation marks: "not pass," "half-pass" (i.e., the problem started appropriately but contained a major error), and "pass" (i.e., the problem was completed with only minor errors).

Predictor variables, response variable, and sample refinement. We determined two classes of ordinal predictor variables based on the instructor's grading scheme. An ordinal variable describes a scale that is logical and ordered, but whose distances between values are

unknown or undefined. First, we developed a four-point scale to assess students' mastery of each standard during the semester (1 = not assessed; 2 = not passed; 3 = partially met; 4 = met). We created a similar four-point scale for students' standard performance on the final exam (1 = not assessed, 2 = not passed, 3 = half-passed, 4 = met). During this process, we removed standard PT from consideration because it was not included in the final exam. We used students' final letter grade in the course as our response variable (i.e., A, B, C, D, F). However, because few students received D's, we collapsed the scale into four levels: Below-C-level, C-level, B-level, and A-level students. Our final sample comprised 39 students and their corresponding final-grade data, which we related to their mastery of two sets of 17 standards (i.e., the semester and final exam).

Data Analysis and Explanation of Statistical Tests and Measures

Our data analysis consisted of two stages: matched-pair analysis to measure students' retention of the ability to meet standards, and the identification and validation of important predictor variables while controlling for small sample size and multicollinearity. We now describe the statistical procedures employed at each stage.

Matched-pair analysis (RQ1). To conduct our matched-pair analysis, we computed the difference between each student's mastery level at the end of the semester and their mastery level on the final exam. We then calculated the mean change in mastery per standard across the observations. A negative change indicated that, on average, students' mastery scores were lower on the final than their semester mastery scores.

Identifying and validating important predictor variables (RQ2). We used several statistical methods to identify significant standards whose semester mastery values predicted students' final grades. Due to the small sample size and multicollinearity (e.g., some standards covered similar topics), we employed four levels of analysis. First, we calculated the Goodman-Kruskal gamma (γ) for each standard and used Fisher's Exact test to identify significant predictors of final grade while ignoring multicollinearity. Second, we examined multicollinearity among these standards by evaluating the correlation matrix, computing variance inflation factors (VIFs), and using alias detection to identify variables that are nearly perfectly correlated. Third, we used the Least Absolute Shrinkage and Selection Operator (LASSO) to remove variables with high multicollinearity and Ridge Ordinal Regression to construct a model to predict students' final grades. We note that we chose to use these modeling methods rather than the standard multinomial model due to our small sample size. Finally, we performed a partial correlation analysis to reconcile paradoxes and determine the independent effect of each standard. Through our analysis, we identified two standards that showed substantial loss of retention between semester-standard mastery and the final exam and were important predictors of final course grades. We report these standards in the results of this paper.

Results

RQ1: Retention Differences Between Semester Mastery and Final Exam Performance

Our analysis of the mean mastery level difference per standard on the final exam and during the semester revealed a negative overall and relative change for all but one standard, implying that students' final exam mastery levels were lower, on average, than their semester mastery level (e.g., 4 = mastery during semester, 3 = half pass on final). We did not find this result surprising. Still, the level of retention loss varied by standard, as evidenced by Tables 3a and 3b below. We draw two conclusions from this data. First, standards related to functions covered early in the semester (e.g., F1, F4–F6) generally exhibited higher retention loss, and standards covered late in the semester (e.g., EL2–EL4, PR3–PR4) exhibited lower retention loss. Second, two standards, EL5 and PR2, were covered late in the semester and showed a high relative loss

in retention. This implied (to us) that standards EL5 (exponential modeling problems) and PR2 (linear modeling and optimization) merited further analysis regarding their impact on final-grade predictability.

Table 3a: Lowest Retention Standards

Standard	Mean Semester Mastery	Mean Final Mastery	Relative Change
F6	3.41	2.68	–21.41%
F1	3.90	3.27	–16.15%
PR2	3.13	2.64	–15.65%
F5	3.49	2.96	–15.19%
F4	3.90	3.34	–14.36%
EL5	3.31	2.84	–14.20%

Table 3b: Highest Retention Standards

Standard	Mean Semester Mastery	Mean Final Mastery	Relative Change
EL2	3.05	2.87	–5.90%
PR4	2.59	2.46	–5.02%
EL4	3.00	2.90	–3.33%
PR3	2.92	2.90	–0.68%
EL3	3.08	3.14	+1.95%

RQ2: Independent Effects of Standards on Final Grade Predictability

Identifying an initial list of important predictor standards. Our initial determination of the Goodman-Kruskal gamma (γ) and use of Fisher's Exact Test for each student's standard semester mastery level in relation to their final semester grade indicated that standards EL2–EL5 and PR2 were likely important predictive standards whose mastery level could help predict final grade. The gamma values for standards F1, F2, and F4 were extremely high (near 1), but Fisher's Exact Test showed these standards were not statistically significant ($p > 0.05$), indicating that these standards could be aliased, meaning they are linearly dependent. In the context of our work, the term "aliased" implies that removing all but one of the aliased standards (i.e., F1, F2, F4) from the model would not affect the model's predictive power.

Addressing multicollinearity with the identified predictor standards. We used three multicollinearity diagnostics to refine further our list of important predictor standards: the correlation matrix, VIF, and alias detection. Through these diagnostics, we determined that standards F1, F2, and F4 were indeed aliased, that standards PR2 and EL5 were highly correlated ($r = 0.795$), and that standard PR2 had a high VIF score (VIF $=$ 5.94; any VIF score higher than 5 is considered significant). Consequently, when we used LASSO and Ridge Ordinal Regression to identify important standards to include in our model, we found a paradoxical result. Specifically, the LASSO model selected standards PR2 and EL5, but the Ridge Ordinal Regression only identified students' semester mastery level for standards EL2, EL4, and EL5 as having significant independent effects on a model to predict students' final grades, claiming that the independent effect of standard PR2 was not significant ($p = 0.119$). Since we considered PR2 and EL5 to be topically different (optimization, exponential modeling) but to share some problem-solving similarities (constructing a function to model a real-world scenario), we elected to further investigate standard PR2 by conducting a partial correlation analysis.

Partial correlation analysis to determine the independent effect of standard PR2. The purpose of the partial correlation analysis was to test whether standard PR2 added unique predictive value to our model while controlling for standard EL5. We previously noted in our analysis that the standard PR2's high VIF score (5.94) indicated that it explained 83.2% of the variance shared with the other LASSO-selected standards (e.g., EL2, EL4, EL5). The variance percentage in the preceding sentence was calculated by converting the standard formula $VIF =$

$\frac{1}{1-R^2}$ to the form $R^2 = 1 - \frac{1}{VIF}$. Additionally, the correlation matrix indicated a strong bivariate correlation between standards PR2 and EL5 ($r = 0.795$, 63.2% shared variance; obtained by squaring r). In partial correlation analysis, a high correlation ($r > 0.5$) indicates that a standard adds unique predictive power to the model, even if it is highly correlated with another standard. In our case, the partial correlation analysis of standard PR2, while controlling for EL5, revealed a high and significant correlation value ($r = 0.598, p < 0.001$). We draw two conclusions from the partial correlation analysis. First, despite the high multicollinearity between PR2 and EL5, the partial correlation analysis confirmed that standard PR2 retains significant independent predictive value ($r = 0.598, p < 0.001$) after controlling for EL5. Second, we conjecture that the Ridge Ordinal Regression conclusion that standard PR2 is not a significant predictor for students' final grades ($p = 0.119$) is likely due to an overly large penalty applied by the analysis to standard PR2 based on its high VIF score, and is not indicative of an incompatibility with predicting final grades. These conclusions led us to estimate the practical effect on students' overall semester grade using an odds ratio based on their performance on four standards: PR2, EL2, EL4, and EL5.

Odds ratio and the relationship between predictor standards and final grade. The odds ratios that we calculated for standards PR2, EL2, EL4, and EL5 estimate a multiplicative increase in the odds that moving up one level on the standard mastery (e.g., moving from a "partially met" to a "fully met" on standard PR2) will increase the students' final grade measure (e.g., moving from a "B" grade to an "A" grade). We note that the odds ratio does not distinguish between starting mastery and starting grade levels and only measures the odds that, given a one-unit increase in mastery, the student's current grade level will also increase.

The following table, Table 4, shows the odds ratios for standards PR2, EL2, EL4, and EL5. For instance, the odds ratio for EL5 indicates that students who improve by one mastery performance level during the semester are 4.41 times more likely to achieve the next higher letter grade category, holding all other standards—including PR2—constant. We make two notes about the odds ratios. First, the high odds ratios for PR2, EL2, EL4, and EL5 reinforce the predictive power of these standards for students' final grades based on current mastery. Second, the overarching topical ideas connecting these four standards, which the odds ratios imply corresponded with an increased final grade in our sample of College Algebra students, included interpreting and solving exponential and logarithmic equations (EL2, EL4, EL5) and modeling real-world phenomena with functions and using these functions to make inferences about quantities and their values (PR2, EL5).

Table 4: Odds Ratio of Select Standards in Relation to Final Grade Improvement

Standard	PR2	EL5	EL2	El4
Odds Ratio	2.73	4.41	3.69	3.16

Discussion and Conclusion

The aim of this paper was to address two ideas related to students' performance in College Algebra coursework: retention of specific topics within the course (RQ1) and the topics whose mastery appeared to have the strongest predictive power for students' final grades (RQ2). Through our analysis, we determined that students' semester-long mastery of topics related to logarithmic functions (EL2: graphing; EL4: algebraic solving), exponential functions (EL4: algebraic solving; EL5: modeling), and quadratic functions (PR2: optimization) had the greatest

predictive power for their final grades. We further determined that students exhibited little relative retention loss on some standards (EL2, EL4) and large retention loss on others (PR2, EL5) on the final exam. Taken together, these results indicate that, for our sample, the most central standards related to retention loss and final grade predictability were PR2 and EL5.

While we do not claim generalizability of our results due to our small sample size, our findings both reinforce and extend prior research on students' thinking about precalculus topics. For instance, our finding that students' lowest relative retention of mastery ability is associated with modeling reinforces Thompson and Carlson's (2017) theoretical discussion of the importance of reasoning about quantities. We extend their results by providing empirical evidence that students may struggle to maintain their ability to quantitatively model real-world scenarios using algebraic rules over time. Additionally, our identification of College Algebra students' mastery of topics related to exponential and logarithmic functions as central to models predicting their final grades reinforces the importance of Ellis et al.'s (2016) and Kuper and Carlson's (2020) work on students' thinking about these topics. Our results extend these findings by indicating that mastering exponential and logarithmic functions is not only important for developing mathematical acumen but may also be more strongly related to students' final grades in precalculus coursework than other topics.

Our results may also have practical implications for teaching College Algebra or precalculus courses. For instance, our findings suggest a potential need for instructors to be intentional when planning instruction on exponential functions, logarithmic functions, and modeling problems. Additionally, our results indicate that students may find it easier to retain the ability to perform basic algebraic or graphing tasks related to exponential and logarithmic functions than to construct (and draw inferences from) quadratic and exponential models for real-world scenarios. Taken together, our results suggest that instructors might consider reflecting on the structure and pacing of precalculus courses, and on revisiting ideas related to modeling, exponential functions, and logarithmic functions.

While our study identifies important potential foci for improving precalculus coursework, it has practical limitations that could be addressed in future research. For example, our sample consisted of 39 students from a single College Algebra section. Consequently, we cannot claim generalizability of our results, and further work is needed to verify and extend the ideas presented in this paper. Additionally, this paper does not provide empirical evidence on how to modify precalculus curricula to improve students' retention of ideas related to modeling, exponential functions, or logarithmic functions. Instead, we highlight the need for further work to identify practical ways to support students' mastery of these topics, particularly at the curricular level. Finally, we collected our data from a course that used AG methods, suggesting the need to verify our results in a traditionally graded course, though this process may be more difficult due to the lack of clearly defined standards. We welcome future research in these areas.

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NAVIGATING THE TRANSITION FROM TWO TO THREE DIMENSIONS: A CASE STUDY OF STUDENT CHALLENGES IN DIFFERENTIAL CALCULUS

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In this case study, we examine the challenges faced by a strong multivariable calculus student when working on semi-structured interview tasks in differential calculus of functions of two and three variables. We use the notion of types of relations between schema components of the Action-Process-Object-Schema (APOS) theory to illustrate specific relations that even proficient students often fail to develop. Results reveal different aspects of directional derivatives, the gradient vector, and their relationship, that were particularly difficult for this student and can thus be expected to challenge most students.

Keywords: Advanced Mathematical Thinking, Calculus, Cognition

Introduction and Literature Review

Multivariable calculus is crucial for modeling a wide range of phenomena. Therefore, it is important to learn how to best support students' understanding of this subject. Here, we are primarily concerned with multivariable differential calculus, which is used for modeling change. Starting with the early work of Tall (1992) and Yerushalmy (1997), a growing body of research has emerged on the teaching and learning of multivariable differential calculus. This includes partial and directional derivatives (Borji et al., 2024, 2025; Martínez-Planell et al., 2015, 2017; Weber, 2015), gradient vectors (Moreno-Arotzena et al., 2020), slopes in three dimensions (McGee & Moore-Russo, 2015), total differentials (Trigueros et al., 2018; Lankeit & Biehler, 2024), and other topics in the multivariable differential calculus (Harel, 2021), including schema construction (Trigueros et al., 2025). While these studies provide information about students' understanding of differential multivariable calculus, much remains to be understood about how students relate the different types of derivatives.

Multivariable calculus introduces new challenges for students transitioning from two to higher dimensions. Part of the difficulty stems from the extra information that must be imagined with the increase in dimension, transforming drawings that would be "close" representations in a two-dimensional (2D) setting into "distant" ones in a three-dimensional (3D) context (Parzysz, 1988). For example, the graphical representation of secant lines to the graph of a one-variable function at a point and how they may approach a tangent line is straightforward on a 2D figure, but it is challenging to visualize, draw, and relate to the symbolic representation in the 3D case of a partial derivative of a two-variable function. This increased cognitive demand can hinder students' visual thinking (Eisenberg & Dreyfus, 1991), which is essential in calculus.

As dimensions increase, the possible interpretations of equations and the cognitive demand to understand their meanings both increase. For example, the equation $x = 1$ can represent a point on the number line, a line in 2D, or a plane in 3D space. The interpretation depends on the dimensions of the geometric context. Another illustration is when students who instinctively think of the equation $x^2 + y^2 = 1$ as a circle on a plane must interpret it as a right circular cylinder in 3D space. In general, students accustomed to performing procedural actions on symbolic expressions often find it challenging to have their mathematical activity depend on a

geometric context that they must now anticipate and imagine. In the case of two-variable functions, the simplest operation a student can perform is to substitute a number for a single variable in an expression of the form $z = f(x, y)$. However, frequently students do not understand the geometric meaning of the result (e.g., Trigueros & Martínez-Planell, 2007, 2010). These researchers also report that students struggle to represent the domain of two-variable functions in 3D space, where students must represent an ordered pair (x, y) as a triple $(x, y, 0)$. The interpretation gap between 2D and 3D also occurs in the differential calculus of two-variable functions. For example, a direction vector of a directional derivative is a vector with two components, while the function itself is represented by a surface in 3D space. Hence, to represent the direction vector of a directional derivative together with the function, the student must interpret the direction vector $\langle d_1, d_2 \rangle$ as $\langle d_1, d_2, 0 \rangle$. This is not obvious to beginning students. Rather, since the surface is a 3D object, students expect a direction vector to be a vector in 3D, possibly with a non-zero z component. Our study underscores situations where students need to bridge the gap between 2D and 3D representations. The following research question guided this study: What can we learn about the 2D and 3D representation gap in the differential multivariable calculus from the case study of one student?

Theoretical Framework

A schema, as understood in APOS theory (Arnon et al., 2014), is a coherent collection of actions, processes, objects, and other schemas dealing with a specific mathematical concept where the components of the collection are interrelated in such a way that an individual can tell when a problem situation falls within the scope of the schema. Schemas are in a constant state of evolution as individuals encounter new tasks and learn more mathematics. To work with schemas, a person may model them as a collection of components and different relations between them. For example, some of the components of multivariable differential calculus are function, slope, tangent plane, directional derivative, partial derivative, total differential, and gradient. Trigueros (2019) introduced three types of relations that group two or more components. These are useful for describing the evolution of a schema. A *correspondence relation* is one established out of habit or repetition, where the individual cannot explain the relation or needs help to establish it. A *transformation relation* involves some explanation or justification for how changes in one component are related to changes in another, but this might not be consistent across different problem situations. A *conservation relation* is fluidly and consistently established across different problem situations. It occurs when students show that related structures can be used interchangeably.

Methodology

This is part of a larger study examining students' understanding of the differential calculus of functions of two variables, which involves 59 students in two sections taught by the same instructor (the second author). Ten students agreed to participate in two, one-hour, semi-structured interviews (conducted by the lead author) about their understanding of multivariable differential calculus. We focus only on Jackson. He earned a course grade of A, was considered to be among the top students in the course, and performed much better in the interview than the other participants. The case study assumes that mathematical situations in which Jackson struggled are likely to challenge other students, as was indeed the case, and so it concentrates on examining these situations. Jackson's class used the digital resource CalcPlot3D (Seeburger, 2016) and 3D-printed surfaces for multiple collaborative, investigative activities. The instructor used a "locally linear approach" (Borji et al., 2024). To follow this approach, she included

explicit discussions of slopes in three dimensions (3D), leading to an exploration of vertical change on a plane symbolized by $\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y$. Then she used vertical change on a plane to discuss the point-slopes equation of a plane, $z - c = m_x(x - a) + m_y(y - b)$, the tangent plane at a point, $z - f(a,b) = f_x(a,b)(x - a) + f_y(a,b)(y - b)$, and directional derivatives, viewed as $D_{\langle u,v\rangle} f(a,b) = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{f_x(a,b)u + f_y(a,b)v}{\|\langle u,v\rangle\|}$.

An interview instrument consisting of nine multi-part tasks was used. For the sake of this case study, we consider and renumber portions of four of these tasks.

Suppose the graph of $z = f(x,y)$ is as follows (Figure 1, top left):

a) State the sign (positive, negative, zero) of $D_{\langle -2,0.5\rangle} f(4,0)$. Justify your answer.
b) Represent the direction vector $\langle -2,0.5\rangle$ in the drawing.
c) If $H(t) = D_{\langle 1,2\rangle} f(t,0)$, draw how the graph of H may look for $2 \le t \le 4$.

The plane in Figure 1 (top right) is tangent to the graph of a differentiable function $z = f(x,y)$ at the point $(1,2,0)$.

d) Use the graph of the given tangent plane to find $D_{\langle 1,1\rangle} f(1,2)$.
e) Can you relate the computation you did in the previous part (a) to Figure 1 (bottom left)?

(For f as in Figure 1, bottom right) Approximately in which direction $\vec{v}$ does $D_{\vec{v}} f(2.5,1.5)$ appear to attain its largest possible value? (Give the answer as an approximate direction vector $\vec{v}$ in algebraic form.) Its smallest possible value?

(For f as in Figure 1, bottom right) Suppose H is defined by $H(x,y,z) = z - f(x,y)$. Let P be the darkened point on the surface, that is, P has coordinates $\big(2.5,1.5,f(2.5,1.5)\big)$. Approximately in which direction does the gradient vector $\nabla H(P)$ point? Explain.

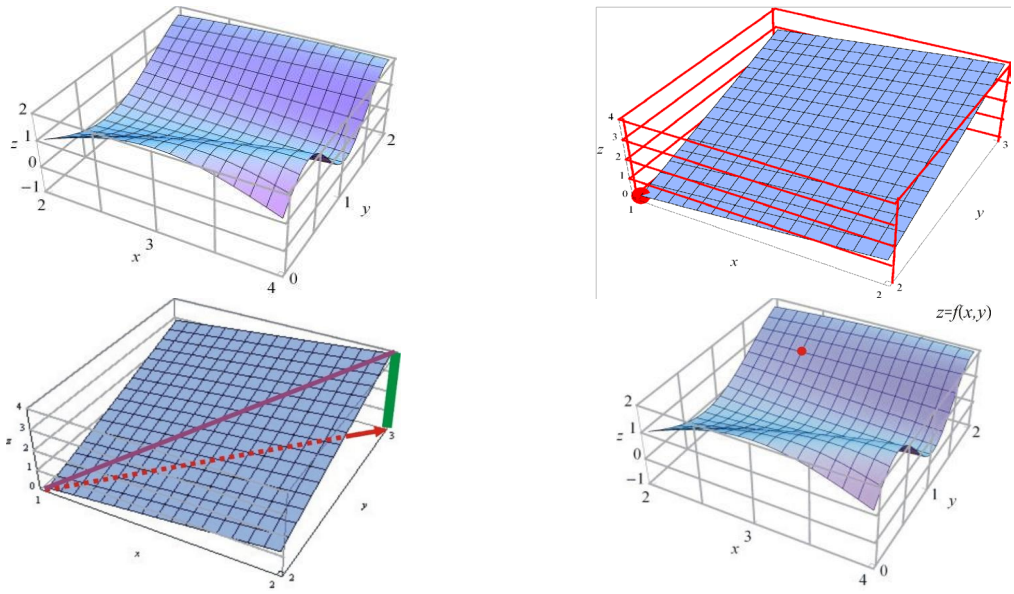


Figure 1: Figures for Task 1, 2, 3, and 4

The tasks were designed to provide insight into students' understanding of different components of the differential multivariable calculus and the relations students established between them. Task 1 was designed to provide opportunities to demonstrate a relation between the components of Function (F) and Directional Derivative (DD), possibly mediated by the Slope (S) component. Task 2 allowed consideration of the relation between the components of tangent plane (TP) and DD, while perhaps including the components of S and Gradient (G). Task 3 allowed further exploration of the relation between the components of DD, F, and G. Task 4 provided an opportunity for the student to relate F and G.

The analysis identified the components students related, which may have differed from researchers' expectations, as well as the type of relation established for each task. The analysis was done independently by two researchers then discussed by all until a consensus was reached.

Results and Discussion

In Task 1a, Jackson was given the graph in Figure 1 (top left) and asked to state the sign of $D_{\langle -2,0.5\rangle} f(4,0)$. He used an available 3D-printed surface corresponding to the one in the figure, rather than using the given graph. He easily determined the sign and justified it by pointing and using gestures on the 3D-printed surface:

> Jackson: I say it's positive … because from where I've got my finger at four, zero, if I were to move to the left in the *x* direction and up in the *y* direction, I'm moving uphill, almost, on this surface. So, I see the directional derivative is positive.

So, in this task, Jackson showed a transformation relation (because of his justification) between the components F and DD. Then, in Task 1b, he was asked to represent the direction vector $\langle -2,0.5\rangle$. Instead of drawing on the given graph, he again used the 3D printed surface.

> Jackson: Okay. So let me get this, alright, right here, and I draw it [did not draw; he physically represented the vector with a pencil] like $-2,0.5$. Something like this [placed the pencil tangent to the surface at the point corresponding to $(4,0,f(4,0))$].
>
> Interviewer: Is that vector pointing up? [since the represented vector had a positive *z* component]
>
> Jackson: Yeah, in the positive *y* direction.
>
> Interviewer: Is the direction vector a vector in two dimensions or three dimensions?
>
> Jackson: Two dimensions. So, okay, so on the surface. So, right here is a vector in two dimensions, where it's pointing up in the positive *y* direction and left negative 2 units [placed the pencil horizontally below the 3D-printed surface in the $\langle -2,0.5\rangle$ direction on what would be the *xy* plane].

Initially, Jackson placed the pencil tangent to the surface with a positive *z* component. Seeing this, the interviewer asked whether the vector was "pointing up." Jackson seemed to have interpreted "up" to mean in the *y* direction, and thus answered "yeah, in the positive *y* direction." However, with further questioning, Jackson realized the nature of the problem with his physical representation ("two dimensions, so okay, so on the surface …") and placed the pencil horizontally, as expected. The relation shown in this task between the components of F and DD was labelled as a correspondence relation because the initial representation of the direction vector as tangent to the surface seemed to be motivated by habit or repetition.

Martínez-Planell et al. (2015) observed that even high-performing students struggle with the geometric interpretation of the direction vector in a directional derivative. Jackson also experienced some difficulty. This suggests that geometrically representing a direction vector in the context of a directional derivative is not trivial for students. Directional derivative at a point was discussed in class in terms of the slope of a tangent line in 3D. This could possibly lead students to confuse the direction vector with the tangent line. From the other interviews in the larger study, it was noted that students often think of directional derivative as a direction or tangent vector.

Using the same surface, in Task 1c, Jackson was asked to describe how $D_{\langle 1,2\rangle}f(t,0)$ would change over the interval $2 \leq t \leq 4$. He ran his index finger pointing in the direction $\langle 1,2\rangle$ over the edge of the 3D-printed surface corresponding to $y = 0$ and was able to solve the task:

> Jackson: (2,0) to (4,0), direction $\langle 1,2\rangle$, going to the right.… It seems like it's gonna be changing sign, almost, yeah, from 2 to 3 changing sign, because when I am at (2,0), for instance, it seems like it's pointing towards negative z values, and by the time I make it to (4,0), it's pointing towards positive z values.

Jackson also drew (an approximation of) the graph of $H(t) = D_{\langle 1,2\rangle}f(t,0)$ correctly (Figure 2). When Jackson says, "it's pointing towards negative z values" and later "it's pointing towards positive z values", he seems to be referring to the tangent line and attending to whether the line points up or down. His argument seemed to be guided by the tactile determination of the change in inclination (from downwards to upwards) in the surface as he moved his finger from the point corresponding to (2,0) to that of (4,0). Jackson related the Function and Directional Derivative components (F-DD) and justified the relation with his argument. This was labelled as a transformation relation. Although one may think that the component Slope (S) is also part of this relation, as we will later see, Jackson seemed to be thinking of directional derivative as a measure of inclination without being aware of the slope of a tangent line as a rate of change.

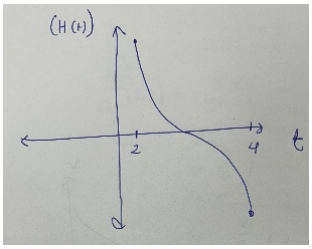


Figure 2: Jackson's work on Task 1c

In Task 2a, Jackson was given the graph of a tangent plane to a function at a point (1,2,0) (see Figure 1, top-right) and asked for the directional derivative of the function in the $\langle 1,1\rangle$ direction. He readily computed the directional derivative using the dot product formula. After computing the slopes of the plane in the x and y directions as $m_x = 1$ and $m_y = 3$, respectively, and finding a unit vector in the direction of the given direction vector $\langle 1,1\rangle$ he solved the task.

However, in Task 2b (discussed below), when asked to interpret his work geometrically, he did not interpret the directional derivative as the slope of a line in the tangent plane. Jackson did not think of the directional derivative as a slope for this task, contrary to what appeared to have happened when he used the printed 3D surface in Task 1c.

Jackson: [Reading the task] Can you relate the computation you did in the previous part to the figure below? (Figure 1, bottom-left) … the purple line represents our gradient, and the red line, the unit [sic] vector $\langle 1,1 \rangle$ maybe, and there's a green bar between the two.

Interviewer: You said the purple line represented the gradient, but the gradient only has two components.

Jackson: Hmm, okay. So, is the red line in the *xy* plane? Yeah, okay, … so that should be the gradient … but the direction also had two components as well. So, what's the purple line?

Interviewer: Well, it's in the tangent plane. That's a hint.

Jackson: It's in the tangent plane. It's not the gradient … So, if there's a change in z…

Interviewer: The tangent plane is made up of tangent lines.

Jackson: … Okay, so that's a tangent line. Alright. So, I'd say, I assume the red line is the gradient, and the purple line is the tangent line.

Jackson did not realize that the "red line" represented the direction vector $\langle 1,1 \rangle$. This showed once more that recognizing the geometric representation of a direction vector was a challenge for him. That is, the relation F-DD he constructed did not include the geometric representation of the direction vector. Furthermore, Jackson was unable to move from knowing that the purple line is a tangent line to thinking about the slope of the tangent line. He seemed not to think of directional derivative as a slope, with vertical change represented by the length of the green segment (vertical change) and horizontal change as the length of the red line segment (magnitude of the direction vector $\langle 1,1 \rangle$). Indeed, he never used the term "slope." That is, the component S was not leveraged as a mediator. So, recognizing that the directional derivative is a measure of inclination, as Jackson did by passing his finger over the edge of a 3D-printed surface in Task 1c, did not immediately translate into him recognizing that it is a quotient of vertical change over horizontal change, as he was expected to observe in the triangular diagram of Task 2b.

In Task 3, given the graph of a function, Jackson was asked to approximate the direction $\vec{v}$ in which the directional derivative $D_{\vec{v}}f(2.5,1.5)$ appears to attain its largest possible value.

Jackson: … So, for its maximum largest value, the $\vec{v}$ would be in the same direction as the gradient. So, at the point on this surface [pointing to $(2.5,1.5)$ on the graph], the gradient, … will be pointing up from that point. Hmm! Now, I'm gonna switch over to the surface to see … right here. Yep, I believe this is where the gradient is pointing, and that direction … would have an *x* and *y* coordinate. So that direction is … $x = 0$ and $y = 1$. Also, put me in the $\hat{\jmath}$ direction or $\langle 0,1 \rangle$ direction, I guess.

Observe that in this situation, Jackson had no trouble going from the graphical representation of a two-variable function to the graphical and symbolic representations of its 2D gradient vector $\langle 0,1 \rangle$. He had previously (in Task 1b) shown some difficulty physically representing the 2D direction vector on a 3D-printed surface, and (in Task 2b) initially represented the gradient as a vector tangent to a given surface (the tangent plane).

Interviewer: Okay, how about the smallest possible value?

Jackson: [The] smallest possible value would be perpendicular to the gradient. So, then I'd say $\langle 1,1,0 \rangle$, maybe at that point.

Interviewer: Why should it be perpendicular to the gradient?

Jackson: Because, I guess, for the directional derivative, it's the gradient dotted with the direction we're moving in, and to obtain the largest value possible for the directional derivative, Del *f* [the gradient] and U [a unit direction vector] would be in the same direction, since we're using the dot operator. And for it to be the smallest value 0, or no! The smallest value would be -1. So, it should be the opposite direction, not perpendicular. Sorry. So, if I said $\langle 0,1 \rangle$ before, it should be $\langle 0,-1 \rangle$ ...

Jackson started by relating the function (F), directional derivative (DD), and gradient (G) components and justified the relation by arguing, "to obtain the largest value possible for the directional derivative, Del *f* and U would be in the same direction, since we're using the dot operator." So, in this task, the relation F-G-DD is established as a transformation relation.

However, note that he initially gave a 3D vector ("then I'd say $\langle 1,1,0 \rangle$") as the direction vector in which the directional derivative is the smallest possible value. This shows an inconsistency in how he represented direction vectors, since he did not use three components in his previous response. He then argued that the smallest possible value occurred when a dot product is 0, before recognizing that the dot product could be -1. His argument to discuss the largest and smallest directional derivatives, as well as his justification of the relation F-DD-G, resulted from an algebraic consideration of directional derivative as a dot product. This suggests that, in this task, Jackson did not relate the component of Slope (S) with the other components in F-DD-G. He seemed to primarily think of directional derivative as a dot product rather than as a slope. In the student's mind, the algebraic consideration took precedence over the geometric interpretation of directional derivative as the slope of a tangent line.

In Task 4, Jackson was given a three-variable function $H(x,y,z) = z - f(x,y)$ and a point P on the surface $z = f(x,y)$ (see Figure 1, bottom-right). He was asked to approximate the direction in which the gradient vector $\nabla H(P)$ points.

Jackson: … Oh, okay. So that if I'm evaluating *H* at *P*, I assume that would mean the *z* [sic] value is just equal to 0, then and $\nabla H(P)$ should point in the same direction [referring to the direction $\langle 0,1 \rangle$ of Task 3] …

Since *P* is on the surface, then $z = f(x,y)$ and so, $H(P) = 0$. However, Jackson confused *H* with the two-variable function *f* and said, "that would mean the *z* value is just equal to 0." In order to stress the difference between *H* and *f*, the interviewer asked:

Interviewer: So, *H* is a function of how many variables?
Jackson: Hmm, okay, sorry. *H* is a function of 3 variables. So, its gradient would have three components. Then, alright. What would that third component be then, hmm? At the point … *z* is equal to 0 [still confusing *H* with *f*]. So, I'm gonna try plotting the point on the graph as best as I can. So, 2.5, 1.5, and the *z* value for *H*, I believe, is 0. [Looking at the graph of $z = f(x,y)$] Hmm, oh, no! [realizing his mistake] First thing I can think of, I feel like the *z* component should be positive [from looking at the graph]. I'm gonna first start by writing ∇H equals, at the point *P*, the first component is 0, *y* component is one, and *z* is what I'm left with to figure out.

When Jackson said, "the first component is 0, *y* component is one," he was still confusing *f* with *H*, as he was using the values previously obtained in Task 3 for f_x and f_y as if they were H_x and H_y. In order to lead him to reconsider, the interviewer asked:

Interviewer:	So, how do you compute a gradient vector?
Jackson:	You take the partial of each component with respect to x, y, z, respectively.
Interviewer:	Okay. Could you compute the partials of H?
Jackson:	Let's see, so H_x should equal $0 - f_x$. So, $-f_x$. H_y should equal $0 - f_y$ … H_z should equal one. Okay. [I'm] just making sure I did that right … I believe ∇H would be $-f_x$, $-f_y$, 1 as its components. So, at the point $(2.5,1.5)$ [it] should be $\langle 0,-1,1\rangle$.

Jackson was eventually able to find the gradient at P ($\nabla H(P) = \langle 0,-1,1\rangle$), but he required scaffolding to distinguish between the gradients of f and H, probably due to the habit of working with two-variable functions. So, we concluded that he was able to establish a correspondence relation between components of functions of two or three variables and gradient. In another interview task, which we do not discuss in this report, Jackson was able to draw the gradient of a two-variable function at a point, given the contour diagram of the tangent plane at the point. At that moment, the interviewer asked:

Interviewer:	… You said that the gradient of *f* was perpendicular to a contour line. Can that help you in the previous problem where you had the function of 3 variables?
Jackson:	… Okay, yeah, so since this is a function of three variables. I guess, instead of level curves, it'd be level surfaces, and the gradient will be perpendicular to the surface. So, at that point, I could try drawing a line directly normal to that point on the surface.

The fact that the interviewer's intervention was needed to help Jackson recall the relation between three-variable functions (and their level surfaces) and the gradient vector suggests that a correspondence relation was established between these two components.

Conclusions

Exploring issues that challenge strong students in multivariable differential calculus can highlight areas where most students need support. Depending on the course and student population, these aspects should be explicitly addressed during instruction. Underscoring some of these challenges is a contribution of this study.

Jackson found challenges with the directional derivative of a two-variable function: representing the 2D direction vector in a 2D rendering of the 3D graph or a 3D physical representation of the graph, interpreting the directional derivative as a slope, and relating the dot product representation of a directional derivative with its geometric representation as a slope.

Jackson encountered challenges with the gradient of a two-variable function: representing the 2D gradient graphically in 3D and recognizing the similarities and differences between 2D contour curves and 3D level surfaces of two- and three-variable functions, as well as their corresponding gradient vectors. This is another contribution of the study, and more so since there is not much written about directional derivative and gradient moving across dimensions.

The interplay between 2D and 3D objects and their representations is key to understanding student challenges. The challenge in representing 2D vectors in a 3D figure or physical manipulative, or of 3D vectors on a 2D rendering of a two-variable function graph, was anticipated by Parzysz (1988). The case of the difficulty in the (3D) graphical representation of a (2D) direction vector was also observed by Martínez-Planell et al. (2015). Similarly, the

graphical representation of a 3D gradient vector relative to a level surface represented in 2D was also a source of difficulty. This needs to be further studied. Relating the symbolic representation of directional derivative as a dot product with its geometrical representation as the slope of a tangent line in a given direction is also not easy for students. The dot product characterization hides the geometric interpretation of directional derivative as a slope. These observations extend the findings of Borji et al. (2025).

A limitation of this study is that we are considering only one student. Our future work will analyze interview responses of other student participants; preliminary analyses suggest that the challenges that Jackson experienced were common to other students. We assumed, although we could have verified, that what was challenging for Jackson was also challenging for most other students. Further studies are needed to corroborate this. Further, there are other aspects that perhaps most students need help with that were not encountered in this case study.

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DIDACTICAL REORIENTATION FOR THE DESIGN OF VARIATIONAL TASKS: THE CASE OF CONTINUOUS EXPONENTIAL COVARIATION

REORIENTACIÓN DIDÁCTICA PARA EL DISEÑO DE TAREAS VARIACIONALES: EL CASO DE LA COVARIACIÓN EXPONENCIAL CONTINUA

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A didactic reorientation is proposed for the design of variational tasks in Calculus courses for engineers, centered on continuous exponential covariation. In contrast to the traditional approach, dominated by the formal treatment of numerical functions, this proposal seeks to recover the dynamic, continuous, and quantitative nature of change. Grounded in Harel's DNR model and in a historical-epistemological analysis, the study introduces Variational Ways of Understanding and Thinking as operational constructs to guide the pre-design, design, and analysis of tasks. Variational and non-variational ways of understanding continuous exponential covariation are distinguished, establishing criteria to promote covariational reasoning. The proposal provides theoretical and methodological foundations to reorient the teaching of Calculus toward a perspective consistent with the dynamic nature of varying quantities.

Keywords: Calculus, Cognition, Curriculum, Undergraduate Education

In university Calculus courses, particularly those aimed at engineering students, the study of variation and change is usually approached through a formal and static treatment centered on real numbers, numerical functions, limits, derivatives, and integrals of numerical functions. This didactic orientation, inherited from Mathematical Analysis, tends to privilege symbolic manipulation over the analysis of the processes of change underlying real phenomena, thereby limiting the development of mathematical thinking for interpreting and modeling situations from the physical and technological world.

In the case of exponential behavior phenomena, previous research has shown that their teaching is commonly structured around formal definitions, prototypical examples, and finally application exercises (Khanh et al., 2020). This instructional pattern represents a break with the historical-epistemological development of Calculus, whose origins lie in the need to understand and quantify continuous change processes, such as non-uniform motion and other variations dependent on time or space (Crosby, 1998; Kleiner, 2001; Jiménez, 2025).

This issue becomes especially relevant in engineering education. Calculus, historically conceived as the mathematics of variation and change, constitutes a fundamental tool for modeling and analyzing physical and technological phenomena. However, pre-university and university teaching of Calculus often follows a formal approach characteristic of Mathematical Analysis, one that is poorly aligned with the practical needs of these students. This discrepancy between curricular objectives and professional demands results in difficulties in understanding, low motivation, and a limited use of mathematical knowledge in applied contexts.

From a historical perspective, the study of variation and change emerged in response to the need to quantitatively understand real processes, such as non-uniform motion, through the analysis of variable quantities and their continuous behavior (Crosby, 1998; Kleiner, 2001). Calculus thus arose as a mathematics oriented toward modeling dynamic phenomena and coordinating covariational relationships between quantities. Nevertheless, this dynamic character

has been diluted in contemporary didactic practices. Variational thinking is frequently reduced to the manipulation of formulas, static definitions of functions, or the construction of graphs through point-plotting procedures, which hinders covariational reasoning and contradicts the continuous nature of change (Jiménez et al., 2022; Vasco, 2003). The dominant curricular conception anchors the study of variation to the treatment of numerical functions, excluding the temporality and dynamicity inherent to phenomena (Jiménez, 2025). From our perspective, this orientation does not foster the genuine development of variational thinking, understood as reasoning about variable quantities within phenomenological contexts.

Several authors have pointed out that the difficulties in developing this type of thinking cannot be attributed solely to individual cognitive factors of students, but are also closely linked to curricular design and prevailing didactic practices (Ímaz & Moreno, 2010). Within this framework, we argue for the need for a didactic reorientation in the design of variational tasks, conceived as a reconsideration of the criteria that organize learning experiences in order to privilege the analysis of variable quantities and their relationships of continuous covariation. In particular, this study focuses on continuous exponential covariation, whose teaching is often constrained by formal and static approaches that fail to reflect the inherent dynamics of the phenomenon (Foti, 2020).

Drawing on Harel's (2008a) conception of mathematical thinking, theoretical constructs are introduced to characterize and promote variational ways of understanding and thinking, with the purpose of providing conceptual and methodological tools to guide the design of learning tasks. The aim of this work is to contribute to the construction of operational criteria for the design of variational tasks on continuous exponential covariation, recovering the dynamic, continuous, and quantitative nature of change. Thus, the research question guiding this study is: What ways of understanding can be characterized as variational, and which correspond to non-variational ways, in the context of continuous exponential covariation?

Theoretical and Explanatory Framework

Harel (2008a, 2008b) states that the primary goal of Mathematics Education is to develop students' mathematical thinking, which consists of a set of mathematical ways of understanding and mathematical ways of thinking. The author defines ways of understanding as the particular cognitive product of a mental act carried out by a person, and ways of thinking as the cognitive characteristics of a mental act performed by a person, revealed through the recurrent actions of their ways of understanding.

This profound reflection developed by Harel regarding the characterization of mathematical knowledge and the fundamental purpose of mathematics teaching, and actively promoted by him in recent years (Harel, 2021), led him to formulate a methodological framework to achieve such a purpose. He called this framework "DNR-based instruction." The acronym derives from the three fundamental principles on which the framework is based: the Duality principle (D), the Necessity principle (N), and the Repeated Reasoning principle (R). Of particular interest here is the principle of Duality.

The Duality principle states that students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they already possess (Harel, 2008a, p. 19). This principle asserts that ways of understanding (concepts and skills) and ways of thinking (practices, dispositions, and beliefs) are interdependent; that is, a bidirectional relationship exists between them.

The theoretical constructs presented in Guershon Harel's view of mathematical thinking provide an operational pathway for promoting mathematical thinking. This perspective has motivated us to define and characterize two new theoretical constructs necessarily related to variational mathematical thinking. These constructs are *Variational Ways of Understanding (VWU)* and *Variational Ways of Thinking (VWT)*, a point that we will revisit later.

Variational Thinking Approach

The approach adopted in this study is distinguished by emphasizing a dynamic way of reasoning about the fundamental ideas that gave rise to Calculus. Among these ideas are the notions of continuous quantity, the understanding of variable quantity, and the treatment of the infinitely small. From this perspective, promoting a type of mathematical thinking linked to variation and change requires clearly specifying its object of study: variable quantities.

The Object of Study of Variational Thinking: Variable Quantities

In various natural and social phenomena, quantities arise that vary over time or space. By their nature, these quantities reflect the inherent dynamism of real-world processes and therefore constitute the central object of variational thinking. Examples of such quantities include time, temperature, pressure, and velocity, among others, all of which can be measured or quantified using the corresponding unit or standard and measuring instrument; that is, variable quantities possess units of measurement (Jiménez et al., 2022).

Variational thinking is concerned with understanding and mathematizing the behavior of these quantities by analyzing how they change, how they relate to one another, and what patterns emerge from their evolution. In this regard, Jiménez et al. (2022) identify four essential characteristics of variable quantities:

- The symbol representing them refers, in the first instance, to the quantifiable quality (temperature, pressure, time, density, etc.).
- This symbol also represents each and every numerical value that the quantity may assume within the phenomenon.
- These numerical values do not change randomly; rather, they follow a coherent temporal or spatial sequence.
- Such a sequence of numerical values may be continuous or discrete, although variational thinking focuses particularly on the continuity of change.

However, mathematical work centered on numerical functions is characterized by an abstract and static treatment. By its nature, a real-valued function of a real variable is conceived as a correspondence relation between two numerical sets, without necessarily requiring reference to real phenomena. Larson and Edwards (2016) express this as follows:

> Let X and Y be sets of real numbers. A real-valued function f of a real variable x from X to Y is a correspondence rule that assigns to each number x in X exactly one number y in Y. The domain of f is the set X. The number y is the image of x under f and is denoted by $f(x)$, which is called the value of f at x. The range of f is defined as the subset of Y consisting of all images of numbers in X. (p. 19)

This definition highlights the purely abstract, numerical, atemporal, dimensionless, and static character of numerical functions. The numerical variables involved merely represent elements of sets, without necessarily being linked to measurements or to the quantification of real-world phenomena. In contrast, the numerical values associated with variable quantities preserve an

intrinsic connection to the phenomenon from which they originate and are inseparably tied to a temporal dimension. Numerical variable values, on the other hand, are selected arbitrarily and lack a meaningful temporal relationship, since within numerical sets the order in which elements are taken is of little importance.

Dynamic Nature of Variational Thinking

Based on a critical analysis of the specialized literature on the subject, we have come to conceive variational thinking as requiring a dynamic way of thinking (Vasco, 2003), one that is directly related to the study of variable quantities. We conceive variable quantities as those perceptible and quantifiable properties or qualities present in real-world phenomena and processes that are intrinsically dynamic by nature. Variable quantities truly vary. They are ubiquitous in the so-called phenomena or processes of variation and change.

Within variational thinking, the leading role is played by variable quantities, which are mathematical objects of a nature essentially different from that of functions. Functions appear only as secondary actors, as one of the emergent tools for the mathematization of variable quantities and of the processes in which they are involved (Jiménez et al., 2022).

In the studies presented by Thompson and Carlson (2017), as well as by Jiménez et al. (2022), the authors conceive variational thinking as consisting of two components: variational reasoning and covariational reasoning, both related to the mathematical conceptualization of variable quantities. In the first case, reasoning and mathematical work are carried out on a single variable quantity along the number line, whereas in the second, simultaneous reasoning and mathematical work are carried out on two variable quantities within the Cartesian plane.

Theoretical Proposal: An Operational Structure for the Development of Variational Thinking

Based on the variational thinking approach and Harel's (2008a) conception of mathematical thinking, we propose that, in order to develop students' variational thinking, learning strategies must be designed to enable them to generate *Variational Ways of Understanding (VWU)* and *Variational Ways of Thinking (VWT)*. We define Variational Ways of Understanding as the particular cognitive product of the mental act of producing dynamic images related to the essential characteristics or properties of variable quantities present in ongoing phenomena of variation and change. Variational Ways of Thinking, in turn, are described as the cognitive characteristics of the mental act of producing dynamic images carried out by the student, revealed through the recurrent actions of their variational ways of understanding.

For example, if a student formulates an interpretation, similar or equivalent to the statement that "the numerical values assumed by the literal representing a variable quantity are associated with a unit of measurement," the student is exhibiting a Variational Way of Understanding, since this constitutes a particular cognitive product of the mental act of interpreting an important characteristic of variable quantities. Furthermore, if the student sustains more than one Variational Way of Understanding, it is likely that they also possess the Variational Way of Thinking that "variable quantities progressively assume numerical values that can be divided into three groups: the current value, previous values, and subsequent values."

For the organization, design, and analysis of instructional strategies or didactic activities, an operational structure of methodological-didactic implications is established, based on three essential stages. This structure has been adapted from Harel's DNR model (2008b).

Stage 1: Predesign

The purpose of this stage is to define the *Variational Ways of Understanding* expected to be promoted in students and to establish the conceptual and cognitive elements that guide the design

of the activities. To identify and characterize the *VWU* to be promoted in students, the following process must be carried out:

- First, the type of covariation to be addressed (linear, quadratic, exponential, etc.) must be characterized, as well as the application contexts in which such covariation manifests itself (physical, biological, economic, technological, among others). This makes it possible to establish the connection between variable quantities and their phenomenological meaning. In other words, the VWU characterize the dynamic development of the process.
- Second, a historical-epistemological analysis must be conducted in order to examine the development of the mathematical concept over time and how it is structured within disciplinary knowledge. This analysis provides a deeper understanding of the ways of reasoning and representing variation that have been significant throughout the history of mathematics.
- Finally, the VWU are formulated as specific configurations of thought expressed through variational language. These VWU are not formulated as formal definitions or declarative knowledge, but rather as more or less precise statements about the quantitative properties of variable quantities. Their function is to represent ways of mentally organizing the quantitative characteristics of the variable quantities involved in situations of change.

In this sense, *VWU* operate as normative orientations for task design, since they guide the selection and elaboration of didactic activities that foster in students a type of mathematical thinking coherent with the continuous, dynamic, quantitative, and relational nature of phenomena of variation.

Stage 2: Design

At this stage, the guidelines that direct the development of didactic activities intended to promote the previously formulated *VWU* are specified. Two fundamental components are defined here: principles for the design of mathematical tasks and principles for the design of technological tasks.

Principles for the Design of Mathematical Tasks

The design of mathematical tasks is guided through the outline of the cognitively logical sequence by which cognitive products involving students' mental acts are obtained, thereby favoring the progressive construction and analysis of the *VWU*. This structuring begins with the identification of the desirable and expected mental acts in students and is translated into statements and actions that guide task development.

Principles for the Design of Technological Tasks

The principles for the design of technological tasks are guidelines focused on coherently integrating digital resources such as GeoGebra, simulators, video clips, among others, in ways that establish a connection between the physical world and the abstract world of mathematics. Technology should facilitate cognitive experiences that not only present information dynamically but also enhance reasoning about the continuous variation of quantities (Swidan et al., 2019).

This approach encourages students to develop the intellectual need to actively interact with dynamic representations, visualizing and manipulating variable quantities in real time, thereby strengthening the construction of variational images coherent with processes of change. In this way, technology becomes a means of fostering the development of variational thinking and modeling phenomena from a visually dynamic perspective.

Once the instructional strategies or didactic activities have been defined, the implementation of the activities with students begins.

Stage 3: Analysis

The purpose of this stage is to evaluate the extent to which the designed and implemented activities succeed, or fail, in promoting the development of *Variational Ways of Thinking* in students within the specific context of study.

The analysis focuses on the mental productions expressed by students in their verbal or written statements, justifications, and arguments. Following Harel (2008a, 2008b), these productions allow the teacher to indirectly infer the common cognitive characteristics that emerge from the mental acts carried out during the development of the activities.

The analysis seeks to obtain solid evidence that students are capable of identifying and formulating the essential quantitative characteristics of the covariation being addressed, based on the joint and simultaneous coordination among the numerical values of the variable quantities involved in the analyzed phenomena.

In this way, the theoretical proposal presented constitutes a new approach to the design of variational tasks, distinct from what has been reported in the literature and inspired by Guershon Harel's pragmatic-pedagogical view of mathematical thinking. Its distinctive character lies in the operationalization of *Variational Ways of Understanding* and *Variational Ways of Thinking*, employed as articulating constructs for the processes of predesign, design, and analysis of tasks aimed at promoting students' variational thinking.

Results

The Predesign Stage as a Methodological Foundation for Identifying Variational Ways of Understanding

In order to address the research question guiding this study, the Predesign stage established within the methodological-didactic implications for the development of variational thinking was carried out. This stage made it possible to construct a solid conceptual and cognitive pathway for identifying, characterizing, and formulating the *Variational Ways of Understanding (VWU)* associated with continuous exponential covariation.

The Predesign stage was developed through three interconnected moments:

1. Characterization of the type of covariation and its phenomenological contexts: the mathematical topic under study, continuous exponential covariation, was defined, and its possible manifestations in physical, biological, and technological contexts were analyzed. This analysis made it possible to establish the connection between variable quantities and their phenomenological meaning, recognizing that the VWU must describe the dynamic, continuous, and quantitative development of the variation process.
2. Historical-epistemological analysis: a study was conducted on the historical development of the concept of exponential variation and of the ideas that structured continuous covariational reasoning. This analysis made it possible to identify ways of reasoning that have historically been relevant for understanding proportional dependence, relative rate of change, and exponential dynamics as a continuous phenomenon. These findings were fundamental for establishing conceptual distinctions between variational and non-variational ways of understanding.
3. Formulation of the VWU as specific configurations of thought: based on the two previous analyses, the VWU were formulated not as formal definitions, but as statements describing quantitative properties of variable quantities in situations of exponential

change. These configurations express how a student might mentally organize the dynamic behavior of a quantity that changes continuously and exponentially. Their normative function is to guide the design of tasks that promote reasoning coherent with the quantitative and continuous nature of variation phenomena.

Variational Ways of Understanding vs. Non-Variational Ways of Understanding Continuous Exponential Covariation

The results of the historical-epistemological analysis led to the identification of a set of Variational Ways of Understanding that may guide the design of variational tasks aimed at enabling students to reason about phenomena exhibiting exponential behavior.

The following distinctions are proposed:

- A Variational Way of Understanding is any form of reasoning centered on variable quantities, their quantitative dependence, and the continuous structure of change.
- A Non-Variational Way of Understanding is any form of reasoning that departs from this criterion by privileging numerical functions or relationships between numerical variables without explicit reference to the quantities involved in the phenomenon.
- Although some non-variational ways of understanding may be mathematically correct, they do not foster variational or covariational reasoning because they do not necessarily capture the continuous dynamics or the quantitative relationships among the quantities involved (see Table 1).

Table 1: Non-Variational vs. Variational Ways of Understanding Continuous Exponential Covariation

Non-Variational	Variational
• Equal arithmetic increments in the independent variable of an exponential function lead to equal proportional changes in the variable y.	• A covarying quantity with exponential behavior, undergoing equal arithmetic variations in the independent variable quantity, leads to equal proportional changes in the dependent variable quantity.
• In an exponential function, if h is the change in the independent variable, then $a^h - 1$ is the proportional change in the dependent variable.	• A covarying quantity with exponential behavior $N(\Delta h)$ undergoing equal arithmetic variations in the independent variable quantity, leads to proportional changes in the dependent variable quantity of the form $N(a^{\Delta h} - 1)$.
• The exponential function is a continuous function whose relative rate of increase depends specifically on the increment produced from any given value of x. Knowing this rate makes it possible to determine the base of the exponential function	• The average relative rate of change of a covarying quantity with exponential behavior, for equal increments in the independent variable quantity, is equal to a nonzero numerical constant.
• Exponential function as a differentiable function satisfying the differential equation $y' = ky$	• A covarying quantity whose instantaneous rate of change is directly proportional to the current value of that variable quantity, $\frac{dy}{dx} = ky$.

Conclusions

The characterization obtained provides a foundation for designing tasks aimed at promoting variational thinking regarding continuous exponential covariation. The identified Variational Ways of Understanding make explicit: the essential quantitative properties of the phenomenon, the continuous structure of exponential change, and the dynamic interdependence between quantities.

In contrast, non-variational ways of understanding reveal forms of reasoning detached from the dynamicity required by variational thinking (Jiménez et al., 2022). That is, they are mathematical ways of understanding centered on the properties of numerical functions and abstract variables; therefore, they are not characterized as variational.

From these distinctions, three implications for task design emerge: (1) prioritizing the analysis of real phenomena that allow the identification of variable quantities whose variation is observable and quantifiable; (2) promoting the construction of meanings grounded in intellectual necessity, in coherence with the Necessity principle of the DNR model; and (3) incorporating dynamic representations and technological tools that support the continuous visualization of covariational behavior.

Taken together, the results show that the Predesign stage not only made it possible to answer the guiding research question, but also provided a methodological framework for translating formal conceptual distinctions between variational and non-variational ways of understanding into operational criteria for the design of tasks on continuous exponential covariation.

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A PATTERN-BASED APPROACH: IDENTIFYING AND LEVERAGING THE PRESENCE AND PREVALENCE OF LEARNERS' WAYS OF THINKING

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A key dimension of classroom teaching is attending to the shared needs of groups of learners—deciding how to use time and resources to support collective learning. It is vital, then, to identify approaches that actually leverage the group nature of classrooms—surfacing patterns in the ways-of-thinking students bring to instruction, which can be used to structure rich reflective discussions and guide classroom activity. This theoretical paper situates and introduces a pattern-based approach*, which comprises a range of classroom activity types—from open-ended* generative activities *to closed-form* pattern-based questions (PBQs)*. We compare the pattern-based approach to Peer Instruction (PI), an instructional innovation that also leverages group-centered design. We then describe a use of the pattern-based approach in Calculus I, showing how instructors can use PBQs to elicit students' ways-of-thinking as resources for instruction.*

Keywords: Assessment, Calculus, Classroom Discourse, Instructional Activities and Practices

Perhaps the most salient feature of classroom learning is that it occurs in a group. However, many educational innovations treat the size and diversity of classroom groups as a *liability* to be overcome, while others see it as an *asset* to be leveraged. Here, we introduce a *pattern-based approach* to integrating group-centered assessment and instruction—which takes the latter view—and we illustrate its application in Calculus I. From the pattern-based perspective, the diversity of thinking in a classroom group offers a rich ecology of ideas, in which robust, multi-faceted ways of thinking can be identified (Stroup et al., 2023). Discussion and responsive instruction that put these ideas in conversation can then allow all learners to develop more robust understandings. Our goal in this paper is to illustrate how the pattern-based approach can complement other "assessment for learning" (Black et al., 2004) methods. To situate our approach, we first review literature on formative assessment and active learning, including work that has directly supported Calculus reform.

Literature Review: Formative Assessment and Instructional Practices

Assessment for learning is now well established. Atasoy and Kaya (2022) note that in recent years "the assessment process has evolved from evaluating students' understanding by comparing their achievement with others to organizing instructional practices to improve teaching or learning in classrooms" (p. 1). This trend follows Cowie & Bell's (1999) proposal to define formative assessment as "the process used by teachers and students to recognize and respond to student learning in order to enhance that learning, during the learning" (p. 32).

Integrating with the Course's Success Metrics, Culture, and Instructional Activities

Still, Hansen (2020) notes how important it is for instructors to be explicit about their intentions with formative assessment practices, including both their place in course grading and in classroom culture. On the culture front, research suggests that assessment practices focusing on *ideas* rather than exposing or embarrassing students are more effective: "For students to feel confident and safe in their classrooms, their contributions (e.g., their responses, ideas, and explanations) should be carefully considered to avoid embarrassment (Chin, 2004; Hodgson,

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2010)" (qtd in Ruiz-Primo, 2011, p. 20). Similarly, Wiliam (2011) finds that classroom feedback is less effective when it "cue[s] attention beyond the task (for example on the self)." (p. 6). Ruiz-Primo (2011) outlined criteria for "effective assessment conversations." Such discussions are "supportive tools of social participation and social cognition" (Ruiz-Primo, 2011, p. 18). Similarly, "in assessment conversations, it is important to focus not only on the 'cognitive' feedback but also on the 'affective' feedback" that students take away from the interaction (Ruiz-Primo, 2011, p. 21). These and other attributes support classroom assessment cultures where students attend to and build upon ideas of others (Sadler, 1989).

Formative assessment activities can be particularly effective when they connect closely to the instruction of the day, leveraging students' elicited "ways of thinking" as resources for instructional interaction. This aligns with key findings from Black & Wiliam (1998) that for formative assessment to be effective, it had to be integrated with classroom instructional practices (Wiliam & Thompson, 2008). In particular, according to Black & Wiliam (2009) well-designed formative assessment offers teachers the opportunity to "creat[e] and capitaliz[e] upon, 'moments of contingency' in instruction for the purpose of the regulation of learning processes" (Black & Wiliam, 2009, p. 12; and cf. Wiliam & Leahy, 2007). In this way "dialogic formative assessment interactions," are "characterized by the conscious discovery of novel information about student understanding" (Ruiz-Primo, 2011, p. 16), for the purpose of using that information to guide teaching and learning.

Active Learning and the Calculus Context

Aligned with assessment for learning, in the literature on Calculus Reform, there are widespread recommendations to integrate *active learning* approaches (Freeman et al., 2014; Hsu et al, 2008; Laursen & Rasmussen, 2019; Laursen et al., 2014; Saxe & Braddy, 2015; Smith et al., 2021; Theobald et al., 2020). Ström et al. (2021) present active learning as a broad term encompassing a range of interventions that foster "teaching methods and classroom norms that engage students in sense-making activities." (p. 147). More specifically, active learning practices aim to promote: (1) students' deep engagement in mathematical thinking, (2) peer-to-peer interaction, (3) instructors' interest in and use of student thinking, and (4) instructors' attention to equitable and inclusive practices. (Laursen et al, 2014; Laursen & Rasmussen, 2019; Rasmussen & Kwon, 2007). It is evident, then, that active learning intersects with formative assessment (items 2 and 3) as well as classroom culture (items 1 and 4).

As Calculus instructors implement active learning approaches, they can encounter barriers in integrating these techniques with the assessment practice of the course (Ehrenfeld & Mark, 2024; Uhing et al., 2021) and with broader instructional patterns (Reinholz, 2015; 2016). Nevertheless, Miller et al. (2006) suggest that even a modest inclusion of good questions and peer discussion can improve Calculus I outcomes, and Cronhjort et al. (2013) present an experimental study showing that active learning can be effective in this context.

A Historical Point of Comparison: Peer Instruction (PI)

Our pattern-based approach supports the four principles above defining active learning, and it does so by adopting a particular focus on the classroom group. This is consistent with, though not necessarily entailed in active learning, and in formative assessment or assessment-for-learning more broadly. We do, however, recognize a focal attention to group-centered learning and teaching (cf. Shepard, 2020) in Peer Instruction (Mazur, 1997), a specific active-learning approach that has had substantial impact. Comparing our approach to PI reveals both similarities and differences, each of which highlights essential features of our pattern-based approach.

Peer instruction was developed in response to a problem of practice that emerged in physics

education in the 1980s and 1990s. Accomplished professors at leading universities administered the Force Concept Inventory, or FCI (Hestenes et al., 1992)—a multiple-choice instrument designed to test for Newtonian understandings of physics—and they were surprised to find that their instruction did not generally alter their students' "common sense beliefs" (Halloun & Hestenes, 1985, p. 1043). Instead, results showed that their students retained ways of thinking consistent with once-dominant paradigms (e.g., Aristotelian perspectives) at odds with normative scientific explanations central to today's physics instruction. Following a notion that such "misconceptions" need to be "corrected early" in instruction, Peer Instruction, or PI (Lasry et al., 2008; Mazur, 1997) was developed as a pedagogical approach that accommodated the constraints of teaching in large-lecture-hall settings common in introductory STEM courses.

Schematic Overview of a Peer Instruction Episode

PI has been implemented widely, with a wide range of fidelity (Vickrey et al., 2015), largely at the postsecondary level, but in a wide range of settings (Crouch et al, 2007; Fakoya et al., 2023; Lasry et al., 2008; Yildirim & Canpolat, 2019; Zendrato et al, 2025) and with various extensions or embeddings into instructional sequences (e.g., Freeman et al., 2014; Michinov et al., 2015). Here, we provide an image of the activity structure of the core innovation, along with the rationale for its components.

A PI instructor poses conceptual questions to the class, at key moments in a lecture (Crouch & Mazur, 2001). These could be the original "ConcepTests" (Mazur, 1997) or other conceptually focused multiple-choice questions. PI questions are designed to provoke a diversity of responses; often, the correct answer does not receive even a plurality of "votes." A publicly displayed bar graph of responses demonstrates to the students that there is no consensus; and the next ("peer") phase of the activity leverages this fact. Students are asked to "discuss their answers" with "someone in the class with a different answer" (Mazur, 2009, p. 51). After the discussion period, the question is repeated, and the new distribution of responses is shown. The instructor may then also choose to engage directly with "misconceptions" associated with incorrect responses. There are several design features of PI that we find notable (and laudable):

Treating students' thinking as significant. PI elicits and engages with the diversity in students' ways of thinking, as a resource to drive instruction. Its questions are designed to evoke diversity, and peer interaction then leverages that diversity. Students are viewed as capable agents with reasonable viewpoints that they can articulate and argue for.

Using aggregate responses as a public artifact to motivate discussion and action. A key feature is the public but anonymous display—which then motivates the instructional sequence. For students, seeing the bar graph creates not only a sense of problem (we do not all think in the same way), but also a sense of community (whether or not I am correct, I am not alone).

Presenting attractors, versus distractors. In the original design, multiple-choice response options are not designed as "distractors" to trick students into answering incorrectly. Rather, like FCI items, they are created as "attractors," appealing to students' stable ways of thinking about phenomena. Thus, in responding, students are identifying with coherent or reasonable conceptualizations. The discussion and debate among alternatives engages with substantive conceptual issues, rather than the "test taking skills" associated with learning to avoid distractors.

The Pattern-Based Approach

This paper presents another group-centered approach to integrating assessment and instruction: the pattern-based approach. This activity structure is tailored for smaller, discussion-oriented classrooms, where it offers strategies to blend what Cowie & Bell (1999) describe as "planned" and "interactive" modes of formative assessment. A key objective is to use

questioning to elicit instructionally relevant patterns among students' ways-of-thinking (cf. Carpenter et al.,1996; Lesh et al., 2013). Below, we describe how this has been achieved with standardized Pattern-Based Questions, or PBQs. Although the pattern-based approach is certainly not restricted to using closed-form questions, a comparison with Peer Instruction's multiple-choice questions helps to illuminate the distinctive nature and potential of PBQs.

What are Pattern-Based Questions (PBQs)?

On its surface, a PBQ is very similar to a multiple-choice, multi-select ("choose all that apply") item. However, PBQs use this format to elicit more information, and more useful information, about groups of learners. Given a question with four options—A, B, C, and D—both multiple-choice, multi-select items and PBQs allow students to select more than one option (e.g., "A and C" or "B, C, and D") permitting 15 combinations (or 16 if "no response" is permitted). Thus, in theory, an optimally designed PBQ could present students with a prompt to classify themselves into one of 16 instructionally relevant patterns in ways of thinking about a topic.

Now, the goal of a *traditional* item is to determine whether or not students are thinking about a topic in the normatively correct manner. Thus, for traditional design, the response-option space contains a single correct pattern (whether a single response-option or an option-combination). In contrast, PBQs aim to make greater use of the combinatorial space of possible responses. They aim to capture the variety of different ways that students are currently thinking about the topic being assessed, in order to put these ways-of-thinking into conversation. In this regard, the design of PBQs have some features in common with Diagnostic Classification Models (Rupp & Templin, 2008; Sessoms & Henson, 2018; Williamson, 2023) and the related Cognitive Diagnostic Assessments (Sun & Suzuki, 2013), which also aim to interpret the response-option space in terms of ways-of-thinking. Importantly, for a given response option in the context of the set of four response options, a student's decision *not* to select that option is potentially as informative as their decision to select it. Thus, every student who responds is contributing four interrelated informative components about their way of thinking (that is, the shorthand "A and C" actually consists of "A, and not-B, and C, and not-D," and the question designer can make use of this combinatorial space to reveal meaningful patterns in the group.

Leveraging the Public Display of Responses

A PBQ invites students to identify ways of thinking that they resonate with, or find compelling, among the response options. The appropriate public display for PBQ responses is thus a bar graph of all possible response combinations of response options (see Figure 1). The 15-bar display depicts the presence and prevalence of ways-of-thinking in the responding group, providing valuable information for both teachers and students. Students see there are others in the class who think the way they did (cf., Minstrell, 2000; Minstrell & Stimpson, 1996). The display also concretely manifests the diversity of thinking present, spurring reflection and discussion. In particular, we have found that students are often interested in the "other bars"—different from their own—and are enthusiastic to conjecture "how someone might be thinking" who selected a given combination of options.

Schematic Overview of a PBQ Discussion

Comparing the pattern-based approach with Peer Instruction reveals both connections and contrasts along the "Features of PI" that we highlighted above.

Treating students' thinking as significant, *but* replacing a misconceptions approach with an asset-based perspective. Like PI, PBQs take students' ways of thinking seriously, evoking diversity in order to encourage students to articulate their thinking and viewing this as a

rich opportunity for learning and identity development. However, PI takes a misconceptions approach, engaging student ideas in order to problematize and ultimately root them out. In contrast, PBQs take an asset-oriented perspective to the diversity of students' ways of thinking, viewing their current conceptions as resources and connecting new concepts to prior thinking.

Using aggregate responses as a public artifact to motivate discussion and action, *but* exploring connections between ideas, rather than casting them as oppositional. The public display of student responses in a PI episode "calls out" persistent misconceptions, and the approach does not highlight how these misconceptions might be hybridized or connected (Smith et al, 1994). This is by design; built into the nature of the questions and the practice of engaging in oppositional debate is the idea that misconceptions must be confronted to be open to change. In contrast, PBQs are designed to enlist students in identifying connections between responses and among the different ways-of-thinking that appear in the class. We describe these patterns and connections with a musical metaphor: conceptual chords.

Presenting attractors, versus distractors, *but* encouraging students to conceptualize connections as routes for growth. PI divides students' perspectives into distinct "buckets"—the options of the multiple-choice question. A Kuhnian (1970) view of incommensurable scientific paradigms underlies this design, paired with a pedagogical theory of conceptual change that holds students' thinking can shift dramatically when confronted with contradictions in their conceptual systems (Strike & Posner, 1982; Longfield, 2009). Polarizing the class's thinking into categories prepares them for debate as a confrontation between paradigms. In contrast, we construct PBQs under a view that learners build new ways of thinking by reorganizing aspects of their prior conceptions into new structures (Piaget, 1972; Vygotsky, 1978). Thus, we view different patterns as milestones in pathways of conceptual growth. The pattern-based approach evokes diverse ways of thinking in order to create opportunities to put students in positions of agency, articulating connections that they see among this diversity.

Instructional Vignettes from Calculus I

In this section, we describe several instructional sequences responsive to student answers to PBQs. These occurred in a Calculus I course taught by the first author (C. B.) in Fall 2025. We focus on examples where students' response patterns revealed different ways of looking at mathematical objects, and on how this diversity could support and structure interactive discussions of patterns of thinking.

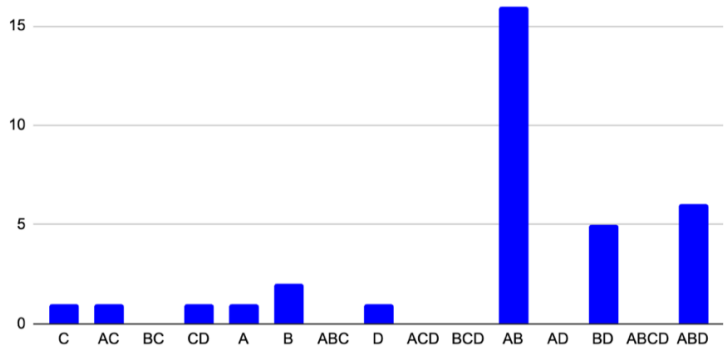


Figure 1: Response Display for a Four-Option PBQ

Note. The target combination, "ABD," is at far right, with other combinations ordered by greater or lesser distance from the target.

For example, given the response display in Figure 1, Author 1 focused first on option "B," shared by all of the four most common combinations (see Figure 2 for the PBQ). The discussion of this bar surfaced two compatible but distinct perspectives on "ending further forward than they started"—an "integral" one, based in accumulating distance, and a "compensation" oriented one, viewing the positive and negative parts of the graph as canceling each other out, with the positive in excess. This distinction helped to reveal the different ways of thinking behind "AB" and "BD." "AB" attends to the meaning of points on the graph as giving readings of velocity at particular times, while "BD" attends to accumulated distance as the signed area between the curve and the horizontal axis, which can be calculated using triangle formulas from geometry. Both are valued perspectives, and though the one associated with "BD" was a focus for the lesson of the day, the discussion helped to stabilize some of the multiple interpretations that can be derived from a velocity graph. One note about the response display is that only one response out of 34 (i.e., "C" only) lacks any correct option. Moreover, discussion of "C" revealed that it operated not *just* as a "distractor"—it included an ambiguity of language, in which "returns to where they started" could be seen to mean, turning around to head back to the starting line.

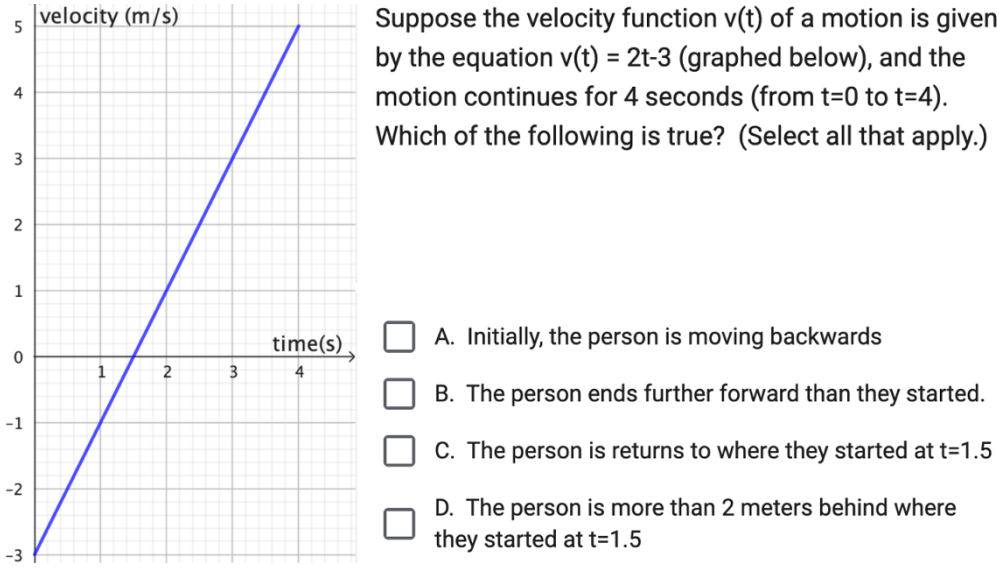


Figure 2: PBQ Producing the Responses in Figure 1

Another facilitation approach we have observed highlights a "popular" response combination that pairs one correct option with one that is incorrect. This was an appropriate strategy for orchestrating discussion of the PBQ responses shown in Figure 3, below (see Figure 4 for the PBQ). Noting that "quite a few of us" chose this combination, the instructor can ask about the reasoning. Any student can respond, either describing their own thinking or speculating what someone who chose this option "might have been thinking." Later, the teacher might solicit connections between this "bar" and another one. Through such discussions, students are invited to express their thinking in relation to others' thinking, as reflected in the display. The discussion is valuable both for students who are struggling with the focal topic, and also for those who have mastered it, who then have opportunities to articulate their current thinking and connect it with

prior ways of thinking. We often use PBQs at the start of an instructional session, since these discussions set the stage for later activities.

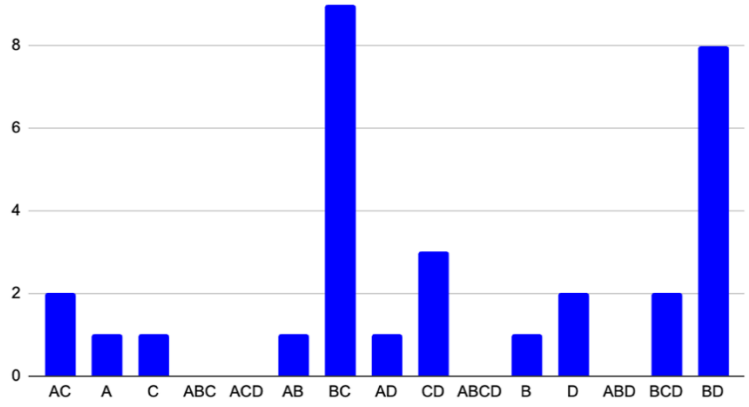


Figure 3: Responses to the PBQ of Figure 4

In the case of this PBQ, the "BC" combination suggests that students were attending to the behavior of the denominator and not of the numerator at x=1. The class had recently covered L'Hôpital's Rule and one student mentioned this as an approach, leading them to select A. Though incorrect, this response reflected noticing something important, and it also gave an opportunity to distinguish between the *value* of the function and its *limit* as x approaches 1. Finally, discussion of option D allowed an opportunity for students to recall the behavior of the logarithm, building on the initial discussion of the numerator's behavior at x=1. In this case, the focus was much *less* on the correct answer and much more on the kinds of thinking needed to detect and describe indeterminate limits and removable discontinuities.

Consider the function f(x) below. Which of the following are true? (Select all that apply.)

$$f(x) = \frac{\ln(|x|)}{x-1}$$

- ☐ A. f(1) = 1
- ☐ B. f(1) is not defined
- ☐ C. f(x) has a vertical asymptote at x=1
- ☐ D. f(-1) = 0

Figure 4: PBQ Producing the Responses in Figure 3

A third discussion approach can leverage situations where the majority of students are finding correct responses, but few are attentive to *all* of them. In Figure 5, below, all of the singleton correct answers have representation, as do the doubletons (see Figure 6 for the PBQ). In this setting, the discussion can work to support students in recognizing and "trying on" multiple different ways of looking at the mathematical object. Often, problem statements can frame mathematical thinking so that one way of looking appears to be the only correct or productive one. Yet having multiple ways at one's disposal can be a powerful problem-solving and sense-making resource. In this case, Author 1 was surprised to see that the bounds on the value of $f(x)$ were the least salient or accessible of the features involved, while even-odd-ness was more salient (especially given the exponentiation involved). Yet this offered opportunities

for students to express how they thought about properties of *products* of functions, across all these features (parity, periodicity, and value).

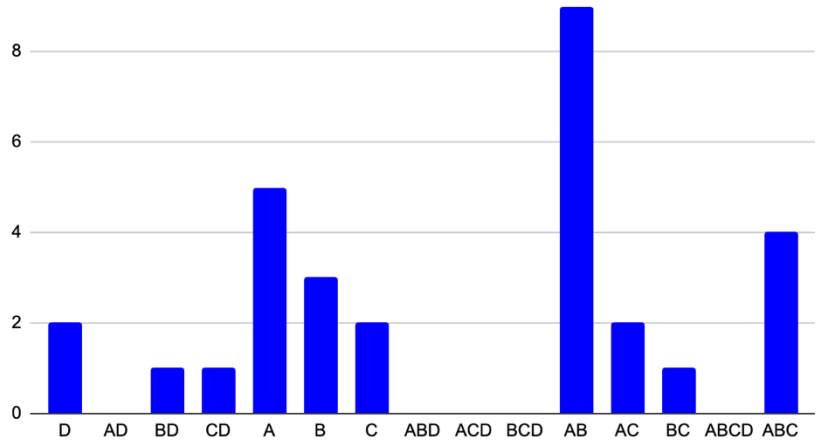


Figure 5: Response Display for the PBQ in Figure 6 (the target response is "ABC")

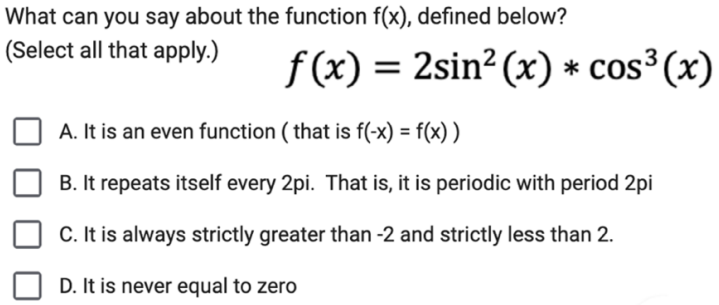


Figure 6: PBQ Producing the Responses in Figure 5

Connection to the Lesson and to Other Pattern-Based Activities

In this Calculus I class, PBQs were the opening activity of each instructional day, so that the "lectures" of the course could often be seen as unpacking and elaborating on the ideas raised in these discussions. Moreover, the PBQs paved the way for more open-ended pattern-based activities, called *generative activities* (Stroup et al., 2005), in which students construct mathematical objects that express given behaviors or features. In the case of the three PBQs above, these were: "Create a velocity graph so that the person ends up exactly where they started;" "Create a function that has a removable discontinuity at x= -3;" and "Create an even function that is the product of two functions that are not even." These generative activities show some of the "power" of the ideas explored in the PBQs, as the target properties are exemplified in entire families of objects, all created by the students.

Discussion and Conclusions

The pattern-based approach operates at the level of *groups*, exploring the presence and prevalence of distinctive patterns of thinking, and orchestrating a dialogue among them. As a result, it builds upon seasoned educators' knowledge and experience of the texture of students'

thinking as they develop increasingly sophisticated understandings. Because PBQ responses reflect *how* students think, rather than only *how correctly* they think, PBQs promise to elicit and convey more information than traditional closed-form assessment tools, as well as information that could be more useful for guiding instruction (cf. Shephard, 2016). The patterns of thought that appear in a group can be directly engaged in discussion, involving students in meaning-making around their own learning and encouraging them to connect their own ways of thinking with those of others in the group. These discussions can give opportunities for more voices to contribute; and animating key conceptual ideas can help to personalize the process of learning.

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ON THE WAYFARING PATHS OF EMERGENT LEARNING IN AN ENACTIVE APPROACH TO UNDERSTANDING POLAR GRAPHING

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The research reported here seeks to make contributions to precalculus education by developing the curricular and pedagogical components of an enactive educational ecosystem that enables the emergence of a graphical understanding of the polar coordinate system (PCS). Students in a doctoral-level course in mathematics education are wayfaring from inquiries to insights about the qualities and behaviors of polar graphing as they reason through pole-based choreographies of movement and material interactions with conventional and invented graphing tools. In this Brief Research Report, we share preliminary findings from analyses of their wayfaring paths which reveal the shape of polar graphing problem solving and suggest configurations of tools, tasks, and practices that enable the emergence of its understanding. These findings have implications for the design of enactive approaches to teaching for emergent learning.

Keywords: Cognition, Curriculum, Design Experiments, Learning Theory, Precalculus

The polar coordinate system (PCS) is typically introduced in Precalculus, where it provides a foundation for undergraduate coursework in mathematics, physics, engineering, and other STEM fields. Regrettably, students' engagement with it is often limited to procedural tasks, such as plotting points, rote memorization of the relationship between the shapes of graphs and the parameters of their polar equations, and converting these equations between Cartesian and polar forms. And too little attention is paid to how radius and angle measure co-vary in the graphs of polar equations (e.g., Moore et al., 2014; Paoletti et al., 2013). This project seeks to remedy the anemia of this conventional approach by developing an enactive (Thompson, 2007; Varela et al., 1991) educational ecosystem for emergent learning that enables learners to graph and reason within the PCS. Prior research affirms the viability of this approach (Greenstein et al., 2024a, 2024b; Olson et al., 2024).

As we conceive of it, a graphical understanding of the PCS (or "polar graphing") entails an understanding of how radius, r, and angle measure, θ, covary to trace out polar curves, as well as a familiarity with polar families and the relation of their parameters to qualitative features of their graphs. The theory of change at work here is as follows: Conceptual knowledge (i.e., know-what; Varela, 1992) of polar graphing can emerge for the learner in fields of promoted sensorimotor interactions (Abrahamson & Sánchez-García, 2016; Abrahamson & Trninic, 2014) with the physical tools that afford embodied doing (i.e., know-how) associated with target concepts. Thus, our objective is to assume an enactive and materially agential approach to knowing in order to determine just what those tools and interactions look like. Through experimental, exploratory, and emergent inquiry, we are testing the conjecture that it is possible to teach polar graphing through an *embodied choreomathematics* that develops one's dynamic understanding of angular motion, rotation, and spatial curvature. The question framing the

inquiry shared here is as follows: *What are the elements of an enactive educational ecosystem that enables the emergence of a graphical understanding of the polar coordinate system?*

Theoretical Framing

The enactive theory of cognition (Maturana & Varela, 1987; Varela et al., 1991; Thompson, 2007) and the theory of inclusive materialism (de Freitas & Sinclair, 2014) frame our investigation of mathematical thinking as a complex and emergent phenomenon constituted in embodied and material engagement (Malafouris, 2013), and in the affect and aesthetics of relational interactions between learners and their environments. These theories reject Cartesian dualisms of mind and body, and take inspiration from phenomenology (Husserl, 1977; Merleau-Ponty, 1962) for its attention to the essence of lived experience. Affirming the compatibility of their epistemological commitments, the enactive approach we are taking is known as "4EA Cognition." 4EA cognition proposes that thinking is inherently *Embodied* (emerging from embodied interactions), *Embedded* (in social, cultural, and contextual situations), *Extended* (beyond bodies into tools, practices, and habits), *Enactive* (in that knowing is a worldly doing of sense-making from which meaning emerges), and *Affective* (driven by an emotional attunement that valences stimuli in terms of meaningful salience; Colombetti, 2013).

From an enactive perspective, knowing is an emergent phenomenon of interactions that arises, is contingent upon, and is shaped through the experiential structures that learners have constructed through histories of environmental interactions. What is critical for discerning the tools, tasks, and practices that lead to PCS understanding when framed from this perspective is that cognition ("thinking") is a distributed and emergent process that draws upon environmental affordances (Gibson, 1986) and the determining structures (Maturana & Varela, 1987) of each learner's embodied being as an active agent. These distributed processes are non-linear and unfolding events where "meanings build on one another, combine, [and] shape and get shaped through interactions" (Proulx, 2019, p. 5) in an emergent logic that is irreducible to any one component, process, or pedagogical practice. In short, if thinking is a distributed ecological phenomenon, then so is polar graphing.

Methods

This project is taking a design-based approach (Brown, 1992; Collins, 1992; DBRC, 2003) to discerning the curricular configurations of tasks, tools, and practices that can engage and develop learners' graphical understanding of the PCS. The exploratory case (Yin, 2014) presented here is of research undertaken in a doctoral-level applied theory course in mathematics education called *Choreomathematics of the Polar Coordinate System*. Students in the course are investigating mathematics as an embodied and aesthetic practice by designing, performing, and analyzing pole-based choreography within polar and cylindrical coordinate systems to develop a dynamic understanding of angular motion, rotation, and spatial curvature. The nine students enrolled in the class are participants in the research. Two (the second and third authors) are participant-researchers on the project, as is the course instructor (first author). In addition to photos and video recordings of students' activity and researchers' field notes (Van Maanen, 1988) about the mediators and phenomenologies of their activity, collected data includes students' journal entries and "line reflections." Journal entries are narratives of students' inquiries (what did you seek to learn?), insights (what did you learn?), and 4EA reflections (what conceptual, social, and material interactions shaped the learning and affect/phenomenology of your inquiry?). Their purpose is for students to draw from their mathematical experiences to cultivate an appreciation

for the embodied and extended interactions that constitute our diverse ways of knowing and learning mathematics.

Line reflections are depictions that students draw of their "path laid down in walking," a poetic phrase that Varela (1987) borrowed to convey the emergent behavior of enactive cognition. Using symbols such as loops, braids, knots, and forks, students' lines are meant to depict the unfolding paths of learning they take in each class as they aim to make sense of the graph of some new polar equation. Their purpose is to elucidate paths of learning that reveal how the dynamic, contingent, and structure-determined interactions of meaningful learning emerge more like entangled trails of wayfaring (Ingold, 2007) than point-to-point transport along a learning trajectory.

We are engaged in thematic analysis (Braun & Clarke, 2006) of the unfolding paths of "PCS knowing" that emerge in evolving assemblages of each group's embodied and extended interactions. The objective of the analysis is to assess and articulate the efficacy of curricular configurations from which PCS understanding can emerge and the pedagogical scaffolds that support them. At the same time, we are attending to students' in-the-moment thinking through phenomenological inquiry using micro-phenomenological methods (Petitmengin et al., 2019) that elicit the embodied, affective, and relational dimensions that often go unnoticed yet are consequential to their learning. What makes the collected data ideal is that learning involves enacted doing, and doing makes thinking visible and open to micro-phenomenological analysis.

Preliminary Findings

In this section, we share excerpts from the journal entries of two students who participate in different groups. Both entries will make reference to Figure 1. The common point of inquiry for the entries is the endeavor to make sense of the polar graph of $r = \theta$. Looking across the entries, we will see mathematical thinking about polar graphing happening across assemblages (de Freitas & Sinclair, 2014) of bodies, tools, diagrams, and discourses. Any insights that students achieve through these mediated interactions lend credence to claims of their efficacy and are thus suggestive of the very tools, tasks, and practices that constitute an effective, enactive educational ecosystem for learning the PCS.



Figure 1: Students Enact the Graph of $r = \theta$ Using a Dance Pole and HDMI Cord (Left), and a Cardboard Disk and Retractable Tape Measure (Right).

Group 1. We did our work at the pole. We wondered how r and θ covary and whether we could *feel* their respective rates of change simultaneously. One of us grabbed an HDMI cord to embody the changing radius, tied it around the pole, walked it out to its full length, which we designated 2π, and marked that point on the floor. We then folded it in half and in half again to locate $\pi/2$, π, and $3\pi/2$. Next, we used tape to make those same markings as θ-values on the floor. Tethered to our *radius*, our goal was to circulate the pole at a constant speed by focusing on *feeling* the constant change in the radius length and arriving at the 2π floor marker at the very same time that we reached 2π on the rope (see Figure 1, left). We each took turn after turn, but no one came close. Looking back and forth between the corresponding markings on the floor and on the rope, I eventually experienced an aha! moment of clarity when I suddenly *feelized* that our step *size* was the missing link! Constant changes in arc length must be reflected in continual increases in step size! This was a *pivotal moment* in our sense-making! We came up with dance steps of increasing size for each quadrant: *baby steps, short steps, normal steps,* and *mega steps*.

Group 2. We did our work on a 6-foot diameter disk that we took to be the polar plane. I began by intuiting that since r and θ were equal, the radius and angle measure would be equal, so then the graph of $r = \theta$ must be a circle. When a group member graphed it in Desmos, I was so confused about why it was a spiral! When a second group member was unable to explain it to me, he proceeded to find a way to embody the relationship using a retractable tape measure. While I held the tape measure at the origin of our polar plane, he extended it outward at familiar increments ($r = 0, \pi/6$, and $\pi/4$) while simultaneously rotating around the disk at arc lengths equal to those same increments (see Figure 1, right). I conceptualized this know-how as "you walk out that much, you turn that much," enacted this concept through my own trace around the disk, and formalized the experience as coordinated increases in distance *from* the origin (r) and rotation (θ) *about* the origin.

What these journal entries reveal are embodied and materially engaged interactions that lead to sense-making. These include forms of movement (e.g., walking, circling, stepping, and rotating) that embody polar equations as dynamic mathematical objects; people-material couplings that enact changing radii and angle measure (via a dance pole, HDMI cord, and floor markings) in addition to polar coordinates and spiral growth (via a large disk, retractable tape measure, and Desmos graph); and conceptual-material entanglements that make the abstract into the actual (as in the "aha!/feelized" and "you walk out that much, you turn that much" moments). There are other affective ("so confused"), discursive (step sizes), and perceptual ("looking back and forth") interactions in there, as well.

Conclusion

The objective of this work is to discern the elements of an enactive educational ecosystem that supports the emergence of a graphical understanding of the polar coordinate system. Preliminary findings of students' wayfaring along pole-based choreographies show how specific choreomathematical configurations foster insights into the qualities and behaviors of polar curves. These findings have theoretical implications relative to how enactive cognition and inclusive materialism can be applied in precalculus through embodied and materially agentive curricular interactions. They also have practical implications that move precalculus curricula beyond procedural PCS instruction toward emergent and embodied learning of polar graphing. As this work continues, we will refine these configurations of tools, tasks, and practices, and supplement these findings with phenomenologies of students' experiences of wayfaring along these emergent paths.

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OBSERVING THE EMERGENCE OF CRYSTALLINE CONCEPTS THROUGH GENERATIVE STRUCTURAL REASONING

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In this brief report, I present an analysis of two students' generative structural reasoning as they worked to analogically develop the quotient ring concept based upon their previous knowledge of group theory. I observed key differences in their reasoning and subsequent creations. Although both students engaged productively with the task, one student began to establish a clearer articulation of what he would need to successfully design the quotient ring concept by appealing to general qualities of quotient objects, thus suggesting that his understanding of the underlying structure was progressing toward a crystalline conception of quotient objects. Implications of this study suggest one mechanism through which students might transition from possessing an instantiated conception of structure toward an emergent a crystalline conception through generative structural reasoning as opposed to analogical transfer alone.

Keywords: Advanced Mathematical Thinking, Reasoning and Proof, Undergraduate Education

In this paper, I present further empirical evidence sharpening previously documented theoretical claims (Hicks et al., 2023) that a student can recreate analogical concepts through processes of comparative and inductive analogy. The specific aim of this report is to delineate the ways in which two students come to observe the rich mathematical structure that underlies the concept of quotient objects in advanced mathematics. This qualitative study seeks to answer the following research questions: *How do two students reason about the underlying formal mathematical structure associated with the quotient ring concept? In what ways does their structural reasoning differ?*

Theoretical Background

This paper takes as its premise that knowledge construction is rarely linear. The development of a rich understanding of a particular concept can take place over time and across a series of (possibly unrelated) episodes that result in a refined understanding of a particular concept. The math education community has documented several cases in which prior knowledge influences new knowledge construction, or vice versa (e.g., diSessa, 1993; Ellis et al., 2022; McGowen & Tall, 2010). The context I describe here involves *generative structural reasoning* through analogy as a contrast to analogical transfer alone. By generative structural reasoning, I refer to reasoning that not only observes and reasons with formal mathematical structure, but also creates formal mathematical structure.

Crystalline and Amorphous Concepts

In a previous report, a colleague and I developed the concept of an *abstracted domain* (see Figure 1) as a theoretical contribution to observing students' structural reasoning through analogy (Hicks et al., 2023). In contrast to a source domain (typically an already known concept) and target domain (typically a new concept being explored), the abstracted domain represents the domain in which the underlying formal mathematical structure of the concepts is formed. Similar to Tall's notion of a concept becoming thinkable, we posited that this domain may not necessarily be recognized by the thinker in the moment of establishing the domain. In this report, I connect our contribution with Tall's notion of a *crystalline concept* to better describe the

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

development of the abstracted domain, and how it might leveraged for analogical concept creation.

A crystalline concept is one that "develops an increasingly sophisticated structure as we compress knowledge and link ideas together." (pg. 3–4) Tall's use of the word "crystalline" purposefully invokes a metaphor stemming from the study of solids possessing highly organized and uniform internal structure. Building on this metaphor, I borrow the term "amorphous" from the study of crystalline solids to introduce the term *amorphous concept*. A solid is called amorphous when it lacks the organizational structure of a crystal, meaning that its internal structure may appear disorganized despite forming a cohesive, observable shape. In the context of mathematics education, I use amorphous concept to refer to formal structures that continue to remain in a stage of development in which connections are yet to be made to become crystalline.

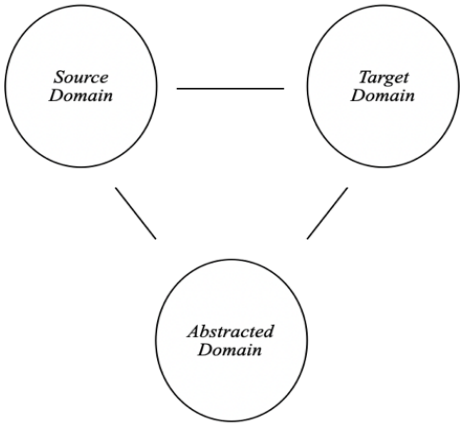


Figure 1: The Abstracted Domain in Relation to the Source and Target

Comparative and Inductive Analogy

Based upon both empirical evidence and theorizing about the nature of the abstracted domain, two forms of generative structural reasoning were identified: comparative analogy, and inductive analogy. Using the relationship displayed in Figure 1 as a template, these forms of reasoning are identified by considering the different activities that might lead to the development of the abstracted domain. In the case of comparative analogy, the abstracted domain is formed by recognizing the underlying structure that exists between two concepts that are compared between the source and target. Inductive analogy, on the other hand, involves identifying the abstracted domain from a single example which is then applied to the target domain to generate new concepts within that domain. Figure 2 displays these relationships using the sub-structure concept as an example.

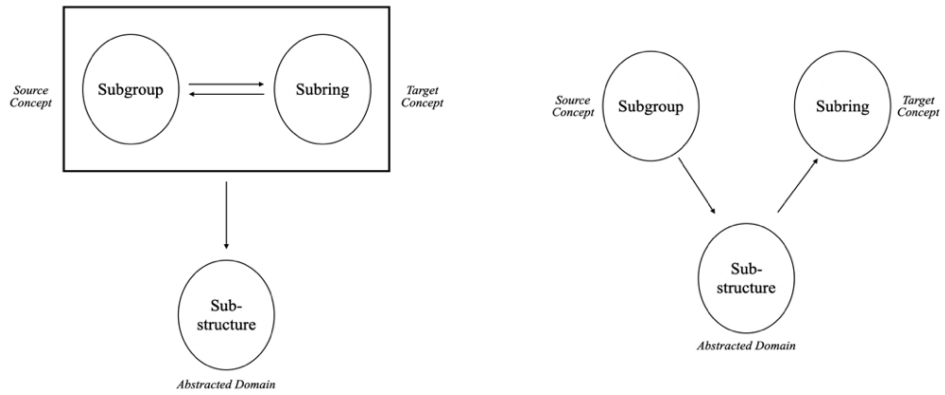


Figure 2: Comparative Analogy (left) and Inductive Analogy (right)

Methods

The data for this study is part of a larger study that was focused on investigating how students reasoned analogically to create structures in abstract algebra. Four students were recruited to participate in a series of five task-based interviews designed to elicit their approaches to constructing several ring-theoretic structures by analogy with what they knew from group theory. Each interview was video- and audio-recorded and subsequently transcribed. The data for this particular study is focused upon just two of the students, and upon just one interview from each student wherein they were asked to construct a structure in ring theory that was analogous to quotient groups in group theory. These students were chosen for this analysis because they exhibited fruitful reasoning associated with developing the quotient ring concept while serving as contrasting cases.

To describe Brandon and Andrew's generative structural reasoning about the quotient ring concept, I scrutinized all relevant events in their interviews in which progress was being made toward developing a formal mathematical structure, including cases of their defining activity, their creation of and checking of examples, and the revision of their ideas based on further exploration and investigation. Profiles of their reasoning were created and then compared with the goal of documenting how and in what ways their reasoning differed significantly. For this brief report, a subset of these results are described below.

Results

As hypothesized at the outset of the study, Brandon and Andrew displayed key differences in their generative structural reasoning. My overarching claim is that Andrew had developed an abstracted domain relevant to quotient objects and displayed evidence of transitioning from an amorphous conception toward a crystalline conception of quotient objects, whereas Brandon's conception remained as an instantiation of a concrete domain, never quite reaching the amorphous state by establishing an abstracted domain.

Variation in Domain Use and Selection

Analogical concept creation requires an established source domain to begin the development of the new concept in the target. In Brandon's case, the selection of source domain was immediately chosen as the quotient group concept, as encouraged by the task design. Afterward, the source domain remained largely static and unchanging throughout the course of the interview. Andrew, however, displayed variation in his selection of source domains. Although Andrew began in the expected manner by attending to the source domain as the quotient group,

he also explored other possibilities to draw upon his construction of the quotient ring concept. In particular, he drew upon the quotient space concept from topology, stating:

> The way I think of it, and this is probably from topology because I took topology first, and there's a thing called a quotient space. And the way we do that is we take points from space and we glue certain points together. That's the way my professor explained it. But I think of it as a quotient group in a similar way. This entire set equals 0, so we're taking all of these numbers and gluing them together to equal some representative number.

Here, not only is Andrew drawing upon another domain, but he is also reasoning conceptually across those domains by adapting ideas in one domain (e.g., "glue certain points" in topology) into the other (e.g., "gluing [numbers] together").

Another case of source domain selection arose in the form of example-based reasoning. I describe this special case here. As suggested by Duit (1991) in the science education community, examples are in and of themselves types of analogies. In fact, each of Brandon and Andrew drew upon the same example to support their reasoning through constructing the quotient ring concept. Despite this, I claim that the nuance in which each of Brandon and Andrew drew upon their examples further reveals differences in their attention to their formation of the abstracted domain and progression toward a crystalline concept.

Brandon's use of examples occurred in two phases. First, as Brandon continued his construction, he began to consider what he could recall about the quotient group $\mathbb{Z}/2\mathbb{Z}$. Almost as soon as he wrote down the example, he scratched it out, stating: "Oh, I'm trying to think about it in groups." The fact that Brandon discarded the investigation of the quotient group example based on the premise that it was related to groups and not rings suggests further evidence that he was not aware, nor becoming aware, of the underlying structure between the quotient group and quotient ring concepts themselves. Furthermore, although Brandon had previously recognized that all rings are examples of groups with a multiplicative operation, he did not leverage that connection here. Upon discarding the example, he returned to his investigation of the quotient ring concept and continued to push the equivalence relation idea further. The second phase of Brandon's use of examples occurred later after finalizing his quotient ring concept, and then being asked to consider the example $\mathbb{Z}/n\mathbb{Z}$ as part of a follow up task. He first considered the meaning of an equivalence class in this context, and chose to investigate the set $\mathbb{Z}_6$. Brandon's inquiry into the nature of the example did not progress from there.

Andrew also investigated the quotient group $\mathbb{Z}/n\mathbb{Z}$ to assist him with his construction. However, Andrew's attention and purpose for using the example was quite different in form and function. First, Andrew attended to the example $\mathbb{Z}/3\mathbb{Z}$ as a quotient group. In doing so, he enriched his recall of key properties of quotient group that he had been struggling to make sense of when working with the general definition, stating: "I always just go back to an example like this [writes $\mathbb{Z}/3\mathbb{Z}$.] So how would I form this? Let's test my definition." Andrew then worked with the specifics of the example to help of the general definition, and in doing so, recalled key properties such as coset, the notion of partitioning, and representative elements. In doing so, Andrew launched his creation of the quotient ring concept by exploring how these recalled properties could be adapted to the ring context. This further evidences the claim that Andrew was successfully coming to establish an understanding of the quotient object structure underlying each of quotient group and quotient ring.

Discussion

This in-depth analysis of Brandon and Andrew's generative structural reasoning further builds upon the ways in which comparative and inductive analogical reasoning might be displayed within students' thinking. In particular, this account of the students' reasoning revealed that, although each student engaged with productive mathematical thinking, Brandon attended only to concrete, instantiated concepts, while Andrew displayed evidence of forming an amorphous conception of the underlying structure of the concept and subsequently progressing toward a crystalline conception of the quotient object concept. A hypothetical categorization thus becomes visible: (a) concrete comparative reasoning in which only the source and target concepts are recognized and separated during the construction of a new concept, (b) emergent inductive reasoning transitioning from comparative reasoning in which an amorphous conception of an abstract domain becomes evident and useful for building new concepts, and (c) fully inductive reasoning in which the abstract domain is realized and leveraged to establish a new concept.

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FROM REPRESENTATIONS TO CONCEPTUAL LAYERS: AN ANALYSIS OF LEARNING OPPORTUNITIES FOR THE DERIVATIVE

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This study examined the learning opportunities provided in a Calculus I class, with a specific focus on lessons related to derivatives. The investigation analyzed a mathematician's classroom instruction and the accompanying textbook to understand the nature of the learning experiences offered to students. We paid particular attention to whether students were given opportunities to explore multiple aspects of the derivative concept. The findings suggest that, based on the lessons and textbook used, students were more likely to have limited opportunities to engage in multiple aspects of the derivative.

Keywords: Calculus, Class Practices, Derivative, Textbooks

Introduction

Calculus is a foundational course for most disciplines in science and engineering, not only in the United States but also worldwide. Due to its importance in undergraduate education, numerous studies have investigated how students learn calculus topics (Hong, 2024; Hong & Choi, 2019; Larsen et al., 2016). Despite the significance of calculus in undergraduate education, across the United States, 28% of students enrolled in Calculus 1 either fail or withdraw from the course (Bressoud et al., 2013). In the United States, three main topics in Calculus 1 are limits, derivatives, and integrals. Of these, the concept of the derivative has been studied frequently (Cooley et al., 2007; Haciomeroglu et al., 2010; Hong & Lee, 2024; Litteck et al., 2024). Many experts in teaching Calculus 1 considered understanding of the derivative as a central concept or a central skill that is fundamental to Calculus 1 (Sofronas et al., 2011). One key finding from these studies is students' inability to flexibly move among different representations of derivatives (symbolic, graphical, and verbal) (Cooley et al., 2007; Haciomeroglu et al., 2010). To better understand the reasons behind students' struggles, two critical areas to explore are classroom instruction and curriculum, which have not been examined often (Larsen et al., 2016). This study is not motivated merely by a lack of prior work, but by the need of deepening our understanding of how opportunities for students to explore different representations are presented in two crucial areas of student experience: the mathematics classroom and the textbook. By situating our analysis within curriculum materials and class practices, the study may contribute to ongoing discussions about how to design learning environments for students to explore multiple representations of derivative. Here are two research questions that guided us.

1. How did the instructional tasks afford opportunities for students to engage with multiple representations and layers of the derivative?
2. How did the tasks in widely used textbook afford opportunities for students to engage with multiple representations and layers of the derivative?

Related Literature

Theoretical Considerations

Duval's theory of semiotic representations (Duval, 2006) states that mathematical objects can only be accessed through semiotic representations. Duval's "registers" refer to systems of semiotic representations, include symbolic, graphical, and verbal forms. Duval distinguishes between two types of transformations involving representations: treatments, which are

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transformations within the same register, and conversions, which involve changing from one register to another without altering the mathematical object itself (Duval, 2006). For example, expanding $(x+3)^2$ to x^2+6x+9 can be a treatment while expressing a quadratic function with an equation, $f(x) = (x+3)^2$, and then graphing it can be a conversion. To build a deeper conceptual understanding, students need opportunities to explore how the derivative can be represented and interpreted across different semiotic registers.

Conceptualizing Layers of the Derivative

In the context of derivatives, students can view the derivative as different layers such as the slope of the tangent at a single point, the ratios, limits, functions (Zandieh, 2000). With the ratio, students can see the derivative as dividing one quantity by another one. When each ratio converges to a single value, you can see the derivative as limit. Finally, when students consider such limits at every point on the interval, the derivative can be viewed as a function. Using this framework, Zandieh (2000) described that many students displayed what she termed "pseudostructural understanding" of derivatives, meaning that students were not able to see derivative as these layers. For example, consider differentiating the function $f(x) = x^3$ twice. When the equation is given, students can compute the first and second derivatives and use them to sketch the corresponding graphs. However, when the problem is presented without the equation, only the graph of the second derivative is provided, they need to interpret the values of the second derivative and relate them to the slopes of the tangent lines of the first derivative, considering how these slopes represent rates of change over an interval. This process allows students to see how limit, ratio and function are related in the context of derivative. Zandieh's framework can be tied to Duval's theory of representations as Duval's theory explains the need of representational conversions in mathematics general, while Zandieh's framework specifies the conceptual structures of the derivative that such conversions need to support.

Method

Data Sources, Setting and Analysis

This study took place at a Midwestern research university in the United States, focusing on the teaching practices of Dr. T, an experienced calculus instructor with a PhD in mathematics. Dr. T had taught Calculus 1 multiple times, and the study aimed to observe her class practices without external influence. There were 25 students in Dr. T's class and each class session was 50 minutes long. Data collection involved video and audio recordings of all 33 calculus lessons she taught. For this study, the analysis concentrated on three lessons about derivative concepts including the derivative as a function, the first and second derivative test and sketching graphs, as other lessons on derivative were dedicated primarily to different differentiation rules such as the chain rule and quotient rule. We were particularly interested in how Dr. T presented graphs of a function and its first and second derivatives. Dr. T is positioned as a critical case (Flyvbjerg, 2006). By situating this analysis within a "critical case"—an experienced mathematician teaching with the most widely adopted calculus textbook in the United States—this study highlights how normative instructional structures may shape derivative learning. The study does not claim that this professor's practices represent all calculus instructors; rather, the aim is to demonstrate pedagogy and challenges that can inform practice in teaching derivative. Dr. T used Calculus 7E (7th Edition) by Stewart (2010), the most widely used calculus textbook in the U.S. (Bressoud, 2011). This study examined two sections called "Derivative as a function" and "What Derivatives Tell Us about the Shape of a Graph." In all, we examined eight implemented tasks from Dr. T's lessons, 27 pages, 15 worked examples and 205 exercise problems from Stewart Calculus (SC). Our analysis involved breaking down each segment—defined as the discussion of one task—and we analyzed worked examples and exercise problems from the sections of

derivative. There are other sections on derivatives, but they are about various differentiation rules, so they were excluded from this study. Each unit of analysis included one worked example, and one exercise problem. In total, we examined eight tasks from Dr. T's class, 220 worked examples and exercise problems from SC. To ensure the reliability of our findings, a fellow mathematics education researcher who is familiar with teaching calculus reviewed the discussions of the eight tasks and 220 textbook tasks for inter-rater reliability. Our initial rating achieved 96.5% agreement (197 out of 204). After further discussion, we reached 100% agreement.

Example of Analysis

Table 1 shows analytic framework that we used and examples of each code. We coded each derivative task using Duval's representations and transformations, and Zandieh's three layers. With these codes, when we coded each task, we examined students' opportunities if they were able to solve them correctly.

Table 1: Analytic Framework and Coding Examples

Task	Presentations/Transformations	Layers
1. Find the derivative of the function using the definition of derivative. $(x) = 3x - 8$	Symbolic (Treatment)	Ratio/Limit/Function
2. If $f(x) = x + \frac{1}{x}$, find of $f'(x)$.	Symbolic (Treatment)	Function
3. Match the graph of each function in (a) – (d) with the graph of its derivative in I–IV. Give reasons for your choices.	Graphical → Verbal (Conversion)	Ratio/Function
4. Let $f'(x) = \sqrt[3]{x}$. If $a \neq 0$, Use Equation 2.7.5 to find $f'(a)$.	Symbolic (Treatment)	Ratio/Limit
5. Sketch the graph of a function that satisfies all of the given conditions. Vertical asymptote $x = 0$, $f'(x) > 0$ if $x < -2$, $f'(x) < 0$ if $x > -2$ $(x \neq 0)$, $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$	Symbolic → Graphical (Conversion)	Ratio/Function
6. Make a careful sketch of the graph of f and below it sketch the graph of f'. $f(x) = Sinx$	Symbolic → Graphical (Conversion)	Function
7. Find where the function $f(x) = 2x^3 - 15x^2 + 24x - 5$ is increasing and where it is decreasing.	Symbolic → Verbal (Conversion)	Function

The first and second tasks in Table 1 engage students primarily in using symbolic representations, and other representations may not be necessary to solve them. However, when solving the first task using the definition of the derivative, students will have opportunity to view the derivative as both a ratio and a limit, whereas the second task can be solved symbolically without explicitly considering these layers. The third task provides opportunities for students to use graphical representations of derivatives. In Task 3, when matching graphs of a function and its derivative, students need to reason about rates of change—that is, where the function is increasing or decreasing—and recognize and consider that infinitely many of these rates over entire interval form a new function. Students would

only implicitly encounter the idea of a limit as they considered rates of change over an entire interval. The fifth task invites students to use both symbolic and graphical representations. To graph the function, they must reason about rates of change (ratios) to describe and visualize its behavior. This is one example of a task that allows students to consider multiple conditions and intervals to think about derivative (Cooley et al., 2007). The sixth example allows students to work with both symbolic and graphical representations, integrating multiple perspectives on the derivative. Although this task involves a symbolic-to-graphical conversion similar to the fifth task, unlike that task, students may not need to consider rate of change when finding the derivative of sin$x$. As a result, these tasks require different types of knowledge. The fourth task primarily involves symbolic representation, and because students were asked to use the definition of the derivative (Equation 2.7.5), they again engage with the ideas of ratio and limit, but it is unlikely for them to consider rate of change over entire interval. Finally, the seventh example requires verbally identifying intervals. Students will need to consider signs of the function to identify the interval without necessarily considering ratio.

Results

Dr. T's First Lesson

Dr. T's first lesson was on derivative as a function. She began the lesson with one problem that students were asked to find the derivative of a function ($f(t) = t^{-1} - t$) and find the value of $f'(5)$—coded as Symbolic treatment and ratio/limit/function layer. This problem was solved using the equation and the definition of the derivative. Then, she pointed out that value of derivative at one point is instantaneous rate of change and that would be the slope of tangent line. Then, she had a task about sketching the graph of the derivative when the graph of a function is given (Figure 1).



Figure 1: One Task from Dr. T's Lesson

She was trying to let her students think about what it means to have instantaneous rate of change.

> At every point there's a tangent. So, at every point there is a slope. So, if this is the graph of x, I should be able to draw the graph of the derivative of the F by figuring out what the different slopes are.

From how she explained this task, we can see that Dr. T was trying to point out how graphs of a function and its derivative are related. Her students had opportunities to think about how instantaneous rate of change can be used and interpreted to sketch the graph of the derivative. Then, she did two more similar problems. In solving both tasks, Dr. T continuously mentioned how the signs of the given function can be used to sketch the graph of its derivative. For example, the third problem was about sketch the derivative of the following function—coded as Graphical to verbal conversion and function/ratio layer.



Figure 2: One Task from Dr. T's Lesson

> So, when you have a corner, the derivative is not defined. So, when we draw a derivative, it should not be defined as zero. Alright? Over here, what's the derivative positive or negative? It's negative because the thing decreasing and over here it's actually very close to zero negative and then it gets steeper and steeper.

Again, Dr. T mentioned how slopes can be used to understand the behavior of the derivative.

Dr. T's Second Lesson

Dr. T's second lesson was on the first derivative test. She drew a graph and stated the following—coded as Graphical to verbal conversion and function/ratio layer.

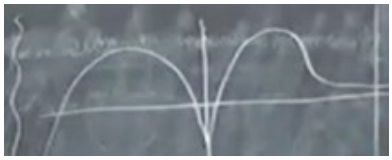


Figure 3: One Task from Dr. T's Lesson

> Here's a function and what we're going to do is design the sign of the derivative, the first, the second derivative everywhere. So, the first thing I would like to do is look at this function graph and decide where the first and or second derivative are either zero or undefined.

Once she gave her students time to work on this, she created a table to check how signs change at each interval and pointed out how the slopes of the function can be used to think about the graphs of its derivatives. After discussion of this task, she solved two more problems where equations are given to them. These problems were solved by differentiating the equation and creating a table.

Dr. T's Third Lesson

She continued with graphing functions and again, students were given equation (Figure 4) coded as Symbolic to graphical conversion and function layer. After discussing this task, she moved on to the next topic, L'Hospital's rule.

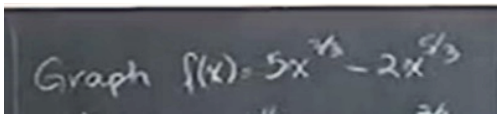


Figure 4: One Task from Dr. T's Lesson

In all, equations were given to students for four out of eight tasks and the graphs of functions were given for other four tasks. When graphs were used, she consistently pointed out how the values of function (the slopes) can be interpreted to find the graphs of derivatives. However, there were no tasks that asked students to consider several conditions and intervals to think about function and its derivatives (fifth example in Table 1) or ones that used by Cooley and colleagues.

Learning Opportunities in Stewart's Calculus

In two sections, there were 15 worked examples. Among them, 13 of them were presented with symbolic representation. For example, Figure 5 shows one worked example from Section 2.8—coded as symbolic to verbal conversion and function layer. Students may consider rate of change but the task does not ask students to consider rate of change explicitly.

> Find where the function $\boldsymbol{f(x) = 3x^4 - 4x^3 - 12x^2 + 5x}$ is increasing and where it is decreasing.

Figure 5: One Worked Example from SC (p. 297)

The solution shows the function can be differentiated and factored to find intervals where it increases. Unlike task in Figure 5, solution of worked example in Figure 6 shows that the graph of function was examined to see how instantaneous rate of change at each point can be translated to the graph of its derivative—coded as conversion—graphical to verbal and function/ratio layer.

> The graph of a function *f* is given in Figure 1. Use it to sketch the graph of the derivative $\boldsymbol{f'}$.

Figure 6: One Worked Example from SC (p. 153)

Figure 7 shows another worked example that does not include equations—coded as symbolic to graphical conversion and ratio/function layer. Solution shows that students need to consider and interpret each condition to see how the function behaves.

> Sketch a possible graph of a function f that satisfies the following conditions:
>
> $$\boldsymbol{f'(x) > 0 \text{ on } (-\infty.1),\ f'(x) < 0 \text{ on } (1,\infty)}$$
> $$\boldsymbol{f''(x) > 0\ (-\infty.-2) \text{ and } (2,\infty)\ f''(x) < 0 \text{ on } (-2,2)}$$
> $$\boldsymbol{\lim_{x\to-\infty} f(x) = -2, \lim_{x\to\infty} f(x) = 0}$$

Figure 7: One Worked Example from SC (p. 301)

Exercise Problems

Table 2 shows that about 60 % of tasks in this section allow students to use symbolic representation and 50% of tasks are about symbolic treatment.

Table 2: Exercise Problems in the First Section

Representation	Layer	Frequency	Percent
Symbolic (Treatment)	Function	2	2.6%
Symbolic (Treatment)	Ratio/function	18	23.6%
Graphical to Verbal (Conversion)	Ratio/function	30	39.5%
Symbolic (Treatment)	Ratio/limit	20	26.3%
Symbolic to Graphical (Conversion)	Ratio/Function	6	7.9%
Total		76	

Figure 8 shows one exercise problem from SC. It provides an equation to estimate the values of derivative at a few points, and students were asked to use these values to guess the formula for $f'(x)$—coded as Symbolic to graphical conversion and Ratio/function layer.

Let $f(x) = x^2$
Estimate the values of $f'(0), f'\left(\frac{1}{2}\right), f'(1)$ and $f'(2)$ by zooming in on the graph of f.

Figure 8: One Exercise Problem from SC (p. 163)

On the other hand, task in Figure 9 shows that students are asked to understand function and its derivative without equations. Without an equation, students need to consider how rate of change over time is related to the graphs of derivatives—opportunities to see conversion between graphical and verbal representations.

The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve and explain your choices.

Figure 9: One Exercise Problem from SC (p. 165)

Table 3 shows distribution of exercise problems in the second section. For this section, about 67% of tasks provided equations to solve derivative tasks. The most frequent task in this section is coded as "Symbolic to verbal conversion and function layer" (Figure 10).

Table 3: Exercise Problems in the Second Section

Representation	Layer	Frequency	Percent
Symbolic to Verbal (Conversion)	Function	54	41.8
Symbolic to Graphical (Conversion)	Ratio/function	34	26.3
Graphical to Verbal (Conversion)	Ratio/function	41	31.8
Total		129	

(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

$$\boldsymbol{f(x) = \frac{x}{x^2 + 1}}$$

Figure 10: One Exercise Problem from SC (p. 306)

In contrast, examples in Figure 11 requires for students to consider various conditions of a function to sketch or identify the graph without using equations—coded as symbolic to graphical conversion and ratio/function layer.

Sketch the graph of a function that satisfies all of the given conditions.

$$f'(5) = 0,\ f'(x) < 0 \text{ when } x < 5$$
$$f'(x) > 0 \text{ when } x > 5,\ f''(2) = 0,\ f''(8) = 0$$
$$f''(x) < 0 \text{ when } x < 2 \text{ or } x > 8$$
$$f''(x) > 0 \text{ when } 2 < x < 8$$
$$\lim_{x \to -\infty} f(x) = 3\ ,\ \lim_{x \to \infty} f(x) = 3$$

Figure 11: One Exercise Problem from SC (p. 306)

Discussion and Conclusion

This study examined how opportunities to learn the concept of derivative were structured across enacted instruction and a widely used calculus textbook. Instead of focusing on students' challenges directly, we investigated the representational and conceptual affordances embedded in instructional tasks. Using Duval's theory of semiotic representations and Zandieh's layers, we analyzed how tasks positioned students to engage with representations and derivative layers. Our findings suggest that the structure of instructional tasks may systematically constrains sustained engagement with the derivative across multiple representational registers and conceptual layers as instructional tasks afford limited opportunities for students to explore multiple representations and layers. While symbolic fluency is essential in calculus, the issue is not just the prevalence of symbolic work, but the nature of representational activity students were expected to perform. Opportunities for conversion between registers—particularly those requiring students to coordinate graphical, verbal, and symbolic representations—were comparatively limited and often episodic. Similarly, when examined through Zandieh's framework, there were limited tasks that positioned students to engage simultaneously with the derivative as ratio, limit, and function. We were able to observe that students may consider ratio or limit implicitly for some tasks. Thus, it can enhance students' learning opportunities if they are asked to consider those layers explicitly. Our findings suggest that such fragmented understandings may not arise solely from cognitive difficulty, but also from the organization of learning opportunities themselves. When tasks privilege symbolic treatments and provide limited occasions for representational conversion and layered coordination, students may have few sustained opportunities to build structurally integrated conceptions of derivative. In conclusion, this study reframes difficulties with derivative not solely as issues of student cognition, but as consequences of how instructional tasks structure representational and conceptual activity. By analyzing opportunities to learn through complementary theoretical lenses, we contribute to ongoing discussions about how calculus instruction might better support deep and integrated understanding of fundamental concepts.

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CREATING A COMMUNITY OF PRACTICE IN MATHEMATICS DEPARTMENTS TO PROMOTE TEACHING THAT EMPOWERS STUDENTS

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Keywords: Undergraduate Education, Professional Development, Instructional Activities and Practices

This brief report examines the development and early outcomes of a Community of Practice (CoP) designed to support mathematics faculty in exploring pedagogical approaches that promote student empowerment. Seventeen participants, representing a range of instructional roles and course contexts, engaged in bi-weekly discussions on topics such as balancing procedural and conceptual understanding, assessment practices, productive failure, technology use, and student engagement. Data were collected through session surveys, monthly reflections, informal gatherings, and an end-of-semester questionnaire. Initial findings indicate that faculty highly valued having a dedicated space for pedagogical dialogue and appreciated learning from colleagues' experiences across diverse teaching environments.

Objectives and Purpose of Study

Mathematicians and mathematics educators both have teaching goals that include empowering students to successfully make sense of mathematics. As faculty members, it is imperative to have opportunities to discuss mathematics pedagogy with others who may be facing similar situations, and we have seen such groups form regionally (e.g., Johnson et al., 2021), but our goal was to start within our department. Therefore, a group of five faculty members came together to create a Community of Practice (CoP) for a mathematics department where participants could explore various pedagogical theories and strategies together through a theme of student empowerment. While studies have considered mathematical CoP's relationship among the members and with administration (e.g., Sack et al., 2016), our CoP's initial goals were to encourage pedagogical discussions among the department. The research questions we developed to examine this CoP include: How do mathematics faculty engage with the pedagogical aspects of teaching mathematics? What aspects of faculty professional development, in the form of a CoP, do participants find helpful? Note that, for this paper, we use the term *faculty* to include any person who could serve as the instructor of record for a mathematics course, which includes tenure-track faculty, clinical faculty, lecturers, and graduate students.

Background and Theoretical Perspective

We entered this work aware that traditional lectures have been the dominant mode of instruction for in mathematics and other STEM courses across many institutions (Stains et al., 2018). While the sole use of traditional lecturing has been questioned for years (e.g., Gibbs, 1982), we also find literature suggesting that it may not be the lecture that is the issue, but the way the lecture is structured (e.g., Kohn, 2017). We were also aware that in our mathematics department, faculty taught or coordinated courses with enrollments ranging from 15 to 3000

students with sections that varied in size. This meant that our colleagues face differing challenges and use a variety of strategies, depending on the type of class they teach. Thus, our intentions were to offer mathematicians opportunities to discuss new and current practices that support empowering students in mathematics and how they might apply to courses with various sizes and settings.

Using Wenger-Trayner and colleagues' (2023) *Communities of practice within and across organizations: A guidebook* as a baseline structure, we considered the key characteristics of a CoP of *domain*, *community*, and *practice*. Within our department, our *domain*, or shared goals and passions, were our love of mathematics and empowering students through the teaching and learning of mathematics. We built *community* through regular formal meetings and opportunities to gather informally and discuss challenges and successes in our teaching experiences as well as to consider other research-based strategies. *Practice* was integrated as the organizers of the CoP led discussions about theories and strategies that could be helpful in addressing the shared challenges with empowering students that we discussed. At times, we included guests from other departments on campus (e.g., Teaching and Technologies and the Center for Instructional Excellence) to share with the community or asked members of the CoP to share specific experiences from their teaching practices (e.g., implementing standard-based grading).

We used the concept of student empowerment as common ground in our CoP for mathematics professors. Empowerment served as a shared goal that resonated with faculty members' inherent desires to see students succeed in mathematics. A focus on empowerment also helped us shift the content of CoP sessions from purely theoretical pedagogical ideas to tangible, student-centered outcomes (Christian et al., 2020). We considered three dimensions of empowerment in mathematics learning: self-empowerment, cognitive empowerment, and social empowerment (Hassi & Laurson, 2015) and the roles these dimensions serve in mathematics teaching and learning.

Methods

To create space for sustained, faculty-driven dialogue around approaches to teaching mathematics, a group of five faculty members (three clinical mathematics faculty, one tenure-track mathematics/mathematics education faculty, and one University Alliance Fellow) met regularly over the course of four months to design a six-session, bi-weekly CoP under the theme of student empowerment (Christian et al., 2020; Hassi & Laursen, 2015). A Math Welcome Luncheon, hosted by the Provost's Office for Teaching and Learning, served as the primary venue for introducing the CoP and initiating recruitment, while engaging the department in dialogue around math engagement and shared challenges related to admissions, instruction, student support, and mathematics placement. Faculty were also invited to participate through targeted email communications. We developed six CoP session topics inspired by our perceptions of the needs of the faculty in our own department. The topics stemmed from discussions of our development teams' own experiences and concerns with math teaching, particularly in entry-level math courses at the university (e.g., Pre-calculus and Calculus I and II). For each session topic, we reviewed research articles to support session development or to share with participants to read before the session. The session titles were: (1) Mathematical Empowerment: Reframing our Instructional Impact (Christian et al., 2020; Hassi & Laursen, 2015); (2) Balancing Procedural and Conceptual Understanding (Khoule et al., 2017); (3) Developing Assessments that Foster Empowerment and Reduce Anxiety (Lewis, 2023; Nilson et al., 2023); (4) Teaching Math through Failure and Disorientation (Dibbs, 2019); (5) Technology

as Pedagogical Leverage; and (6) Empowering Students through Engagement (Lugosi & Uribe, 2022).

We collected data from the CoP using end of session surveys, end of month surveys, informal meetings and discussions with participants, and end of semester questionnaire. At the end of each session, participants scanned a QR code and completed a short survey including questions like: "What is one idea, insight, or resource from today's session that you found thought-provoking or useful? To what extent did today's session feel relevant to your current teaching practice or course design?" At the end of each month, participants received an email with a link to a survey regarding the sessions of that month, including questions like: "What, if anything, have you applied—or considered applying—from recent CoP sessions in your own teaching or course planning?" At the end of the six sessions, participants received an end of semester survey with questions like: "Which aspect(s) of the CoP were most impactful for you (e.g., session discussions, shared tools, facilitation structure, peer collaboration)? Why?" Additionally, participants were invited to an informal luncheon for fellowship and to hear about the next steps for the CoP. At the luncheon, they were able to give input through a virtual platform (i.e., Mentimeter) on their takeaways and ideas for the future.

Seventeen faculty members committed to the CoP with a mixture of tenure-track and clinical faculty, lecturers, and graduate students with a variety of teaching roles (e.g., coordinating a calculus course for 3000 students; teaching three sections of 30 students). Because of the various opportunities for feedback, every participant gave written feedback at some point throughout the semester. We are still in the process of analyzing all collected data, but we share some interesting initial findings below and look forward to sharing these in more detail in the future.

Initial Results and Discussion

Participants revealed various insights through the end-of-session surveys. For example, session two, regarding balancing procedural and conceptual understanding, included participants bringing and sharing mathematics problems that they deemed procedural and items deemed conceptual. The discussion revealed that faculty held varied understandings of these constructs, as evidenced by the collaborative analysis of whether "Compute $\cos^{-1}\left(\cos\left(\frac{6\pi}{5}\right)\right)$" was a procedural or conceptual mathematics question. After the session, one participant wrote: "It was interesting to hear how we all think of procedural vs conceptual a bit differently … and not just our definitions but whether or not questions have to be one or the other or can be both." This finding highlights the importance of professional development addressing the procedural-conceptual distinction, which Khoule et al. (2017) demonstrated significantly impacts student anxiety and performance. When faculty develop shared understanding of these constructs through collaborative dialogue, they are better positioned to design instruction and assessments that intentionally allow students to develop both types of knowledge.

Participants had more time to reflect on what value the CoP was bringing to their practice before completing the surveys at the end of the month. Several participants shared the value of simply having this space to discuss teaching with other faculty, as this is not something that had been regularly prioritized in the department. For example, one participant shared, "The discussion has been most valuable … I like hearing about what others are experiencing and implementing in their classrooms, even if it isn't practical for my classes."

Our end-of-semester survey data showed that faculty found the CoP highly valuable, with average ratings of 4.2 out of 5 for worthwhileness and 4.3 out of 5 for likelihood to recommend to colleagues. The space for discussions continued to be most important to CoP members. In the

end-of-semester survey, we asked "Which aspect(s) of the CoP were most valuable to you?", and 87.5% of the respondents who answered that question indicated that discussions were the most valuable. The high value participants placed on discussions aligns with research demonstrating that communities of practice support faculty innovation through shared knowledge and resources (Johnson et al., 2021; Sack et al., 2016).

Conclusion

These CoP sessions with a group of university mathematics faculty allowed for meaningful discourse and revealed current practices that some faculty have tried for empowering students in mathematical thinking in both large lecture calculus and small section courses. Faculty attending these sessions learned about strategies their colleagues were using and, at times, shared their own successful strategies

Our initial analysis shows that participants clearly valued having a formal space for discussing pedagogy with colleagues and appreciated having a theme focused on student empowerment, likely due to the presence of *domain*, *community*, and p*ractice*, key characteristics of a CoP (Wenger-Trayner et al., 2023). In the future, participants recommended more time for practical implementation planning, collaborative development of course policy pilots, and continued focus on topics like assessment practices, mastery-based approaches.

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TEACHING AND LEARNING ASSISTANTS' PERSPECTIVES ON ACTIVE LEARNING IN CALCULUS AND PRECALCULUS RECITATIONS

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Institutions of higher education in the United States have been making significant efforts to reform introductory calculus courses to improve student retention and learning quality (Reinholz et al., 2021; Talanquer, 2014; Wu et al., 2018). Reform efforts such as active learning strategies, Teaching Assistants' (TAs) training, and the integration of Learning Assistants (LAs) have been shown to positively affect student learning, retention, and confidence (Clements, 2025; Williams & Fowler, 2014; Wilson & Varma-Nelson, 2016). However, little is known about LAs' and TAs' perspectives on these changes. To address this gap, we explore the research question: *How do LAs and TAs perceive the efficiency of the active learning recitations?*

This study took place at a mid-sized public university in the Northeastern U.S., focusing on recitation sections in Precalculus, Calculus I, and Calculus II. Each recitation was led by one TA, who divided students into small groups to work on problems provided by the course instructor, with one LA, who assisted students with problem-solving. Data were collected during group meetings with all LAs and TAs, held twice per semester across four semesters. The data included written feedback and group discussion summaries on aspects of the recitations that TAs and LAs found most helpful for students' learning, effective support strategies, and students' study skills. We used qualitative content analysis to identify key themes in the data (Mayring, 2021).

The results suggest that LAs and TAs perceived problem-solving in small groups as one of the most helpful aspects of the recitations. They felt students valued collaborating in a low-stakes, low-pressure, and judgment-free environment. TAs and LAs also observed that students appreciated having a brief review of the main mathematical ideas before group work. Among suggestions for efficiently facilitating group work, LAs and TAs mentioned using randomized grouping, sitting at students' eye level, knowing students' names, and allowing themselves to be vulnerable by admitting they did not know all the answers and using that as an opportunity to model problem-solving. At the same time, LAs and TAs noted that limited algebra skills and reliance on memorization negatively affected student confidence and decreased engagement.

These outcomes suggest that LAs and TAs view the active learning aspects of recitations positively and have developed effective strategies to support student learning before and during group work. LAs and TAs also perceived that students' challenges stemmed from insufficient prior knowledge and limited self-regulation skills, rather than resistance to active learning (Theobald, 2021; Tolman & Kremling, 2023). These results are significant because TAs and LAs are on the frontline of educational support in Calculus courses. Their perspectives can help unify instructional efforts to create cohesive learning environments.

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INVESTIGATING THE LIMITS OF NOTHING: CALCULUS STUDENTS' CONCEPTIONS OF ZERO

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Keywords: Calculus, Mathematical Representation, Number Concepts and Operations

Zero's deceptive simplicity and importance in mathematics affords it the benefit of being useful across all levels of mathematics. It may not be bold to assert that students engage with zero in some way everyday they enter a classroom. Research on zero is rather limited, but it has emerged as a topic of interest in research on parity (Levenson et al., 2007), integer division (Grouws & Reys, 1975; Tsamir, 1996), and algebraic reasoning (Carpenter & Levi, 2000). Many of these studies found that students conceptualize zero as nothing but that this conception may be more nuanced than previously believed (Seidelmann, 2004; Vest & Alibali, 2024). Still, the existing body of research suggests that students' and teachers' understanding of zero is limited or incomplete (Inhelder & Piaget, 1969; Tsamir et al., 2000; Seidelmann, 2004). These studies have classified understanding and conceptions of zero, but they have yet to consider how a student's conception of zero may be a fragmented combination of the different conceptions previously proposed. Reframing previous frameworks on conceptions of zero into an overlapping scheme may allow for a greater understanding of the nuanced conceptions that previous studies have suggested.

This report and the associated poster concern work related to two research questions: *in what ways do calculus students conceptualize the concept of zero and its many roles in mathematics?* and *in what ways do calculus students employ varying conceptions of zero to navigate tasks and mathematical situations?* Zero's central role in calculus and the likelihood that students have encountered many mathematical contexts by the time they reach a calculus class has informed my decision to study calculus students. Furthermore, exploring calculus students' conceptions presents the opportunity to witness the negotiations that take place when a students' traditional conception of zero does not align with their understanding of algebra and calculus. For this study, 50 high school students and 84 undergraduates responded to a digital survey meant to uncover how they conceptualize zero as an isolated object and in the context of mathematical problem solving. The survey responses were analyzed using a Braun & Clarke (2006) thematic analysis, through the lens of the Nature of Zero (NZ) and a second time through the lens of the Uses of Zero (UZ). The analyses uncovered a diverse set of perspectives from students and while the most common responses were associated with the idea of nothing, students' descriptions of zero and its use uncovered further beliefs about what it means to be nothing and the complicated relationship nothing has with existence, especially in real-world examples.

A subset of 12 students, six from each sample, were invited to participate in follow-up interviews to further explore their conceptions of zero. These interviews have yet to be fully analyzed but initial findings suggest that students are comfortable using zero in mathematical situations but may not have a strong understanding of why zero behaves as it does. When some students describe zero, they find that their conception does not align with how they use zero to operate in domains such as multiplication, division, and limit calculations. Furthermore, many students report a belief that zero is simply an abstract representation of nothing, but this belief brings into question zero's existence, its nature, and its behavior in mathematics.

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EXPLICATING TWO DIFFERENT USES OF PROXY QUANTITIES

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Keywords: Cognition, Mathematical Representations, Modeling

Quantitative reasoning—constructing measurable attributes and reasoning about relationships between those attributes (Thompson, 2011)—has emerged as a critical way of thinking for STEM concepts across all grade levels (Czocher et al., 2022; Ellis et al., 2015; Johnson, 2015; Jones & Ely, 2023; Karagöz Akar et al., 2022; Kularajan et al., 2024; Rodriguez et al., 2019; Steffe & Olive, 2010). Because quantities range in their complexity, the mental operations an individual can enact for one quantity as compared to another vary. For instance, with respect to length, an individual can physically construct a unit-length, disembed that unit-length from a whole, and iterate it end-to-end to measure a whole (Steffe & Olive, 2010). With pressure, however, physically constructing a unit, disembedding that unit, and iterating it is not possible.

One way in which differences among quantities is handled is using *proxy* quantities. Defined generally, a proxy is an agent that represents or stands for something else (Floridi, 2015). In the context of quantitative reasoning, a proxy involves using a quantity in place of another quantity. In this poster, I distinguish between two forms of proxy quantity use. Both forms are for the purpose of moving from complicated quantities to those more familiar or convenient (e.g., from pressure to length) to better understand or explore some quantitatively complex situation. Yet, the two forms are distinct in their cognitive activity.

In one form, a proxy quantity is introduced because it is in a relationship with another quantity with respect to the context under consideration. An example of this includes substituting time in for a distance quantity, often when they are in a proportional relationship (Olsho et al., 2022; Zimmerman et al., 2024). More generally, if someone understands a relationship between quantities A and B, they might use that as a starting point when exploring A and C—adapting it based on how B and C are connected. Quantity B's relationship with A and simultaneous role as an intermediary makes it a proxy for C. This form of proxy use is common in science practice (Guttinger & Love, 2025; Olsho et al., 2022), and it occurs *within* a context.

A form of proxy use left somewhat implicit in educational research and teaching is that which involves using alternative quantities to represent quantities from some other context for the purpose of affording quantitative reasoning. For instance, mercury thermometers use the proxy quantity of length to represent temperature, thus allowing an individual to reason with length in place of temperature. More generally, this form of proxy use underpins a quantitative use of coordinate systems to represent covarying quantities from a context (Lee et al., 2020; Moore, 2021). For instance, an individual could use an instrument to collect the pressure and temperature of a sealed container full of gas and represent that data in a Cartesian coordinate system. Using the Cartesian representation, they could conclude that as the temperature increases, the pressure increases linearly. In drawing this conclusion, their enacted quantitative reasoning entails lengths along the coordinate axes acting as proxy quantities for pressure and temperature in the context.

As illustrated in this poster, these two forms of proxy use have important differences with respect to students' enactment of quantitative reasoning and thus provide a possible clarifying orientation for future explorations of students' meanings across STEM concepts.

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PRE-SERVICE TEACHER EDUCATION

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

EXPERTISE OF PROSPECTIVE TEACHER NOTICING BASED ON MODELS OF STUDENTS' THINKING

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We focused on the degree to which prospective teachers demonstrated expertise in distinguishing between concrete and abstract aspects of students' thinking when noticing, as well as in individual versus pattern-focused noticing, both while generating models of students' thinking. Data were collected through simulations as part of a longitudinal study of 18 prospective teachers enrolled in three postsecondary mathematics content courses. Overall, most prospective teachers demonstrated less expert-like characteristics in their simulation interview responses, providing mostly a description of the simulated students' strategies and with limited connections to broader aspects of problem-solving strategies or mathematical thinking. Analysis shows that noticing of students' mathematical thinking occurs while model building, yet there is a significant difference in the interpretation of the students' mathematical thinking.

Keywords: Model Building, Noticing, Representations, Simulations, Students' Thinking

Researchers have emphasized the importance for teachers to attend to and interpret students' mathematical thinking (Jacobs et al., 2010; König et al., 2024; van Es & Sherin, 2008). This focus has been emphasized internationally in recent decades, with an astounding number of research papers on noticing (see König et al., 2017), including entire edited volumes exclusively on the topic (e.g., Dindyal et al., 2021; van Es & Amador, 2025; Weyers et al., 2024). At the same time, researchers have long studied key tenets of noticing (i.e., reasoning about students' thinking) and have focused on developing models of students' mathematical thinking (e.g., Norton, 2018; Steffe & Thompson, 2000; von Glasersfeld, 1981). Although in vernacular, these two foci (i.e., noticing, model building) within the mathematics education literature differ, we argue that the honing on students' mathematical thinking that is required to build a conceptual model of how a student thinks is parallel (in many ways) and complementary to the *specificity* (van Es & Sherin, 2021) needed to notice students' thinking at more extended levels (van Es, 2011). We contend that for a teacher to truly reach an extended level (i.e., van Es, 2011) of noticing in which they can attend to relationships between entities, including students' mathematical thinking, and demonstrate expertise of noticing (i.e., Bragelman et al., 2024), they must also be able to create a mental model of a particular student's mathematical thinking.

To better understand how prospective teachers notice and model students' mathematical thinking, we immersed 18 prospective teachers in a series of teaching simulations, characterized as malleable representations of practice (Amador et al., 2026), and asked them what they noticed as they interacted with a simulated student. We examined how prospective teachers demonstrated characteristics of expertise in two dimensions and answered the following research questions: When asked about how a student solved a problem, to what extent were prospective teacher responses (a) *concrete* versus *abstract,* or (b) *individual* versus *pattern-focused* with respect to noticing?

Theoretical Framing

Researchers have studied the expertise of teaching and teacher education (e.g., Berliner, 1988), with an often unintended comparison and analysis of the skills those more expert-like exhibit versus those more novice (Borko & Livingston, 1989; Carter et al., 1988; Sabers et al., 1991). In such efforts to describe teaching expertise, the lack of skills of those more novice stands in stark contrast to those with expertise (Bastian et al., 2022; Huang & Li, 2012; Jacobs et al., 2010; Stahnke & Blömeke, 2021; Wolff et al., 2016; Wolff et al., 2017); however, we work to intentionally describe expertise without assuming a deficit view of those without such skills. Teaching expertise is a central yet distinct consideration for noticing expertise. Lachner et al. (2016) note that the notion of teacher expertise may be a precursor to understanding that serves as a backdrop for a more refined focus on the expertise of noticing. As teachers demonstrate expertise in noticing, it is essential to capture the nuance of what it means to be expert-like, as the skills required for noticing expertise are more refined than skills regularly referenced for teaching expertise. Due to the refinement of noticing as a part of teaching, the expertise of noticing should be studied to articulate the granularity of knowing how to attend to and interpret students' mathematical thinking (i.e., Jacobs et al., 2010; van Es & Sherin, 2008; van Es, 2011). We employ a framework (i.e., Bragelman et al., 2024) that examines prospective teachers' ability to distinguish between *concrete* and *abstract* aspects of students' thinking, as well as *individual* versus *pattern-focused* noticing.

Related Literature

Noticing and Expertise

Jacobs and Spangler (2017) well document the importance of noticing and responding to students' mathematical thinking. In fact, they claim that mathematics teachers "need noticing expertise" to attend to and interpret instructionally important features in the classroom (p. 771). They contend that noticing is a core practice for teaching, and noticing expertise has been linked to other positive outcomes (e.g., task modification, see Choppin, 2011). Additionally, Jacobs and Spangler claim that noticing expertise—and they deliberately use the term "expertise"—can enable teacher learning and is a learnable skill for prospective teachers (e.g., Fernández et al., 2012; Schack et al., 2013). Although Jacobs and Spangler (2017) describe noticing as conceptualized through attention, interpretation, and decisions about the next instructional steps, they fall short of clarifying the specific dimensions of the expertise concept they invoke, which raises questions about specific components of noticing, such as model building and expertise.

Model Building and Expertise

Even though noticing and model building have traditionally been treated as distinct areas within mathematics education literature—in this paper they are theoretically linked—the precision required to construct conceptual models of student thinking parallels and complements the specificity necessary for extended noticing (van Es & Sherin, 2021), particularly for noticing where teachers attend to relationships between students' mathematical thinking and instructional strategies (van Es, 2011). Building models of students' mathematical thinking—which involves attending to and interpreting student reasoning to inform instructional decisions as outlined in seminal noticing research (Jacobs et al., 2010)—demands a level of expertise that remains understudied (e.g., Bragelman et al., 2024; Copur-Gencturk & Tolar, 2022; Bastian et al., 2022, 2024; Jacobs et al., 2024; Lee, 2018), particularly regarding prospective teachers. Given that noticing represents an aspect of teaching where expertise can be demonstrated (Jacobs et al., 2024) and significantly impacts students' mathematical thinking, and considering model building's role as a necessary component of advanced noticing, further research is essential to

characterize noticing expertise among prospective teachers, specifically their capacity to develop models of students' mathematical thinking that are both explanatory and predictive.

Context and Method

In prior research (i.e., Amador et al., 2026; Castro-Superfine et al., 2025) we presented a framework for different representations of practice, categorizing a representation as either *static*, *dynamic,* or *malleable* (Castro Superfine et al., 2025). The affordances of simulations, a malleable representation, include providing a safe, risk-free environment in which prospective teachers can engage in repeated practice, practice emerging skills, and explore different instructional strategies (Howell & Mikeska, 2021; Shaughnessy & Boerst, 2018). Through their questioning, prospective teachers engage in iterative cycles of attending to and interpreting students' thinking, two central components of teacher noticing (van Es & Sherin, 2002), and the process by which we argue model building (i.e., Norton, 2018) can occur.

Data for this study were collected as part of a longitudinal study of a cohort of 18 prospective teachers across three mathematics content courses. Over the course of three semesters, prospective teachers engaged in five curricular units focused on attending to, interpreting, and deciding how to respond to students' thinking aligned with the mathematical topics covered in each course. Each unit concluded with a teaching simulation in which a prospective teacher played the role of a teacher, eliciting the mathematical reasoning of a student, played by a research team member, on a particular math task. Teaching simulations were conducted outside of class time on a voluntary basis, and participants were incentivized for their participation.

Prospective teachers had the opportunity to participate in five teaching simulations, each involving a nonstandard strategy for a routine mathematics task. The simulations took place over Zoom and were immediately followed by an interview in which prospective teachers shared their understanding of the simulated student's approach to the task. Before each simulation, prospective teachers completed the focal math task independently and studied a written sample of student work from the task. During the teaching simulation, prospective teachers were instructed to elicit the strategy and reasoning used by the simulated student, as played by a member of the research team. After the prospective teacher felt they had elicited sufficient information from the simulated student, another member of the research team asked them a series of questions about their understanding of the student's strategy. We used the transcripts of these interviews as the basis of our analysis. In this paper, we present results from two simulations, Column Addition and Constant Difference. In the Column Addition simulation, a child is asked to add three two-digit addends <29+36+18>. The child adds the tens column first as a single digit, next adds the ones column, notating in two digits. Finally, the child combines the two tens digits to arrive at the correct solution. In the Constant Difference simulation, the student correctly applies a process for solving a multi-digit subtraction problem <784 – 315> that is used in many European and South American countries by adding 10 to the top number in the ones place and 10 to the bottom number in the tens place.

Data Analysis

As we were interested in exploring the intersection of prospective teachers' expertise in noticing with their model building, we built from a prior analysis of data from these simulations in which we coded prospective teachers' models of student thinking for the extent to which they were explanatory, predictive, and generalizable, characteristics which are necessary for an effective model (Castro Superfine et al., 2025). Responses were coded as *explanatory* if they accurately reflected the standardized student's strategy, *predictive* if they could produce an expected result on a similar task, and *generalizable* if the prospective teacher could characterize

the cases in which the strategy would work. We limited our analysis of expertise to responses coded as explanatory and predictive to ensure that the dimensions of expertise in prospective teachers' noticing relate to the simulated student's thinking. Table 1 summarizes the number of prospective teachers who had explanatory and predictive models for the Column Addition and Constant Difference simulations. We do not include generalizable as a delimiter on this analysis because it was the most challenging dimension for prospective teachers, and while generalizability is a key dimension of a full model of student thinking, we contend that prospective teachers are able to demonstrate dimensions of expertise related to student thinking without a full grasp of the mathematical generalizability of that thinking.

Table 1: Number of Prospective Teachers with Explanatory and Predictive Models

Simulation	Explanatory	Predictive	Total
Column Addition	13 (72%)	12 (67%)	18
Constant Difference	8 (100%)	8 (100%)	8

The co-authors first analyzed a subset of the Column Addition simulation data for the Concrete vs. Abstract and Individual vs. Pattern Focus dimensions of noticing expertise from the Framework for the Expertise of Noticing (Bragelman et al., 2024). We then coded the interview responses for all prospective teachers with explanatory and predictive models for the Column Addition and Constant Difference simulations using the codes in Tables 2 and 3. Four members of the research team coded each response individually and discussed divergences until 100% agreement was met.

Table 2: Coding for Concrete vs. Abstract Characteristics of Noticing

Less Expert-like	Developing	More Expert-like
Concrete Knowledge Response includes a declarative or absolute interpretation (that the student is thinking, more evaluative).	Response includes an interpretation that may contain declarative or absolute statements, but begins to indicate some "adaptive process" OR connects to problem-solving principles specific to children's mathematical thinking.	**Conditionalized Knowledge** Response includes interpretation "indicative of an adaptative process" with functional abstractions; suggests more robust problem-solving principles specifically to children's mathematical thinking.

Table 3: Dimension of Expertise for Individual vs. Pattern Focus in Noticing

Less Expert-like	Developing	More Expert-like
Object Perspective Response includes moments of children's mathematical thinking as discrete examples	Response includes features and patterns of children's mathematical thinking and	**Pattern Perspective** Response includes features and patterns of children's mathematical thinking beyond

for one or multiple students; however, does not yet connect these moments across students or within a student's progression of children's mathematical thinking.	begins to connect these moments across students or within a student's learning progression, but the connection is not fully developed.	explicitly stated, either about the learning progression of one student OR across multiple students (e.g., multiple students are thinking similarly about X).

Results

Concrete Versus Abstract Noticing

The following addresses the first part of the research question, which focuses on the extent to which prospective teachers' responses were concrete versus abstract regarding noticing. Table 4 centers on the emergent expertise of participants as their concrete versus abstract noticing was characterized as less expert-like, developing, or more expert-like, across the two simulations.

Table 4: Emergent Expertise in Concrete Versus Abstract Characteristics of Noticing

Simulation	Less Expert-Like	Developing	More Expert-Like
Column Addition	7	5	0
Constant Difference	4	3	1

For the Column Addition simulation, five of the 18 prospective teachers (~28%) exhibited emergent expertise in Concrete vs. Abstract characteristics of noticing. For Constant Difference, four of the 8 prospective teachers (~50%) exhibited emergent expertise in noticing. Prospective teachers with less expert-like responses were able to elicit from the hypothetical student the full strategy employed and use it predictively with a similar problem; however, their responses were limited to descriptions of the strategy. The following response is indicative of this more descriptive approach to describing the hypothetical student's process:

> He said that he started with the ones place, and he knew that he couldn't subtract four from five. So he knew he had to borrow for the ones place, but he didn't exactly know how to tell me where he borrowed from. He just knew he needed more for the ones place for that four, and he just said that he came up with the ten, that he added [ten]. Then he said that in order to keep it equal, that he had to add to the top number and to the bottom number, so he added another ten to that bottom one which made it two. And he said that you have to do that to keep the difference. And then after he said he just subtracted normally.

At the time of the simulation, the prospective teachers had not yet learned the constant difference strategy for subtraction, so this experience with the strategy was entirely novel. In other words, prospective teachers with less expert-like responses were not contextualizing the novel strategy against other problem-solving strategies.

In contrast, for prospective teachers exhibiting emergent expertise, their responses included adaptations to more abstract problem-solving strategies. For example, the single prospective teacher who exhibited more expert-like characteristics of noticing offered the following response when asked how the student found the difference:

> Well, for me, I had to do a completely different problem. I had to write a problem similar to it and see if I could do it just like he did it. I realized that it's almost the same as borrowing, but instead of taking eight and turning into seven, he's taking one and adding another one to

it, making it two. Eight minus two is still six. If you took um, borrowed, took the eight away and turned it into seven, it'd still be seven minus one; it'd still get six. That digit does not change.

In this response, the prospective teacher compares the strategy employed to the traditional regrouping strategy, offering insight into the efficacy of the strategy. *What* was noticed changed significantly as this prospective teacher interpretation of the process includes comparing the novel process to one the prospective teacher understood and so also included evaluating the mathematical validity of the strategy.

Individual Versus Pattern Noticing

The following answers the second part of the research question, which focuses on the extent to which prospective teacher responses were individual versus pattern focused regarding noticing. Table 5 centers on the emergent expertise of participants as their concrete versus abstract noticing was characterized as less expert-like, developing, or more expert-like.

Table 5: Individual Versus Pattern Focused Characteristics of Noticing

Simulation	Less Expert-Like	Developing	More Expert-Like
Column Addition	5	7	0
Constant Difference	3	4	1

For the Column Addition simulation, seven of the 18 prospective teachers (~39%) exhibited emergent expertise in Concrete vs. Abstract characteristics of noticing. For Constant Difference, five of the 8 prospective teachers (~63%) exhibited emergent expertise in noticing. Prospective teachers with less expert-like responses were able to articulate mathematical ideas or concepts understood by the hypothetical student; however, their responses were limited to a single problem. The following response is indicative of this approach:

> All I really got is that he was talking about how he had to keep it pretty much equal for the top and bottom, and he had to keep the difference. So. I think that adding that one to make it five, that would be him keeping the difference. If I need to elaborate more, I don't feel like I got enough from the interaction from him to do so.

In contrast, for prospective teachers exhibiting emergent expertise, their responses included aspects of children's thinking beyond explicitly stated. This could include within-student learning progress, or across students, such as comparing or contrasting cases. For example, the single prospective teacher who exhibited more expert-like characteristics of noticing offered the following response when asked what mathematical concepts the student understands:

> He would understand that five is equal to fifty or five tens, so he has a pretty good understanding of place value. He just he just he has a difficult, it seems like he has a difficult time grasping the concept of borrowing. <What evidence from the interaction makes you think that?> Because he said he had to add ten ones to make the four into fourteen and then he had to add one ten to the one to make it into two, so he knew that both times he was adding a ten, the only difference being that he was adding one ten versus ten ones. Around this age hear "whatever you do to the top you do to the bottom" when discussing fractions and that that can be greatly misconstrued amongst students, when it comes to stuff like this.

The response compares two strategies, particularly colloquialisms common in mathematics teaching and learning, and suggests the prospective teacher holds skepticism about whether employing such colloquialisms as problem-solving strategies correlates to mathematical understanding. In other words, both examples show *that* noticing of students' mathematical thinking occurs while also showing a significant difference in *the* interpretation of the student's mathematical thinking.

Discussion

Our aim in this paper was to better understand the extent to which prospective teachers demonstrate noticing expertise. We analyzed data from two teaching simulations to understand the nature of prospective teachers' noticing expertise in relation to their ability to develop models of students' mathematical thinking, focusing on responses from prospective teachers who could create a model of students' mathematical thinking. We then focused on the degree to which those prospective teachers demonstrated expertise in distinguishing between *concrete* and *abstract* aspects of students' thinking, as well as in *individual* versus *pattern-focused* noticing, both key dimensions of noticing expertise (Bragelman et al., 2024). Overall, most prospective teachers in our study demonstrated less expert-like characteristics in their simulation interview responses, providing mostly a description of the simulated students' strategies and with limited connections to broader aspects of problem-solving strategies or mathematical thinking.

In relation to the concrete versus abstract dimension of noticing expertise, most prospective teachers provided less expert-like responses, with slightly fewer prospective teachers providing responses that evidenced characteristics of developing expertise. While most prospective teachers' responses were limited to descriptions of the students' strategies, others described the students' strategies and provided some, albeit superficial, connections to other problem-solving strategies. Indeed, noticing students' mathematical thinking is often challenging for prospective teachers, particularly for those in the early stages of their teacher preparation coursework (e.g., Tekin-Sitrava et al., 2022). Prospective teachers tend to focus on other aspects of classroom environments rather than students' thinking with much granularity (Kaiser et al., 2017). Only one prospective teacher in our study demonstrated more expert-like characteristics in their simulation responses. Thus, while prospective teachers were able to accurately describe a student's strategy and predict how the student would solve a different problem, prospective teachers generally did not demonstrate emergent expertise in connecting students' mathematical thinking to other problem-solving strategies or principles.

Similarly, in relation to the individual versus pattern-focused dimension of noticing experience, some prospective teachers provided less expert-like responses, whereas slightly more provided responses that evidenced characteristics of developing expertise. Prospective teachers could connect parts of students' strategies, or certain steps in students' problem-solving process, to a broader progression of students' mathematical thinking, albeit not in a robust way. Several prospective teachers mentioned conceptions such as place value in their responses but did not elaborate much beyond that. This finding is similar to that of Callejo and Zapatero (2017), who found that prospective teachers could describe many mathematical elements of students' thinking but could not connect those mathematical elements to students' current understanding. Given noticing depends on knowledge of teaching and learning (Stürmer et al., 2013), findings are unsurprising given prospective teachers' lack of opportunity for instructional interactions.

Taken together, these findings raise two broader questions about prospective teachers' development of noticing expertise. First, is it reasonable to expect prospective teachers, particularly those who are in the early stages of their teacher preparation, to develop and

demonstrate expertise in noticing? It is reasonable to expect prospective teachers to express moments of noticing expertise but perhaps not demonstrate such expertise with great frequency or regularity, at least initially in their preparation. We agree that prospective teachers need to understand students' mathematical thinking, begin to interpret what children's thinking implies for their current understanding, and learn to decide where children need to go next in order to advance their learning (Jacobs et al., 2010; van Zoest & Stockero, 2012). However, developing such skills takes time and can be challenging for prospective teachers, who often lack experience and knowledge gained through classroom interactions with students. Yet the prospective teachers' responses analyzed in our study were able to build accurate models of students' thinking and to predict how students would solve similar problems. Perhaps one's ability to build an accurate and predictive model of students' thinking is necessary, but not sufficient for developing expertise in noticing. Indeed, in our prior work involving prospective teacher noticing in simulations (Castro Superfine et al., 2025), we argued that prospective teachers need more support for asking questions in ways that elicit and shape (van Es & Sherin, 2021) relevant information during interactions with students to allow for attending and interpreting that is responsive. Teaching simulations position prospective teachers to influence what elements of students' mathematical thinking become visible in their interaction with the simulated student, similar to how a teacher might do so through questioning (Webb et al., 2014). With limited instructional knowledge from which to draw, prospective teachers likely need more support in learning how and what questions are most efficacious to elicit students' thinking.

These findings raise a second question for characterizing the noticing expertise of prospective teachers. How can researchers capture the moments of expertise prospective teachers express? When presented with the opportunity to interact with a simulated student using a novel mathematics strategy, prospective teachers who are able to develop explanatory and predictive models of the student's thinking also express moments of noticing expertise, even amidst the noise of a university classroom, years before the prospective teachers begin their practice. As the findings show, these moments of skill expression are inconsistent within and across teaching simulations. Yet moments of expert-like skill expression in teacher noticing occurred in our sample of prospective teachers. As such, characterization of prospective teacher skill expression in noticing requires a fine-grained instrument as opposed to instruments that characterize noticing behavior of practicing teachers (e.g., Jacobs & Empson, 2021). The Framework for the Expertise of Noticing (Bragelman et al., 2024) offers one way to capture and describe such moments of emergent expertise, where prospective teachers begin to consider a student's thinking in broader, more abstract contexts or against the patterns of students' development.

By offering a framing of prospective teachers as capable of emergent expertise in noticing, we move the mathematics education field towards a new framing of prospective teachers, one that does not position them as deficient in a variety of skills but elevates them as demonstrating moments of expertise. We contend that researchers and mathematics teacher educators more broadly consider how they frame prospective teachers by acknowledging the expertise prospective teachers bring to the teacher preparation programs.

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ELEMENTARY PRESERVICE TEACHERS' ENDORSED INDIVIDUAL NORMS FOR DISCUSSION OF AN OPEN MATHEMATICS TASK

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While sociomathematical norms (Yackel & Cobb, 1996) understandably receive considerable attention in mathematics education research, social norms are also necessary for mathematical activity, including engagement in productive mathematical discourse. Here, we explore the social norms endorsed by 58 elementary preservice teachers (PSTs) through self-reporting after they engaged in discussion of an open, justification-oriented mathematics task in their mathematics methods course. We drew on Yackel et al.'s (1991) norms for individual activity as we analyzed PSTs' responses to survey questions focused on (1) what they deem important for discussing and justifying solutions and (2) their actual and anticipated reactions to disagreeing with others' solutions or justifications. Findings suggest patterns and complexities of PSTs' endorsed social norms that have implications for mathematics teacher educators.

Keywords: Classroom Discourse; Affect, Emotion, Beliefs, and Attitudes; Preservice Teacher Education

In recent decades, Yackel and colleagues' seminal works describing the largely unspoken yet taken-as-shared social norms (Cobb et al., 1989; Yackel et al., 1991) and sociomathematical norms (Yackel & Cobb, 1996) that regulate mathematical discourse in classrooms have been one of the most significant influences on the field's understanding of mathematical discourse (Herbel-Eisenmann et al., 2017). Much of that research has focused on sociomathematical norms rather than social norms (Herbel-Eisenmann et al., 2017). This focus is understandable given that sociomathematical norms—such as what counts as mathematical evidence or what constitutes a mathematically different strategy—are specific to mathematics classrooms. Social norms—which include expectations for activity such as collaboration and persistence—apply to any classroom context. While social norms that support productive classroom discourse in general do not guarantee productive mathematical discourse specifically (Kazemi & Stipek, 2001), social norms are necessary for and "inextricably related to mathematical activity" (Yackel et al., 1991, p. 397). Social norms also have direct implications for students' opportunities to learn (Yackel et al.).

In addition to a focus on sociomathematical norms over social norms, studies of norms in mathematics education typically describe norms based on teacher interviews and observations of class sessions and do not seek the perspective of students (Levenson et al., 2009). Asking students what norms they see as existing in the classroom (perceived norms) and what norms the students themselves value (endorsed norms) provides a more nuanced and holistic representation of norms than observing enacted norms alone.

Our study seeks to further emphasize the centrality of social norms and students' individually endorsed norms in mathematics contexts by examining the endorsed norms of a specific population: elementary preservice teachers (PSTs). Teacher preparation guidelines emphasize the need for PSTs to experience the types of mathematical discussion with which they will engage their future students (Association of Mathematics Teacher Educators [AMTE], 2017; Conference Board of Mathematical Sciences [CBMS], 2012). The importance of these experiences have been particularly emphasized for elementary PSTs, as this way of engaging with mathematics likely

contrasts with the opportunities they had in their own schooling (e.g., Ball, 1990). By investigating PSTs' endorsed norms within the context of a mathematical discussion of a task that requires justification of one's ideas, we ascertain critical information about what norms PSTs are and are not taking up. This information has implications for PSTs' own opportunities to learn as well as the opportunities they will provide for their future students.

Our research questions are:

1. What norms do individual PSTs report endorsing after a discussion of an open, justification-oriented mathematics task?
2. How do PSTs describe actual or anticipated reactions to disagreeing with others' solutions and justification?

Theoretical and Conceptual Framings

Work on social and sociomathematical norms is situated within the emergent perspective, which in turn is built on the tradition of symbolic interactionism (Herbel-Eisenmann et al., 2017). The emergent perspective coordinates psychological and social understandings of learning and focuses on the reflexive relationship of individual students and the learning community. Yackel and colleagues' work is largely representative of this perspective, as it is the most cited in terms of research on mathematical discourse in recent decades (Herbel-Eisenmann et al., 2017). Their foundational studies (e.g., Yackel et al., 1991; Yackel & Cobb, 1996) took place in the context of a second-grade classroom in which the teacher sought to teach in a manner consistent with inquiry-based instruction.

In Yackel et al.'s (1991) examination of social norms, the authors identified norms that guided mutual construction of ideas, including the importance of cooperation, persistence, meaningful activity, and coming to a consensus. These norms can apply in any classroom context, not just to mathematics classrooms. Yackel and Cobb (1996) built on the idea of social norms by describing sociomathematical norms, which are specific to mathematics classrooms. Sociomathematical norms include what "counts" as mathematically different, sophisticated, or efficient and what constitutes an acceptable mathematical explanation. In an earlier description of the study, Cobb et al. (1989) foreshadowed the naming of social and sociomathematical norms by describing students as having two problems: the "mathematical problems" (for which sociomathematical norms are relevant) and "the negotiation of a viable cooperative relationship that would make it possible for them to solve their mathematical problems" (social norms) (p. 129). In other words, productive sociomathematical norms are only possible in the context of established social norms conducive to that productivity.

In addition to social norms for collaborative work, Yackel et al. (1991) identified social norms for individual activity. Their three identified norms frame our work: (1) "children figure out solutions that are meaningful to them," (2) "explain their solution methods to their partner," and (3) "try to make sense of their partner's problem-solving attempts" (p. 398). While social norms for individual activity are inextricably intertwined with Yackel et al.'s (1991) norms for social cooperation, norms for individual activity focus on the individual's responsibility within the social context of the discussion.

Importantly, social norms—including those for individual activity—have direct implications for students' opportunities to learn. Yackel and Cobb (1996) described opportunities to learn, which include the opportunity to apply what is learned from others' solutions to one's own, reconceptualize a problem in the process of working through an incorrect solution method, and broaden one's own understanding in order to make sense of others' solutions. These

opportunities to learn are vital to consider in terms of individuals' endorsed norms, as the norms the individual endorses contribute to or inhibit the individual's opportunities to learn.

Methods

This study was part of a larger exploratory study focused on PSTs' engagement in open mathematics tasks in their elementary mathematics methods course at a large public university in the Mid-Atlantic United States. We report here on the norms for individual activity that PSTs in three sections of the course reported endorsing regarding their discussion of a Which One Doesn't Belong? (WODB; Danielson, 2016) task.

Context and Participants

WODB is an instructional activity in which learners are presented with four mathematically interesting images or mathematical objects and asked to choose which of the four does not belong and why. The task is designed so that each of the four objects can be justified and in multiple ways. As such, the necessity of justifying one's response is built into the task. Learners are reminded of this goal with the statement: As long as your claim is true, it is correct.

The task with which PSTs engaged in this study was chosen in collaboration with the course instructor to bring out key ideas related to addition and subtraction. The task presented PSTs with four addition expressions (see Figure 1). Common responses to the task included attention to even and odd addends (e.g., top left expression, all addends are even), sums and multiples (e.g., top right expression, only sum that is a multiple of ten), number of digits (e.g., bottom left expression has a double-digit addend), and number of addends (e.g., bottom right expression has four addends). While multiple responses and justifications were possible, disagreement could nevertheless take place when a PST found evidence that disputed another PST's claim.

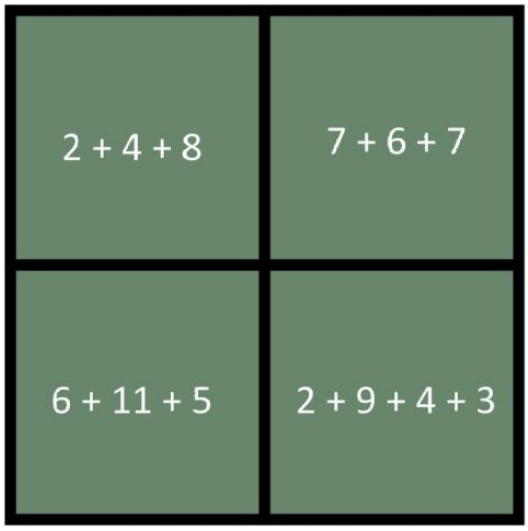


Figure 1: WODB Task (Heon, n.d.)

Across the three sections of the methods course, 58 PSTs participated in the study and were in attendance on the day of the WODB discussion ($n = 15$, $n = 21$, and $n = 22$). Of these PSTs, 54 PSTs presented as female and 4 PSTs presented as male. The discussion of the WODB task took place just before the midpoint of the semester. Participants had been introduced to the routine through a brief example, but had not previously engaged in a full discussion. After the

instructor introduced the task, PSTs had 2–3 minutes to record their thinking in writing. Then, the instructor invited PSTs to share their responses, encouraging them to connect with and build upon peers' responses rather than engaging in a parallel "show-and-tell" approach. To scaffold this conversational goal, the instructor supplied prompts for engaging with one another's thinking (see Figure 2). The discussions themselves ranged from approximately 10–20 minutes.

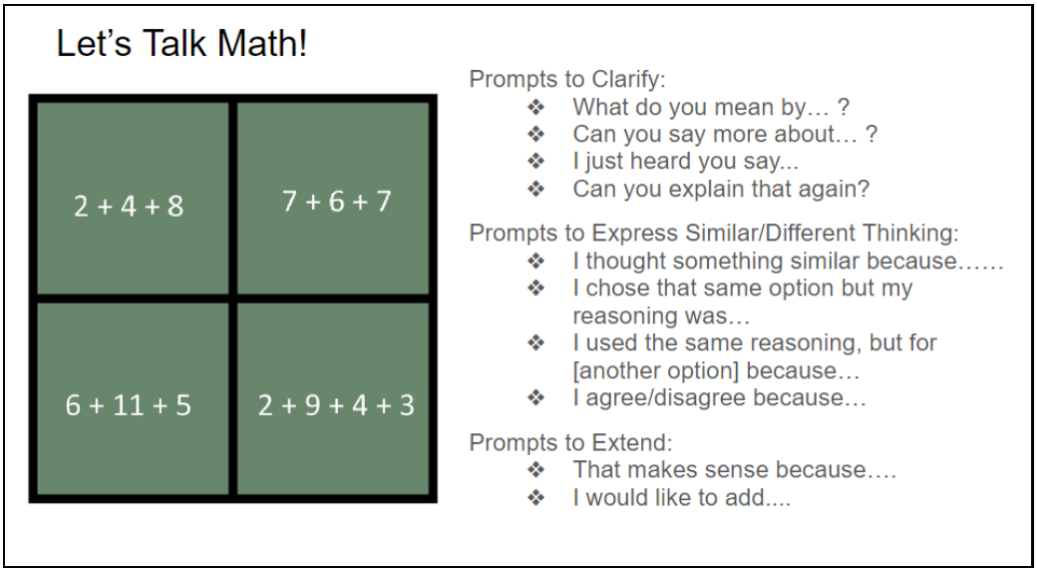


Figure 2: Prompts as Displayed During WODB Discussion

Data Collection and Analysis

Following the discussion, PSTs completed a brief open-response questionnaire that asked them about their engagement with the task and with one another's ideas. For the purposes of this report, we focused on participants' responses to survey prompts that asked about their views on discussion and disagreement: (a) Describe what you thought was important as you discussed and justified your solution and/or in others' discussions and justifications, and (b) What did you do (or what would you have done) if you disagreed with any of the solutions or justifications given during the discussion?

We engaged in a multi-step coding process that utilized both deductive and inductive coding (Miles et al., 2014) and analytic memos (Saldaña, 2021). Yackel et al.'s (1991) norms for individual activity served as *a priori* codes for deductive coding: generating meaningful solutions, explaining solution methods, and attempting to make sense of others' strategies and approaches. Additionally, inductive codes were developed according to patterns that emerged in the data. We used this strategy to collaboratively code one section of PST responses during a research meeting, which allowed us to discuss emerging inductive codes, develop a code book, and refine our coding strategy. Through this process, we noticed that only a small number of PSTs (n = 2, 3%) mentioned aspects of Yackel et al.'s norm related to figuring out a solution that is meaningful to the individual (p. 398). We speculated this was due to the wording of the survey question, which focused on discussing and justifying solutions, rather than the process of arriving at a solution and subsequently decided not to include this code in our analysis. We also noticed the need for an inductive code focused on factors external to individual norms that PSTs deemed important when explaining and justifying solutions and the need for an inductive code

related to reactions PSTs took or reported they would take when disagreeing with a solution or a justification (see Table 1 for a codebook excerpt). We individually coded the two remaining sections of participant responses. During research meetings, we compared codes to confirm agreement and to discuss any discrepancies until agreement was reached. Our analysis of the data illuminated patterns and complexities in PSTs' endorsed norms of mathematical discussions, which we share next.

Table 1: Codebook Excerpt

Code	Description	Example from PST Responses
Explaining	PST mentioned an aspect related to explaining their own solution method to their peers.	"I thought it was important to make sure your explanation worked for the option you chose" (Danielle, Section 2).
Attempting to Make Sense	PST mentioned an aspect related to making sense of their peers' solution methods.	"I felt that it was important keep an open mind as my classmates shared their thoughts" (Jenny, Section 1).
External Factors	PST mentioned an aspect important to them that was external to themselves or their peers (e.g., teacher, classroom environment).	"Having a safe classroom environment where discussion is welcomed is incredibly important, especially when discussing different opinions" (Hannah, Section 3).

Findings

Analysis of PSTs' survey responses revealed patterns and complexities surrounding the individual norms for mathematical discussions related to discussing and justifying solutions and disagreeing with others' solutions and justifications.

Patterns in Discussing and Justifying Solutions

PSTs' responses to the first question of interest on the survey focused on what they deemed important in discussing and justifying solutions. These classroom norms for individual activity (Yackel et al., 1991) were salient in PSTs' responses: explaining solutions to others and trying to make sense of others' problem-solving attempts (p. 398). PSTs placed a greater emphasis on explaining solutions (n = 32, 55%) than on making sense of others' approaches (n = 23, 40%). Table 2 displays the frequency of these norms in PSTs' responses.

Table 2: Aspects Important to Discussing and Justifying Solutions

Norms for Individual Activity	Number of PSTs			
	Section 1 n = 15	Section 2 n = 21	Section 3 n = 22	Total n = 58
Explaining Solutions to Others	8 (53%)	15 (71%)	9 (41%)	32 (55%)
Trying to Make Sense of Others' Attempts	7 (46%)	5 (24%)	11 (50%)	23 (40%)
External Factors	3 (20%)	5 (24%)	4 (18%)	12 (21%)

Note. Percentages do not equal 100%, as more than one norm was present in some responses.

Specific to explanations, PSTs mentioned the importance of being "able to back up how you got what you got and what you did" (Remy, all names are pseudonyms), "explaining your thought process fully and thoroughly" (Elsa), and having "an observable justification for my claim" (Aspen). There were also PSTs who stressed the importance of terminology and language within explanations. For example, Katherine mentioned that explanations should "walk the line between making sense to the larger group while still using appropriate terminology," and Susan noted that "use of vocabulary … [and] being clear in your reasoning" were important to explanations. These responses suggest that when endorsing the norm of explaining their solutions to others, PSTs place importance on supporting their solutions and using terminology and language they deem appropriate within their explanations.

PSTs' responses also mentioned aspects of attempting to make sense of their peers' strategies (n = 23, 40%). In doing so, PSTs mentioned the importance of being open to different perspectives. For example, Blair asserted, "It's important to look at it from the lens of your classmate. Think about how they solved it and look at it from that angle." Similarly, Lola noted, "I think it was important to take what others were saying and try to see that perspective. I think this is important so you can expand your thinking." There were also PSTs who made connections to their own work as they made sense of others' solution strategies. One of these PSTs remarked, "I tried to understand their thinking in a way that I could understand, relating their thinking to my thinking and solutions" (Serena). Marie also noted how the discussion "allowed us as students to extend on our own thinking and deepen our understanding." These responses suggest that PSTs saw their attempts to make sense of their peers' solution strategies as opportunities to extend their learning through seeing different approaches and making connections to their own thinking.

Interestingly, over 20% of PSTs (n = 12, 21%) mentioned external factors that they deemed important to discussion in their responses. These factors were often related to the teacher's role in the classroom, such as providing "a safe classroom environment" (Hannah) and providing "wait time" (Chris). Addy mentioned the importance of the instructor "reframing our thinking on the board" during the discussion, and William pointed out that providing reasoning in explanations "helped the teacher write your ideas down more clearly." As an additional external factor, there were two PSTs, both in Section 2, who valued external validation from their classmates. Hermione confided, "I'm not confident in math, so having others agree with differences I noticed was important," and Wendy thought it was important that "some other students used the same types of reasoning as me after I shared my thinking." The use of external factors suggests that some PSTs might not have seen themselves as active members of the discourse community.

Complexities When Disagreeing with Others' Solutions or Justifications

PSTs also indicated what they did or would do if they disagreed with any of the solutions or justifications provided during the discussion of the task. Similar to the norms they self-reported endorsing after the discussion, Yackel et al.'s (1991) norms of explaining solutions to others and trying to make sense of others' solution attempts (p. 398) were prevalent in PSTs' responses (see Table 3). Across PSTs, emphasis was placed on explaining their own solution versus attempting to make sense of others' solutions and justifications in response to disagreeing with others. That is, when disagreeing, PSTs emphasized their own solution strategy rather than trying to understand how others approached the problem.

Table 3: Norms Endorsed when Disagreeing with Others' Solutions or Justifications

Norms for Individual Activity	Number of PSTs			
	Section 1 $n = 15$	Section 2 $n = 21$	Section 3 $n = 22$	Total $n = 58$
Explaining Solutions to Others	6 (40%)	13 (62%)	10 (45%)	29 (50%)
Trying to Make Sense of Others' Attempts	6 (40%)	5 (24%)	10 (45%)	21 (36%)
Anti-norm	3 (20%)	6 (29%)	3 (14%)	12 (21%)

Note. Percentages do not equal 100%, as more than one norm was present in some responses.

Complexity in endorsed norms. When they disagreed with others' solutions or justifications, the PSTs who endorsed explaining their own solutions reported actions such as "contribut[ing] my opinions to the discussion" (Elsa) and "stat[ing] that I disagreed and explain[ing] why I disagreed" (Amelia). Similarly, Olivia mentioned, "I would have stated that I respectfully disagreed, as well as provided reasoning for why I disagreed that way they understood my point of view." These responses suggest a focus solely on their own solutions without trying to make sense of how others approached the problem. In contrast, there were PSTs who attempted to make sense of others' solutions and justifications during disagreements. These PSTs reported actions such as "try[ing] to consider another perspective" (Molly) and "open[ing] it up for a discussion to try to understand my classmate's thinking" (Ann). When Lola disagreed with her classmates' perspectives, she noted, "I just tried to look at it in a different perspective and see where they were coming from." Similarly, Jenny reported, "I feel that I would question it. I would want to try to understand that person's way of thinking about the task." These responses illustrate how PSTs also endorsed Yackel et al.'s (1991) norm of trying to make sense of others' approaches.

Although an equal number of PSTs endorsed norms related to explaining their own solution and attempting to make sense of others' solutions during disagreements in Section 1 and Section 3, explaining their own solution when disagreeing with others' solution approaches was most prevalent in Section 2. Additionally, in Section 2, almost 30% of PSTs ($n = 6$, 29%) reported reactions to disagreeing with colleagues that could be described as anti-norm reactions, actions that conflict with Yackel et al.'s (1991) norms. Anti-norms were present in each section and reflected PSTs' impassiveness when disagreeing with others. Examples of anti-norm statements about disagreeing include PSTs who "probably wouldn't have said anything" (Wendy) and who would have "just kinda brushed it off" (Laura). Anti-norm reactions might have been the results of PSTs not feeling "comfortable directly saying one of my classmate's answers was wrong" (Kelly) or not feeling confident enough to disagree. As Lily confided, "I would not say anything about disagreeing because everybody thinks differently and it probably would have made sense to other people even if it did not make sense to me."

Complexity in reactions. The different norms that were endorsed by PSTs connected to various actual and anticipated reactions to disagreeing with others' solutions and justifications. These reactions were active and public ($n = 34$, 57%), active and internal ($n = 10$, 17%), and passive ($n = 12$, 21%). When reactions were active and public, PSTs vocalized aspects of their disagreement by stating their disagreement, sharing their way of solving the problem, and asking questions to explore others' perspective, as previously discussed. PSTs also "used the prompts that were given [for discussion]" (Maria) as they vocalized aspects of their disagreement.

Approximately 10% of PSTs ($n = 6$) explicitly referenced using the provided discussion prompts (see Figure 2) in their reactions to disagreements. There were also PSTs who actively disagreed through internal grappling with others' perspectives, such as Hannah, who "listened to the other justifications and tried to understand the reasoning behind them." Similarly, Susan mentioned, "I just let it sink in and tried to understand where they were coming from," and Mae noted, "I just listened and tried the problem out in my head." These responses suggest that even when no public action was taken, some PSTs were actively engaged in attempting to understand others' solutions and justifications. In contrast, there were also PSTs who reported passive reactions to disagreeing with their classmates. These reactions were the anti-norm reactions to disagreeing with others, which included "remain[ing] quiet" (Calleigh) and doing "nothing" (Lynn).

Discussion and Future Directions

In addressing our first research question, related to the norms that individual PSTs endorsed through self-reporting after a discussion of a justification-oriented mathematics task, responses suggest that PSTs endorsed the norm related to explaining their own solutions with a greater frequency than the norm related to attempting to make sense of others' solutions and justifications. This pattern was present in PSTs' responses to both questions of interest on the survey. This suggests that as mathematics teacher educators prepare elementary PSTs to engage school students in meaningful mathematical discussions (AMTE, 2017; National Council of Teachers of Mathematics [NCTM], 2014), emphasizing the critique of others' reasoning is necessary as PSTs develop their skills "to engage all students in mathematical discussions involving explanation *and* critique as well as encourage the belief that each student can make valued contributions" (AMTE, 2017, p. 86, emphasis added). The development of this social norm has the potential to increase learning opportunities for PSTs and the elementary school students with whom they work as both PSTs and their students attempt to make sense of others' reasoning (Yackel et al., 1991).

As we addressed our second research question, we noticed complexities in PSTs' descriptions of their actual and anticipated reactions to disagreeing with others' solutions or justifications. These complexities include the presence of internal grappling while disagreeing with others in ways that might not outwardly signal engagement as well as the presence of anti-norm and passive behaviors. This suggests that as mathematics teacher educators position PSTs as active members of the discourse community (e.g., NCTM, 2014), intentionality must be used to help PSTs make their thinking public. We see the use of discussion prompts (see Figure 2), which 10% of PSTs ($n = 6$) explicitly referenced, as one strategy for encouraging the development of Yackel et al.'s (1991) social norms.

The patterns and complexities related to social norms we report here reflect PSTs' perspectives after engaging in a Which One Doesn't Belong task. In exploring the norms endorsed by elementary PSTs, future work could investigate whether and how these norms continue to develop as PSTs experience additional open-justification-oriented mathematics tasks. Additional future work could also explore the impact of different types of open tasks (e.g., How Many? [Danielson, 2018], Notice and Wonder [Fetter, 2013]) on the PSTs' endorsed norms as well as what instructional strategies (e.g., discussion prompts) impact PSTs' endorsed norms. It will also be important to compare and identify the relationship between PSTs' endorsed norms and enacted norms within the context of the discussion as well as the extent to which enacted norms (and anti-norms) afford or constrain PSTs' opportunities to learn. This study provides a necessary foundation in terms of endorsed social norms for individual activity on which future work will build.

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TRAJECTORY-BASED ANALYSIS OF INSTRUCTIONAL BELIEF CHANGE IN PRE-SERVICE MATHEMATICS TEACHERS

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This study examines how pre-service elementary mathematics teachers' instructional beliefs change over time during a course discussing constructivism. Adopting a longitudinal perspective, the study captures heterogeneous and nonlinear belief trajectories using data from 80 participants collected at four time points over a 15-week semester and analyzed through distance-based longitudinal clustering. Four distinct trajectories emerged, differing in initial levels, timing, and stabilization, with teacher-centered beliefs playing a more decisive role than student-centered beliefs. Most participants showed substantial shifts during the early phase of the course, suggesting a critical window for change, after which trajectories stabilized. These findings demonstrate that mean-level analyses obscure key dynamics in teacher learning and underscore the value of trajectory-based approaches in mathematics teacher education.

Keywords: Teacher Belief, Elementary Teacher Education, Longitudinal Clustering Method

Introduction

Research on pre-service teachers' instructional beliefs has long emphasized the importance of shifting from teacher-centered to student-centered approaches in mathematics classrooms. Within this body of work, belief change is often conceptualized as a desirable outcome of teacher education programs grounded in constructivist perspectives (Cobb & Yackel, 1996). Consequently, many empirical studies have examined whether instructional beliefs change before and after an intervention, typically reporting mean-level differences between pre- and post-measurements. While such findings provide valuable evidence regarding overall program effectiveness, they rest on an implicit assumption that belief change unfolds in a linear and homogeneous manner across individuals.

However, growing research suggests that teacher learning processes are rarely uniform or linear. Pre-service teachers enter teacher education programs with well-established instructional beliefs shaped by years of prior schooling, and exposure to constructivist ideas does not necessarily lead to coherent or consistent belief change. Instead, new instructional ideas often coexist with prior beliefs, resulting in tension, partial adoption, or selective reinterpretation. From this perspective, focusing solely on mean-level change risks obscuring important differences in how instructional beliefs evolve over time.

To address this limitation, the present study adopts a longitudinal perspective on instructional belief change among pre-service mathematics teachers participating in a semester-long course discussing constructivism. Rather than focusing on overall gains in student-centered beliefs, the study foregrounds individual trajectories of change, examining how different patterns of belief change emerge, when major shifts are most likely to occur, and which belief dimensions play a central role in differentiating these patterns. Using a distance-based longitudinal clustering method applied to repeated measures collected at four time points, the study aims to provide a more nuanced account of instructional belief change that moves beyond traditional pre–post designs.

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Theoretical Framework

Instructional Beliefs in Mathematics Teacher Education

Instructional beliefs refer to teachers' conceptions about the nature of teaching and learning, including the roles of teachers and students in classroom activity. In mathematics education, these beliefs are often discussed along a continuum between teacher-centered and student-centered orientations (Pajares, 1992; Fang, 1996). Teacher-centered beliefs emphasize explanation, control, and the efficient transmission of mathematical knowledge by the teacher, whereas student-centered beliefs highlight students' active participation, exploration, and meaning construction.

Research has consistently shown that instructional beliefs are deeply rooted in teachers' prior learning experiences and cultural contexts (Pajares, 1992). Pre-service teachers, in particular, enter teacher education programs with relatively stable beliefs shaped through years of observing traditional classroom instruction, a process commonly described as the "apprenticeship of observation" (Lortie, 1975, as discussed in Pang, 2003). These beliefs function not merely as abstract preferences but as practical frameworks that guide how teachers interpret instructional situations and make pedagogical decisions (Fang, 1996).

Coexistence and Tension Between Teacher-Centered and Student-Centered Beliefs

Rather than forming a simple dichotomy, teacher-centered and student-centered beliefs often coexist and interact in complex ways. Previous studies have shown that teachers may endorse student-centered principles at a normative level while continuing to rely on teacher-centered practices in actual classroom settings (Pang, 2003; Fives & Buehl, 2016). This gap between adoption and enactment suggests that instructional beliefs are not unified or internally consistent but are structured across multiple levels, including normative beliefs about what ought to be done and practical beliefs about what is feasible in real classrooms.

In the Korean mathematics education context, this tension has been repeatedly documented. Pang (2003) demonstrated that many elementary mathematics teachers expressed strong support for student-centered instruction while maintaining explanation-driven, teacher-led practices. Such findings indicate that belief change cannot be understood solely as the acquisition of new student-centered ideas but must also account for the persistence and reorganization of existing teacher-centered beliefs.

Beyond Linear Models of Instructional Belief Change

Despite recognition of the complexity of instructional beliefs, much empirical research has conceptualized belief change using linear or stage-based models, often relying on pre–post comparisons. While these approaches provide useful information about overall trends, they implicitly assume that belief change progresses in a uniform direction and at a similar pace across individuals.

Recent studies on teacher learning challenge this assumption by emphasizing the nonlinear and dynamic nature of change. Drawing on professional development research, Bae and colleagues (2022) argued that changes in teachers' instructional orientations are better understood as generative processes involving disruption, reorganization, and stabilization rather than gradual accumulation. Similarly, Clarke and Hollingsworth's (2002) interconnected model conceptualizes teacher change as an interactive process among beliefs, practices, outcomes, and reflection, highlighting the possibility of uneven and recursive change pathways. These models collectively suggest that belief trajectories are not reducible to a single developmental arc; rather, distinct patterns of disruption and stabilization may emerge depending on the interaction between individual belief configurations and instructional experiences. The present study operationalizes

this perspective by applying trajectory-based analysis to capture such differentiated pathways empirically.

A Trajectory-Based Perspective on Instructional Beliefs

Building on these perspectives, this study adopts a trajectory-based framework for understanding instructional belief change. From this viewpoint, belief change is not evaluated solely by its direction (e.g., toward student-centered instruction) but by the patterns through which change unfolds over time. This framework foregrounds three key dimensions: initial belief configurations, the timing of major shifts, and subsequent stabilization processes.

First, initial configurations matter because pre-service teachers begin teacher education with differing degrees of teacher-centered and student-centered beliefs, shaped by their prior schooling experiences (Pajares, 1992; Pang, 2003). Second, the timing of change is critical. Research suggests that belief change may concentrate in particular periods when existing beliefs are challenged by instructional experiences that cannot be easily assimilated (Bae et al., 2022). Third, stabilization refers to how beliefs settle into relatively coherent patterns after periods of disruption, potentially leading to qualitatively different outcomes across individuals (Franke et al., 2001).

Research Questions

The purpose of this study is to examine how pre-service mathematics teachers' instructional beliefs change over time within a shared constructivist-oriented course context. Rather than treating belief change as a uniform or linear process, the study adopts a longitudinal perspective to explore whether qualitatively different patterns of change emerge across individuals.

Guided by this perspective, the study addresses the following research questions:

1. What distinct trajectories of instructional belief change emerge over time among pre-service mathematics teachers?
2. How do these trajectories differ in terms of the timing and magnitude of change across the course?
3. Which dimensions of instructional beliefs contribute most strongly to differentiating these trajectories?

Methods

Participants and Context

Participants were pre-service elementary mathematics teachers enrolled in a semester-long constructivist-oriented mathematics education course at a university in South Korea. The course was designed not to train specific instructional techniques but to encourage critical reflection on participants' existing instructional beliefs through readings, discussions, and writing activities centered on constructivist perspectives on mathematics teaching and learning.

Instructional belief data were collected from 132 participants across four measurement points (Weeks 1, 6, 9, and 12). To ensure the reliability of longitudinal trajectory analysis, only participants who provided valid responses at three or more time points were included in the analysis. This resulted in a final analytic sample of 80 pre-service teachers.

Measures

Instructional beliefs were measured using a previously validated scale (Hwang et al., 2022) that distinguishes between teacher-centered beliefs (TC) and student-centered beliefs (SC). Teacher-centered beliefs reflect emphasis on explanation, control, and efficient knowledge transmission by the teacher, whereas student-centered beliefs emphasize students' active

engagement, exploration, and meaning construction. All items were rated on a five-point Likert scale. Internal consistency reliability was satisfactory to high (Cronbach's $\alpha = .761–.943$) across all measurement points for both belief dimensions.

Analytic Approach

To examine heterogeneous patterns of instructional belief change over time, the study employed a distance-based longitudinal clustering method using the SFclust algorithm implemented in the FKmL package. Unlike variable-centered approaches that focus on average change, SFclust is designed to identify groups of individuals who share similar trajectories across multiple time points.

SFclust was selected for three main reasons. First, the method does not impose a priori assumptions about the functional form of change (e.g., linear or quadratic), making it well suited for capturing nonlinear belief trajectories. Second, it allows multiple belief dimensions to jointly contribute to cluster formation, enabling an integrated examination of teacher-centered and student-centered beliefs over time. Third, the method is particularly appropriate for relatively small samples with repeated measurements, as it emphasizes pattern similarity rather than parameter estimation.

Prior to clustering, longitudinal data were organized in long format, with time represented as a numeric variable. Partial missing data were addressed using linear interpolation to preserve the continuity of individual trajectories. To prevent scale differences from disproportionately influencing distance calculations, belief variables were standardized using their respective standard deviations. An L1 constraint was applied during clustering to regulate variable contributions and enhance interpretability of cluster structures.

Determination of the Number of Clusters

The number of clusters was determined by considering both cluster stability and interpretability. Solutions with two to six clusters were evaluated through repeated analyses with different initial seeds. Cluster stability was assessed using the Adjusted Rand Index (ARI), and the minimum cluster size was examined to ensure that each cluster remained interpretable. Based on these criteria in Table 1, a four-cluster solution was selected as providing the best balance between stability and meaningful differentiation of belief trajectories.

Table 1: Cluster Stability Indices Across Different Numbers of Clusters

Number of Clusters	Mean ARI	SD of ARI	Minimum ARI	Maximum ARI	Minimum Cluster Size (Median)	Minimum Cluster Size (Minimum)
2	0.713	0.211	0.484	1.000	36.5	31
3	0.573	0.192	0.374	1.000	22.5	13
4	0.501	0.187	0.267	1.000	15.0	10
5	0.466	0.168	0.247	1.000	10.0	4
6	0.471	0.167	0.218	0.860	7.0	2

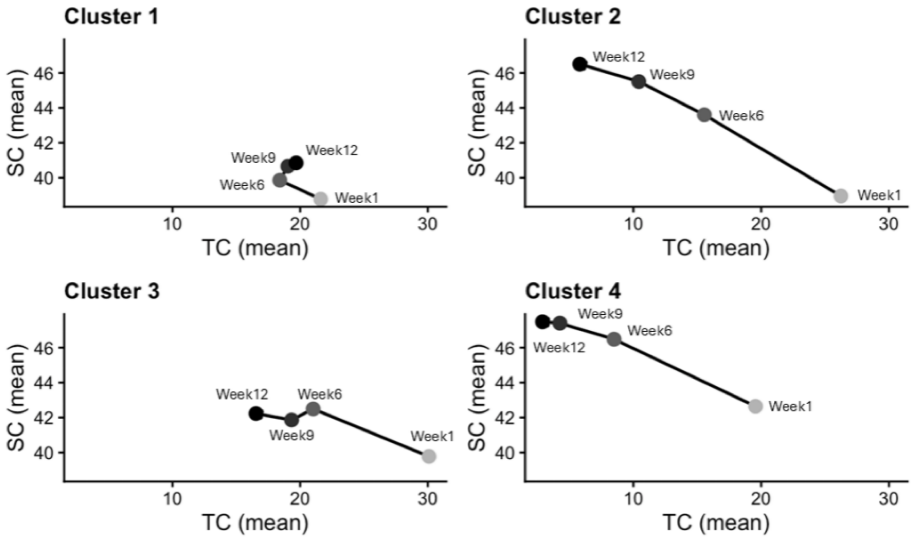
Results

The SFclust-based longitudinal clustering analysis identified four distinct trajectories of instructional belief change among pre-service mathematics teachers. These trajectories differed not only in the direction and magnitude of change but also in their initial belief configurations and temporal patterns of disruption and stabilization. Importantly, these differences were not

visible in mean-level analyses, which suggested a general decrease in teacher-centered beliefs and an increase in student-centered beliefs across the cohort.

Across the four trajectories, teacher-centered beliefs (TC) played a more decisive role than student-centered beliefs (SC) in differentiating clusters. Variable weight estimates indicated that TC contributed substantially more to cluster formation than SC, suggesting that patterns of weakening, persistence, or reorganization of teacher-centered beliefs were central to understanding heterogeneous belief trajectories.

Analysis of longitudinal trajectories revealed four qualitatively distinct patterns of change in instructional beliefs over time (Figure 1). These trajectories differed not only in the direction and magnitude of change but also in their initial belief levels and temporal dynamics. In particular, variation across clusters was more pronounced for TC than for SC, indicating that patterns of weakening or persistence of TC played a central role in differentiating trajectories.



Cluster (n)	Teacher-Centered Beliefs: Change Pattern	Student-Centered Beliefs: Change Pattern	Trajectory Type
Cluster 1 (13)	Initial decrease followed by stable maintenance	Gradual increase	Stable Maintenance
Cluster 2 (20)	Sharp decrease following a high initial level	Sustained and pronounced increase	High-Start Rapid-Shift
Cluster 3 (22)	Highest initial level with gradual decrease	Slight increase followed by stabilization	Gradual Transition
Cluster 4 (25)	Low initial level followed by a drop to the minimum	Continuous increase from a high level	Low-Start Consolidation/ Transition

Figure 1: Characteristics of Instructional Belief Change Across Clusters

With respect to teacher-centered beliefs, clusters exhibited markedly different change patterns. One trajectory was characterized by a modest initial decline followed by relative stability, indicating limited reorganization beyond the early phase of the course. In contrast,

another trajectory showed a steep and sustained decline from a high initial level, reflecting substantial weakening of teacher-centered beliefs across the semester. A third trajectory began with the highest initial TC levels and demonstrated a gradual but consistent decrease, suggesting a slower and more incremental adjustment process. The final trajectory was marked by relatively low initial TC levels and a rapid early drop to minimal levels, followed by sustained stability, indicating an early and decisive shift away from teacher-centered beliefs. Taken together, these findings demonstrate that teacher-centered beliefs did not decline uniformly across participants; rather, the timing and intensity of TC change varied substantially, resulting in distinct patterns of belief reorganization.

In contrast, student-centered beliefs showed more homogeneous patterns of change across clusters. All trajectories were characterized by either gradual increases or consistently high levels of SC over time. Differences across clusters were primarily reflected in the rate and timing of increase rather than in overall direction. Some clusters exhibited modest growth in SC that paralleled relatively stable TC trajectories, whereas others showed sharp and sustained increases in SC following early declines in TC. In one trajectory, SC increased slightly and then stabilized, while in another, SC remained high from the outset and continued to increase steadily throughout the course. These patterns indicate that increases in student-centered beliefs alone were insufficient to differentiate belief trajectories.

Across clusters, the most substantial changes in instructional beliefs occurred during the early phase of the course, particularly between Weeks 1 and 6. During this period, pronounced declines in teacher-centered beliefs were observed in several trajectories, whereas one trajectory exhibited only modest early change followed by stabilization. After the initial phase, belief trajectories tended to stabilize along divergent paths. For some clusters, teacher-centered beliefs converged toward relatively stable end states, while others continued to show gradual change. This temporal pattern suggests that early instructional experiences played a critical role in shaping subsequent belief trajectories, with later phases of the course serving primarily to consolidate rather than fundamentally reshape instructional beliefs.

To further examine how teacher-centered and student-centered beliefs interacted over time, trajectories were visualized in a two-dimensional belief space (Figure 1). The resulting patterns revealed distinct movement paths across clusters. Some trajectories showed limited overall movement, characterized by modest growth in student-centered beliefs without substantial disruption of teacher-centered beliefs. In contrast, other trajectories exhibited pronounced shifts marked by rapid declines in teacher-centered beliefs accompanied by strong growth in student-centered beliefs. Still others followed more gradual transition paths involving moderate TC decline and relatively stable SC levels. These joint trajectories underscore that instructional belief change cannot be reduced to a single dimension or directional shift; rather, qualitatively distinct patterns emerged through different configurations and interactions of teacher-centered and student-centered beliefs over time, even within a shared instructional context.

Discussion

The present study examined instructional belief change among pre-service mathematics teachers using a longitudinal, trajectory-based approach. Guided by three research questions, the discussion focuses on (a) the identification of distinct trajectories of change, (b) differences in the timing and magnitude of change across trajectories, and (c) the belief dimensions that most strongly differentiated these trajectories. Together, the findings underscore that instructional belief change is best understood as a heterogeneous and dynamic process rather than a uniform shift toward student-centered instruction.

Distinct Trajectories of Instructional Belief Change

Addressing the first research question, the analysis identified four qualitatively distinct trajectories of instructional belief change within a shared course context. These trajectories revealed substantial heterogeneity in how pre-service teachers' beliefs evolved over time, despite exposure to the same constructivist-oriented instructional experiences. This finding challenges assumptions implicit in mean-level analyses that belief change follows a single developmental pathway and instead highlights the coexistence of multiple patterns of reorganization.

Importantly, the identified trajectories differed not only in direction but also in their initial belief status and overall shapes. Some trajectories reflected early and decisive shifts away from teacher-centered beliefs, whereas others showed more gradual adjustment or relative stability after an initial change. These results suggest that instructional belief change involves qualitatively different developmental processes across individuals, rather than varying degrees of the same process.

Timing and Magnitude of Change

The second research question concerned how trajectories differed in the timing and magnitude of change. Across most trajectories, the most substantial changes occurred during the early phase of the course, particularly between Weeks 1 and 6. During this period, several clusters exhibited pronounced declines in teacher-centered beliefs, indicating that early instructional experiences played a critical role in disrupting existing belief structures.

Following this early phase, belief trajectories tended to stabilize along divergent paths. For some pre-service teachers, early disruption led to sustained reorganization, whereas for others, beliefs stabilized with only modest change. The existence of a trajectory characterized by limited early change followed by stability further illustrates that exposure to the same instructional context does not guarantee comparable belief transformation. These findings highlight the importance of attending to temporal dynamics in teacher education research and practice, suggesting that when change occurs may be as consequential as whether change occurs.

Central Role of Teacher-Centered Beliefs

The third research question examined which belief dimensions contributed most strongly to differentiating trajectories. The findings indicate that teacher-centered beliefs played a more decisive role than student-centered beliefs in shaping distinct patterns of change. While student-centered beliefs generally increased or remained high across all trajectories, they were insufficient on their own to distinguish among patterns of belief development.

Instead, trajectories were primarily differentiated by how teacher-centered beliefs weakened, persisted, or were reorganized over time. This finding suggests that instructional belief change cannot be adequately conceptualized as the accumulation of student-centered beliefs. Rather, meaningful change appears to involve varying degrees of destabilization and reconfiguration of existing teacher-centered beliefs, which often remain deeply rooted in pre-service teachers' prior schooling experiences and cultural norms.

Conclusion

Taken together, these findings have important implications for mathematics teacher education. First, they suggest that programs should attend explicitly to pre-service teachers' existing teacher-centered beliefs, particularly during the early stages of coursework. Simply promoting student-centered instructional principles may be insufficient if entrenched beliefs about teaching, authority, and knowledge transmission remain unexamined. Early instructional activities that surface, challenge, and problematize these beliefs may be especially influential in shaping longer-term belief trajectories.

Second, the presence of multiple trajectories within the same instructional context points to the need for differentiated support. Pre-service teachers enter teacher education with varying initial belief configurations, and these starting points appear to condition how individuals respond to instructional experiences. Recognizing this heterogeneity may help teacher educators design learning environments that are more responsive to different patterns of resistance, adjustment, and stabilization.

Methodologically, this study demonstrates the value of trajectory-based analytic approaches for investigating instructional belief change. By identifying heterogeneous and nonlinear patterns of change, distance-based longitudinal clustering offers insights that remain hidden in traditional pre–post designs. Future research could extend this work by incorporating qualitative analyses to examine how pre-service teachers interpret instructional experiences at key transition points, or by linking identified belief trajectories to subsequent instructional practices during teaching practica or early career teaching. Applying trajectory-based approaches to larger and more diverse samples would further clarify how instructional belief change unfolds across contexts. It should also be noted that distance-based clustering results may be sensitive to the choice of clustering parameters and initial seeds; while multiple runs and ARI-based stability checks were used to mitigate this concern, replication with independent samples remains an important direction for validating the identified trajectory structure.

In conclusion, this study advances understanding of instructional belief change in mathematics teacher education by aligning belief development with multiple trajectories rather than a single linear pathway. By directly addressing the research questions, the findings highlight the existence of distinct belief trajectories, the critical role of early phases of change, and the central importance of teacher-centered beliefs in differentiating patterns of development. Adopting a trajectory-based perspective offers a more nuanced account of teacher learning and provides a foundation for designing teacher education programs that more effectively support meaningful and sustained instructional belief change.

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USING TUTORING TO SUPPORT ELEMENTARY TEACHER CANDIDATE CONFIDENCE

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This study explored changes in 48 senior elementary teacher candidates' confidence relating to planning and adjusting mathematics lessons, assessing student understanding, supporting critical thinking, motivating low-interest students, establishing routines, and working with families after completing a tutoring project as part of their math methods course. Paired sample t-tests of survey data measuring confidence before and after the tutoring showed significant gains in all areas. Qualitative results indicated tutoring activities supported candidate confidence in pedagogical knowledge for breaking up standards, creating and implementing organized yet flexible lessons, assessing student understanding beyond correct answers using activities and questioning, and finding or adapting appropriate resources. One significant theme across all data was the improved confidence for working with parents and families.

Keywords: Elementary School Education, Preservice Teacher Education

Litster, Ramirez, and Terry (2025) note that in order to maintain quality in elementary content and methods courses over time, there must be a process of ongoing review and refinement of courses that keeps in mind the needs of the teacher candidates and their professors. Active learning tasks should be one of those considerations. Mathematics teacher educators (MTEs) have a variety of pressures in the content and processes they include in their methods course designs. Litster et al. (2000) warn that "active learning does not mean adding content to an already limited schedule" (p. 635). Instead, they note that MTEs must carefully choose which types of tasks or projects the TCs will engage with that best support mathematics teaching and learning expectations through active learning experiences.

We advocate that a tutoring project is one such active learning experience that provides multiple benefits to TCs' confidence in teaching and learning mathematics as well as their pedagogical content knowledge (PCK). Thus, the purpose of this study was to explore how refining a mathematics content course to exchange simulated teaching experience projects with a tutoring project influenced changes in teacher candidates' (TCs') confidence for teaching and learning mathematics.

Theoretical Framework

This study is grounded in Situated Learning Theory (Lave & Wenger, 1991). This theory posits that learning cannot be acquired solely in isolation; rather, teacher candidates must engage in socially authentic contexts that are real, low-risk, and include active rather than passive involvement. Darling-Hammond et al. (2002) stated that TCs' "sense of preparedness is by far the strongest predictor of teaching efficacy" (p. 294). They noted that programs in their study that helped TCs feel better prepared were those that interwove practicum experiences with coursework. Experiential learning helps teacher candidates connect with communities, apply practical knowledge, and apply theory to practice (Williams & Sembiante, 2022).

Multiple research studies have confirmed that real contexts, such as early field experiences that interweave experiences and coursework, can provide low-stress practice that supports TCs' self-efficacy and identity as a teacher. For example, Iwai (2025) found that early field

experiences helped students improve their self-confidence in instructional strategies, student engagement, and classroom management. The only area in which students did not increase in confidence was family engagement, due to a lack of experience in this area during the early field experience.

Litster et al. (2025) found that the implementation phase of an interview project, where TCs worked directly with a child, showed the highest percentage of evidence across multiple institutions for specialized content knowledge and pedagogical content knowledge. Abbott et al. (2026) showed improved confidence, identity and classroom readiness from site-based methods courses. These improvements were primarily based on positive perceptions of "real" settings, opportunities for curricular development, and development of TCs' roles as teachers.

While students can learn from simulated experiences such as peer demonstrations or computer-based classroom simulations, the level of realism matters (Theelen et al., 2019). When preservice teachers (PTs) perceive the simulated or peer behavior as unrealistic or exaggerated, engagement decreases and confidence gains are limited. While Artificial Intelligence (AI) and technology continue to improve realism, working directly with a K-5 student allows TCs to elicit and use evidence of student thinking (National Council of Teachers of Mathematics [NCTM], 2014), practice noticing skills (Jacobs et al., 2010), and develop pedagogical content knowledge (Ball et al., 2008).

Methods

This study used a convergent-parallel mixed methods design (Creswell & Plano Clark, 2011), gathering quantitative and qualitative data concurrently and merging them using complementary techniques (Teddlie & Tashakkori, 2006) to answer the research question: How did teacher candidates' confidence for teaching mathematics change after participating in a tutoring project?

Participants were 48 teacher candidates in their third of four professional semesters, enrolled in the only mathematics methods course in their program. During this course, students engaged in a tutoring project where they conducted a pre-assessment, designed and implemented 25-minute math lessons with one or two tutoring students twice a week for approximately eight sessions based on tutoring data, and met with parents to share student outcomes as well as recommend resources.

K-5 students were recruited from districts and private schools near the university. Parents or family members brought the students to the university twice a week for free hour-long tutoring sessions in math and literacy (25 min each).

Quantitative data included a self-efficacy survey measuring candidates' confidence in nine areas of mathematics teaching before and after participating in the tutoring project. The survey was adapted from Tschannen-Moran & Woolfolk Hoy's (2001) teacher efficacy survey. Teacher candidates rated themselves on a scale of 1–9 in each area. Paired samples *t*-tests were computed using SPSS to identify significant changes in TCs' confidence before and after the tutoring project.

Qualitative data included four open-response questions evaluating change in confidence related to (1) identifying and breaking up elements of a math standard, (2) assessing student thinking, reasoning, and understanding, (3) lesson planning and teaching, and (4) finding and using math resources. An additional question asked candidates to reflect on their overall impressions of the tutoring project. Descriptive and pattern coding were used to analyze TC responses for patterns and themes (Saldaña, 2015).

Results

We first present the quantitative paired samples *t*-test statistics and results (see Table 1). Second, we present qualitative themes from the open-ended responses.

Table 1: Paired Sample *t*-Test for Teacher Candidate Self-Efficacy Survey

	Statistics		Paired Differences							
						95% CID				Sig.
Self-Efficacy Criteria	MB	MA	M	SD	SEM	L	U	*t*	df	(2-t)
Plan and Adjust Math Lessons	4.98	7.38	–2.396	1.233	.178	–2.754	–2.038	–13.460	47	<.001
Gauge Student Comprehension	5.31	7.69	–2.375	1.496	.216	–2.810	–1.940	–10.996	47	<.001
Design and Ask Questions	5.60	7.63	–2.021	1.550	.224	–2.471	–1.571	–9.030	47	<.001
Encourage Critical Thinking	5.27	7.33	–2.063	1.535	.222	–2.508	–1.617	–9.306	47	<.001
Provide Alternative Explanations	5.33	7.50	–2.167	1.464	.211	–2.592	–1.742	–10.257	47	<.001
Motivate Low-Interest Students	5.77	7.68	–1.915	1.427	.208	–2.334	–1.496	–9.200	46	<.001
Establish Routines/Transitions	5.71	7.38	–1.667	1.243	.179	–2.028	–1.306	–9.286	47	<.001
Assist Families for Home Support	4.81	7.29	–2.479	1.624	.234	–2.951	–2.008	–10.576	47	<.001
Plan and Meet with Parents	2.38	4.04	–1.660	1.128	.165	–1.991	–1.328	–10.084	46	<.001

As shown in Table 1, all areas of the self-efficacy survey demonstrated statistically significant gains at the p <.001 level for change in teacher candidates' confidence after engaging in the tutoring program. The lowest areas of student confidence before (MB) starting the tutoring program were planning and meeting with parents, assisting families for home support, and planning and adjusting math lessons.

Teacher candidates showed a growth (M) in confidence of approximately 1.7 points in establishing routines/transitions and meeting with parents, and approximately 2 more points in the other areas. The greatest gains were in planning and adjusting lessons (2.4) and assisting families for home support (2.5). The highest areas of confidence after tutoring (MA) were gauging student comprehension and motivating low-interest students (7.7).

Qualitative results revealed multiple themes across five categories: (1) finding and breaking up standards; (2) planning and teaching; (3) assessment of student thinking, reasoning, and understanding; (4) finding and identifying resources; and (5) overall impressions of the tutoring project.

Two common themes relating to teacher candidate's confidence in finding and breaking up standards were (1) improvements in TCs' ability to align lessons to the standards and (2) breaking up elements of a standard into more time-appropriate or manageable chunks. One TC's reflection best illustrates the first theme relating to alignment.

> At the beginning of the tutoring project, I honestly felt unsure about breaking down a math standard because the wording sometimes felt confusing and overwhelming. After receiving feedback, I realized how important it was to stay aligned to the exact standard instead of adding extra components. This helped me better understand how standards guide instruction. By the end of the tutoring sessions, I felt much more comfortable analyzing standards and designing lessons directly from them.

Other TCs echoed these statements, citing the use of state resources to clarify alignment (I used websites like Georgia Inspire to clarify what the student needed to know.) and being intentional about activities and resource (Over time, I learned to break the standard into clearer parts. This shift helped me design more intentional activities; Overall, I feel more confident now in

choosing resources that actually match the learning goal instead of just picking whatever looks good.).

Breaking standards into smaller chunks also helped TCs manage time. As two TCs reflected, "At the beginning, looking at a full math standard felt overwhelming because there was so much packed into just one sentence. As I started planning lessons, I learned how to break the standard into smaller, manageable skills." and "Breaking it down helped me feel more confident making lessons that build understanding step by step." Breaking down the skills also supported student comprehension. Other TCs noted, "breaking the skill into smaller chunks made it easier for [student] to master each step and helped build his confidence;" "I noticed that when I slowed down and focused on one element at a time, the student understood the content better;" and "This experience showed me that breaking standards into smaller parts makes instruction more effective and less overwhelming for both students and myself."

Three common themes relating to planning and teaching were (1) structured organization to improve pacing and time management; (2) importance of flexibility during implementation; and (3) finding the right balance between fun, efficient, and rigorous. In regards to organization, students often noticed that the timing for their activities did not always match their original planning (I learned that what I planned on paper did not always match how long activities actually took or how engaged the students would be.). However, over time they were able to provide clearer organization to improve their pacing and time management. For example, one TC reflected, "My first few lessons were not as structured, and transitions felt rushed. Over time, I began organizing sessions into clear segments. This structure helped the sessions run more smoothly and increased my confidence." In another example, "Lesson planning and teaching required a lot of time and organizing instructions into clear, manageable steps to build upon the prior knowledge of what she already knew."

Teacher candidates often reflected on how the project reinforced the importance of flexibility and helped build this skill. One TC's reflection illustrates this theme:

> At first, lesson planning felt stressful because I worried about choosing the right activities and making sure everything made sense. However, teaching the lessons helped me realize that flexibility is just as important as planning. For instance, during the measurement lessons, I had to adjust pacing when we ran out of time measuring with paper clips. Each session made me feel more comfortable leading instruction and making decisions in the moment. By the end of tutoring, I felt much more confident in both planning and teaching.

The importance of flexibility was echoed by other TCs, such as these three examples: "These adjustments helped me feel more prepared and flexible as a tutor, rather than feeling stuck to a rigid plan." "In the beginning, I tried to plan every minute of the session, but I discovered that being responsive to [student's] energy and engagement made the lessons more effective." "I noticed that my teaching does not always follow the lesson plan exactly. At first, this made me feel unprepared, but I began to realize that this flexibility is actually responsive teaching."

Many TCs gained confidence in finding the right balance of fun, efficiency, and rigor. Some students noticed they started having fun, but not necessarily learning—"At the beginning of tutoring I wanted her strictly to have fun, but I noticed I was more focused on the activities rather than the learning goals. Once I realized my fault I made corrections to make it fun but also efficient." Other TCs had the opposite problem, but were still able to find balance—"From day one, my lessons included too much content, but I began to make the tutoring session change from just paperwork to more movement and engaging." Many TCs came to enjoy seeing students grow while also having fun (My favorite lesson was our answer and match puzzle. Both my

students were very engaged during these lessons and were showing great progress in their subtraction work.).

Three common themes relating to assessment of student thinking, reasoning, and understanding were: (1) difference between being correct and understanding; (2) identifying and responding to misconceptions; and (3) using probing questions. At least half of TCs reflected that when they started assessing students, they focused on being correct (At first, I mainly paid attention to whether the answer was correct; At first, I mainly looked for correct answers as evidence of understanding; At the beginning, I focused on whether they got the question right or wrong). However, over time this changed from "correct" to "understanding" (Over time, I realized that understanding how she got the answer mattered much more.). As TCs' focus changed, so did their assessment practices. For example, some TCs started using hands-on tasks as assessments, "instead of just asking for answers, I began asking him to show me with base ten blocks and explain what he was doing." Other TCs used conceptual questions (This showed me the importance of asking follow-up questions like, "How do you know?" or "Can you show me another way?" I began incorporating think-alouds to better assess conceptual understanding.) These changes increased TCs confidence in understanding students (I became more confident in using questioning, observation, and modeling to truly understand his thinking; I felt more confident analyzing student responses and adjusting instruction based on their reasoning rather than just accuracy; Working with [student] taught me that effective teaching starts with understanding how students think.).

Student were able to gain confidence in using their assessment strategies to identify misconceptions and respond with appropriate adjustments. These four examples illustrate this theme: "Observing how she positioned the hour and minute hands helped me identify misconceptions about half-hour concepts. These instances allowed me to change instruction immediately through modeling and guiding questions." "Hearing their explanations made it easier to catch misconceptions and adjust instruction in the moment." "I would have my students explain the reasoning behind their answers. This would eventually help them with changing their own mistakes the more I did it." "Asking questions allowed me to adjust my instruction and provide immediate support when misunderstandings occurred."

While conceptual questions allowed TCs to assess student understanding, probing questions helped prompt discourse and elicit student responses. Many TCs noted that at the beginning of the project, students did not talk a lot (e.g., "Especially with Kindergarten, kids do not always verbalize their misconceptions, struggles, thoughts, etc." "Initially, I found it challenging to effectively assess how well students understood the material, as they would sometimes provide incomplete or unclear responses."). However, questioning prompted the discourse (e.g., I learned to ask probing questions like, "What does your equation represent?"; I learned to ask probing questions and give students the opportunity to explain their reasoning.). These opportunities directly impacted TCs' confidence (My level of confidence increased in assessing student thinking, reasoning, and understanding because I became more comfortable asking guiding questions during lessons.)

Three common themes relating to finding and identifying resources were: (1) appreciation for accessible recourses, (2) shift from worksheets to hands-on activities, and (3) modifying resources to align with standard objectives. TCs appreciated having access to resources such as physical manipulatives from a nearby supply closet (I was able to get any physical resources I needed from the math room, which was very helpful.) or collaborative colleagues (Most of my activities were figured out through consultation with my classmates and professors). On day one

of the semester, the professor had TCs explore the resources in the supply closet to document pros and cons of similar manipulatives. TCs used these resources in almost every lesson (I chose to use fraction tiles, fraction circles, white boards, erasers, paper, and dry erase markers. I used these materials throughout all my lessons.), especially after seeing the benefits such as this TC, "The first day, the student would not really want to share with me how she did it. It wasn't until the second day when we used the manipulatives that the student was able to explain to me how she got her answers and show me many different ways to solve the problem." On the first day, TCs worked in groups to identify potential grade-specific activities. Some TCs continued this practice through the tutoring whereas others called on their colleagues as needed "At the start of the tutoring project, I relied heavily on activities I witnessed in my practicum field experience last semester. These did not always work for my student's needs so I asked my professor for help and ideas."

Many students started with worksheets, which is what they have experienced themselves and seen in the classroom ("At first, I relied mainly on worksheets;" "At the start of this project, I wasn't sure what types of resources would best support multiplication beyond worksheets."). However, by the end of the project, most students gained confidence in moving beyond worksheets based on their experiences with students. For example, one TC reflected:

> At first, I relied mostly on basic worksheets, but I realized that [Student A] and [Student B] learned better with interactive tools like counters, number lines, ten-frames, and simple subtraction games. I also learned to adapt resources rather than using them as-is, such as turning a worksheet into a game. This helped me feel more confident in choosing resources that matched both the standard and the learning needs of my students.

These additional examples also illustrate this shift: "I soon discovered that hands-on and interactive materials made the lessons more meaningful for [Student]." "I definitely got more creative during tutoring. I found myself being more resourceful with my teaching tools and this allowed for more creative lessons." "Later, I intentionally chose concrete models, interactive challenges, and even a scavenger hunt to make learning more meaningful."

The final theme brings together ideas prior themes relating to standard alignment and intentionality of resources as TCs reflected on their increased confidence in adapting resources to meet their needs. Two TC stated:

> Finding and using math resources was one of the areas in which I grew most confident. Because I clearly understood the purpose of each activity, modifying or designing resources felt more focused and intentional. This process strengthened my confidence in choosing materials that align with both the math standard and the student's need.

> I learned to evaluate resources more critically, ensuring they aligned directly with the math standard and learning goal. By the end of the project, I felt more confident locating high-quality math materials and adapting them to meet the specific needs of my student.

Although we cautioned students from paying for any resources during this course, several TC did find them useful, with adaptations: "Not everyone's favorite, but I love finding ideas on Pinterest and teachers pay teachers. I know these aren't 'educational' sites, but if you review them and tweak them, they can give you some great ideas." "Resources can be tailored to specific learning goals."

Three common themes stemming from TCs' overall impressions of the tutoring project were appreciation for: (1) the ability to practice planning and implementing creative mathematics

tasks, (2) opportunities to work and talk with parents, and (3) better understanding of how to be flexible and adaptive.

One TC reflected, "I got to see how effective my lessons were and self reflect on my teaching abilities. I also got to be creative and find ways to engage my students. I enjoyed seeing my students grow by the end of the sessions." Many students used creativity by incorporating student interests ("I got out of it was to find out what students like and really implement that into your lesson."), integrating real world contexts ("Finding different ways get the kids engaged and actually putting what they learn to real life"), and making it fun ("I think I was able to grow as an individual as well as a future educator because I got to learn how to plan for *one* student and how to make it a fun but educational time." "Instead of using a number line to explain a math concept on the table, I had them get up and "hop" when using the number line on the whiteboard.").

Another TC reflected, "I think what I got most from this experience was learning how to prepare for talking with parents. It was something we don't talk a lot about in our classes, so it was nice to get experience in a big aspect of teaching." This sentiment was echoed by at least half the TCs. For example: "I was nervous about the parent conferences but I feel like those outlines really helped a lot!" "I think the parent conference really boosted my confidence in talking to a child's parent." "One positive I got from this is to talk to parents and plan for conferences." "I have more confidence to meet with parents."

Many students reflected on improvements to flexibility and adaptability. For example: "The ability to be flexible and to account for different learners' strengths and weaknesses more fluidly." "I learned how to tweak and plan for personalized instruction based off a students' strengths and weakness and how to better support/scaffold within areas of struggle." "I really enjoyed getting to give assessments and base my teaching based off of the data. I was scared of my ability to do this." "I feel like I really felt what it is like as a teacher for your plans to never go the way you want them to. I felt the real feeling of defeat when you think that you aren't doing a good job but then realizing that things aren't nearly as bad as I thought. If this is going well, I can change it right now."

A few other reoccurring themes related to enjoying seeing the positive effects on student learning ("I really enjoyed watching my student grow educationally."), increased confidence to apply strategies to larger classroom settings ("Working with a student one-on-one and figuring out their learning needs. This could help me get an idea on where to start with each student in my classroom."), and better organization of resources ("It showed me that organization is *super* important when setting up lessons and unit plans.") and time ("Time management!") when working with students.

Discussion

The results support literature that active experiential learning opportunities through early field experiences that combine authentic contexts with coursework can support teacher candidate confidence (e.g., Abbott et al., 2026; Williams & Sembiante, 2022). Many early field experiences are classroom-based and do not provide TCs with opportunities to engage with parents and families, despite this being an important aspect of teaching (Iwai, 2025). This may explain why supporting and working with families were the two lowest areas of TCs' confidence in this study before the tutoring project.

A significant convergent result from this study demonstrated that an after-school tutoring project, in which TCs engage directly with parents or families, can increase confidence in planning and meeting with parents as well as assisting families with home support. Although meeting parents had the lowest post-score overall (improving from 2.38 to 4.04), assisting with

home support showed the largest gain (4.81 to 7.29).

Results also indicated improvements to TCs' confidence in applying pedagogical content knowledge (Ball et al., 2008), specifically relating to planning and adjusting lessons. TCs reported increased confidence in identifying and breaking up standards into time-appropriate and developmentally appropriate components as they sequenced their lessons across specific units of instruction for their students. They also improved their ability to organize activities, materials, and transitions to directly align with those standards and their students' needs. Finally, practice adapting lessons during and between tutoring sessions supported an increase in confidence for flexibility when planning and teaching mathematics.

TCs indicated an improved confidence in using noticing skills to identify and use evidence of student thinking (Jacobs et al., 2010; NCTM, 2014). This included confidence in gauging comprehension beyond correct answers by using probing questions to encourage critical thinking, identify misconceptions, and verify understanding of the "why" behind the "how". In addition to confidence in the content and pedagogy, TCs demonstrated increased confidence in their ability to motivate and engage students. By the end of tutoring, many shifted from relying primarily on worksheets to incorporating hands-on conceptual explorations, real-world connections and story problems, and engaging practice games or activities.

In conclusion, community tutoring is a structure that differs from traditional field experiences. As one student said, "Lesson planning for tutoring looked a little different, as it was a lot more focused within the time allotted for the specific subjects, with certain areas being focused on such as representing learning in multiple ways, prompting student questioning and higher-order thinking, and the implementation of literacy into math, all working together to not only assess the student, but to build instruction around their strengths and weaknesses and support them in acquiring the skills and knowledge necessary to improve." Similar to Darling-Hammond et al. (2002), the interweaving of experiential learning and coursework allowed for structured application with specific ongoing support from the professors. We highly recommend that teacher education programs consider incorporating early community-based field experiences, such as tutoring clinics within their methods courses to support teacher candidate confidence in teaching mathematics.

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COGNITIVE SHIFTS IN PRE-SERVICE SECONDARY TEACHERS' CONCEPTUAL AND PROCEDURAL COORDINATION ABOUT ASYMPTOTIC BEHAVIOR

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We present the results of a qualitative study that investigated the cognitive mechanisms through which pre-service secondary mathematics teachers (PSMTs) coordinate their conceptual and procedural knowledge about the asymptotic behavior of rational functions. While literature establishes that PSMTs often favor procedural proficiency over conceptual depth, the internal reorganization of these knowledge forms remains underexplored. Grounded in Piaget's theory of reflecting and reflected abstraction, we analyzed how participants' schemes evolved during task-based clinical interviews. Findings reveal that disrupting routine procedures with anomalous function behaviors triggers reflected abstraction, leading participants to integrate structural explanations with procedural execution. These results suggest that teacher education should engender cognitive conflict to foster specialized content knowledge in PSMTs.

Keywords: Preservice Teacher Education, Cognition, Learning Theory, Undergraduate Education

Shulman (1986) proposed that effective teaching goes beyond procedural competence, requiring teachers to understand the concepts and principles of a discipline (*substantive structures*) and comprehend the means by which they are established (*syntactic structures*). Building on this perspective, Tallman (2021, 2023) argued that teachers' subject-matter content knowledge is most effective when their mathematical schemes are coherent and coordinated. Tallman further proposed that a teacher's awareness of the mental actions and conceptual operations organized within these schemes is crucial to developing content knowledge with the potential to inform effective pedagogical action.

Star and Stylianides (2013) demonstrated that teachers' application of conceptual and procedural knowledge profoundly influences how they assess and respond to students' work, underscoring the practical significance of understanding how teachers might be supported in integrating their procedural and conceptual knowledge structures. Empirical findings further suggest that pre-service secondary mathematics teachers often perform procedures successfully while demonstrating limited conceptual justification (Tchoshanov & Monarrez, 2024). However, less is known about the cognitive processes through which procedural action schemes are destabilized and reconstructed into structurally grounded explanations.

This gap is particularly salient in the context of teachers' knowledge about the asymptotic behavior of rational functions, where teachers' fluent application of common procedures—such as comparing degrees or identifying zeros of the denominator—often dissuade the need for conceptual justifications that connect limiting behavior with the algebraic structure of rational expressions. Drawing on Piaget's notions of *action schemes*, *reflecting abstraction*, and *reflected abstraction*, this study examines how procedural schemes are enacted, disrupted, and reorganized to develop deeper conceptual understanding. Specifically, this study was guided by the research question: *How do pre-service secondary teachers' mathematical schemes shift as they coordinate procedural and conceptual knowledge of asymptotic behavior in rational functions throughout their participation in task-based clinical interviews*?

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Procedural and Conceptual Knowledge of Asymptotic Behavior

Yerushalmy (1997) proposed that asymptotes can be understood in a meaningful way or at a surface level, which depends heavily on recall of declarative knowledge. For instance, a vertical asymptote was meaningfully understood by participants of Yerushalmy's study as "a vertical line that exists only at points where the function is not defined and the function's values approach negative or positive infinity" (Yerushalmy, 1997, p. 8). Participants conceptualized a horizontal asymptote as "a horizontal line, parallel to the *x*-axis, that comes closer to the curve at infinity" (Yerushalmy, p. 12), and oblique asymptotes were characterized by the function's linear end behavior without being limited to finite values (Yerushalmy, p. 8). Building on these meanings, we outline the basic procedural and conceptual knowledge of the horizontal, vertical, and slant asymptotes of rational functions in Table 1.

Table 1: Basic Procedural and Conceptual Knowledge of Asymptotes

	Procedural Knowledge	Conceptual Knowledge
Horizontal Asymptote	A horizontal line found by comparing the degrees of numerator and denominator of a rational expression.	A horizontal line represents the limit of a function's values as $x \to \pm\infty$; the function may cross its horizontal asymptote. The student understands that the difference between the function's values and the value of the limit can be made as small as desired by increasing *x* beyond a threshold defined in terms of the desired error bound.
Vertical Asymptote	A vertical line found by setting the denominator of a rational expression equal to zero and solving for the independent variable.	A vertical line that a graph of a rational function approaches but never touches, indicating where the function is undefined. The student understands that the function's values tend to $\pm\infty$ because the denominator of a function defined as a quotient approaches zero while the numerator approaches a non-zero value, making the quotient approach $\pm\infty$. Such as: $\lim_{x\to a^-} f(x) \to \pm\infty$, and $\lim_{x\to a^+} f(x) \to \pm\infty$.
Slant (oblique) Asymptote	A slanted line found by performing polynomial long division and discarding the remainder.	Describes the function's linear end behavior as $x \to \pm\infty$ but does not restrict finite values. The student understands that the end behavior of a rational expression behaves linearly. More specifically, the student understands that as *x* increases or decreases without bound, the function's value $f(x)$ varies at essentially a constant rate with respect to *x*.

The conceptual characterizations of asymptotes expressed in this table also allow participants to reason about how input and output quantities change together. In this sense, the forms of reasoning we classify as conceptual knowledge align with descriptions of *quantitative reasoning* (Thompson, 1990, 1994, 2011) and *covariational reasoning* (Carlson et al., 2002; Thompson & Carlson, 2017) reported in the literature. Quantitative reasoning refers to the mental actions involved in conceptualizing situations in terms of quantities and quantitative relationships

(Thompson, 2011; Tallman, 2021). Covariational reasoning refers more specifically to reasoning about how two quantities change together, such as how changes in x correspond to changes in $f(x)$ simultaneously (Carlson et al., 2002). In this study, we applied the theoretical frameworks of quantitative and covariational reasoning to supplement our broader perspectives of Piagetian abstraction and radical constructivism to analyze participants' conceptual knowledge and the transition from procedural to conceptual reasoning.

Theoretical Perspective

We leveraged Piaget's notions of reflecting and reflected abstraction as analytical constructs to describe participants' integration of procedural and conceptual knowledge structures (Piaget, 2001; Tallman, 2021; Tallman & Uscanga, 2020; Tallman & O'Bryan, 2024). Reflecting abstraction involves three phases: (1) *differentiation*: the individual dissociates an action or coordinated sequence of actions from the outcomes they produce; (2) *projection*: the individual constructs an internalized representation of the actions, and (3) *reorganization*: the individual integrates these internalized actions into coherent mental structures that can be applied to a generalized class of objects. Once these actions are internalized and reorganized, the abstraction becomes reflected through the subject's retroactive operation on these actions at the level of thought, where the internalized actions constructed through reflecting abstraction are organized within cognitive schemes (Tallman, 2021, p. 4). This retroactive operation leads to a higher level of awareness of and coherence in an individual's mathematical conceptions, which enables the individual to strategically apply their meanings in a range of novel contexts.

We also applied Piaget's notion of an action scheme in our analysis. An action scheme is a cognitive structure that enables an individual to carry out a goal-directed sequence of actions in response to a situation perceived as paradigmatic (Tallman et al., 2021). Such schemes often support efficient performance but remain tied to the limited contexts that reflect the recognition criteria of the scheme. In this study, the enactment of action schemes characterizes procedural knowledge, particularly when participants fluently apply rules for determining asymptotes without fully articulating structural justifications. Cognitive shifts are therefore identified as moments when these procedural schemes are disrupted, examined, and reorganized into structurally grounded explanations of asymptotes, marking the coordination of procedural actions with conceptual understanding.

Methods

Participants

The participants in this study were two pre-service secondary teachers, Mia and Kathy (pseudonyms), both in their fourth year of a secondary education program with a concentration in mathematics at a large Midwestern R1 university. Mia served as a learning assistant in an entry-level mathematics course, tutored courses through Calculus III for three semesters, and completed her student teaching in an 8th-grade Pre-Algebra class. Kathy served as a learning assistant for Calculus I for three semesters and completed her student teaching at a high school where she taught Precalculus, Trigonometry, Geometry, and AP Calculus AB.

Setting and Tasks Interview

We designed this research as an exploratory case study (Yin, 2009) and structured it around a three-part data collection process to elicit participants' reasoning and support their reflection.

Introductory interview (25 minutes). During the introductory interview, we focused on each participant's experiences and beliefs about teaching rational functions, particularly focusing on how they envisioned explaining the concept of asymptotes to their future students.

Task-based clinical interview (75 minutes). We conducted the task-based clinical interview (TBCI) with each student immediately after their introductory interview. The structure of the clinical interview and our analysis of the data followed the experimental and analytical approaches, respectively, as described by Goldin (1997, 2000) and Clement (2000). The tasks focused on provoking participants' reasoning about the asymptotic behavior of rational functions and were designed to investigate how they coordinated their procedural and conceptual knowledge (Table 2).

Final interview (20 minutes). Two days after the TBCI, we conducted a follow-up interview to examine each participant's reflections on their prior reasoning. Participants revisited their earlier work and described how their thinking had shifted and how they coordinated their procedural and conceptual knowledge across tasks. In total, participants' time commitment for the study was approximately two hours.

Table 2: Three Interview Tasks with Rationale and Analytical Focus

Tasks	Rationale and Analytical Focus
Task 1: Find the domain of each function and identify any horizontal and vertical asymptotes: a. $\frac{1-5x}{1+2x}$ b. $\frac{2x^2}{x+1}$ c. $\frac{3x+1}{x^2+x+9}$	This task is primarily procedural. Although it invited procedural enactment, we prompted participants to explain the meaning of asymptotes as they worked. Our analysis focused on whether participants' action schemes were conceptually grounded or merely enacted as isolated rules and procedures.
Task 2: Define a rational function with domain $(-\infty, k)\cup(k, 5)\cup(5, \infty)$ with $k < 5$ such that the function has only one vertical asymptote at $x = 5$.	Following Task 1, we designed this task to disrupt the action scheme of setting denominators equal to zero to identify vertical asymptotes. Our analysis focused on how participants' mathematical schemes had shifted.
Task 3: Define $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials. Define p and q such that $y = 2x + 1$ is an oblique asymptote.	We designed this task to disrupt routine procedural approaches. For this task, reliance on procedures alone is likely insufficient, prompting participants to coordinate a procedural and a conceptual approach. Our analysis focused on how participants reasoned when their action schemes become unstable, which allowed us to document the extent to which participants integrated their procedural and conceptual knowledge structures, and the extent to which these structures were enriched through this integration.

Data Analysis

During the interviews, we engaged in iterative cycles of *responsive and intuitive interaction* and *analytical interaction* with each research participant. These forms of interaction are essential to the teaching experiment methodology (Steffe & Thompson, 2000) but are appropriate in the

context of task-based clinical interviews for constructing a viable second-order model of the students' mathematical understandings and ways of reasoning.

After completing data collection, we transcribed and analyzed all interviews using Braun and Clarke's (2006) thematic analysis approach to identify patterns of meaning across tasks. The coding process was theoretically informed by our Piagetian perspective. We characterized procedural knowledge by the enactment of action schemes, particularly when participants relied on memorized rules and fixed sequences of steps in familiar contexts without being able to provide conceptual justifications. We identified participants' application of conceptual knowledge through evidence of reflecting abstraction, such as when participants examined their prior procedures, articulated why those procedures worked, and extended their reasoning beyond rule-based enactment. We then identified participants' cognitive shifts through behavioral indications of reflected abstraction, marked by moments in which previously enacted procedures were disrupted or overgeneralized, and participants reorganized their schemes into more stable and conceptually grounded structures. In these instances, procedural and conceptual knowledge were coordinated, and their coherence became part of participants' awareness.

Results

Due to space limitations, we present and discuss only Mia's case, as her expressed reasoning more compellingly demonstrated a shift from procedural enactment to conceptual reorganization. Mia's trajectory also provided a clearer illustration of the cognitive shifts identified in our analysis.

Mia's Conceptions of Asymptotic Behavior in the Introductory Interview

Mia identified asymptotic behavior as the central idea of rational functions and emphasized the importance of teaching this concept by building on students' prior knowledge, stating, "It starts with breakdown, kind of, what they do know foundationally." She believed teaching vertical asymptotes effectively entails linking new ideas to students' understanding of "undefined slopes" and "exploring how graphs approach but never touch the vertical boundaries." For horizontal and slant asymptotes, Mia supposed that letting students compare the degrees of the numerator and denominators would be easier for them to start exploring these concepts. Then, Mia suggested framing rational functions as "top heavy," "bottom heavy," or "balanced" to convey the criteria by which horizontal asymptotes exist. When explaining slant asymptotes, Mia was only able to associate a characteristic of the symbolic representation of a rational function with the existence of a slant asymptote, as she stated, "the numerator is one higher than the denominator." Mia struggled to articulate a conceptual justification for this condition during the interview.

During the introductory interview, Mia articulated an intention to connect instruction on rational functions to students' prior knowledge and expressed a desire to teach the asymptotes conceptually. However, her explanations provided limited insight into her own personal meanings for asymptotic behavior and into how those meanings related to the procedures she described to determine horizontal, vertical, and oblique asymptotes. The central focus of the TBCIs that followed the introductory interview was to elicit observable products of Mia's meanings for asymptotes so that we might construct and refine a viable model of her mathematical schemes.

Mia Enacted Stable Procedural Schemes with Limited Conceptual Justification

When engaged in Task 1, Mia effectively applied action schemes but demonstrated little inclination to move beyond uncritical enactment toward meaningful conceptual understanding. She accurately identified the vertical asymptote of each function by setting the denominator

equal to zero and solving for the independent variable. Describing her solution to Part (b), Mia stated, "It's *x* equals negative one, because I put *x* plus one equals zero. And so that's my vertical asymptote." After she explained her solution, we asked Mia to elaborate on the meaning of the vertical asymptote she had obtained. Mia connected her answer with visual intuition, saying, "If you're thinking visually left to right, like what would be impeding that, that would be a vertical asymptote." We interpret Mia's comment to indicate her conception of vertical asymptotes as impediments to the generation of a continuous curve representing the correspondence of a function's input and output values. Mia's remark suggests that she might have envisioned the action of sketching a function's graph, moving from left to right, and approaching a vertical asymptote, which acts like a kind of barrier that forces the curve to approach positive or negative infinity. This interpretation reflects Mia's conceptual understanding of vertical asymptotes; however, there is little evidence that this interpretation was meaningfully connected to the procedure she applied to determine the location of the vertical asymptotes for each function in Task 1, nor to the algebraic structure of a rational expression. Specifically, Mia did not articulate how the function changes as *x* varies near the function's undefined value.

Mia used a degree comparison method to identify the horizontal asymptotes in Task 1: "First, you need to look at your polynomial on the numerator and denominator. Identify the polynomial of highest degree […] I would call this balance." While responding to Part (a), she circled the leading terms "$-5x$" and "$2x$" in the numerator and denominator of the rational function $y = \frac{1-5x}{1+2x}$, and labeled the case as "balanced," indicating that the degrees of the numerator and denominator are equal. She then correctly concluded that the horizontal asymptote is determined by the ratio of the leading coefficients and obtained $y = -5/2$ as the solution.

Mia used metaphors such as "top-heavy" and "bottom-heavy" to describe different cases while responding to Parts (b) and (c). We interpreted her use of these metaphors as a possible indication of reasoning quantitatively about the behavior of a rational function, as she seemed to compare the relative sizes of some amount represented in the numerator and denominator. However, Mia did not fully explain why the ratio of leading coefficients determines the horizontal asymptote of a rational function when the degrees of its numerator and denominator are equal, or how changes in *x* simultaneously affect the ratios of leading terms in the numerator and denominator. In particular, Mia did not express her intuitive understanding that a horizontal asymptote is the constant function $y = k$ such that the values of the rational function *f* can be made as close to *k* as desired by making $|x|$ sufficiently large. Mia's work on Task 1 established her procedural fluency, showing that her rule-based enactments functioned as her dominant mathematical schemes.

Task 2 prompted Mia to define a rational function with a domain restriction satisfying only one vertical asymptote (see Table 2). Aware that neither *k* nor 5 are in the domain of the function, Mia acknowledged, "I know that I have funky things going on here with my domain." She considered how the function should behave within intervals of its domain, rather than plug in values of *x* to get a sense of the graphical behavior of the function. Mia then attended to the distinction between holes and vertical asymptotes, noting: "I want the vertical asymptote at five, I need the hole to be somewhere else, if the hole's somewhere else, then I have to have something that cancels in the numerator and the denominator." Mia's expressed reasoning demonstrates her familiarity with the procedure for identifying a hole, with some evidence of conceptual understanding of the distinction between removable and infinite discontinuities. About Mia's answer ($\frac{(x-2)(x-1)}{(x-5)(x-1)}$), we asked: "Where did the two [in the numerator] come from?" She responded: "I just put in a different number than five because I didn't want a hole at five, so

two is whatever k is. A hole is not a vertical asymptote." Mia's explanation provides further support for her understanding that a common linear factor in the numerator and denominator of a rational expression corresponds to a removable discontinuity, and that a hole is distinct from a vertical asymptote.

Task 2 disrupted Mia's zero-denominator action scheme by requiring her to construct a rational function that satisfied both a domain restriction and the condition of having only one vertical asymptote. Unlike in Task 1, where she identified vertical asymptotes by applying memorized rules, while completing Task 2 Mia coordinated factorization, domain restriction, and discontinuity type to define a function that satisfied a domain restriction with one vertical asymptote. Mia's expressed reasoning on Task 2 suggests emerging reflecting abstraction, as previously enacted procedures were reorganized into a more structurally coherent scheme.

Reflected Abstraction Through the Structural Reorganization of Action Schemes

Task 3 prompted Mia to construct a rational function with a slant asymptote of $y = 2x + 1$. She initially relied on a familiar rule, explaining that "p is one degree higher than q's highest degree," reflecting her knowledge of the condition for the existence of a slant asymptote. Mia then defined a function satisfying this condition such that the quotient was equal to the slant asymptote, producing $f(x) = (2x^3 + x^2)/x^2$ (Figure 1, left). Although this function met the degree condition, that it simplifies to $f(x) = 2x + 1$ (for $x \neq 0$) yields a linear function with a hole, rather than an irreducible rational function *approaching* a slant asymptote. This initial solution for f revealed a limitation in Mia's conceptual understanding as she correctly applied the structural condition for the existence of a slant asymptote but did not initially recognize that a slant asymptote describes the approximately linear end behavior of a non-linear rational function.

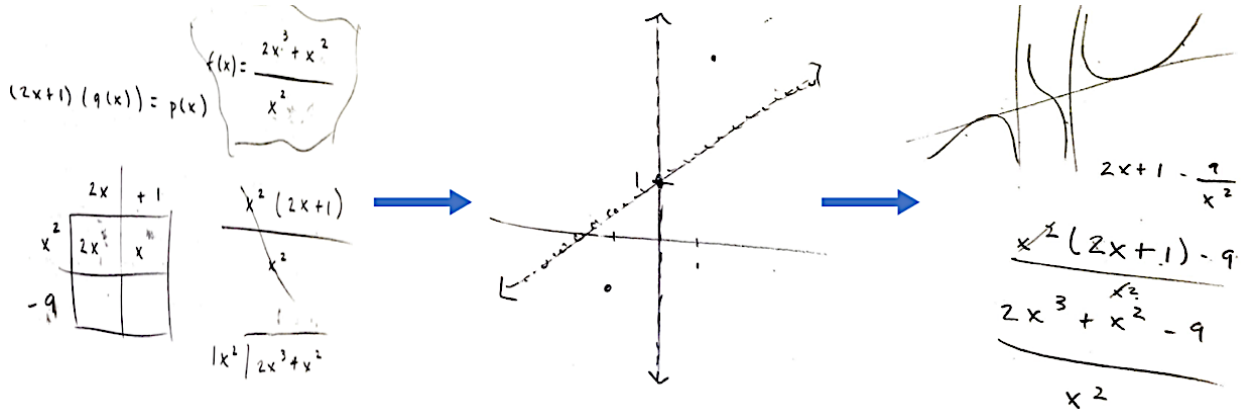


Figure 1: Mia's Reasoning Shifted, Refining Her Initial Solution in Task 3

When asked to graph the function she had constructed, Mia appeared uncertain and identified the line $x = 0$ as a vertical asymptote, marking it with a dotted line on her graph (Figure 1, middle), without recognizing that the function actually has a hole at $x = 0$. To sketch the graph, she evaluated specific values of x (e.g., $x = 1$ and $x = -1$). Although this strategy reflects an effort to understand the function's behavior, all computed points lie on the line $y = 2x + 1$. Notably, Mia did not initially recognize that the graph of the function she defined was linear, with domain R\ {0}, even though she appeared to define f to simplify to $2x + 1$.

After completing the sketch of her graph, we asked Mia to identify the slant asymptote on the graph. At that moment, a critical shift occurred: Mia factored x^2 in the numerator of $\frac{2x^3+x^2}{x^2}$ and

cancelled with the x^2 in the denominator. Noticing that the expression in the denominator cancels with a factor in the numerator, and recognizing that all values of a rational function's input for which its denominator equals zero are excluded from the domain, Mia questioned, "Did I make that a hole instead of a vertical?" This question marked a transition from procedural enactment to reflective examination of her prior actions. By recognizing the removable discontinuity and acknowledging that f simplified to a linear form, Mia began to think more conceptually about how to construct a different example. We asked her to explain how a slant asymptote affects the graph of a rational function. Mia's response demonstrates her attention to the global behavior of a rational function, explaining that as one considers values from negative infinity to positive infinity, the term of highest-power in a rational expression dominates while additional terms become insignificant as the input increases or decreases without bound. She refined her example to $f(x) = 2x + 1 - 9/x^2$ (Figure 1, right), explaining that the term "$-9/x^2$ becomes negligible as $x \to \pm\infty$." This explanation reflects Mia's coordination of algebraic structure with limit-based reasoning about a rational function's end behavior, indicating reorganization of her prior action schemes into a more conceptually grounded and structurally coherent understanding. Mia not only revised her prior solution but explained why $2x + 1 - 9/x^2$ approaches $2x + 1$ as $x \to \pm\infty$. Mia's previously enacted procedural schemes were reorganized and integrated into a single structurally coherent scheme linking algebraic form and end behavior.

Mia's Reflection on the TBCI During her Final Interview

Mia acknowledged that in comparison to Task 1 from the TBCI, Task 2 was more challenging because Task 1 could be "completed procedurally." Although she successfully solved Task 2, Mia reflected that she often relied on a "plug-and-chug" procedural approach and recognized that Task 2 disrupted her uncritical application of these procedures. She expressed more excitement when reflecting on Task 3, emphasizing that its novelty helped her reason about why slant asymptotes occur.

Discussion

Mia's comments in the introductory interview anticipated her reasoning in the task-based clinical interview. She demonstrated stable enactment of action schemes while working through Task 1. Procedural approaches dominated her mathematical thinking as she relied on rule-based identification of asymptotes while offering limited structural justification in terms of limits and end behavior. This result supports Tallman's (2021) claim that procedural accuracy does not guarantee conceptual clarity. When Mia's application of routine procedures was disrupted in Tasks 2 and 3, she began to reorganize her scheme. She demonstrated a shift toward coordinating algebraic form with asymptotic behavior, suggesting the emergence of structural coherence and development of her conceptual understanding of asymptotes.

Cognitive shifts like the one Mia demonstrated are not just personal psychological developments. When they become a repeated experience of pre-service secondary teachers, these shifts have the potential to challenge the procedural status quo in secondary mathematics classrooms. We argue that mathematics teacher education programs should involve disrupting PSMTs' procedural action schemes and supporting them in becoming teachers who prioritize conceptual depth over rote memorization.

Our results suggest that mathematics teacher education curricula and instruction should explicitly include anomalous tasks (like Tasks 2 and 3) as a standard pedagogical tool to elicit PSMTs' cognitive conflict and engender the reorganization and refinement of their mathematical schemes.

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REPRESENTATIONAL AFFORDANCES IN FRACTION MULTIPLICATION REASONING: PRESERVICE TEACHERS' CONTEXTUAL AND VISUAL MODELS

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We examined how middle school and elementary preservice teachers' (PSTs') visual and contextual representations evidenced their fraction multiplicative reasoning prior to instruction. The contextual representations showed PSTs successfully constructed contexts that represented the multiplicand in relation to a clearly defined whole, but connecting both fractions to the original whole was challenging. PSTs' visual models demonstrated an emerging understanding that the multiplier should operate within the multiplicand. When additional partitioning of the referent unit was required, some PSTs relied on familiar procedures (e.g., creating common denominators), reflecting non-multiplicative or algorithmic reasoning. Building on PSTs' demonstrated strengths and areas for growth, we offer instructional recommendations for mathematics teacher educators (MTEs) to support multiplicative reasoning with fractions.

Keywords: Mathematical Representations, Preservice Teacher Education, Rational Numbers

Background and Study Purpose

The National Council of Teachers of Mathematics (NCTM, 2014) has identified that effective mathematics teaching should offer opportunities for students to connect multiple mathematical representations, use those representations to problem-solve, and develop procedural fluency grounded in conceptual understanding. Using and connecting representations supports learners in unpacking and relating different constructs. Visual representations in particular support the development of foundational fraction concepts because "the area model is more convenient for part–whole meanings of fractions" (Toluk-Uçar, 2009, p. 174). In addition to supporting concept development, representations demonstrate learners' mathematical reasoning. For example, preservice teachers' (PSTs') written contextual problems for given symbolic expressions reveal the meanings they attach to those expressions (Toluk-Uçar, 2009), such as establishing relationships between the multiplier and multiplicand (i.e., referent unit) in fraction multiplication. Thus, engaging PSTs in representing fraction multiplication visually and contextually potentially reveals how they transfer meanings across representations.

Research has suggested PSTs find some representations more accessible than others. PSTs were more successful at converting symbolic expressions into word problems than at transforming word problems into appropriate visual representations (Toluk-Uçar, 2009). Moreover, engaging with multiple representations, including word problems, helped PSTs to construct meaning from contextual scenarios (Sosa-Martín et al., 2024) and enhanced their conceptualization of abstract mathematical ideas. Thus, creating visual models for contextual problems and posing story problems for given representations arguably supports PSTs' understanding of partitioning and units, ideas that are often abstract in fraction multiplication (Mack, 2001). Unpacking PSTs' representations and abstractions also informs Mathematics Teacher Educators' (MTEs') instruction related to fraction multiplication.

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Building on the research summarized above, we investigated how PSTs' contextual story problems and visual models reveal their conceptions of multiplicative reasoning and partitioning, an area that warrants further investigation. We highlight patterns of how PSTs drew upon previously learned concepts of fractions, fraction addition, partitioning, referent units, and multiplicative reasoning to construct contextual and visual models for fraction multiplication.

Theoretical Considerations: Representations for Fraction Multiplication

Both mathematicians and mathematics educators have identified the flexible use of multiple representations as a key mathematical practice (NCTM, 2014; Wasserman, 2023), underscoring how mathematical knowledge is learned and discovered by establishing relationships between multiple representations. Fluency with representations reflects teachers' pedagogical content knowledge (Ball et al., 2008), evidenced in "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9).

Barmby and colleagues (2009) proposed the *Representational-Reasoning* model to highlight how connections between representations reflect learners' mathematical understanding. Within this model, understanding is framed as a network of internal representations connected through explicit reasoning and expressed through external representations (i.e., physical, symbolic, verbal, contextual, visual [NCTM, 2014]). When learners engage with multiple external representations, they can build and strengthen connections among their mental (internal) representations, which are "abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience" (Pape & Tchoshanov, 2001, p. 119). Because learners' internal representations cannot be assessed directly, researchers rely on external ones to infer potential connections learners established between underlying ideas. Moreover, because internal representations may stem from memorized algorithms or real-life experiences linked to mathematical concepts (Barmby et al., 2009), using this model to examine PSTs' contextual and visual representations may reveal the types of internal reasoning they may be drawing on. This, in turn, allows MTEs to better support PSTs in constructing coherent connections between multiple representations and developing a holistic understanding of fraction multiplication.

Conceptual understanding of fraction multiplication involves multiple concepts, including the meaning of a fraction and multiplicative reasoning. Multiplicative reasoning occurs through understanding the concept of unitizing, which involves "operating with singleton units to coordinate composite units" (Singh, 2000, p. 273). There are potential obstacles in learning unitizing as students typically learn fraction addition and whole number multiplication prior to fraction multiplication (Common Core, 2010). Similarly, PSTs bring experience with fraction addition, whole number multiplication, and even tendencies to rely on recalled procedures (Lovin et al., 2018). This may mean students and PSTs rely on understanding multiplication as repeated addition by interpreting 3×4 as $4 + 4 + 4$, and these interpretations are not the same as interpreting 3 times 4 as "3 fours" (Götze & Baiker, 2021, p. 263). Interpreting multiplication as repeated addition of the same-sized groups does not readily translate to fraction-by-fraction multiplication (Thompson & Saldanha, 2003), which requires learners to identify the distinct roles of the multiplicand and the multiplier. Moreover, PSTs who are familiar with the multiplication algorithm may attempt to map the algorithm directly onto a representation (Kang et al., 2025), demonstrating basic but not advanced reasoning connections (Barmby et al., 2009).

To address these challenges of multiplicative reasoning, we explored PSTs' potential internal representations externalized in their contextual and visual models. This exploration revealed the networks of reasoning among internal representations and their sources, and highlighted implications for framing instruction to support PSTs' representational reasoning in fraction

multiplication. We asked PSTs to construct visual and contextual representations (with explanations) for given symbolic and contextual problems, which enabled us to investigate how they coordinated internal and external representations in reasoning about fraction multiplication (Barmby et al., 2009). We asked the following research question: What did PSTs' contextual and visual representations reveal about their fraction multiplication reasoning prior to instruction?

Methods

Context and Participants

This study is part of a larger project conducted across four U.S. universities—three Midwestern and one Southeastern. The four authors taught math content and methods courses at their respective institutions. The larger project aims to develop, implement, and study assessment tasks that reveal nuances in PSTs' conceptions of fraction multiplication and to design instructional sequences to strengthen it. For this manuscript, we selected 31 PSTs (6 middle school and 25 elementary) from four institutions, representing 50% of the PSTs enrolled in their respective courses. PSTs at each institution varied in the number of mathematics content courses they had taken. We implemented a sequence of activities focused on fraction multiplication: (1) whole number × fraction, (2) fraction × whole number, (3) unit × non-unit fraction, (4) unit × unit fraction, (5) non-unit × non-unit fractions, and (6) fractions greater than 1 × non-unit fraction. Across these tasks, we emphasized using area models and creating contextual problems.

Assessment, Data, and Analytic Methods

We invited PSTs to construct (1) visual models for two given fraction multiplication story problems and (2) a contextual story problem for a given fraction multiplication statement (see Table 1). For this study, we used only the pretest data, as we aimed to understand what types of representations PSTs demonstrate for fraction multiplication prior to instruction. All the PSTs' contextual and visual representations for these tasks were the data for this study.

Table 1: Assessment Tasks to Collect PSTs' Visual and Contextual Representations

Solve the Following
Use a visual model and symbolic notations to solve tasks 1 and 2 and provide a brief narrative explanation that will help the reader understand how you solved the problem. 1. Alex is working on designing a garden. He wants to plant 3/5 of the garden with tomatoes and 2/3 of the tomatoes will be cherry tomatoes. How much of the garden will be planted with cherry tomatoes? 2. Tegan is also working on designing a garden. He wants to plant 4/5 of the garden with potatoes and 2/3 of the potatoes will be red potatoes. How much of the garden will be planted with red potatoes? 3. Create a fraction multiplication story problem that requires students to find 1/2 of 3/4.

In the first round of data analysis, we used open and axial coding (Saldaña, 2021) to identify the types of representations PSTs used, which helped us develop separate codebooks to analyze PSTs' multiplicative reasoning with fractions demonstrated through their visual and contextual representations. Both the codebooks had common codes related to: (a) mathematical operations represented; (b) whether a clearly defined referent unit was present and operated on; and (c) computations used and mathematical correctness. In addition, the codebook for contextual problems captured the relationships between the given fractions and the whole; whether the context involved discrete (counts) or continuous (e.g., area, volume) representations; and how

the unknown was framed (Table 2). The codebook for visual representations categorized how models were drawn, labeled, and described (Table 3). In the second round of analysis, we categorized PSTs' contextual and visual representations into five categories of multiplicative reasoning: Integrated, Partially Integrated, Transitional, Foundational, and Uninterpretable. Each category is described differently for visual and contextual representations (see Tables 2 and 3 for descriptors). In both rounds, two coders independently coded the data and resolved discrepancies through discussion.

Findings

We describe evidence of PSTs' understanding of fraction multiplicative reasoning first with contextual representations and then with visual representations.

Fraction Multiplication Reasoning Evidenced in PSTs' Contextual Representations

About one-third (29%) of PSTs' story problems were Integrated representations (Table 2). They demonstrated evidence of correct use of multiplicative reasoning and the referent unit. Thus, their representations reflected an interpretation of the given fraction statement as ½ of ¾ (i.e., ½ × ¾). As such, they wrote story problems that appropriately represented ½ × ¾ as *a half of three-fourths of a single whole*. For example, Elena wrote the following story problem: "Audrey is painting a wall. She want[s] to cover ¾ of the wall with a mural, but she want[s] ½ of the mural to be in black and white. How much of the wall will be in black and white?"

Table 2: Integration of Contextual Representations Across PSTs

Integration of Representations	Descriptors	Participants *n* (%)
Integrated	• Context represents multiplicative reasoning • Mathematically correct stories depict the given symbolic statement • Multiplicand is interpreted as the referent unit • Clear and correct referent whole • The unknown in the story clearly asks for fraction multiplication	9 (29%)
Partially Integrated	• Context represents multiplicative reasoning • Mathematically correct stories, but the context switches the multiplicand and multiplier in the given symbolic statement • Multiplier is interpreted as the referent unit • Clear and correct referent whole • The unknown in the story clearly asks for fraction multiplication	2 (7%)
Transitional	• Context does not consistently represent multiplicative reasoning (e.g., subtraction, addition, and/or division are represented) • Mathematically correct stories, but they do not represent the given symbolic statements • Referent units are not correctly used (e.g., counts used) • Clear and correct referent whole	5 (16%)
Foundational	• Context does not represent multiplicative reasoning • Stories are mathematically incorrect and/or require clarification because one or more of the following occur: operations are unclear or mixed, wholes are not specified, referent units are misused, or the unknown in the story problem is not clearly defined	15 (48%)
Uninterpretable	• Did not write a story problem	0 (0%)

About 7% of the PSTs' contextual problems showed evidence of their multiplicative reasoning; however, they flipped the order of the multiplier and multiplicand (i.e., ½ as the referent unit) as seen in Jimena's story that belonged to the Partially Integrated category:

> Lori wanted to bake a cake for the birthday party coming up, so she decided to do 2 flavors. Lori wanted ½ of the cake with vanilla and ¾ of the vanilla to be with strawberry filling. How much of the cake will be with strawberry filling?

About 16% story problems were themselves mathematically correct but did not consistently represent multiplicative reasoning (Transitional Integration). Instead, they represented additive reasoning or mixed representations. Karl wrote a fraction addition problem for ½ of ¾: "Bob ate ½ [of] a pizza. Jim ate ¾ of another pizza. How [much] did the[y] eat all together?" This story reflected Karl's additive reasoning, though they did not specify if the pizzas were the same size.

About half (48%) of the PSTs' representations were Foundational. These stories were mathematically incorrect or required clarification because they either used multiple operations (e.g., starting with multiplication and shifting to division), used multiple units (e.g., beginning with fractions and switching to counts without specifying the total), or treated fractions as separate wholes rather than composite units. Emma used subtraction ("leaving her 3 slices left"), multiplication ("½ of the remaining 3 slices"), and treated the numerator and denominator of ¾ as separate wholes: "Katie had 4 slices of pizza. She ate 1 slice, leaving her 3 slices left. What is ½ of the remaining 3 slices?" Another PST Kylie wrote: "There were bins of marbles, one was ½ full and the other was ¾ full. How many bins were there to fit all the marbles?" We marked this representation as Foundational because they asked "how many" without providing a total count.

Fraction Multiplication Reasoning Evidenced in PSTs' Visual Representations

We utilized the same five categories for PSTs' visual representations (Table 3) to unpack their visual models for the contextual problems ⅔ × ⅗ and ⅔ × ⅘ holistically.

Table 3: Integration of Visual Representations Across PSTs

Integration of Representations	Descriptions	Participants *n* (%)
Integrated	• Models represent multiplicative reasoning for both tasks • Models demonstrate correct products • Multiplicand identified as the referent unit for both tasks • Operation within the identified referent unit for both tasks • Product relates back to the original whole	1 (3%)
Partially Integrated	• Models represent multiplicative reasoning for both tasks • Models demonstrate correct products • Multiplier identified as the referent unit for at least one task • Operation within the identified referent unit for both tasks • Product relates back to the original whole	2 (7%)
Transitional	• Models represent multiplicative reasoning • One of the factors identified as the referent unit for each task • Began to operate within identified referent units, but could not complete because either the referent units were represented separately or not partitioned correctly for at least one task	14 (45%)
Foundational	• Models do not represent multiplication for either of the tasks • Use of referent units not shown in both tasks	9 (29%)
Uninterpretable	• No models or models are uninterpretable for both tasks	5 (16%)

In the top panel of Figure 1, Cristino's description indicates that they first operated within the referent unit and then extended the lines to determine the size of the newly formed units relative to the whole. In contrast, Charlotte's model on the bottom panel of Figure 1 shows that they did not necessarily operate within the referent unit when partitioning. Although these models appear similar, we classified them as Integrated and Partially Integrated based on their visual representations and/or explanations concerning the principle that the multiplier should operate exclusively within the referent unit representing the multiplicand.

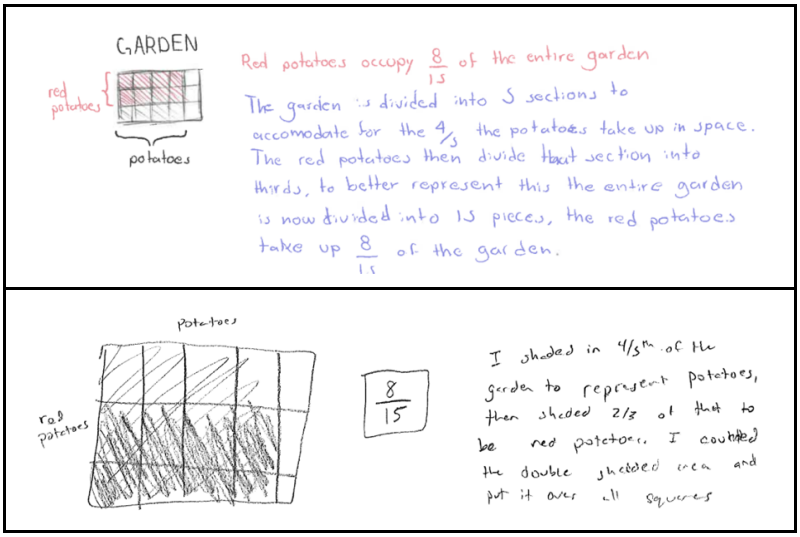


Figure 1: Visual Representations for ⅔ × ⅘ by Cristino (top) and Charlotte (bottom)

About a half (45%) of the PSTs exhibited some understanding of the referent unit (Transitional Integration). However, their models did not show the evidence of the concept related to (a) partitioning the referent unit to represent the multiplier or (b) how partitioning the referent unit relates to the whole to make sense of the newly formed unit fractions. As a result, PSTs often represented the first story problem meaningfully, as it did not require additional partitioning, whereas the second task was more challenging because additional partitioning was needed to represent the multiplier. In Figure 2, Amelia's visual representations illustrate that she correctly identified the referent unit and operated (or attempted to operate) in both cases but had difficulty representing ⅔ within ⅘ for the second task, as it required additional partitioning.

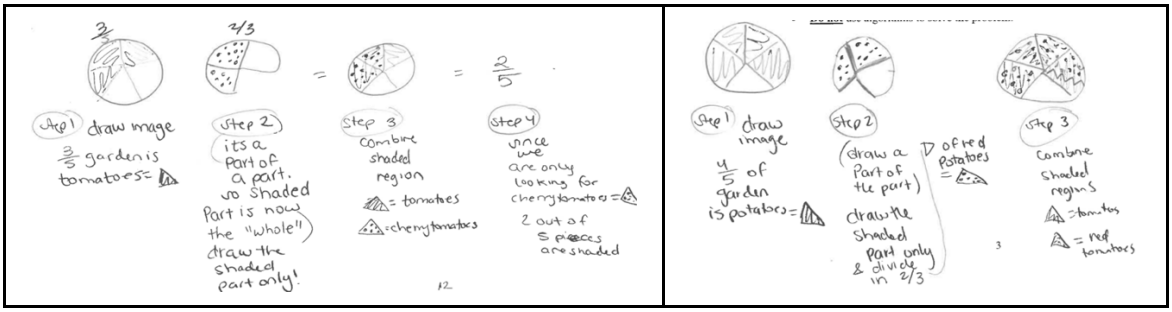


Figure 2: Amelia's Visual Representations for ⅔ × ⅗ (left) and ⅔ × ⅘ (right)

Kylie was able to operate within the referent unit (Transitional Integration) but created a single rectangular model for each problem. Similar to Amelia, they successfully represented Task 1, wherein no additional partitioning of the referent was required (Figure 3, left), and grappled with additional partitioning and did not complete Task 2 (Figure 3, right).

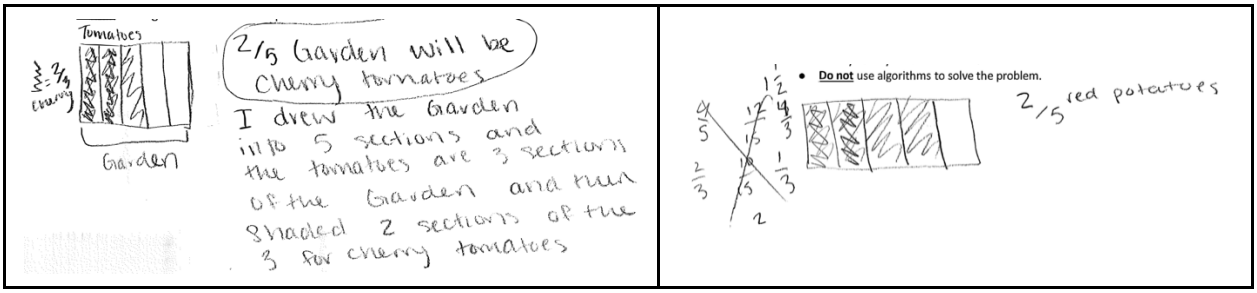


Figure 3: Kylie's Visual Representations for ⅔×⅗ (left) and ⅔×⅘ (right)

About one-third (29%) of the PSTs' representations fell into the Foundational Integration category, indicating that fraction multiplication was not represented in their models and/or models did not show the concept of the referent unit. In Figure 4, the left model illustrates that Liana attempted to use subtraction, created a common denominator, and tried to depict the fractions visually. In the final category, labeled Uninterpretable Integration, PSTs' models could not be meaningfully interpreted, as they provided no clear indication of which fractions or operations were represented, nor how the referent units were identified, as shown in Taylor's model (Figure 4, right).

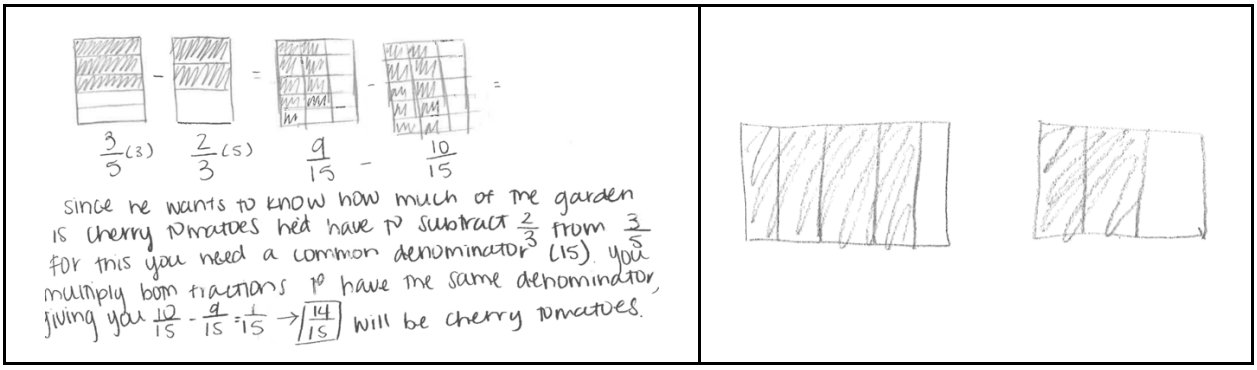


Figure 4: Visual Representations by Liana (⅔ × ⅗; left) and Taylor (⅔ × ⅘; right)

Discussion and Implications

Overall, our findings indicate that, prior to instruction, many elementary and middle school PSTs demonstrated a fundamental use of multiplicative reasoning, referent units, and fraction concepts, and they expressed these ideas through both contextual and visual representations. For example, more than half of the PSTs demonstrated an understanding that the multiplier must be operated within the model constructed for the multiplicand. We were encouraged by the depth of reasoning exhibited prior to explicit instruction. Meanwhile, our findings showed areas where PSTs' existing understandings need strengthening. We frame these areas not as shortcomings of

PSTs, but as productive starting points that can guide MTEs in designing targeted instruction to create opportunities for PSTs to refine and extend their reasoning, so they are better prepared to teach fraction multiplication to K-12 students.

Although many PSTs visually represented referent units for both tasks, the implied connections of their mental models of partitioning within the referent unit seemed comparatively underdeveloped. This was evident in the second task in our study, where they needed to partition the referent unit. As PSTs grappled with partitioning to represent the multiplier within the referent unit, many attempted to create common denominators or relied on algorithms. This suggests that when their reasoning about partitioning was not fully developed, they drew on other internal models, such as fraction addition rules (i.e., making common denominators) or memorized algorithms (i.e., creating equivalent fractions, keep-change-flip, etc.) to construct external representations. They seemingly believed that creating the same denominator would help them partition correctly. This aligns with Barmby and colleagues' (2009) idea that basic reasoning among internal representations may limit the construction of external representations that reflect conceptual understanding. Also, PSTs' prior understanding about partitioning was potentially informal and did not support the conceptual complexities needed (Mack, 2001). Accordingly, we suggest that MTEs should highlight the concept of partitioning within a fraction of the whole and support PSTs in examining how the size of newly formed unit fractions change in relation to the original whole when beginning instruction on fraction multiplication.

PSTs' contextual and visual representations also suggested that PSTs may need more opportunities to understand multiplication as operating on single units and that the composition of multiple units can form a new single unit (Singh, 2000). Prior studies have shown that using the concept of repeated addition may not automatically transfer to the development of multiplicative reasoning, which is closely associated with unitizing (Götze & Baiker, 2021), and we see similar evidence among our participants. We found PSTs using additive reasoning to create contextual or visual representations or not treating the multiplicand as a single unit. For example, some PSTs interpreted ¾ as three separate pieces rather than a single unit to be operated on multiplicatively. This again suggests that MTEs should be intentional about how PSTs use multiplicative reasoning, and that using precise language—such as "3 fours" for 3×4 instead of repeated addition (Götze & Baiker, 2021, p. 263).

Insights from PSTs' written story problems also provide helpful guidelines for MTEs. The participants who found it challenging to write coherent stories tended to grapple with two representational connections: (a) maintaining a clear relationship between the two fractions and the original whole, and (b) consistently using the same unit within a single story. For instance, some stories began with one whole (e.g., a garden area) and then shifted to another context without a clear whole (e.g., the number of flowers). Other stories shifted between units within a single story, such as volume and surface area. This pattern suggests that, alongside operating within a referent unit, PSTs would benefit from explicit attention to preserving the original whole and consistently using the same unit throughout a problem context.

Our study portrays PSTs as already having meaningful and generative ideas about fraction multiplication. To follow, we will look at how these reasonings shift after instruction and how they vary with the number of mathematics courses PSTs completed. By building on these ideas and intentionally supporting connections among referent units, partitioning, unit consistency, and unitizing, MTEs can help PSTs develop robust understandings of fraction multiplication.

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ELEMENTARY PRESERVICE TEACHERS' JUDGMENTS IN AI-MEDIATED DATA SCIENCE LESSONS

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This exploratory study investigates how elementary preservice teachers (n=8) enrolled in elementary math methods at a public university designed AI-mediated data science lessons and how their confidence and instructional judgment developed through this process. Guided by an AI-Mediated Teaching Framework, they engaged in collaborative lesson planning supported by AI tools. Data sources included pre- and post-surveys, lesson plans, AI decision logs, peer evaluations, collaborative transcripts, and reflections. Results indicated increased confidence across all framework constructs, with the largest gains in ethical reasoning and equity-centered design. Qualitative findings showed that PSTs used AI as a design partner by accepting low-stakes suggestions, adapting outputs for developmental appropriateness and coherence, and rejecting ideas that lacked data science alignment.

Keywords: Preservice Teacher Education, Technology, Learning Environment Design, Statistics, Probability, Data Science

In mathematics education, Artificial Intelligence (AI) has been positioned as something that can be helpful for generating tasks, differentiating instruction, and significantly reducing planning time (Trust et al., 2023; Berryhill et al., 2024). Current research suggests that AI-generated instructional materials can vary significantly in quality and almost always require careful human judgement to protect rigor, conceptual coherence and alignment with instructional goals (Sapkota & Bondurant, 2024; Gattupalli et al., 2023). These concerns are particularly prevalent among preservice teachers (PSTs) who may be vulnerable to accepting AI-generated materials at face value, or unsure about how to integrate AI into the instructional planning process (Guan et al., 2025; Abualrob, 2025). Data science has also become an area of focus for elementary mathematics across the United States (Institute of Education Sciences, 2025), especially in California (California Department of Education, 2023). Research supports that when data science is integrated into early mathematics, it can support students' sensemaking, reasoning, and engagement with real world contexts (Institute of Education Sciences, 2025). Currently there is little empirical evidence at how PSTs design and evaluate data science lessons, specifically when AI tools are used in the planning process (Shin et al., 2021; Rosenberg & Jones, 2024). However, less is known about the instructional choices that help PSTs move beyond using AI for idea generation toward using AI with professional judgment. In particular, research has not yet examined how PSTs evaluate, adapt, and reject AI-generated suggestions while designing elementary data science lessons.

This exploratory study examined how elementary PSTs engaged with AI while designing a data science mathematics lesson. AI was positioned as a pedagogical thought partner whose input must be evaluated, revised, and aligned with instructional goals (Luckin, 2018; UNESCO, 2024). This study addressed the following research questions: (1) How do elementary preservice teachers design an AI-mediated data science lesson plan in mathematics when supported by an AI-Mediated Mathematics Teaching Framework? (2) In what ways do their confidence and instructional judgment related to AI-mediated data science lesson planning change before and

after engaging in framework-guided lesson planning? (3) How do they evaluate, adapt, and reject AI-generated instructional ideas during collaborative lesson planning?

AI-Mediated Teaching Framework

The AI-Mediated Teaching Framework (Figure 1) builds from mathematics teacher education, technology integration, and AI literacy scholarship. It foregrounds teachers' responsibility to evaluate mathematical accuracy, cognitive demand, equity, authorship, bias, and alignment with learning goals. The framework extends prior work by making AI-specific planning practices explicit, including prompt design and decisions to accept, adapt, or reject AI-generated suggestions.

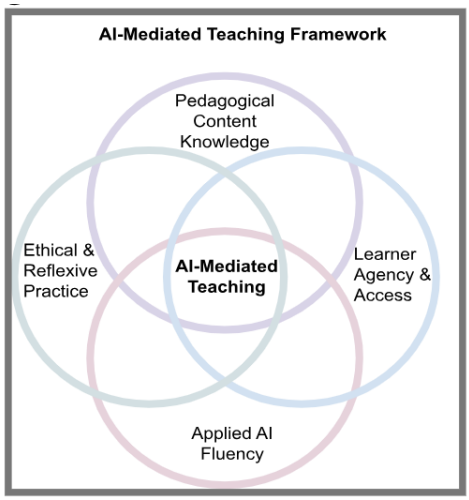


Figure 1: The AI-Mediated Teaching Framework (Shields, 2026)

Pedagogical Content Knowledge (PCK). AI is positioned as a provisional design partner whose suggestions are evaluated through the lens of PCK. PSTs are encouraged to use AI to generate examples, representations, or task ideas while using their own judgement to assess accuracy, cognitive demand and alignment with learning goals.

Learner agency and access. This component draws on Universal Design for Learning (CAST, 2018), Multi-Tiered Systems of Support (McIntosh & Goodman, 2016), Culturally Sustaining Pedagogy (Paris & Alim, 2017), Multilingual and English Language learner supports to design tasks that offer multiple entry points, honor students' funds of knowledge, promote inquiry and encourage collaboration. The framework positions AI-generated materials as starting points that need refinement to ensure inclusive and student-centered learning experiences.

Applied AI fluency. This component involves the practical knowledge and skills needed to use AI tools responsibly and effectively during lesson planning. This includes understanding AI affordances and limitations, crafting intentional prompts, and iteratively refining outputs (Luckin, 2018; Holmes et al., 2021). Research suggests that while PSTs may express confidence in using AI, they often lack structured opportunities to evaluate AI-generated content (Abualrob, 2025). Within the framework applied fluency is developed through collaborative prompting, documentation of AI interactions, and explicit evaluation of outputs.

Ethical and reflexive practice. This component grounds the considerations of bias, data privacy, authorship, transparency in alignment with AI literacy frameworks (Stanford Commons,

n.d.; UNESCO, 2023, 2024). PSTs are prompted to reflect on how AI shapes their instructional decisions, whose perspectives are centered in lesson contexts, and how AI-generated materials may reinforce or challenge inequities. Reflection is not used as an add on, but a core part of professional judgment in AI-Mediated Teaching.

Operationalizing the Framework Through a Rubric

The AI-Mediated Teaching Framework was used through a lesson-planning rubric that structured PSTs' engagement with AI during data science lesson design. The rubric scored the four constructs of the framework while anchoring content specific criteria in the California Mathematics Framework (CMF) and youcubed data science progression (California Department of Education, 2023; Boaler, 2021). PSTs used a 1–4 rubric to evaluate AI suggestions and make explicit accept, adapt, or reject decisions based on alignment with learning goals, student needs, and ethical considerations. The rubric also functioned as a peer-evaluation tool. Table 1 shows a concise version of the rubric.

Table 1: Concise Rubric for AI-Mediated Teaching Framework (Shields, Under Review)

Construct	Rubric Criteria
PCK	(1) Learning target and question are unclear or not grade-appropriate. Activities don't really match the target. Misconceptions and success criteria are missing or very superficial. (4) Learning target and statistical investigative question are clear, grade-appropriate, and rooted in a meaningful context (school/community). Activities build logically from data collection to representation to analysis. Anticipated misconceptions and success criteria are specific and realistic.
Learner Agency & Access	(1) Lesson is mostly one-size-fits-all, with few chances for students to make decisions or share their perspectives. Little or no attention to language supports, scaffolds, or cultural relevance. (4) Lesson offers meaningful student choice and voice (e.g., input on data context, representations, or discussion ideas). Multiple representations (tables, graphs, verbal/written explanations) are built in. Clear ELL/MLL supports, MTSS scaffolds, and culturally sustaining connections to students' communities and languages.
Applied AI Fluency	(1) Little or no evidence of intentional AI use. Accept/Adapt/Reject is missing or not explained. The lesson seems copied from AI or, conversely, AI is barely used despite the assignment. (4) AI prompts are purposeful (different prompts for tasks, data sets, questions, etc.). The AI Decision Log clearly shows Accept / Adapt / Reject choices with strong rationales tied to PCK, agency/access, or ethics. The pair improves the lesson by revising AI output rather than copying it.
Ethical & Reflexive Practice	(1) Little or no sign that AI outputs were checked for accuracy, bias, or privacy. Harmful or misleading content may still be present. AI's role in the lesson is not explained or is presented as "always correct. (4) The pair clearly checks and corrects AI's math, data, and claims. They avoid or edit out biased, sensitive, or PII-related content. They explicitly name how

	they will explain AI's role to students (authorship, bias, privacy) and make visible who created what. Reflection shows honest critique of AI limits.
Overall Rigor & Coherence	(1) Lesson feels fragmented; rigor is mostly procedural with little reasoning or evidence-based discussion. (4) Lesson feels coherent from warm-up to cool-down. Activities, representations, and questions all support the learning target and investigative question. Students are asked to reason, justify, and use evidence, not just compute.

Methods

This study combined quantitative survey data with qualitative analysis of lesson artifacts, AI decision logs, peer evaluations, collaborative transcripts, and written reflections.

Participants and Context

Participants were eight multiple subject PSTs enrolled in an elementary math methods course at a public university that the author taught. All participants had limited prior experience designing data science lessons or using AI tools for instructional planning. The lesson planning activity was embedded in the regular course instruction and aligned with course learning objectives related to data science integration, equity, and instructional design.

Instructional Sequence and Task Directions

The lesson-planning task was implemented as a structured instructional sequence within the elementary mathematics methods course. PSTs first discussed the role of data science in elementary mathematics, with attention to statistical investigative questions, data collection, representation, and interpretation. They were then introduced to the AI-Mediated Teaching Framework and the accompanying rubric. The instructor modeled how AI-generated instructional suggestions could be evaluated for mathematical accuracy, developmental appropriateness, equity/access, and ethical considerations. PSTs then worked in pairs to design a grade-level data science lesson using one generative AI tool of their choice. Rather than being given a fixed prompt, pairs were asked to develop and revise their own prompts, document AI outputs, and record whether they accepted, adapted, or rejected each suggestion. The final lesson plan, AI decision log, peer evaluation, collaborative transcript, and individual reflection served as study artifacts.

AI-Mediated Lesson Planning Task

The participants worked in collaborative pairs to design a data science lesson plan that engaged students in a statistical investigation appropriate for their grade level. They chose from three generative AI tools (i.e., ChatGPT, Gemini, Magic School) as a planning support to generate lesson ideas, task contexts, questions, and representations. Rather than assigning prompts, the activity emphasized iterative prompting and evaluation. They developed, refined, and revised their own prompts in alignment with instructional goals and documented their interactions with AI throughout the planning process. The documentation process indicates how they accepted, adapted, or rejected the AI suggestions during lesson development.

Data Sources

Pre/post survey. A 12-item Likert scale survey (*1 = strongly disagree, 4 = strongly agree*) evaluated PSTs' confidence and understanding across the four framework components. Survey items were grouped as follows: (1) Applied AI Fluency (Q1–Q3): Ex. "I can create prompts for AI tools that generate lesson ideas aligned with my instructional goals." (2) Ethical & Reflective Practice (Q4–Q6): Ex. "I can identify and address potential bias in AI-generated instructional materials." (3) Pedagogical Content Knowledge (Q7–Q10): Ex. "I know how to align lesson

content with the PK–5 California Common Core Math Standards and California Mathematics Framework." (4) Learner Agency & Access (Q11–Q12): Ex. "I know how to adapt lesson content to meet the diverse learning needs of all students."

Peer evaluation. Each PST individually evaluated a lesson plan produced by another pair. Because each lesson was evaluated by two different PSTs, evaluation scores provide an external validity check on lesson quality. The rubric was used to evaluate the four constructs (PCK, Learner Agency & Access, Applied AI Fluency, and Ethical & Reflective Practice) on a Likert scale from 1–4.

Qualitative data. The qualitative sources included: (1) collaborative planning transcripts: Audio transcripts captured PSTs' real-time collaborative planning conversations, providing insight into how instructional decisions were negotiated. (2) AI decision logs: The log documented how AI outputs were evaluated and revised. (3) Lesson plans: A complete K–8 data science lesson plan (learning goals, task description, representations, discussion prompts, and scaffolds) focused on asking statistical investigative questions. (4) Individual post-collaboration reflections: Post-collaboration reflections captured PSTs' perceptions of their learning, confidence, and remaining questions.

Data Analysis

Pre–post survey data were analyzed descriptively to examine changes in preservice teachers' self-reported confidence across the four framework constructs. Paired-sample effect sizes were calculated using Hedges' g^z to account for the small sample size.

Qualitative data including: lesson plans, AI decision logs, collaborative transcripts, and written reflections, were analyzed by mapping evidence to the four constructs of the AI-Mediated Teaching Framework. Additional inductive coding was conducted to capture patterns in preservice teachers' interactions with AI, including common rationales for adaptation, acceptance, or rejection of AI-generated suggestions. Peer-evaluation data were analyzed using both descriptive statistics and interrater reliability measures. Because each lesson was independently evaluated by two preservice teachers, percent agreement and quadratic weighted Cohen's kappa were calculated for each construct to assess consistency across raters.

Results

Survey Results

Table 2 shows pre-post mean and standard deviation scores for PSTs' (n=8) self-reported confidence, which shows growth in each of the constructs after the collaborative lesson planning process. The largest gains were seen in the ethical and reflexive practice ($g^z = 1.05$) and learner agency & access ($g^z = 1.27$), both showing large effect sizes. These results show growth in PSTs' self-reported confidence with ethical reasoning about AI use and designing equitable, learner centered instruction. Moderate gains were seen in PCK ($g^z = 0.84$) showing that teachers reported increased confidence in designing data science math lessons. Applied AI fluency showed positive improvement ($g^z = 0.65$) indicating that PSTs' self-reported confidence using AI tools and professional judgment increased. Across all framework constructs, standard deviations decreased from pre to post surveys, suggesting increased convergence in PSTs' confidence following the instructional sequence.

Table 2: Pre/Post Mean Survey Scores

Construct	Pre M (SD)	Post M (SD)	Mean Diff	Hedges' g^z
PCK	2.84 (0.63)	3.38 (0.35)	0.53	0.84
Learner Agency & Access	2.88 (0.58)	3.50 (0.46)	0.63	1.27
Applied AI Fluency	2.63 (0.77)	3.17 (0.53)	0.54	0.65
Ethical & Reflexive Practice	2.79 (1.01)	3.75 (0.39)	0.96	1.05

Qualitative Results

Qualitative analysis of lesson plans, AI decision plans, peer evaluations, collaborative transcripts, and reflections showed how the PSTs used instructional judgment while lesson planning with AI. Findings are organized by themes associated with the four framework constructs. Table 3 summarizes how PSTs used instructional judgment across the four components of the framework while planning data science lessons. Across artifacts, decision logs, and peer evaluations, PSTs consistently evaluated and revised AI generated outputs. AI was used as a thought partner that PSTs decided whether to refine or simplify outputs that did not fit the instructional goals. Lessons in which AI outputs were explicitly revised were rated most highly in peer evaluations, reinforcing the importance of refinement in AI-mediated planning.

Table 3: Preservice Teacher Reflection Quotes

Framework Construct	PST Reflection
PCK	*"Data science feels less like an add-on and more like a lens I can build into what I already do in math." (PST reflection)*
Learner Agency & Access	*"When students see the purpose, they're more willing to talk about their thinking and explain their ideas."*
Applied AI Fluency	*"AI gave us a lot of ideas, but we still had to decide what actually worked for our students."*
Ethical & Reflexive Practice	*"Just because AI suggests something doesn't mean it's right. I felt responsible for checking the math and thinking about how it would affect the students."*

Accept, Adapt, Reject Decision Patterns

In the analysis of the AI decision logs, adapting the AI outputs was the most common response. Table 4 shows a representation of AI output decisions that teachers did during their lesson planning activity. The table shows evidence that PSTs used critical thinking through evaluation, adaptation, and justification in refining their prompts while planning their data science lessons.

Table 4: PST AI Output Decision Percentages

Decision	% of Decision Made
Adapt	61.1%
Accept	22.2%
Reject	16.7%

Table 5 highlights excerpts from collaborative planning conversations that illustrate PSTs' accept, adapt, and reject decisions when evaluating AI-generated instructional ideas. Across data sources, PSTs positioned AI as a design partner. AI generated outputs were most often adapted, and in some cases, rejected when they were not aligned with instructional goals. Direct acceptance of AI outputs was limited and generally restricted to low stakes components such as wording or brainstorming.

Table 5: PST Collaboration Quotes Paired with Decisions

Decision	Instructional Move	Quote from Collaboration Transcript
Reject	Identifying misalignment with data science goals	*"Wait, this feels kind of Physical Education-focused. Where's the data science in this?"*
Adapt	Refining investigative question for sense-making	*"We can keep the context, but we need to change the question so students are actually analyzing the data."*
Adapt	Simplifying AI-generated supports for usability	*"This is way too much for one lesson. Let's cut this down so students can focus on reasoning, not steps."*
Accept	Using AI for low-stakes idea generation	*"This part works as a starting point, we can keep the wording and build from there."*

Table 6 shows peer evaluation mean scores across design pairs on five constructs aligned with the study framework. Three pairs (Pairs #2, #3, and #4) received consistently high ratings (M = 3.5–4.0) with little to no variability (SD = 0.00 in most cases), indicating strong inter-rater agreement and perceived instructional coherence. In contrast, Pair #1 received comparatively lower ratings (M = 2.5–3.5) with moderate variability (SD = 0.71), suggesting more mixed peer perceptions. Applied AI Fluency showed the greatest differentiation across groups, particularly for Pair #3 (M = 2.00), highlighting variability in the depth of AI integration despite otherwise strong pedagogical and ethical design elements. Peer evaluations reflected high levels of rigor and coherence, with applied AI use emerging as the primary area of distinction among groups.

Table 6: Peer Evaluation Scores

Lesson	Rater	PCK	Agency & Access	AI Fluency	Ethical & Reflexive	Overall Coherence
1	PST 1 / 3	4 / 4	4 / 3	3 / 4	4 / 4	4 / 4
2	PST 5 / 7	3 / 2	2 / 3	4 / 3	4 / 3	4 / 3
3	PST 2 / 6	4 / 4	4 / 4	4 / 4	4 / 4	4 / 4
4	PST 4 / 8	4 / 4	3 / 3	2 / 2	4 / 4	4 / 4

Interrater reliability shown in Table 7 was calculated from the peer evaluation scores shown in Table 6. Because each lesson plan was independently evaluated by two preservice teachers using the same rubric, percent agreement and quadratic weighted Cohen's kappa were calculated for each construct across the four collaboratively developed lesson plans. Quadratic weighting was used due to the ordinal nature of the 1–4 rubric scale. Percent agreement ranged from 50%

to 100%. Weighted kappa values indicated substantial agreement for Pedagogical Content Knowledge (κ = .75) and Applied AI Fluency (κ = .64), moderate agreement for Learner Agency & Access (κ = .43), and perfect agreement for Overall Coherence (κ = 1.00). Ethical & Reflexive Practice showed high raw agreement (75%), though kappa was reduced due to restricted variance across raters, a known statistical limitation when most scores cluster within a narrow range. The lower agreement observed for Learner Agency & Access appears to reflect differences in how raters interpreted the specificity and depth of equity-oriented supports within the lesson plans.

Table 7: Interrater Reliability Results of Peer Evaluation

Construct	Percent Agreement	Weighted Kappa (κ)	Agreement
PCK	75%	0.75	Substantial
Learner Agency & Access	50%	0.43	Moderate
Applied AI Fluency	50%	0.64	Substantial
Ethical & Reflexive Practice	75%	0.00*	Restricted Variance
Overall Rigor & Coherence	100%	1.00	Perfect

Shown in Table 8 are PSTs' reflective comments illustrating how the AI-Mediated Teaching Framework shaped their instructional decision-making during AI-mediated data science lesson planning.

Table 8: PST Reflection on AI-Mediated Teaching Framework

PST Reflection on AI-Mediated Teaching Framework
"Before this, I saw AI as either a complete replacement for planning (which felt unethical) or a tool I shouldn't use at all (which felt naïve given where education is heading). The framework gave us a middle path—using AI as a collaborative partner while maintaining our professional judgment and commitment to students."
"One way the framework improved rigor and coherence was through the use of pedagogical content knowledge (PCK), which allowed me to focus the lesson on a specific teaching framework with clearly defined learning objectives. This ensured that the lesson generated from AI would actually follow those objectives."
"The framework helped me realize that just because AI suggests something doesn't mean it's correct or appropriate. I felt more responsible for checking the math and thinking about how it would affect students… One of the hardest parts was realizing how specific you have to be with AI. The framework helped me understand what to ask for and what to push back on when the output didn't match what we needed."
"The collaboration part of this assignment was extremely helpful. Having another future educator working through the framework with me made the process of using AI much smoother and more thoughtful."

Limitations and Next Steps

Several limitations should be addressed when interpreting these findings. This exploratory study involved a small sample size within a single course context using self-reported confidence ratings, limiting generalizability. In addition, the analysis focused on lesson planning and design artifacts rather than enacted classroom instruction. Future research should examine how AI-

mediated planning translates into classroom practice and how students engage with these lessons using a much larger sample size to better assess for scalability. Further work is also needed to explore how PSTs' AI fluency and professional judgement develop over time with repeated opportunities for AI mediated planning and reflection. Building on these findings, the instructional sequence and AI-Mediated Teaching Framework will be refined and implemented in PK-5 math methods courses in two different sites this spring semester, which will allow for the examination of the framework across several PST classrooms and instructional contexts.

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PROSPECTIVE TEACHERS' UNDERSTANDING OF DIVISION INVOLVING RATIONAL NUMBERS

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The purpose of this qualitative study was to explore prospective elementary teachers' (PTs') understanding of why we multiply by the reciprocal when dividing by a rational number. We analyzed a sample of 44 PTs' written responses on pre- and post-unit assessments, class homework, end-of-semester exam questions that involved division by rational numbers, and task-based interviews. We coded data deductively and inductively to investigate PTs' conceptual and procedural understanding, and their misconceptions about rational number ideas. We found that PTs showed (a) continued reliance on procedures across the semester, (b) growth in conceptual understanding, and (c) a decrease in misconceptions about rational number division. We also discuss the implications of this research in mathematics teaching and mathematics teacher education programs.

Keywords: Conceptual Understanding, Procedural Reliance, Prospective Teachers, Rational Numbers, Problem Solving

Rational number operations remain one of the most persistently challenging areas in elementary mathematics, for both students and the teachers who teach them (Lamon, 2007, 2020; Newton, 2008; Siegler & Pyke, 2013). This challenge is crucial because rational number reasoning is one of the gateways to algebraic thinking that make up a large portion of the mathematics that students take in middle and secondary school (Empson & Levi, 2011; Wu, 2011). PTs, therefore, are tasked with preparing their future students to be competent in rational number thinking so that the students can adapt to algebraic thinking in middle and secondary school. To achieve this task, PTs need to develop the necessary mathematical knowledge for teaching rooted in both procedural and conceptual knowledge about rational numbers. Most PTs, however, enter teacher preparation programs with deep procedural knowledge of rational number operations and with less conceptual knowledge, and if not well prepared, they risk reproducing the same knowledge patterns in their own students (Ball, 1990; Tirosh, 2000). There is a need, therefore, for PT preparation programs adopting teaching methods that prepare PTs to teach effectively. One instructional approach that holds theoretical promise for closing this gap is teaching through problem solving. While teaching through problem solving is among NCTM's (2000) recommendations for effective mathematics instruction, empirical evidence directly examining how this approach shapes PTs' conceptual understanding of specific rational number procedures, such as fraction division, remains limited. This study addresses that gap. We have two aims for this study: to investigate how teaching through problem solving can support PTs'

conceptual understanding of division and multiplication involving rational numbers, and to examine whether and how PTs' conceptual understanding, procedural fluency, and misconceptions change over the course of such instruction. We hope that findings from this study will include arguments of how procedural and conceptual knowledge development occurred in a problem-solving set-up, and how misconceptions tied to both forms of knowledge can be addressed.

Theoretical Framework

Our study is conceptually grounded in three areas of literature: (a) Mathematical Knowledge for Teaching (MKT), (b) teaching through problem solving, and (c) rational number learning.

Mathematical Knowledge for Teaching

Ball and colleagues (2008) introduced the MKT framework and defined it as the knowledge required by teachers to teach mathematics effectively. They adopted Shulman's (1986) subject matter knowledge as a category of MKT and divided it into three categories: (a) common content knowledge (CCK), which is the mathematical knowledge known by both mathematicians and non-mathematicians, such as the ability to perform procedures; (b) specialized content knowledge (SCK), which is the deep mathematics content knowledge required specifically by mathematics teachers; and (c) horizon content knowledge, which is the knowledge of how different mathematics topics are related. They also adopted pedagogical content knowledge from Shulman (1986). Our study is grounded in the MKT framing, specifically two forms of subject matter knowledge: CCK and SCK.

Procedural and Conceptual Knowledge as MKT

Procedural knowledge is the ability to perform procedures efficiently, and as such, this knowledge may be known by both mathematicians and non-mathematicians, as it may at times involve memorizing and reproducing procedures. On the other hand, conceptual knowledge is the ability to comprehend and understand underlying mathematical principles and relationships. Researchers such as Hiebert and Lefevre (1986), Hurrell (2021), and Salifu (2021) argued that both procedural and conceptual knowledge are important for mathematics teachers to teach effectively. Procedural knowledge can be considered a form of CCK, whereas conceptual knowledge can be considered a form of SCK (Ball et al., 2008; Newton et al., 2022). Research has shown that most PTs join teacher preparation programs with prior knowledge rooted in procedural fluency with limited conceptual understanding (Ball, 1990; Li & Kulm, 2008; Ma, 2010; Tirosh, 2000), which raises the need for teacher preparation programs to develop the PTs' conceptual knowledge while at the same time attending to their procedural knowledge. One teaching approach that we have used to achieve this goal is teaching through problem solving (Masingila et al., 2025).

Teaching Via Problem Solving: Procedural and Conceptual Understanding

Teaching via problem solving involves engaging students in tasks without predefined solutions to help them develop reasoning and adaptability (Lesh & Zawojewski, 2007). In previous studies, we found that teaching through problem solving can support students' understanding of why mathematical procedures are useful (Caviness & Masingila, 2023; Masingila & Wambua, 2015; Xie & Masingila, 2017). Such findings are consistent with Hiebert and Lefevre's (1986) definition of conceptual understanding. We contend that teaching via problem solving can enhance students' understanding of procedures and the rationale behind such procedures, that is, procedural and conceptual knowledge, respectively, as we have found in earlier studies (Caviness & Masingila, 2023; Masingila & Wambua, 2015).

Division Involving Rational Numbers

One key area of mathematics that researchers have shown interest in is the study of conceptual versus procedural knowledge in operations involving rational numbers. The NCTM (2000) defined rational numbers as numbers that can be expressed as the quotient of two integers, with the denominator nonzero. Rational numbers can be expressed in various forms, such as fractions, decimals, and percentages (Van de Walle et al., 2013). Researchers like Perry (2023) and Herreros-Torres et al. (2026) argued that fraction understanding remains the most challenging for PTs. Using Hiebert and Lefevre (1986) and Salifu's (2021) call, we assert that students need both procedural and conceptual knowledge about rational number operations.

One fraction operation that has proved challenging for most PTs is fraction division (Copur-Gencturk, 2021). Petit et al. (2015) defined fraction division as the process of determining how many times a fraction fits into another. Siegler and Pyke (2013) and Olanoff et al. (2016) noted that when dividing fractions, most students use the invert and multiply algorithm, a procedural shortcut where the divisor is inverted and multiplied by the dividend. Researchers have consistently shown that while PTs may use the algorithm correctly, they are not able to justify its rationale (Kang, 2022; Nillas, 2010). The conceptual knowledge attached to the invert and multiply procedure is that inverting the divisor shows how many quantities of the divisor fit in one unit, and multiplying therefore means taking the number of divisors in one unit and multiplying that by the dividend (Masingila, 2022b), which is similar to finding how many of the divisors fit into the dividend (Petit et al., 2015).

Across the studies reviewed, it is evident that most PTs have procedural knowledge of fraction division, most of which is rooted mainly in the use of the invert and multiply rule, with less conceptual understanding, which makes it hard for them to transfer their knowledge to other fraction representations. Sahin and colleagues (2020) showed that PTs can develop conceptual understanding when involved in small and whole-class discussions, which is a promising finding that builds on in our study. Consequently, the imbalance between procedural and conceptual knowledge has been proven to cause misconceptions about fraction division.

A misconception refers to a student's conception that produces a systematic pattern of errors (Smith et al., 1993). The conceptions may be tied to students' past or current knowledge that is incomplete to fully account for either procedural or conceptual knowledge. Smith and colleagues argued that mathematics education research should "document misconceptions in as many subject-matter domains as possible" (p. 123). Therefore, there is a need to, in the process of discussing procedural and conceptual knowledge, also address PTs' conceptions that lead to errors with respect to the two forms of knowledge.

Research Questions

In our review of the literature, we found that most PTs hold more procedural knowledge and less conceptual knowledge on fraction division, and that they also hold some misconceptions that need to be addressed. While teaching through problem solving has been shown to be promising in addressing this gap, we did not find evidence on how teaching through problem solving impacts PTs' development of both procedural and conceptual knowledge, and whether or not it affects the misconceptions they hold about fraction division in particular. We anticipate that answering this question will help address some of the unpreparedness or under preparedness faced by mathematics teachers in the area of rational numbers. In particular, our study aims to answer this research question: What appears to be the impact of teaching and learning via problem solving on PTs' procedural and conceptual understanding of why we multiply by the

reciprocal when dividing by a rational number, and what misconceptions resurface and get addressed in the process?

Methods

Our research design was a basic qualitative inquiry (Merriam & Tisdell, 2016). We chose a basic qualitative design because our goal was to understand how PTs make meaning of fraction division in the context of teaching and learning via problem solving, without constraining ourselves to a named qualitative tradition.

Participants and Context

The participants in this study were 44 PTs enrolled in the second course of a mathematics content course taught through problem solving at a university in the northeastern United States during the Spring 2025 semester. The course emphasized topics such as rational numbers, geometry, and measurement. During the study period, instruction focused on rational numbers and operations with rational numbers. Classes met twice a week for 80 minutes and centered on small-group problem solving rather than lectures. Four mathematics education graduate students served as interns in the course. They supported PTs' learning via problem solving.

Data Collection and Analysis

We used multiple data sources to explore PTs' understanding of rational numbers: (a) a pre-unit assessment consisting of three items administered before instruction on rational numbers, with each item having two parts; (b) group classwork from the two problem-solving class sessions each week focused on rational number reasoning; (c) PT's weekly written homework; (d) the end-of-semester course examination that included rational number items; (f) a post-unit assessment parallel to the pre-unit assessment administered after three weeks of instruction; and (g) end-of-unit interviews with five participants designed to probe procedural and conceptual reasoning and to assess how PTs' understanding of rational number ideas had changed. The data were collected over a span of six weeks. Overall, all the data artifacts used in this study were designed to help us answer our research question in line with our theoretical framework of MKT.

We began analysis with a deductive, theory-driven thematic approach (Boyatzis, 1998) by drawing initial codes from prior research on prospective teachers' rational-number understanding. Our initial codes included procedural reliance and conceptual understanding, extracted from Desai (2025). We then applied inductive coding to capture additional codes emerging from the data, such as misconceptions about rational number ideas. We coded in two stages. In Stage 1, each researcher independently coded the pre-unit assessments, and then we discussed the coded data segments as a group to refine code definitions and ensure consistency. In Stage 2, we coded the remaining data, followed by regular weekly team meetings to resolve differences and reach consensus on final codes.

Findings

Our aim in this study was to explore the impact of teaching and learning via problem solving on PTs' understanding of why we multiply by the reciprocal when dividing fractions. In our findings, we use the pseudonyms PT 1 to PT 44 to refer to the 44 PTs. Of the 44 PTs: (a) 33 completed the pre-unit assessment, with five PTs leaving the questions unanswered, (b) 28 PTs handed in their classwork to be used as data, (c) 40 PTs completed the post-unit assessment, with one PT leaving the questions unanswered, and (d) all 44 PTs completed the final exam. We present here our findings under three different themes: conceptual understanding, procedural reliance, and misconceptions in rational number understanding across different data sets.

Conceptual Understanding

Pre-unit assessment. In the pre-unit assessment, two PTs, PT 3 and PT 25, demonstrated conceptual understanding. PT 3 drew a number line and marked 12 one-fourth segments to visualize $3 \times 4 = 12$, illustrating the reciprocal as counting groups of the fraction that fit into one whole. The PT's work shows that each group of one whole would have four one-fourths, hence 12 one-fourths in three whole units.

Classwork and homework. When solving ongoing class homework, six PTs demonstrated conceptual understanding in tasks requiring division by a rational number. In responding to one of the homework questions, "Why do we multiply by the reciprocal when dividing fractions?," PT 6, stated that "*Dividing by a number is the same as multiplying by its reciprocal [since] you're asking how many of that fraction fit into the first one.*" The PT's reasoning demonstrated conceptual understanding because inverting a fraction tells how many of those fractions fit into one whole. Five other PTs also reasoned like PT 6 in solving homework tasks.

Post-unit assessment. In the post-unit assessment completed at the end of the rational number unit, one PT, PT 3, demonstrated conceptual understanding. In explaining $5 \div \frac{1}{3}$, the PT drew a rectangular strip and divided it into five sections, each representing a value of three one-thirds. The strip had a total of 15 one-thirds. The PT's conceptual understanding stems from seeing the three one-thirds as one whole and then drawing five one-wholes.

Final exam. On the final exam completed at the end of the semester, 18 PTs demonstrated conceptual understanding when responding to a question asking why dividing a number by $\frac{4}{13}$ is the same as multiplying the number by $\frac{13}{4}$. PT 10 explained that *"… when you multiply by a reciprocal, it tells how many times that fraction fits into the number. Multiplying by the reciprocal flips the division into multiplication, which gives the same value."* The PTs' reasoning aligns with our definition of conceptual understanding of why the invert and multiply rule works.

Interviews. In task-based interviews we conducted with five PTs at the end of the course, three PTs (PT 36, PT 13, and PT 35) demonstrated that they had developed conceptual knowledge of fraction division. We share below their explanations.

PT 36 noted that dividing $\frac{3}{2}$ by $\frac{1}{4}$ requires switching the division to multiplication and inverting $\frac{1}{4}$ to $\frac{4}{1}$. She applied this procedure to have $\frac{3}{2} \times \frac{4}{1} = \frac{12}{2} = 6$. When we probed to explain why this procedure works, she said:

> Because in order to divide, you have to change it to multiplication because you're trying to see how many one-fourths go into three halves, so you have to multiply. And in order to multiply a division problem, you have to switch it over to four over one because one over four times four over one equals one whole. So that means four over one, there's four whole pieces and one fourth pieces, so that gives you one. So, if you have one fourth and four over one, when you multiply those together, that gives you one. So that's why you have to flip 'em [them] over and then you multiply across and simplify. (PT 36, Interview)

Her reasoning about $\frac{3}{2} \div \frac{4}{1}$ meaning how many groups of one-fourths are in three halves, and that $\frac{1}{4}$ multiplied by $\frac{4}{1}$ equals one whole shows that PT 36 had developed partial conceptual understanding as she was moving towards recognizing that $\frac{4}{1}$ shows how many groups of $\frac{1}{4}$ are in one whole; hence the need to multiply to get the number of one-fourths in $\frac{3}{2}$. Her explanation satisfies our definition of conceptual understanding.

Growth in conceptual understanding. In summary, the above findings across our data sets show that the number of PTs who demonstrated conceptual understanding in the pre- and post-

unit assessments was low (1 or 2). However, the larger number of PTs who demonstrated conceptual understanding on the final exam (18) shows that PT growth in reasoning beyond why procedures work increased gradually across the course. The interviews with five PTs also allowed PTs to give more justifications of why the invert and multiply procedure is a valid method, with three of the five PTs interviewed demonstrating conceptual knowledge. Also important to note is that the three PTs who demonstrated conceptual understanding in the interviews had started the unit relying solely on procedures.

Procedural Reliance

Pre-unit assessment. In the pre-unit assessment, 23 PTs demonstrated procedural reliance. Twenty-two of the PTs inverted the divisor and switched division to multiplication, noting that they used the "keep, change, flip" method, as that is the way they have memorized the "invert and multiply" rule. They attributed the procedure to what they were taught in elementary school: keep the first number, change the division sign to multiplication, and flip the fraction.

Classwork and homework. In completing class homework, 21 PTs demonstrated procedural reliance in their work. Most PTs simply noted that the invert and multiply rule works because dividing by a number is the same as multiplying by its reciprocal. Other PTs reasoned that since division is the inverse of multiplication, changing division to multiplication implies inverting the second fraction. PT 1, for instance, wrote, "*It [dividing by a rational number] is the same as multiplying by its reciprocal [because] doing the opposite procedure on the opposite number yields the same product*." Whereas most PTs reasoned this way, their reasoning did not justify why the invert and multiply rule works.

Post-unit assessment. In the post-unit assessment administered towards the end of the course, a total of 34 PTs demonstrated procedural reliance in their work. As in the pre-unit assessment, all the 34 PTs used the invert and multiply rule without a clear justification as to why the rule works. PT 28, like 32 other PTs, wrote, "*I did keep change flip*." PT 29 similarly wrote, "*When dividing fractions, it's multiplying by the reciprocal*." Other PTs just write *"KCF"* to denote "keep change flip," while others wrote *"reciprocal"* to illustrate multiplying by the inverse.

Final exam. On the final exam done at the end of the semester, 22 PTs showed procedural reliance in solving a question on division by a rational number. Similar to findings from the pre- and post-unit assessment, most PTs stated that it is possible to invert and multiply because of the "*keep-change-flip [rule]*." PT 7 wrote, "*Because multiplication and division are opposites, when reversing the fraction, the operation is also getting reversed*." Apart from stating "*keep change flip*" as the reason why the invert and multiply rule works, other PTs reasoned like PT 7 on the final exam.

Interviews. In the interviews, two of the five PTs interviewed appeared to rely solely on the invert and multiply procedure, and when probed to justify why the procedure was valid, their responses appeared to restate the procedure rather than to justify why they used it. The two PTs both solved the question as $\frac{3}{2} \times \frac{4}{1} = \frac{12}{2} = 6$. PT 10 explained her reasoning as follows:

> I kept three halves. I changed the divisions and multiplication, and then I flipped the numerator and the denominator. So I got three halves times four over one, and then I just multiplied it across. So, three times four is 12, two times one is two, and then divided the 12 by the two and got six. (PT 10, Interview)

PT 38 reasoned in a similar way to PT 10. Both PT 10 and PT 38's explanations relied primarily on procedural descriptions of the invert and multiply rule, with limited explicit reference to the conceptual meaning of fraction division.

Continued reliance on procedures across the unit. In summary, the above findings demonstrate that from the pre-unit assessment through to the post-unit assessment, the final exam, and the interviews, procedural reliance remained dominant across all data sources. Initially, 22 PTs relied on procedures like keep-change-flip without explaining their rationale. This trend persisted throughout the semester, as seen in classwork (22 PTs), final exam (22 PTs), post-unit assessments (34 PTs), and interviews (two of five PTs).

Misconception

Misconceptions were evident in how the PTs did the work procedurally and in their explanation of why such procedures work. As a result, some misconceptions were tied to procedural knowledge, while some were tied to conceptual knowledge.

Pre-unit assessment. At the start of the unit, four PTs showed misconceptions in their work. PT 5 used multiplication for division and wrote "$3 \times \frac{1}{4} = 0.75$." We interpreted this as the PT using the invert and multiply rule without inverting the fraction. PT 44 solved the question as "$3\times \frac{4}{1} = 12$," but referred to this process as cross multiplication instead of the invert and multiply process. This showed a misconception in the PTs' view of division as cross multiplication. PT 3 and PT 38 made an error similar to PT 5 by performing multiplication without inverting. All the misconceptions in the pre-unit assessment can be attributed to the PTs' developing procedural knowledge about fraction division.

Post-unit assessment. We observed four cases of misconceptions about division by rational numbers in the post-unit assessments. For example, PT 5 multiplied five by $\frac{1}{3}$ instead of dividing, meaning the PT probably used multiplication for division or was using the invert and multiply rule, but did not invert the fraction. PT 16 inverted $\frac{1}{3}$, but instead of multiplying, she added the numerators. The PT seems to have misconceived multiplication for addition. The two other PTs solved the question correctly but noted that they used cross multiplication. We interpreted that the PTs confused the invert and multiply rule for cross multiplication. Similar to the pre-unit assessment and the classwork, the misconceptions observed in this case can all be attributed to the four PTs' reliance on procedural knowledge.

Final exam. On the end-of-the-semester exam, three PTs showed misconceptions when explaining whether division is the same as multiplying by the reciprocal. PT 41 wrote, "*Dividing by the reciprocal is the same as asking how many times the whole fraction goes into the denominator of the other fraction*." This was a misconception that can be attributed to conceptual knowledge of fraction division. To be correct, she would have stated that the reciprocal of a fraction shows the number of fractions that fit in one whole, which changes the division to multiplication. Although her answer was not clear, she appeared to think of division as asking how many times a fraction goes into another.

Persistence in misconceptions about division involving rational numbers. The findings above show that instances of misconceptions were evident across all stages of data collection but remained low. Overall, misconceptions involved confusing division for multiplication, using the invert and multiply rule but failing to invert the divisor, or inverting the divisor and performing addition instead of multiplication.

Discussion

Our goal was to investigate the effect of teaching and learning via problem solving on PTs' understanding of why, when dividing a number by a fraction, we change division to multiplication and invert the divisor. In general, we found that at the start of the study, most PTs had deep procedural knowledge of fraction division and could apply the invert and multiply rule well but could not justify the reasoning behind the procedure. Additionally, a few PTs showed conceptual knowledge about fraction division, while some also had misconceptions tied mainly to their reliance on procedural knowledge. These initial findings compare with what other researchers like Kang (2022), Kang and Breiten (2024), Ma (1999), Morano and Riccomini (2020), and Tirosh (2000) found: most PTs can perform the standard invert and multiply rule but may have challenges justifying why the rule works. As the study progressed, the number of PTs who developed conceptual knowledge kept fluctuating, while at the same time, the number of those who exhibited procedural knowledge remained high. A notable change was towards the end of the study when the number of PTs demonstrating both procedural and conceptual knowledge of fraction division was almost equal, at 22 and 18, respectively. The interviews also allowed us to deeply assess some PTs' learning, and we found that three of the five PTs had developed conceptual knowledge, while two had developed procedural knowledge of fraction division. Our findings indicate that from the start to the end of the study, the course helped the PTs develop both procedural and conceptual knowledge. Our findings respond to Hiebert and Lefevre (1986), Hurrel (2021), and Salifu's (2021) recommendations that both procedural and conceptual knowledge are needed to prepare effective mathematics teachers.

In the process of developing procedural and conceptual knowledge, we noted cases of incomplete conceptions that produced errors, which we termed as misconceptions. All the misconceptions observed at the start and during the progress of the study were attributed to procedural knowledge, whereas two of the three misconceptions realized at the end of the study were attributed to conceptual knowledge. In total, we had 10 procedural misconceptions and one conceptual misconception. Examples of procedural misconceptions involved: (a) inverting the divisor and then performing addition instead of multiplication (such as PT 16 in the post-unit assessment), (b) changing division to multiplication without inverting the divisor (such as PT 5 in the pre-unit assessment), (c) applying the procedure correctly but then stating that the procedure is called cross multiplication (such as PT 44 in the pre-unit assessment), and (d) using whole-number reasoning to claim that dividing a number by a fraction increases/decreases the number (such as PT 15 in classwork). The misconceptions identified in this study reflect knowledge and instructional gaps shown by earlier studies (Alghazo & Alghazo, 2017; Powell & Nelson, 2020). We found one misconception linked to conceptual knowledge, which involved PT 41 reasoning that dividing fractions means finding how many groups of the divisor go into the dividend. This finding illustrates how promoting conceptual knowledge can be useful in PT learning, given that PTs' learning to justify mathematical procedures reduces the number of misconceptions they may hold about fraction division.

Based on our findings, we recommend that mathematics teacher preparation programs design activities that require PTs to reason, discuss, and defend their mathematical thinking rather than simply memorizing and practicing procedures. For sample activities that can be used in teaching and learning via problem solving, see Masingila et al. (2022a, 2022b). We also suggest that teacher education programs embrace teaching and learning via problem solving to promote both conceptual understanding and procedural accuracy. For more details on how a teaching and learning via problem solving course can be designed and enacted, see Caviness and Masingila

(2023), Masingila and Wambua (2015), and Xie and Masingila (2017). Finally, we recommend targeted instruction that addresses specific procedural and conceptual misconceptions that PTs have about fraction division, which is also a future direction for our study.

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ELEMENTARY PRESERVICE TEACHERS' SPATIAL VISUALIZATION THINKING AND APPLICATION TOWARDS EXPLAINING ASTRONOMICAL PHENOMENA

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This sequential explanatory mixed-methods study builds upon research examining in what ways preservice teachers (PSTs) visualized astronomical concepts they are expected to teach. We utilized a Spatial Visualization theoretical framework to investigate: (a) What do PSTs scores on a lunar phases and seasons assessment show about their spatial awareness regarding astronomical phenomena? (b) In what ways do PSTs utilize their spatial thinking towards applying to real life phenomena? Quantitative data consisted of students completing a content survey assessing cause of phases, cause of seasons, and visualization of Earth/Moon/Sun scale. Qualitative data involved interviews concerning 2D/3D Earth/Moon/Sun modeling of lunar phases and Earth/Sun modeling of seasons. In comparing Earth-based and Space-based spatial perspectives, PSTs tended to perform better on Earth-based items concerning cause of seasons.

Keywords: Preservice Teacher Education, Geometry and Spatial Reasoning

Objectives

This study builds upon prior research examining how elementary preservice teachers (PSTs) visualize and conceptualize astronomical phenomena they are expected to teach. Sixty PSTs from an elementary methods class were assessed on their understanding of cause of lunar phases and seasons as well as understanding the scale of the Earth/Moon/Sun system. This research investigated how elementary PSTs navigated Earth/Moon/Sun geometric configurations, motions, and scale. Specific spatial domains measured were periodicity, geometric spatial visualization (a Space-based perspective), cardinal directions, and spatial projection (an Earth-based perspective).

Spatial Visualization Theoretical Framework

In terms of astronomy, spatial visualization of celestial bodies (i.e., the Moon, planets, stars) and their heavenly motions are more complex than most people realize; it is inherently abstract. The necessary spatial skills vary by astronomy topic, but spatial ability contributes to understanding astronomical topics such as cause of lunar phases and cause of seasons. Learners are introduced to such ideas in the form of 3D physical models as well as 2D representations with the idea that these ways of understanding will translate to mental visuo-spatial abilities where they mentally manipulate the scale, orbit, and geometric orientation of the celestial bodies at play (Miragliotta & Baccaglini-Frank, 2018; Makamure & Jojo, 2021).

The Next Generation Science Standards (NGSS lead states, 2013) and Common Core State Standards-Mathematics (National Governors Association, 2010) discuss the importance of student understandings related to spatial-scientific learning (i.e., scale, patterns, and modeling). Research literature has connected increased spatial reasoning ability, including Earth-based and Space-based visualizations, with an increased understanding of concepts such as lunar phases and seasons (Cole et al., 2018; Plummer et al., 2014; Wilhelm et al., 2018). The NGSS state that students should learn "patterns of apparent motion of the Sun, the Moon, and stars in the sky… and [can be] explained with models" and "Earth's spin axis is fixed in direction over the short-term but tilted relative to its orbit around the Sun. The seasons are a result of that tilt…"

Predominant misconceptions that students hold to explain cause of the Moon's phases include Earth's shadow, Sun's shadow, and object blocking notions (Wilhelm et al., 2018). In addition to this, students often are not aware that the Moon orbits the Earth and if they do, they do not know the orbital time period (Trundle et al., 2002). Similarly, students who display incomplete understanding about the cause of seasons tend to hold some of the following perceptions: (1) the Earth is closer to the Sun in summer, (2) the direction of the Earth's axis during orbit around the Sun changes throughout the orbit, (3) incorrect time duration of one Earth rotation on its axis, and (4) visualizing the Sun revolves around the Earth (Atwood & Atwood, 1996; Bakas & Mikropoulos, 2003; Trumper, 2006; Wild & Trundle, 2010).

Research Questions and Participants

The purpose of this research was to examine how well PSTs understand astronomical concepts (requiring spatial visualization) that they are expected to teach. Specifically, we examined PSTs' spatial awareness of astronomical phenomena from both Earth-based and Space-based perspectives. Research questions included: (a) What do PSTs scores on a lunar phases and seasons assessment show about their spatial awareness regarding astronomical phenomena? (b) In what ways do PSTs utilize their spatial thinking towards applying to real life phenomena. These questions were explored with 60 undergraduate preservice teachers from one south-central university in the United States. The participants in this study were PSTs enrolled in an elementary science methods and mathematics methods course taken during their senior year. The participants completed two courses in mathematics and one course in physics designed for elementary and middle school PSTs during either their sophomore or junior year. Of the 60, 57 PSTs were female (1 Hispanic American, 1 Asian American, 4 African American, and 51 White) and 3 were male (1 Native American, 2 Decline to answer).

Methods

For this sequential explanatory mixed methods study, data collection involved administering the Astronomical Assessment, a combined inventory of the Lunar Phases Concept Inventory (LPCI; Lindell & Olsen, 2002), Earth/Moon/Sun scaling questions, and a Seasons Concept Inventory (SCI; Krall et al., 2023). The LPCI is a 20-item multiple choice instrument that assesses eight science domains and four spatial domains (see Table 1). Four scaling questions were added to the lunar phases portion of the Astronomical Assessment to gauge PSTs' conceptions of the relative size and distance between the Earth, Moon, and Sun. The SCI was constructed using 14 questions from previously published concept inventories (Bakas & Mikropoulos, 2003; Roald & Mikalsen, 2001; Trumper, 2006; Wild & Trundle, 2010). Questions were selected to address four domains presented in Wild & Trundle (2010). The domains included: (1) Earth's axis is tilted, (2) Earth orbits Sun, (3) Tilt of Earth's axis and its orbit around Sun causes variable amounts of sunlight hitting different Earthly regions, and (4) day and night are caused by Earth's rotation on its axis.

The lunar phases questions were coded as Earth-based or Space-based. In addition, each question was spatially coded as Periodic Patterns (PP), Geometric Spatial Visualization (GSV), Cardinal Direction (CD), Spatial-Projection (SP), or SC (Scale). Table 1 outlines the definition of each code. The Seasons questions were also coded as Earth-based or Space-based. In addition, each Seasons question was coded as: Domain 1: Axis tilt, Domain 2: Earth Orbits Sun; Domain 3: Variable Sunlight, Domain 4: Rotates on Axis, and Domain 5: Cumulation of Domains 1–3.

The sequential process for data collection occurred in two phases. In Phase 1, PSTs volunteered to take the assessment in their elementary methods course, taking students

approximately 29 minutes to complete. Phase 2 was conducted within two weeks of taking the Astronomy Assessment where five students volunteered to be interviewed regarding their understanding of cause of lunar phases and seasons. These interviews were semi-structured and designed to probe deeply into the PST volunteers' thinking. Protocol questions included requests to illustrate both 2D and 3D explanations for why we see lunar phases and what causes seasons. The average interview time was approximately 25 minutes.

Table 1: Lunar Phases Concept Inventory Spatially Coded

LPCI Scientific Domains	Spatial Domains
A- Period of Moon's orbit around Earth B – Period of Moon's cycle of phases	**Periodic Patterns - PP** (*regular intervals of time and/or space*)
C – Direction of the Moon's orbit around Earth E – Phase due to Sun/Earth/Moon Positions G – Cause of lunar phases	**Geometric Spatial Visualization - GSV** (*visualizing the geometric spatial features of a given system as it appears in space*)
D – Moon motion from Earthly perspective F – Phase-location in sky-time of observation	**Cardinal Directions - CD** (*object's vector direction in space as a function of time from a given position*)
H – Effect of lunar phase with change in Earthly location	**Spatial Projection - SP** *(projecting oneself into a different location, confined to the surface of Earth, and visualizing from that global perspective*)
I – Relative size of and distances between Earth/Moon/Sun	**Scaling - SC** (*relative sizes and distances between objects*)

Results

Lunar Phases Survey Findings

Of the 60 PSTs who took the assessment, 27.4% had a scientific explanation of the cause of lunar phases, that is, phases are caused by the changing geometric positions of the Earth/Moon/Sun, which affect how much of the Moon's illuminated surface is visible to a viewer on Earth. Other PSTs held common misconceptions with more than half (53.2%) holding the Earth-shadow explanation and 19.4% holding a Sun-shadow explanation. The lunar phases questions that were coded as Earth-based perspective questions showed PSTs scoring 32.3% correct while the coded Space-based perspective questions showed PSTs scoring 31.3% correct. Figure 1 shows PSTs scoring best on those questions addressing Earth/Moon/Sun scale (SC; 44.2% correct), periodicity of lunar phases and orbit (PP; 40.7% correct), and viewing phases from various Earthly locations (SP; 40% correct). Questions that dealt with cardinal direction of the Moon for phases at different times showed the least understanding.

Seasons Survey Findings

Of the 60 PSTs, nearly half (47.6%) scored correctly on those Seasons questions coded as Earth-based perspective. However, only 28.1% of the PSTs scored correctly on the Seasons questions coded as Space-based perspective. Figure 2 shows PSTs scoring best on those questions that focused on the effects of Variable Sunlight on Earth (48.3% correct). For questions regarding the concept that Earth rotates on its axis, PSTs scored 38.3% correct. The questions with the least understanding concern the concept that Earth orbits the Sun and the effect of Earth's tilted axis.

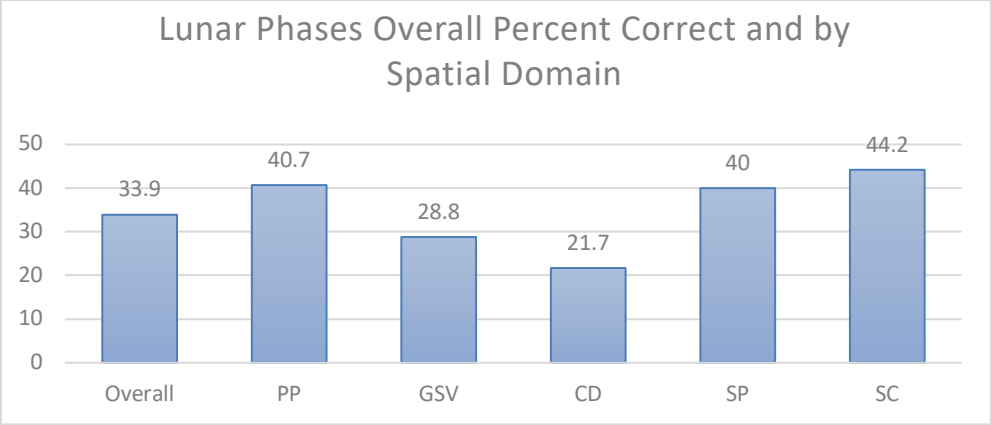


Figure 1: Lunar Phases Survey Overall Results and by Spatial Domain

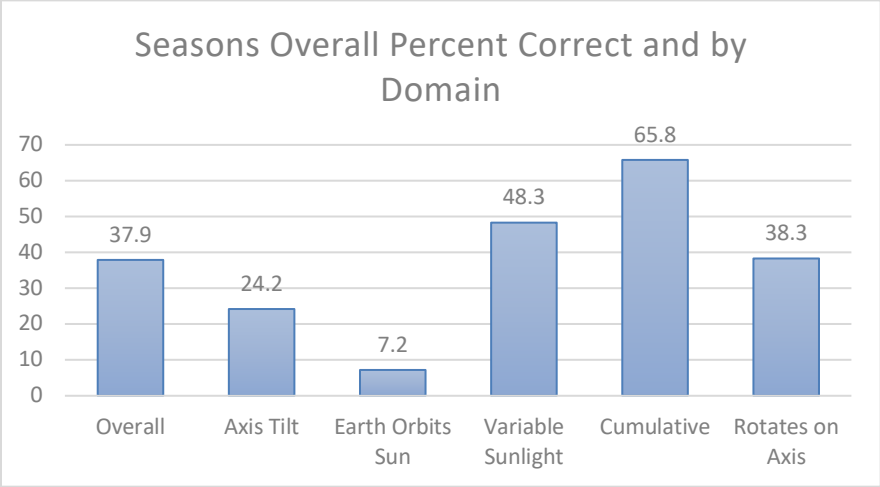


Figure 2: Seasons Survey Overall Results and by Domain

Interview Findings

We highlight two of the five interviews that display the lunar phases and season's understandings and visualizations of the PSTs. These two examples are representative of the five interviews. The two interviewees highlighted were Phoebe and Fiona (pseudonyms).

Phoebe. Phoebe stated that she rarely looked at the Moon. When asked to sketch any way she might have seen the Moon, she drew sketches of the Moon during a full Moon phase and crescent phase (3 similar crescent shapes) as shown in Figure 3.

Researcher:	[after having drawn a full Moon] Would you mind drawing the other ways that the Moon can be perceived? [She drew three crescent shapes.] With your drawings, can you explain a little why they appear differently?
Phoebe:	So, this would be a full Moon [pointed to circle of full Moon], then the crests and, like, throughout the month they will appear differently.
Researcher:	Why does the Moon look different (at different times)?
Phoebe:	I guess it's just how the Sun is hitting it, we portray it different [sic] on Earth.

Researcher: And do you know a little bit more about why this happens?
Phoebe: I'd say it has to do with something the way the Earth is on its, like, tilt and it rotates. We just see it differently.

Phoebe seemed to understand that perspective from Earth might be a reason we see the Moon differently at different times when she claimed, "it's just how the Sun is hitting it, we portray it different on Earth." However, Phoebe incorrectly followed up with the statement that it has something to do with the Earth's tilt.

For the Seasons questions, Phoebe sketched a drawing of the Sun and the Earth positioned at different locations around the Sun in what appeared to be an elliptical orbit (see Figure 3). Phoebe described this orbit in terms of how the Earth "rotates" around the Sun. She stated, "So, here's the Sun. Here's the way that Earth rotates around the Sun. So, then summer would be here [position where Earth is closest to the Sun] because we're closer and then winter would be here because it's farther away." Phoebe held the misconception that the Earth is closer to the Sun in the summer and concluded that would be the reason why it is warmer in the summer.

Fiona. When Fiona sketched Moon phases she recollected observing, she drew a full, a crescent, and a gibbous Moon, although she did not name them as such (see Figure 3).

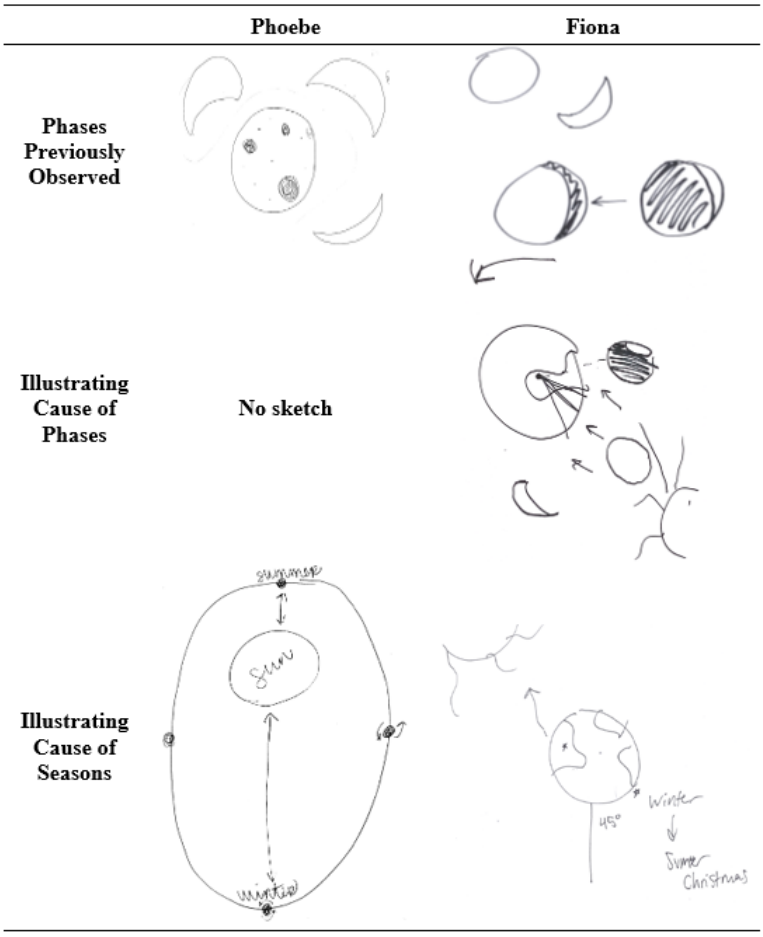


Figure 3: PSTs' Spatial Sketches of Cause of Lunar Phases and Seasons

Below is an excerpt of Fiona explaining why the Moon looks different at different times:

Fiona:	So, if I was right here, and I was looking up at the Moon, like I said, I always assume it's going to be right up here. [Stating Moon is always found in the same location.] Now this is the part that it could be wrong, but I think it's okay if I'm wrong. I think the Sun is going to be like here. … And then depending on where I am, and how much Sun is coming through, is what I would see. But I don't know exactly like what Moon phase I would see.
Researcher:	When you say the Sun is coming through, what do you mean?
Fiona:	From the Sun and the energy, I guess, but specifically the light.
Researcher:	But when you say "through," you mean?
Fiona:	Around and probably, yeah, around.
Researcher:	Okay. So, but you're saying for this one [pointing at the Moon Fiona drew between the Sun and the Earth in her cause of Moon phase sketch in Figure 3], you're not sure, what Moon phase?
Fiona:	I think it'd be a full Moon in the picture I drew but it could be wrong.
Researcher:	If say that were a full Moon, where would the Moon be for us to see something that looks like that [pointing to her crescent Moon phase she had sketched].
Fiona:	Here, and then for the flipped version, but I don't even know if that's actually a thing, it would be here and then that would be covered. Does that make sense?
Researcher:	Okay, so how would that be covered? Like covered with what?
Fiona:	That's a good point. I don't know. I guess it would not. It would be covered but I'm not sure what is covering it. But that's like what I would see from that point.

In Figure 3, under Fiona's sketch of Earth/Moon/Sun geometric positions that would result in viewing a full Moon or a crescent Moon from an Earthly perspective, Fiona first sketched the Earth (farthest left circular figure) and the Sun (bottom right figure with Sun's rays shown with squiggle lines). She sketched the Moon between the Earth and Sun. When asked what Moon phase we would observe with the Moon in that position, Fiona stated that she was not sure, but it was probably a full Moon. She was then asked to show where the Moon would be located for us to observe a crescent Moon, and she drew a crescent Moon on the lower left side and then she said the flipped version would be on the other side (the Moon that is shaded except for a small patch facing away from the Sun). She indicated that the Moon was covered except for the small patch. When asked what the Moon was covered by, Fiona indicated that she did not know. Fiona did not seem to be aware that the illuminated side of the Moon should be facing the Sun.

For the Seasons portion of the interview, Fiona was asked why it was warmer in summer and colder in winter in Lexington. She replied:

Fiona:	Because of our axis, and like how we are, not rotated, but positioned, I guess. And where we are in relativity to the Sun…

Fiona's initial answer suggested that she believed the position of Earth relative to the Sun and Earth's axis has something to do with it. She was asked to draw a sketch to illustrate her meaning. She drew the Earth and explained that it was tilted on its axis at 45°. She then drew a

half circle to illustrate the Sun, explaining that Earth "rotated" around the Sun [gesturing with her hands an orbital movement]. She elaborated further.

Fiona: And it is our summer when we are directly in front of it [Sun], and our rotation is closest to the Sun. And that is why Australia would have winter at that time. [Drawing stars to show that Australia was located on the opposite side of Earth from Lexington.] But, if you flipped it [shows rotation with her hands], then they [Australia] would have summer, and we [Lexington] would have winter at Christmas [and Australia would have Christmas during their summer].

Fiona believed that Earth's rotation on its axis caused the seasons rather than the combination of Earth's tilt on its axis and its orbit around the Sun.

Later in the interview, the researcher asked questions to clarify Fiona's thinking about the causes of seasons and the motion of Earth and the Sun.

Researcher: Okay, so does anything move? Like, does the Sun move, or does the Earth move, or both?

Fiona: Okay, well, I don't know about the Sun. I know the Earth moves. It rotates like this [demonstrates motion of rotation on an axis], as well as like this [demonstrates orbiting around the Sun]. I know our Moon moves around us, but I don't think the Sun moves, but I could be wrong. And then yeah, we rotate around the Sun… [uses the 3D models to show Earth orbiting Sun].

Researcher: So, you said that the Earth moves around the Sun. How long does it take to go around this?

Fiona: It's a month? I don't know. I think a month.

Researcher: And does that affect anything about the seasons?

Fiona: Oh, my immediate reaction says yes, but I don't think so.

Fiona believed that Lexington is warmer in summer because of the closer proximity of Earth to the Sun due to its rotation (Earth's rotation on its axis) toward the Sun. As the Earth rotates away from the Sun in the fall it becomes cooler until Lexington experiences winter when that side of Earth has rotated to be directly opposite the side facing the Sun. It is interesting that Fiona understood that Earth orbits the Sun, but she did not believe this movement affected the seasons experienced on Earth. Further, she believed the path Earth follows around the Sun would take about a month to complete. Although she did not mention a time frame for the complete Earth rotation on its axis, her answer suggests the rotation would take one year to complete. The three interviews not highlighted showed interviewees with the same way of spatially thinking regarding the cause of seasons, that is, claiming Earth's rotation on its axis caused the change of seasons where the side facing the Sun would be summer. Visualizing seasons in this manner also illustrates PSTs' lack of understanding the cause of day and night.

Discussion and Conclusion

This study examined PSTs' spatial awareness of astronomical phenomena from both Earth-based and Space-based perspectives. Regarding RQ1, performance patterns suggest that PSTs' spatial reasoning is fragile across perspectives and particularly vulnerable when tasks require mental transformation into a Space-based frame of reference. On the lunar phases assessment, approximately one-third of PSTs answered correctly for both Earth-based (32.3%) and Space-

based (31.3%) items. The similarity in performance indicates that even when questions are grounded in the more familiar Earth-based viewpoint, PSTs struggle to accurately coordinate Earth/Moon/Sun relationships.

In contrast, results for seasons revealed a clearer divergence. PSTs performed better on Earth-based items (47.6% correct) than on Space-based items (27% correct). This pattern supports the interpretation that Earth-based questions may allow PSTs to draw upon experiential or phenomenological reasoning (e.g., temperature changes, daylight length), whereas space-based questions require coordinated spatial transformations—mentally visualizing axial tilt, orbital motion, and the angle of solar incidence. The substantially lower performance on Space-based items suggests limited facility with perspective-taking and dynamic spatial reasoning.

Findings from RQ2 further illustrate the nature of PSTs' spatial thinking. Interview data revealed that participants such as Phoebe and Fiona could sketch recognizable lunar phases, demonstrating some competence with static 2D representations. However, their explanations exposed incomplete or incorrect causal reasoning. Fiona's explanation that phases occur because of "how much Sun is coming through" similarly suggests a partial but unintegrated understanding of illumination. These responses indicate that PSTs may possess emerging representational skills (e.g., drawing phases) without fully developed mental models capable of supporting accurate causal explanations. Taken together, the quantitative and qualitative findings suggest that PSTs' spatial reasoning is often compartmentalized: they may reproduce familiar images yet struggle to mentally manipulate celestial bodies in 3D space across time. This disconnect aligns with prior research showing that misconceptions in astronomy persist even when learners can produce correct diagrams (Sağdıç & Şahin, 2023; Galperin & Raviolo, 2019). The results highlight the importance of explicitly addressing perspective-taking, dynamic modeling, and the coordination of multiple frames of reference within teacher preparation.

Overall survey scores of 40% or below, coupled with interview evidence of persistent misconceptions, underscore the need for more intentional development of spatial reasoning within elementary teacher preparation. In particular, findings point to the importance of incorporating sustained space-based interventions that require PSTs to shift perspectives, physically model Earth/Moon/Sun systems, and engage in iterative mental simulations. While misconceptions may persist even after instruction, as noted by Tsai and Chang (2005), deeper engagement with spatial representations can move learners toward more sophisticated, though still evolving, understandings.

Recent practice-based and modeling-focused interventions (e.g., Türk & Kalkan, 2018; Galano & Testa, 2025) reinforce that strengthening foundational astronomical knowledge must occur alongside the development of spatial visualization skills. Echoing physicist Wolfgang Pauli's reflection that deeper understanding often involves "misunderstanding at a deeper level," this study identifies a critical starting point rather than an endpoint. The goal is not merely correct answers, but increasingly coherent mental models that integrate geometry, motion, and perspective. Developing PSTs' ability to mentally rotate, transform, and coordinate celestial bodies in both Earth-based and Space-based frames is essential if they are to teach these concepts with clarity and conceptual integrity. Future research should examine longitudinal impacts of explicit spatial interventions and investigate how improvements in spatial visualization translate into classroom practice and student learning outcomes.

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DECOMPOSING LEARNING OBJECTIVES TO SUPPORT PRESERVICE TEACHERS

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This theoretical paper introduces a framework that uses decomposition of practice to support preservice teachers in developing and refining learning objectives within lesson design tasks. The framework focuses on learning objective construction as a central component of planning that can be separated, examined, and intentionally developed. By unpacking the parts of a learning objective, the framework makes the instructional decisions embedded in objective design visible. The aim is to bring greater clarity and coherence to lesson planning and to strengthen preservice teachers' ability to design learning objectives that are clearly focused, well-aligned with tasks, and grounded in meaningful mathematical understanding.

Keywords: Instructional Activities and Practices, Preservice Teacher Education

Relevance of Lesson Planning in Instruction Practices

Instructional practices in mathematics classrooms involve far more than delivering content; they require teachers to skillfully lead discussions, elicit student thinking, and guide learners toward deeper conceptual understanding (Lampert & Graziani, 2009). Mathematics teacher education has directed increased attention to helping preservice teachers (PTs) engage in structured opportunities to rehearse and analyze complex instructional routines. Kazemi et al.'s (2009) exemplifies this with four instructional activities designed to assist preservice elementary teachers' instruction delivery. One of the instructional activities includes a strategy-sharing routine in which the teacher poses a computational problem, helps with different solution strategies, and facilitates a mathematical discussion that highlights connections among solution paths. The work of Kazemi et al. (2009) shows that instructional practice is not something teachers simply implement; rather, it is developed through active engagement in carefully designed teaching activities. The instructional activities from their work help PTs understand how instructional goals shape and connect to the mathematical discussions that occur in the classroom.

Research suggests PTs develop more robust instructional practices when they engage in cycles of participation, analysis, practice, and implementation (Fitriati et al. 2023). In the instructional activities developed by Kazemi et al. (2009), preservice elementary teachers first experienced the activity as mathematics learners, then collaboratively analyzed the work of teaching using a teacher-educator guide as a tool for noticing key instructional moves. The preservice elementary teachers subsequently wrote a lesson plan, practiced the activity with targeted feedback, and finally implemented their plans in classrooms. The findings from this design highlighted a strong link between practice and classroom implementation, demonstrating that structured opportunities for instructional practice enable PTs to analyze how learning objectives influence the depth and richness of classroom discussions.

The emphasis on learning through implementation in practice aligns with broader research on PTs' development, including work on problem posing (Crespo, 2003), leading mathematical discussions (Ghousseini, 2009), and responsive questioning (Nicol, 1998). Across these studies, instructional practices are made learnable by breaking them into smaller, more manageable components that can be examined, rehearsed, and refined. This emphasis is further reinforced by

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research on rehearsals in teaching (Lampert et al., 2013) and on core practices within practice-based teacher education (Forzani, 2014). which collectively emphasizes the importance of structured practice in developing pedagogical competence.

In delivering instruction in the classroom, different planning and decision-making processes are necessary for effective student learning (Fitriati et al., 2023). Lesson planning has been seen as a complex, goal-oriented professional practice that shapes the structure and content of classroom instruction (Cevikbas et al., 2024). To plan lessons, teachers consider students' interests, instructional objectives, resources and materials, and assessment methods, highlighting the importance of teachers developing their own learning objectives, activities, and goals. Building on this foundational view, Superfine (2009) conceptualizes lesson planning as a dynamic problem-solving activity in which teachers navigate multiple interacting elements, including curricular demands, pedagogical beliefs, prior experiences, and reform-oriented approaches to teaching and learning. Within this process, learning objectives serve as a critical anchor, shaping how teachers interpret curriculum materials, select tasks, and anticipate student thinking. The model proposed in the work of Superfine (2009) highlights that teachers often encounter challenges in aligning learning objectives with instructional activities and student understanding, revealing lesson planning as a cognitively demanding practice that involves coordinating multiple components in relation to clearly defined learning goals. These perspectives position lesson planning not as a linear procedural task, but as an integrative process in which the development and refinement of learning objectives are central to effective instructional design.

Research further emphasizes the vital role of lesson planning in developing teacher skills and ensuring instructional quality (Cevikbas et al., 2024). Lesson planning is widely recognized as essential for effective teaching across subject areas and as a foundational component of initial teacher education (Cevikbas et al., 2024; König et al., 2021; Munthe & Conway, 2017). However, as Morris and Hiebert (2017) demonstrate, planning skills do not automatically transfer from teacher education coursework into classroom practice, pointing to a persistent gap between learning about teaching and implementing it. This gap reveals the need for more intentional and structured support that helps PTs not only understand the components of lesson planning but also apply them in meaningful ways. In this sense, strengthening how PTs learn to develop and use learning objectives becomes particularly significant, as it can support the alignment, clarity, and effectiveness of their instructional decisions in practice.

Role of Learning Objectives in Task Design

In mathematics education, tasks are considered the main drivers of learning, with students spending most of their learning time involved in structured activities guided by clearly defined objectives (Hammer & Ufer, 2023; NCTM, 2014). Coletta and Norris-Bauer (2017) describe objectives as clear statements of student learning outcomes that direct task design and align instruction. They emphasize structured objective writing based on cognitive taxonomies such as Bloom’s Taxonomy and ongoing alignment between objectives and activities. The work of Coletta and Norris-Bauer (2017) positions lesson planning as a crucial area where PTs must learn to turn broad curriculum goals into clear learning objectives and well-organized instructional tasks. This makes lesson planning an important area for supporting the development of learning objectives in teacher training. It also emphasizes the need for structured approaches that help PTs connect learning objectives directly to instructional tasks and student thinking, ensuring greater coherence in lesson design.

Learning objectives serve as the central element of lesson planning, guiding task selection, instructional strategies, and assessment decisions (Cevikbas et al., 2024). As noted in Cevikbas et al. (2024), "…a lesson plan can be designed to effectively engage learners and meet desired learning outcomes" (p. 108). PTs often struggle to create learning objectives that are clear, measurable, instructionally meaningful, and well-integrated into lesson planning (Abadi & Ekawati, 2018). Objectives are frequently written as vague content statements or activity descriptions rather than precise descriptions of intended student learning, which limits their effectiveness in guiding instruction and assessment and contributes to their struggle to integrate task design in the lesson plan (Gonzalez et al., 2020). This issue highlights the broader, limited structured support for translating curricular goals into pedagogically coherent and actionable learning objectives within teacher preparation programs. As a result, although clearly defined learning objectives can strengthen coherence and alignment across planning components (Chen & Zhang, 2019). But there remains a gap in understanding how a focus on objectives can be balanced with attention to students' thinking as it unfolds during instruction. This study addresses that gap by introducing a decomposition-based framework that makes this process visible and teachable.

This study responds to the identified gap in the literature by positioning the development of learning objectives as a specific aspect of teaching practice that requires explicit and structured support. Drawing on the concept of decomposition of practice, the study introduces a framework that breaks down the process of writing learning objectives into smaller, learnable components that can be examined, practiced, and refined. This approach is significant because it moves beyond treating objectives as routine statements and instead makes the instructional reasoning behind them visible, supporting PTs in developing clearer, more aligned, and instructionally meaningful lesson plans. The central argument of this study is that learning objectives, when treated as a decomposable element of practice, can be more effectively taught and learned, thereby improving PTs' competence in constructing objectives that guide coherent instruction while remaining responsive to students' thinking. Guided by this argument, the theoretical paper investigates the extent to which decomposing learning objectives supports PTs in writing clear, measurable, and meaningful learning objectives.

Existing Theoretical Framework

Decomposing Learning Objectives as a Form of Pedagogical Practices

Decomposition of practice offers a theoretical approach that breaks teaching into essential, learnable components, making key elements visible so PTs can study, practice, and refine them before integrating them into full instruction (Grossman et al., 2009; Resnick & Kazemi, 2019). It includes developing learning objectives and organizing lesson plans, before integrating them into coherent instruction (Tyminski et al., 2014). Prior research shows this approach has been used to break down practices like leading discussions and managing classroom interactions into components that can be rehearsed (Ghousseini & Herbst, 2016). Extending this idea, writing learning objectives can be treated as a specific, learnable part of lesson planning that requires clear thinking about what students should learn. When this clarity is missing, the resulting lesson is often less coherent and less effectively organized (Gonzalez et al., 2020).

In the decomposition of practice framework, a learning objective means identifying and studying its essential parts before asking PTs to design a full lesson. A learning objective can be broken into at least three components: (1) the specific content or concept students are to learn, (2) the cognitive action students are expected to perform (e.g., explain, justify, compare), and (3) the level of understanding or quality of reasoning that demonstrates success (Anderson &

Krathwohl, 2001). PTs may also need to examine how the objective connects to tasks and evidence of learning. To support this work, teacher educators could engage PTs in analyzing sample objectives, revising vague objectives into clearer ones, distinguishing between activities and learning goals, and aligning objectives (Sztajn et al., 2020).

Research shows that decomposition of practice supports PTs' learning by breaking complex instructional work into clear, manageable parts and organizing learning around those elements. For instance, Cheng (2014) structured lesson study so PTs focused on key planning tasks, such as selecting topics, setting learning goals, and designing instruction, one at a time, which helped deepen their understanding of lesson design. Similarly, Achinstein and Fogo (2015) demonstrated that separating teaching into idea-focused and action-focused components supported the development of pedagogical content knowledge. In mathematics education, decomposition has also been applied to practices like leading discussions, interpreting student thinking, and selecting tasks (Ghousseini & Herbst, 2016; van Es, 2011; Stein et al., 2009). Across these studies, a common pattern emerges: identifying key components, studying them in focused ways, and providing PTs opportunities for practice before integrating them into full classroom instruction.

Building on this understanding of decomposition, this study applies the approach specifically to learning objective writing by positioning it as a distinct and learnable component of lesson planning that requires focused attention. Rather than assuming PTs already inherently know how to write clear objectives, the framework guides them in examining key elements, what students are expected to learn, how learning is demonstrated, the level of understanding required, and how it will be assessed. It can often begin with analyzing examples to distinguish meaningful learning statements from simple activity descriptions. PTs are also supported in connecting objectives to student thinking and instructional tasks, similar to how Cheng (2014) helped clarify lesson focus. By breaking objectives into clear components for study and practice, this approach highlights their role in shaping coherent and aligned lesson design. The theoretical approach positions decomposition as a key tool for PTs to develop the professional skills needed to create meaningful, coherent, and aligned learning objectives (Sztajn et al., 2020).

Decomposing Learning Objectives to Support Preservice Teachers

The reviewed literature consistently positions learning objectives as a central component of lesson planning that shapes instructional decisions, task selection, sequencing, and attention to student thinking (Chen & Zhang, 2019; Fitriati et al., 2023; González et al., 2020). However, studies also show that PTs often struggle to write objectives that are clear, measurable, and meaningfully connected to instruction and assessment (Lim et al., 2018). Effective objectives typically include identifiable elements such as content, observable student actions, expected levels of understanding, and evidence of learning, as well as assessment (Anderson & Krathwohl, 2001; Orr et al., 2022; Rodriguez & Albano, 2017). These elements directly influence the coherence and quality of lesson design. This supports the argument that decomposing practice offers a productive framework for making these elements visible and learnable. Drawing on these findings, this study contributes to the ongoing discussion by arguing that learning objectives should be understood as a decomposable component of teaching practice that benefits from focused analysis and structured support. By breaking objectives into key elements, PTs can better understand how objectives guide instructional decisions, shape learning, and support coherent, mathematically meaningful lesson development.

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CAN AN AI STUDENT FEEL REAL? PRESERVICE TEACHERS' PERCEPTIONS OF AUTHENTICITY IN AI SIMULATED PRACTICE

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This design-based research study reports how preservice teachers (PSTs) perceived the authenticity of AI conversational agents simulating children with mathematical misconceptions. We designed and implemented a chatbot within mathematics methods courses to provide PSTs with structured opportunities to practice questioning strategies aimed at eliciting and building on students' ideas in a low-stakes, practice-based environment. The chatbot simulated one-on-one teacher-student interactions and served dual roles: an elementary student with misconceptions generating child-like mathematical responses and a mentor providing feedback on PSTs' questioning. Two iterations were conducted using traditional and generative AI models We compare PSTs' perception of authenticity across two iterations. Results inform the design of AI-mediated practice tools to support the development of pedagogical responsiveness of PSTs.

Keywords: Preservice Teacher Education, Teacher Noticing, Technology

Purpose and Theoretical Framework

Responsive teaching requires teachers to attend to the substance of students' mathematical thinking and to use that thinking productively in the moment (Robertson et al., 2016). Central to this practice is strategic questioning not only to elicit students' ideas, but to interpret and build on those ideas to advance conceptual understanding. Developing this form of responsiveness is challenging because it depends on engagement with students' often unpredictable reasoning and explanations.

Practice-based teacher education has sought to address this challenge through approximations of practice that provide structured, low-stakes opportunities for rehearsal (Grossman et al., 2009, Lampert et al., 2013). Recent advances in conversational AI extend these approximations by enabling student-emulating agents capable of generating child-like mathematical responses, including common misconceptions (Lee, et al., 2025). Such AI-mediated simulations allow PSTs to repeatedly practice formulating questions, interpreting responses, and determining instructional next moves in real time (Cao et al., 2023).

On the other hand, emerging scholarship raises concerns about authenticity and contingency in AI-generated interactions (Lee, et al., 2025; Zheng et al., 2025). If student responses appear scripted, overly polished, or insufficiently responsive to teacher questioning, the simulation may fail to approximate the complexity of classroom discourse or sustain participants' suspension of disbelief.

This brief research report examines the authenticity of AI-generated student responses across two iterations of chatbot development. By comparing iterations, we identify design trade-offs that inform the development of AI-based approximations of practice in mathematics teacher education.

Method

Research Design

This study used a design-based research approach to examine how preservice teachers' (PSTs') perceptions of the authenticity changed as an AI-based student-emulating agent developed through two iterations—first with traditional AI and second with generative AI. PSTs engaged in a simulated one-on-one interaction with a virtual student named *Noah* through a fraction comparison task. The chatbot was designed to serve dual roles: the virtual student who responded to PSTs' questions with child-like reasoning about fractions and the mentor who provided instant feedback on the PSTs' questioning. In each iteration, PSTs first watched a video of Noah solving a fraction problem and explaining his thinking. They then interacted with the AI-based student-emulating agent in a simulated dialogue. Following the interaction, PSTs completed a survey to report their perceptions of the chatbot's responses and the perceived authenticity of their experiences. Data were collected across the two iterations of the chatbot intervention implemented in different semesters with different groups of PSTs. The purpose of comparing iterations was to examine how changes in the chatbot training approach influenced PSTs' perceptions of authenticity.

Context and Participants

Participants were preservice elementary teachers enrolled in an undergraduate teacher education program at a four-year public university in the Midwestern United States. The chatbot simulation was embedded within a mathematics methods course. The program included a sequence of two mathematics methods courses.

The first iteration of the study was conducted in Spring 2025 in the second method course across included five course sections, with 97 PSTs volunteering to participate. The second iteration was conducted in Fall 2025 across the first method course in a course sequence, with 54 PSTs completed the survey. Participation was voluntary and survey responses were collected following PSTs' completion of the simulation.

Chatbot Development

The chatbots were developed across two iterations using different AI chatbot platforms. The first iteration used IBM Watsonx Assistant, a rule-based AI platform that relied on predefined responses that the system would select based on an interpretation of the user's intent, utilizing technologies such as Natural Language Processing, Deep Learning, and Machine Learning. This allowed controlled dialogue grounded in specific misconceptions, while the chatbot's flexibility in responding to diverse questions. The second iteration was redesigned using OpenAI's GPT-4o, a generative AI model. This iteration used prompt-based instructions to produce student-like responses in real time, enabling more fluid and context-sensitive dialogues. Across both iterations, the chatbots were designed to serve dual roles: virtual student with fraction misconceptions and virtual mentor with appropriate feedback on PSTs' questioning.

Data Sources: Survey Instrument

The data presented here represent a subset of a larger dataset that includes PSTs' conversations with the chatbots as well as analysis of responsive teaching practices before and after their experiences with the chatbot. This brief report focuses specifically on PSTs' perceptions of authenticity.

Survey data were collected immediately after PSTs interacted with the virtual student Noah. The survey included Likert-scale and open-ended items designed to capture perceptions of the chatbot's authenticity and pedagogical usefulness. This brief report focuses specifically on authenticity. PSTs responded to the Likert item, "*The responses from the virtual student were*

authentic," and to the open-ended prompt, "*Please explain why the virtual student's responses were authentic or not*." These items elicited perceptions of whether Noah's reasoning appeared realistic, conversational, and responsive to their questioning.

Data Analysis

Likert-scale responses were analyzed using descriptive statistics to compare response distributions across iterations. Percentages were calculated for each category (Strongly Agree to Strongly Disagree) to examine shifts in perceived authenticity following revisions to the chatbot training approach (Table 1).

Open-ended survey responses were analyzed using an inductive coding process. First, responses were reviewed to identify recurring patterns related to authenticity. Codes were iteratively refined through comparison across responses, resulting in a final codebook representing themes aligned with three dimensions of authenticity: linguistic authenticity (e.g., conversational tone, repetitiveness), cognitive authenticity (e.g., student-like reasoning, misconception errors), and interactional authenticity (e.g., responsiveness, inflexibility). Codes were then applied to all responses, and frequencies were calculated to compare the prevalence of themes across iterations. The analysis focused on identifying relative differences in PST-reported perceptions of authenticity across the two chatbot training approaches.

Results

As shown in Table 1, perceptions of authenticity improved modestly in Iteration 2. Agreement (Strongly Agree + Agree) increased from 51.55% in Iteration 1 to 61.12% in Iteration 2, while disagreement decreased from 23.71% to 18.52%. Neutral responses also declined slightly. Overall, these descriptive trends suggest a positive shift in perceived authenticity following revisions to the chatbot training approach.

Table 1: Likert Scale Responses to the Item "The responses from the virtual student were authentic"

	Iteration 1	Iteration 2
Strongly agree	7.22%	5.56%
Agree	44.33%	55.56%
Neither agree nor disagree	24.74%	20.37%
Disagree	23.71%	16.67%
Strongly Disagree	0.00%	1.85%
N	97	54

Qualitative findings provide a more nuanced view of these shifts (Table 2). From Iteration 1 to Iteration 2, both linguistic authenticity and interactional contingency improved. In Iteration 1, participants frequently noted repetitiveness (19.6%) and unnatural tone (15.5%). In Iteration 2, both concerns declined substantially (3.7% and 7.4%, respectively), suggesting improved conversational flow and a less programmed or scripted feel.

Interactional breakdowns also decreased. In Iteration 1, unresponsiveness (29.9%) and inflexibility (11.3%) were commonly described, such as, "the student has a limited number of possible responses, and the student tends to repeat verbatim the same answers he has given previously to vastly different questions. The bot also doesn't understand context and only replies based on language that is overly precise, rather than context based like a child." Inflexibility

referred to instances in which responses were unrelated to the users' questions or the intent was misread. In Iteration 2, unresponsiveness dropped to zero and inflexibility declined (7.4%), indicating stronger contingency and responsiveness to PST questioning.

In contrast, cognitive authenticity showed regression. In Iteration 1, many PSTs described student-like reasoning (28.9%) and realistic misconceptions (6.2%), noting that "Noah's responses felt authentic because he explained his thinking the way a real student might," with few references to non-student-like responses (2.1%). In Iteration 2, although student-like reasoning remained present (25.9%), references to non-student-like responses increased substantially (25.9%). Participants described the chatbot as having "stronger understanding … then the Noah in the video," or being "smarter than the video," raising concerns about the plausibility of the mathematical thinking demonstrated by the chatbot.

Table 2: Comparison of Authenticity Dimensions Across Two Chatbot Training Iterations

Dimensions	Key Indicators (Codes)	Iteration 1 ($N = 97$) (f, %)	Iteration 2 ($N = 54$) (f, %)
Linguistic Authenticity	Natural tone	26 (26.8)	13 (24.1)
	Unnatural tone	15 (15.5)	4 (7.4)
	Repetitiveness	19 (19.6)	2 (3.7)
Cognitive / Mathematical Authenticity	Student-like reasoning	28 (28.9)	14 (25.9)
	Non-student-like responses	2 (2.1)	14 (25.9)
	Misconception-like errors	6 (6.2)	2 (3.7)
	"I don't know"	8 (8.2)	0 (0)
	Nonsensical responses	9 (9.3)	1 (0)
Interactional Authenticity (Contingency)	Responsive	20 (20.6)	14 (25.9)
	Unresponsive	29 (29.9)	0 (0)
	Inflexibility	11 (11.3)	4 (7.4)
	Forgetfulness	1 (1.0)	0 (0)
	Reset	1 (1.0)	1 (1.9)

Overall, Iteration 1 was characterized by stronger cognitive authenticity but frequent interactional breakdowns, whereas Iteration 2 demonstrated improved contingency and conversational stability but reduced perceptions of student-like reasoning. This suggests that different models for AI-based chatbots have different affordances; future research should account for different dimensions of authenticity and could explore how authenticity might be improved across these dimensions to create a more robust approximation of practice for PSTs.

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MATHEMATICS TEACHER EDUCATORS' ORIENTATIONS TOWARD AI USAGE FOR INSTRUCTIONAL PLANNING WITH A COMMON MATHEMATICAL TASK

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This study begins to identify distinctions in mathematics teacher educator's (MTE's) orientations toward generative AI as a critical friend in instructional planning. As four MTEs from different universities, we individually interacted with AI to plan our usage of a common mathematical task in content and methods courses for prospective teachers. The Orange Juice proportional reasoning task provided a backdrop of "sameness" against which we could perceive differences in our decisions and actions. We identified two primary orientations: AI as a "thought partner" for pedagogical dialogue and AI as a "helping hand" for instructional resource generation. Implications for use of AI as a critical friend in collective MTE inquiry are offered.

Keywords: Teacher Educators, Preservice Teacher Education, Professional Development

Mathematics teacher educators (MTEs) design and facilitate courses to support prospective teachers in enacting task-based, student-centered instruction; developing an understanding of the mathematics they will encounter in their classroom; and recognizing how students' thinking might progress in relation to specific mathematical ideas (Zaslavsky & Leikin, 2004). In our prior work, we collaboratively developed and studied a collective MTE inquiry routine (Kochmanski et al., 2025) with the goal of structuring our own professional learning. This routine consists of consists of five elements: (a) the selection of a common mathematics task, (b) one-on-one planning conversations between each participating MTE and a critical friend, (c) the implementation of the common task, (d) one-on-one debriefing conversations between each participating MTE and a critical friend, and (e) a collective debriefing. The one-on-one planning and debriefing conversations can be challenging to schedule due to the significant demands on MTE time. In this study, we explore how MTEs might use generative AI technologies in a coaching or critical friend capacity during the second element of the collective MTE inquiry routine. This exploration examines distinctions in MTEs' orientations toward generative AI during instructional planning. If AI can serve as a critical friend in MTE instructional planning, then it may prove useful for making structured forms of collective inquiry more accessible to the broader MTE community.

Theoretical Background

As mathematics educators grapple with rapid advancements in AI and the place it should have in teaching and learning, the majority of research has focused on the capabilities and limitations of generative AI in doing mathematics (Zhuang et al., 2025). There are also a growing number of empirical studies by MTEs related to their own prospective teachers' use of AI as a lesson planning assistant (Gurl et al., 2024; Yanar & Ergene, 2025) and a problem solving assistant (Yilmaz at al., 2025). Frambaugh-Kritzer and Stolle (2025) examined the

affordances and limitations of AI as a critical friend within self-study, reporting that AI asked probing questions and provided meaningful feedback. Nishide and colleagues (2025) noted the importance of prompt engineering and the difficulty of building a flow within their communications with AI about teaching practice. They further argued that while ChatGPT as a critical friend could serve as a catalyst for deeper inquiry, it could not replace the situated, relational qualities of dialoguing with a human.

Collective MTE Inquiry Routine

The collective MTE inquiry routine integrates evidence-based aspects of self-study (Brandenburg, 2008), artifact-enhanced collegial inquiry (Van Zoest & Levin, 2021), lesson study and coaching (Gibbons & Cobb, 2017). Participants take an action research stance (Gellert et al., 2012), as they investigate the efficacy of their instruction. This routine is designed to be inclusive of the many different institutional contexts and types of courses that MTEs might teach. In this study, we examine the purposeful use of generative AI as a critical friend in the second of the five elements of MTE inquiry. This element emphasizes a one-on-one planning conversation between each MTE and a peer MTE. We position AI in the role of peer MTE in this element. The following question guided our investigation: How do MTEs use generative AI technologies as they plan to implement a common mathematics task with mathematics teachers?

Methods

We chose the Orange Juice task (Lappan et al., 2009) from Illustrative Mathematics as our common task because it elicits a range of representations of proportional reasoning strategies (See Figure 1). It is appropriate in both elementary and secondary teacher education contexts, allowing for a wide range of MTE learning goals related to teachers' content knowledge, interpretations of student thinking, and effective pedagogies. Each of the four MTEs had a conversation with a generative AI tool of their choice about their implementation of the common mathematics task in a course for pre-service teachers. The uniqueness of the contexts contributed to the richness of the conversations between the MTEs and AI in varied uses of the task.

Terrie planned to use the Orange Juice Task in her STEM teaching methods course for prospective secondary math and science teachers. Her goal for conversation with ChatGPT 5.2 was to generate ideas for modeling and reflecting on posing purposeful questions as an effective teaching practice (National Council of Teachers of Mathematics, 2014), She sought ideas for emphasizing the importance of proportional reasoning and tying together questioning frameworks from both disciplines (Cartier et al., 2013; Smith et al., 2018). The product of their conversation was a crosswalk of questioning strategies and sets of transcripts of student work with varied questioning approaches for pre-service teacher analysis.

Alee planned to use the task in a methods course for preservice secondary mathematics teachers. She engaged with Gemini to support her in figuring out how to integrate the task into a lesson focused on development of and alignment between unit goals and the unit assessment. During the conversation, Alee queried Gemini about ways to use the Orange Juice task that connected to the unit they taught from, while also leveraging the task as a learning opportunity. After back-and-forth dialogue about different pedagogical structures for the lesson, she decided on using the task to surface multiple different representations for solving with the purpose of informally assessing the problem for connections to unit goals.

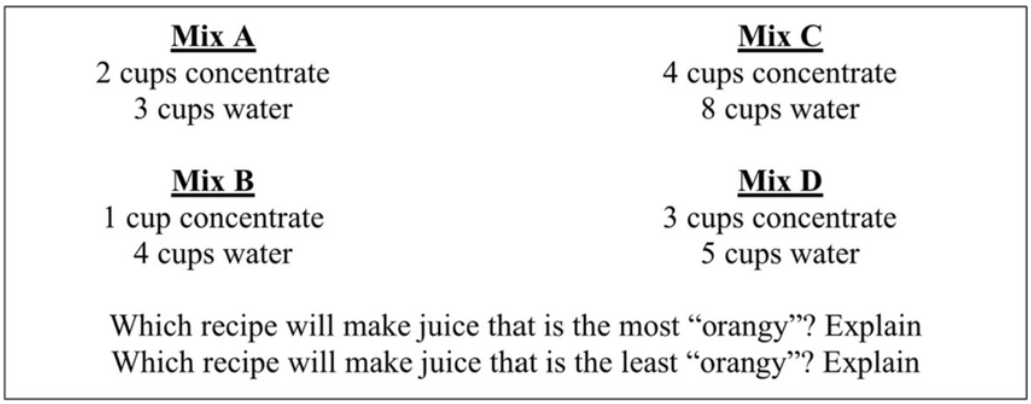


Figure 1: Orange Juice Task (Lappan et al., 2009)

Jonathan teaches two content courses (one undergraduate and graduate) that explore concepts related to proportional reasoning. He planned to use the task in both classes to support his students in the areas of anticipating student solution strategies and analyzing student thinking. Given these goals, he asked Google Gemini 3 to generate a list of potential student solution strategies. Jonathan was familiar with the many common strategies, but he was interested to see if Gemini would generate any additional plausible strategies. Additionally, he asked Gemini to create student solution strategies that involve common errors or misconceptions, such as using additive reasoning. Jonathan knows several common errors/misconceptions, but he was interested to see if Gemini would produce any "novel" ones. For both requests, Gemini provided five potential solution strategies, and after prompting, it generated several additional strategies.

Nick planned to implement the task in a content course for middle grades mathematics teachers that primarily focused on essential understandings related to ratios and proportions. His goal for his conversation with ChatGPT 5.2 was to produce a set of students' work that he could use in the class, as well as produce evidence of pedagogical reasoning that students might make sense of and critique. Nick used ChatGPT 5.2 to first produce five ways in which students might solve the Orange Juice Task. He then asked ChatGPT 5.2 to select three of the five strategies to feature in a discussion and put them in a sequence, explaining why the AI made the decisions it did. In the class, Nick first asked the mathematics teachers to solve the Orange Juice Task on their own. Then, having discussed their strategies, he shared ChatGPT's work with the teachers in his class. He asked the teachers to look for similarities and differences in the strategies the class produced as compared to ChatGPT. He also asked the class to critique the AI's decisions related to selecting and sequencing strategies for a discussion.

We engaged in remote, collaborative qualitative analysis (Sauder et al., 2023) of screenshots of the human–AI dialogues and transcripts of our Zoom conversations. Each MTE first conducted an open coding of their own AI interaction, attending to how AI was taken up in instructional planning and the purposes guiding each prompt. We then met synchronously to share initial coding and identify recurring patterns across cases. Through iterative discussion and constant comparison, we developed and refined two overarching categories that captured qualitatively different ways of engaging with AI during planning. Disagreements were resolved through dialogue, with attention to grounding inferences in specific excerpts from the data.

Findings

We identified two broad orientations for the use of AI in instructional planning. Terrie and Alee treated AI as a "thought partner" who could surface alternative ideas and support pedagogical sensemaking. Jonathan and Nick treated AI as a "helping hand" who could generate student strategies and streamline lesson planning. Below, we provide two representative examples of these different approaches.

Alee started her conversation with AI by prompting, "I would like you to brainstorm with me for a lesson plan." She provided her purpose, a picture of the task, and course objectives related to assessment. Subsequent interactions resembled the exchange of ideas between two human MTEs. In one such interaction, AI responded with a set of "Conceptual Learning Goals" and offered to "brainstorm 2–3 specific Check for Understanding questions." Alee clarified that "learning goals do not start with 'I can,'" highlighting that learning goals are written for teachers. She positioned AI as a "thought partner" that she could challenge and redirect.

In contrast, Jonathan and Nick treated generative AI in a "helping hands" manner, asking it to produce work to increase the efficiency of their lesson planning. Jonathan uploaded the task to AI and prompted, "Please solve this problem using as many different strategies as you can. Identify the strategies that you use". Each of the subsequent prompts by Jonathan elicited a deeper explanation or an elaboration on an AI response, most often taking the form of another question. Prompts included, "What are some other possible student solution strategies that involve common errors or misconceptions?" and "What would be a logical follow-up question to this task?" AI offered to create a practice quiz and demonstrate how to use a ratio table to help students to visualize errors. These patterns of asking questions and reviewing answers along with MTE acceptance of AIs offers to create resources were consistent with a "helping hands" orientation toward use of AI.

Discussion and Conclusions

These two orientations towards AI reflect different perceptions of the potential role of AI in instructional planning. Terrie and Alee elicited pedagogical dialogue in their interactions with AI as a "thought-partner," while Jonathan and Nick looked to obtain resources by positioning AI as a "helping-hand." In both cases, MTE expertise shaped the productivity of the interactions, and all MTEs acknowledged that these orientations could shift depending on instructional goals.

Our findings suggest that AI can support the one-on-one planning phase of the collective inquiry routine, but the depth of this support depends less on the tool itself and more on how MTEs engage with it. Positioning AI as a "thought partner" more closely aligns with the peer coaching element of the inquiry routine because MTEs actively challenge and refine AI responses. By contrast, using AI as a "helping hand" offers fewer visible opportunities for deeper pedagogical inquiry than conversations with a critical friend might offer.

As research on how MTEs study their own practices, contexts, and development in their work to support prospective teachers' learning continues to gain traction (e.g., Castro Superfine & Pitvorec, 2021; Medová et al., 2020), the usefulness of AI as a support for professional learning needs further exploration. While this study focused on how AI might support an MTEs' instructional planning, we also noted that each of us gained new insights into how we might purposefully use AI with our own students. As AI becomes increasingly integrated within mathematics teaching and teacher education, understanding how MTEs deliberately and reflectively engage with AI will strengthen our readiness to meet the needs of tomorrow's teachers.

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EXPLORING THE ASSOCIATION BETWEEN AI INTERACTION LEVELS AND MATHEMATICAL MODELING PERFORMANCE IN PRESERVICE SECONDARY TEACHERS

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This study examines how preservice secondary teachers interact with a conversational Artificial Intelligence (AI) tool during a mathematical modeling task and how interaction relates to performance. Nineteen participants first completed a shipping box design problem independently and then revisited the task using Khanmigo. AI interactions were classified as foundational, constructive, or creative, reflecting the depth of reasoning and questioning. Analysis focused on changes in modeling performance and patterns of AI use. Results indicate that overall performance improved after AI engagement, with higher scores associated with constructive interaction. No students demonstrated creative engagement. These preliminary findings suggest that structured AI interactions can enhance mathematical modeling performance and highlight opportunities to cultivate reflective AI literacy in teacher preparation.

Keywords: Modeling, Preservice Teacher Education, Technology

In the field of mathematics education, advances in generative artificial intelligence (AI) have fueled interest in how new tools can support students' reasoning (e.g., Joung & Kim, 2026; Yoon et al., 2024), teachers' decision-making (e.g., Yilmaz et al., 2026), and programmatic goals in teacher education (U.S. Department of Education [USDOE], 2023). The National Council of Teachers of Mathematics (NCTM) argues that AI has legitimate potential to personalize tasks to students' interests, surface partial understanding, and assist teachers in assessment and planning (NCTM, 2024). At the same time, it is critical that teacher agency and autonomy remain central, and that AI use is guided by transparency, ethical awareness, and attention to bias in outputs (Dykema, 2023). In this way, AI can amplify both students' thinking and teachers' professional judgment while supporting equity, privacy, and instructional integrity.

In teacher education, emerging studies call for developing AI literacy to help preservice mathematics teachers (PMTs) interrogate AI outputs and design learning environments where AI augments, rather than automates, mathematical reasoning (Yilmaz et al., 2026). Mathematics teacher educators (MTEs) are exploring ways to integrate AI tools into PMT coursework, shaping not only what mathematics is learned but how PMTs reason, reflect, and position themselves as future teachers (Aqazade et al., 2026; Joung & Kim, 2026). Structured engagement with AI-supported modeling tasks, coupled with rubric-guided reflections, has been shown to allow PMTs to experience AI as a thought partner (Kim et al., 2026; Naresh & Gunpinar, in press; Sawyer, 2024). Building on this work, the current study reports preliminary findings from one semester of a larger, multi-semester research program examining how structured interactions with a conversational AI tool, Khanmigo, influence PMTs' mathematical reasoning and engagement. This study addresses the research question: To what extent is the level of AI interaction associated with mathematical modeling performance among preservice secondary teachers?

Theoretical Lens

AI tools used in mathematics education range from answer-oriented utilities, such as camera calculators and symbolic solvers, to adaptive platforms and intelligent tutoring systems that support guided dialogue (Vaerenbergh & Pérez-Suay, 2022). Generative AI tools add a conversational layer, explaining procedures, responding to questions, and engaging in interactive problem solving (AI for Education, n.d.). Investigating the role of AI in mathematical problem-solving raises questions about how learners position AI within their activity.

The Instrumental Genesis perspective (Sinclair et al., 2023) describes how digital tools become integrated into mathematical practice. The distinction between a tool as an artifact and as an instrument emphasizes how learners develop schemes of use through purposeful activity. As learners appropriate the tool, it organizes both action and understanding, highlighting the role of teachers in coordinating tools, tasks, and interactions to support conceptual understanding. This lens draws attention to how design features and AI affordances invite particular representations and actions and underscores the interplay between performing techniques and articulating mathematical ideas (Trouche, 2020).

Analytical Frameworks

Two conceptual frameworks guided analysis. The first, from the Maryland Comprehensive Assessment Program (MCAP), conceptualizes mathematical modeling as integrated acts of sense making, including framing the situation, selecting and coordinating information, and articulating reasoning that can be tested and refined (Maryland State Department of Education, 2023). The second framework describes PMTs' engagement with AI along three levels: foundational engagement treats AI as a source of recall and procedural guidance, constructive engagement treats AI as a partner for explanation and strategy development, and creative engagement treats AI as a collaborator for exploration and novel generalizations (Naresh et al., 2024a). These frameworks allow examination of both the depth of mathematical reasoning and the quality of AI interaction.

Methods

The overall study employed a mixed methods research design and the data reported in this paper were analyzed descriptively. The study was conducted in a university mathematics content course for undergraduates pursuing secondary teacher certification. This course was taught by the second author. Twenty-two PMTs were enrolled, and analyses reported here include the 19 participants who consented to research use of their work under institutional review protocols. The authors have been engaging PMTs in their respective classroom contexts in the use of AI tools for mathematical problem solving. Based on lessons learned from prior semesters (Naresh & Gunpinar, in press; Naresh et al., 2025), a key pedagogical decision in the semester under study was to implement a planned intervention that structured how PMTs engaged with AI tools. This intervention was designed to support more intentional mathematical learning by having PMTs work purposefully with a generative AI (Khanmigo) tool during mathematical modeling activities during class time rather than using it only as a supplemental tool for out-of-class projects (Naresh et al., 2024b).

Data were collected across three phases: pre-intervention, intervention, and post-intervention. In this paper, we report on a part of data collected during the pre-intervention phase. At the start of the semester, PMTs completed a candy container package design task independently during class time. This task involved designing a rectangular box to hold 16 identical cylindrical candy containers while minimizing cardboard use (Virginia Department of Education, n.d.). The task

was selected because it (a) elicited decisions about container arrangement and packing efficiency, (b) required attention to dimensional constraints, and (c) encouraged comparison of layouts using surface area as a measure. Additionally, the task provided opportunities for PMTs to frame, refine, and validate solution paths. PMTs were asked to state and justify assumptions, provide diagrams, and include calculations supporting their proposed dimensions. After submitting their independent solutions, PMTs then solved the same problem using Khanmigo. The primary data sources included PMTs' independent and AI-supported solutions, transcripts of their interactions with AI, and follow-up reflections on their interactions with Khanmigo.

Data analysis focused on both changes in PMTs' mathematical modeling performance and patterns of AI engagement. Scores were assigned according to the MCAP four-point rubric, and AI interactions were categorized as foundational, constructive, or creative. Performance on the box design task was compared between the first independent attempt and the second AI-supported attempt. Patterns of PMT-AI engagement were then examined in relation to performance scores to identify whether more sophisticated engagement was associated with higher mathematical modeling outcomes. Additionally, illustrative excerpts from selected PMT-AI interactions were analyzed to characterize the nature of foundational and constructive engagement.

Findings

Among the 19 participants, mathematical modeling performance improved substantially following AI interaction. Initially, only 16% of students achieved the highest score of three points, while 42% scored one point, 37% scored two points, and one student scored zero (Table 1). On the second attempt, no students scored zero or one point, 53% earned two points, and 47% earned three points. Overall, the data show that all students either improved or maintained their performance, with nearly half achieving the highest performance level after AI-supported engagement.

Table 1: Math Modeling Performance Change from First Attempt to Second Attempt with Khanmigo (*n* = 19)

Math Modeling Performance	First Attempt	Second Attempt
0 point	1 (5%)	0
1 point	8 (42%)	0
2 points	7 (37%)	10 (53%)
3 points	3 (16%)	9 (47%)

In Table 2, a clear pattern emerged between AI engagement and performance. Students with foundational AI interaction predominantly earned two points, responding briefly and following AI prompts without extending reasoning. Students with constructive engagement showed stronger performance overall, with 37% achieving three points, using AI to explain, justify, and refine their solutions. No students reached the creative engagement level, indicating that novel exploration was not observed in this phase.

Table 2: The Relationship between AI Interaction and Mathematical Modeling Performance (*n* = 19)

Redo with Khanmigo	Foundational	Constructive	Creative
2 points	9 (47%)	1 (5%)	0
3 points	2 (11%)	7 (37%)	0

Illustrative examples further highlight these patterns. One student with foundational engagement responded to AI prompts with short statements such as "two rows of eight" or "6," following Khanmigo's guidance without elaboration. In contrast, a student demonstrating constructive engagement proposed detailed arrangements, pressed the AI for clarification about efficiency, and questioned the reasoning behind alternative layouts, reflecting more sophisticated mathematical reasoning.

Discussion and Conclusions

These preliminary findings suggest that structured engagement with conversational AI can enhance mathematical modeling performance among preservice secondary teachers. Constructive engagement, characterized by active reasoning, iterative questioning, and explanation, was associated with the highest performance levels, while foundational engagement supported moderate improvement. The absence of creative engagement suggests that additional scaffolding or task design may be needed to elicit novel problem-solving strategies with AI. Overall, these findings underscore the potential of conversational AI as a tool to support mathematical reasoning and highlight opportunities to cultivate reflective and critical AI literacy in teacher preparation. As part of a larger, ongoing study, these results provide initial insights into the interplay between AI engagement and student performance.

In our ongoing analysis we intend to examine patterns of engagement across a larger sample and additional modeling tasks, including fine-grained coding and qualitative analyses of interaction sequences and specific prompting moves. We will also draw on the instrumental genesis perspective (Sinclair, 2023; Trouche, 2020) to examine how PMTs appropriate conversational AI within their mathematical activity and how patterns of engagement reflect emerging schemes of use. These analyses will also strengthen interpretations of the relationship between PMTs' AI use in mathematics modeling in the current study and inform future iterations of AI-supported modeling tasks in teacher education.

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CROSSING THE DIVIDE: A DISSERTATION STUDY OF COMPUTING INTEGRATION IN MATHEMATICS TEACHER EDUCATION

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This study examines how mathematics teacher educators (MTEs) with computing-related expertise conceptualize and situate computing within mathematics teacher education programs. As computing becomes more visible in K–12 policy, standards, and professional discourse, teacher preparation programs are beginning to integrate computing in varied ways. This qualitative multi-case study draws on semi-structured interviews with six MTEs across institutions in the U. S. as well as associated curricular artifacts from their teacher education courses. Guided by Schoenfeld's (2011) goal-oriented decision-making framework, preliminary cross-case analyses reveal substantial variation in MTEs' goals, orientations, and resource landscapes—including varied policy contexts—and differences in how computing is represented, positioned, and integrated within mathematics teacher education programs.

Keywords: Teacher Educators, Preservice Teacher Education, Computing and Coding, Computational Thinking

Introduction

Mathematics and science are becoming increasingly computational, with computational thinking (CT) gaining prominence in professional practice and national standards (ACM Education Policy Committee, 2014; NGSS Lead States, 2013). Research has shown that computing-integrated tasks can support mathematical reasoning, modeling, and problem solving when designed in mutually supportive ways (Bih Fofang et al., 2020; Weintrop et al., 2016). Integration, however, does not automatically yield such mutual support; computational activities can also drift into disconnected coexistence with mathematical goals rather than deepening disciplinary understanding.

Preparing teachers to navigate this complexity remains an open challenge. Preservice teachers often enter methods courses with emerging but underdeveloped understandings of computing and CT, and relatively little is known about how best to support their development in these areas (Moon et al., 2023; Yadav et al., 2014). Although innovative modules and embedded coursework have begun to address this need, less attention has been paid to how MTEs themselves conceptualize and situate computing within teacher preparation programs.

This study examines computing integration in mathematics teacher education from the perspective of MTEs and the curriculum materials they design and provide. Guided by Schoenfeld's (2011) goal-oriented decision-making framework, the study addresses (RQ1) how MTEs with computing-related expertise from different universities across the United States (a) envision the integration of computing within mathematics teacher preparation programs and (b) interpret and evaluate existing approaches for integration, and (RQ2) how computing is represented, positioned, and integrated within the curriculum materials associated with those programs. By attending to how MTEs conceptualize and situate computing within mathematics teacher preparation programs, this study illuminates emerging models of integration and the structural and pedagogical conditions shaping this work.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Conceptual Framework

This study draws on Schoenfeld's (2011) goal-oriented decision-making framework to examine how MTEs conceptualize and situate computing within teacher preparation. Schoenfeld argued that instructional decisions can be understood through three interrelated constructs: goals, resources, and orientations. These constructs interact in particular ways as individuals interpret a situation, draw on available resources, and act in ways aligned with their goals and beliefs.

Applied to mathematics teacher education, this framework provides a lens for examining how MTEs' approaches to computing integration are shaped by the goals they hold for preservice teachers (e.g., disciplinary depth, innovation, equity, policy alignment), the resources available to them (e.g., curricular materials, institutional structures, time, technological infrastructure), and their orientations toward computing (e.g., enrichment, core disciplinary practice, pathway to access). Importantly, state and institutional policy contexts are conceptualized as part of the resource landscape that enables or constrains integration, influencing how MTEs interpret what is possible or necessary within their programs. This framework guides analysis of interview data (RQ1) by foregrounding the rationales underlying MTEs' visions alongside their interpretations and evaluations of existing approaches.

To examine how computing is situated in the MTEs' teacher education practice (RQ2), curriculum artifacts were analyzed along three dimensions. First, the analysis attended to how computing is represented—that is, the scope of activities or practices treated as computational (e.g., programming, robotics, algorithmic reasoning, or unplugged activities). Second, the analysis considered where computing is positioned within program structures (e.g., embedded in teaching methods courses, located in educational technology courses, or offered as stand-alone computer science (CS) courses). Third, the analysis examined how computing is integrated relative to mathematics—whether it is framed primarily as a tool for mathematical exploration or as disciplinary content in its own right. Together, these dimensions provide a structured lens for understanding variation in how computing integration takes shape, or is situated, across institutional contexts.

Method

Research Design

This study employed a qualitative multi-case design to examine how MTEs conceptualize and situate computing integration within mathematics teacher education programs. The study focuses on two primary data sources: semi-structured interviews with MTEs (RQ1) and associated curricular artifacts (RQ2). The design allows for examination of individual institutional contexts while supporting cross-case comparison of integration models.

Participants, Context, and Data Sources

Participants include six MTEs with computing-related expertise from large public universities across the United States. Institutions were selected to encompass variation in how computing is integrated within mathematics teacher preparation, including programs that embed computing within methods or educational technology courses and programs that offer stand-alone computer science coursework. While all participants have experience engaging with computing in some capacity, the extent and structure of integration vary across sites. All participant names used herein are pseudonyms.

Data consist of (a) two semi-structured interviews with each MTE (45–60 minutes); and (b) curricular artifacts provided by participants, including syllabi, assignments, and instructional materials from courses that entailed computing. Interview questions probed participants' goals, orientations, and resources related to computing integration in mathematics teacher education

with attention to varied state and institutional policy contexts. Curriculum artifacts provided evidence of how computing is represented, positioned, and integrated within coursework.

Data Analysis

Interview transcripts were analyzed using thematic analysis (Braun & Clarke, 2006) through an inductive–deductive approach. Initial open coding identified emergent themes, followed by focused coding informed by Schoenfeld's (2011) constructs of goals, orientations, and resources. Institutional and policy contexts were examined as part of the resource landscape shaping decision-making. Analytic memos were developed to support cross-case comparisons and theory building with analysis remaining iterative as patterns were refined across cases.

Curriculum artifacts were analyzed through document analysis guided by three dimensions: (a) representations of computing, (b) how computing is positioned within program structures, and (c) how computing is integrated relative to mathematics. Cross-case comparisons were conducted to identify patterns and variation in integration models across institutional contexts. Findings reported here reflect preliminary analyses.

Findings

Guided by Schoenfeld's (2011) framework, this section is organized around key findings in relation to variation in MTEs' goals, orientations, and resource landscapes (RQ1), and the implications of this variation for how computing is represented, positioned, and integrated within curriculum artifacts (RQ2). Preliminary cross-case analysis reveals distinct models of computing integration emerging from the interaction of these constructs.

Finding 1 (RQ1a)

MTEs articulate distinct goals for computing integration in teacher preparation. Across cases, MTEs articulated distinct primary goals for computing integration in mathematics teacher education. These goals functioned as organizing logics for instructional and programmatic decisions. One participant emphasized preparation for future learning, arguing that PSTs should leave with a flexible conceptual framework that allows them to incorporate emerging tools, without computing being a barrier. Another MTE had a goal of computing being an epistemic extension of mathematical practice, insisting that integration must deepen mathematical reasoning rather than function as a technological demonstration. A third framed computing as preparation for disciplinary evolution toward data science, emphasizing modeling practices, workforce relevance, and teacher self-efficacy in CS. A fourth described computing as a means of making mathematical reasoning explicit and testable, while attending to affective safety and broadening participation.

These distinct goals shaped how participants evaluated the purposes, depth, and necessity of computing within their programs.

Finding 2 (RQ1a + RQ2)

MTEs' orientations toward computing shape what counts as integration and how it appears in coursework. Participants differed in how they conceptualized computing, reflecting distinct orientations toward the role of technology in mathematical practice. For some, computing encompassed a broad spectrum of tools used to make mathematical processes visible. For others, computing was defined more narrowly as interaction with technological systems that function as representational media for exploring, generalizing, and modeling. Another orientation centered computing on data analysis and iterative modeling as core disciplinary practice. Still another emphasized computing as abstraction and communicative precision, requiring learners to formalize procedures in ways that make reasoning explicit and testable.

Despite this variation, a shared throughline was that computing becomes valuable when it makes mathematical structure visible, revisable, and accountable. For example, when preservice teachers programmed regrouping logic in a non-base-ten system, errors in their code immediately revealed misconceptions about place value, requiring them to revise their reasoning rather than rely on informal explanation.

These orientations have direct implications for RQ2. Preliminary artifact analysis suggests that differences in definition of computing correspond to differences in representation (e.g., robotics modules, coding-based representational tasks, data-modeling sequences), positioning (embedded in methods, located in educational technology courses, or structured as standalone CS pathways), and integration (computing framed as an exploratory tool, disciplinary practice, or parallel content area).

Finding 3 (RQ1b)

Resource landscapes mediate feasibility, sustainability, and programmatic placement. Participants consistently described integration as shaped by resource conditions, including funding, institutional structures, credit-hour constraints, disciplinary expertise, and policy context. Statewide infrastructure, grant funding, and collaborative networks enabled sustained integration in some contexts, while shrinking coursework, overloaded methods classes, enrollment pressures, and accountability systems constrained depth in others. Legislative pathways and endorsement structures influenced whether computing could be embedded, required, or offered as standalone coursework. Several participants noted that reliance on local "champions" can both enable innovation and produce inequitable distribution of opportunities. As one MTE reflected, "Right now it works because I'm here and pushing it. But I don't know if it would survive without someone advocating for it." This comment underscores how integration is often sustained through individual initiative rather than structural guarantees.

In Schoenfeld's terms, these resource landscapes shape how MTEs interpret what is possible and necessary within their programs. Decisions about where computing is positioned (embedded vs standalone), whether it is required or elective, and how extensively it is integrated reflect not only pedagogical commitments but structural feasibility.

Cross-Case Synthesis

Integration models emerge from the interaction of goals, orientations, and resources. Taken together, these findings suggest that computing integration in mathematics teacher education is best understood not as a single reform effort, but as a set of contextually mediated models shaped by interacting goals, orientations, and resource landscapes. Attending to these interactions provides a framework for understanding why integration takes different forms across programs—and what structural and pedagogical conditions may support more coherent and equitable approaches moving forward.

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PROSPECTIVE TEACHERS' USAGE OF GENERATIVE ARTIFICIAL INTELLIGENCE TO CREATE IMAGES FOR MATHEMATICAL STORY PROBLEMS

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We discuss our first-time implementation of Adobe Firefly into a mathematics content course for prospective elementary and middle grades teachers (PTs). Our intention was to respond to calls from professional organizations and support PTs in learning one new way to integrate generative artificial intelligence in the mathematics classroom. Throughout the course, PTs occasionally engineered images to accompany mathematical story problems. On the final examination, they designed multiple images to accompany a single problem, and they discussed visual features that offered elements of challenge or support for solving the problem. We analyzed the features PTs identified and found that they attended to a variety of visual elements when considering levels of support and challenge for the images they created.

Keywords: Number Concepts and Operations, Technology, Artificial Intelligence, Imagery

Artificial intelligence (AI) is in vogue, and professional organizations like the National Council of Teachers of Mathematics (Dykema, 2023) have tasked mathematics educators with determining how to integrate it into our profession; at this time, exactly how this should be accomplished remains largely an open question. Presently, we characterize our field as being in a state of exploration regarding the possibilities permitted by AI—mathematics educators are examining what these tools can do and how to use them in educational spaces.

Walkington (2025) synthesized four main uses of GenAI in the field: (a) mathematics problem-solving, (b) mathematics tutoring and feedback, (c) adapting mathematical tasks to learner needs, and (d) supporting mathematics teachers in planning. When reading about the work that researchers are currently doing, we noticed that some researchers are examining how AI can do the present work of teaching more efficiently. We certainly find importance in this work as teachers' time is valuable and maximizing the impact of that time is critical. We also think it is important to examine new pedagogical possibilities afforded by AI regardless of their time-saving status, which is the focus of our present work. In other words, we are interested in new ways to support learning that would not be possible or practical without AI (Hardison & Goméz-Marchant, 2025). For example, in this report, we focus on how prospective elementary and middle grades teachers designed images to accompany mathematical story problems and the features they considered in assessing the levels of support and challenge offered by the images they designed. We address the research question: What visual features do prospective teachers consider when designing AI-generated images to accompany division story problems?

Background

We link our project to the work of Steven Greenstein and Pat Thompson. From Greenstein and colleagues (2023), we draw on a perspective that positions prospective teachers as agents of curricular change. In this view, teachers are framed as designers of classroom artifacts. For example, Greenstein et al. (2023) supported prospective teachers as they designed and 3D-printed fraction manipulatives to use in mathematically pedagogical interactions with children. Here, rather than focus on 3D-printed manipulatives, we focus on images PTs designed using Adobe Firefly, which is a GenAI that takes natural language inputs for generating imagistic

outputs. Drawing on Thompson (2002), we are particularly interested in artifacts that teachers design as didactic objects, which are simply artifacts intended to support reflective mathematical discourse. Thompson notes that objects become didactic when they are mentally coupled with anticipations of discussing particular mathematical ideas. We consider these images to be didactic objects precisely because prospective teachers considered factors that might support or challenge students mathematical thinking as they solved the accompanying story problem. We discuss our methods below.

Methods

The project focused on our first-time implementation of Adobe Firefly into two sections of an undergraduate course for prospective teachers (PTs). The course focused on number and operations in the elementary and middle grades and included, for example, types of story problems (Carpenter et al., 1999) for supporting children's reasoning about the four basic operations. Throughout the course, PTs occasionally used Adobe Firefly to generate images to accompany mathematical contexts.

For the present report, we focus on data from 23 PTs who were enrolled in one section. Specifically, we report on final exam responses wherein PTs were asked to (a) author a story problem for the division $6 \div 4 = \square$, (b) engineer three accompanying images offering varying levels of support and challenge, and (c) to discuss the visual features accounting for these levels.

Data analyzed include the problem PTs wrote, the prompts PTs submitted (three prompts per PT), and the rationales indicating features impacting the relative support or challenge within each image. Here, we focus exclusively on analyzing the rationales PTs provided. To analyze this data, we engaged in open coding. We began by independently analyzing PTs responses individually in descriptive terms. Then, we came together to discuss our analyses and worked to collapse our descriptive terms into broader features that were reflective of responses across PTs. Eventually, the codes reached a point of stability wherein no further changes occurred

Results

We begin by presenting an example to elaborate our current categories and corresponding features that PTs identified as impacting levels of support and challenge (Table 1).

Table 1: An Example of Problem, Prompts, and Images produced by PT1

Problem: There are six chocolate chip cookies to be split among four kids. How many cookies will each kid get?		
Low Support, High Challenge Prompt: Many chocolate chip cookies of different sizes with a jar of cookies in the background	**Medium Support, Medium Challenge Prompt:** 6 separate chocolate chip cookies on a plate	**High Support, Low Challenge Prompt:** In the middle of the picture, have 6 chocolate chip cookies all on top of one plate; on the edges of the picture, have four empty plates; five plates total in entire picture
		

The block quotation below presents a partial rationale offered by this prospective teacher:

> I consider a picture to offer low support if the quantity does not match what is specified in the problem. There are six cookies grouped more so together in the middle, but there are a few beside them and in jars in the background. These cookies being stacked it would be more challenging to picture splitting them each up from the picture alone. I liked how there is a bit of support with one cookie facing mostly forwards to show the circular shape which is a nice shape to be able to divide.
>
> The medium level support picture is more straightforward and connects well with the problem to offer good support. There are exactly six cookies each spaced apart from one another to be distinct. The view from above helps to show all of their circular shapes clearly. There are many ways students can interact with the picture itself to solve the problem. This is in the medium level because there aren't the four children in the problem accounted for.
>
> The high-level support expands off the adequate second picture to account for not just the quantity of cookies from an above view but also the four children for them to be split amongst. For higher optional support, there are four different colored empty plates surrounding the six cookies that students can think of distributing how they choose onto each plate to solve the problem. I edited them to be different colors to help for thinking of each of the four kids. I liked their placements making them similar still by them all surrounding the middle filled plate.

We note PT1 attended to several features as they designed these three images to offer varying levels of support and challenge. For example, PT1 indicates purposefully designed images with considerations of whether image dividend and divisor match the story problem. PT1 explained in their rationales that "the quantity does not match what is specified in the problem" in the first image and "there aren't the four children" in the second image. PT1 also attended to the background and whether items are easy to see and share in the generated images. For the second image, PT1 claimed that "There are exactly six cookies each spaced apart from one another to be distinct. The view from above helps to show all of their circular shapes clearly."

We abstracted 9 different features PTs attended to when considering images' levels of support and challenge (Table 2). We retroactively organized these features into two categories: (a) features that account for individuals' initial interpretation of the problem and (b) features that account for individuals' partitioning and sharing actions. For example, 61% of PTs attended to whether and how items were partitioned in the image, which is important for the given number choices since a fair share will necessarily involve a fractional part of the dividend quantity. Some PTs (39%) described that representations of the quotient in the image impacted the level of support and challenge for the image. Most PTs (52%) considered whether the dividend quantity represented in the image matched the numerosity specified in the problem and nearly half of PTs (48%) similarly considered whether the divisor quantity was appropriately represented. Finally, some PTs used labels (9%) to indicate partitioning or to assign shares of the dividend quantity.

PTs also attended to items related to individuals' initial interpretation of the problem—attending to the qualities of the background (4%) or attending to whether items were easy to see (17%). We were intrigued that so many prospective teachers (26%) considered whether the dividend quantities were homogeneous or heterogeneous. In other words, this code was assigned when prospective teachers considered whether the dividend quantity items were equivalent (cookies of the same shape) or distinguishable (e.g., cookies of different shapes). Shape was one of several features PTs considered with respect to homo- and heterogeneity. For example, some

prospective teachers distinguished items along lines of size or color. Finally, some PTs (9%) also attended to whether some shapes might be more difficult to think about sharing in the first place (e.g., triangular shapes might be more challenging to share than circular shapes).

Table 2: PT-Indicated Features Impacting Level of Support and Challenge

Category	Feature	Number of PTs (%)
Interpretation of the Problem	Background	1 (4%)
	Obscured or easy-to-see items	4 (17%)
	Homogeneous vs heterogeneous dividend quantity	6 (26%)
	Some shapes easier to share than others	2 (9%)
	Partitions	14 (61%)
Partitioning and Sharing Action	Quotient represented	9 (39%)
	Whether image dividend quantity matches problem	12 (52%)
	Whether image divisor quantity matches problem	11 (48%)
	Measurements or labels	2 (9%)

Discussion and Conclusion

In this report, we discussed aspects of our first-time implementation of Adobe Firefly into a course for prospective elementary and middle grades teachers. In the above sections, we have presented an accounting of the features that PTs identified as impacting levels of support and challenge offered by images accompanying mathematical story problems. The rationales offered by the PT reflect their considerations of visual features salient in the didactic objects they created.

We noted in our study that many prospective teachers attended to aspects of the image that we associate with the partitioning and allocation process. In other words, they did not view these images as simply decorative pictures. Future research may more closely investigate how prospective teachers can use image GenAI to produce illustrations that align with models of students' thinking from extant research.

We also note that in this report we focused on static imagery. Generative AI models are now capable of generating video from natural language input. Therefore, future research might also investigate how prospective teachers can generate their own dynamic videos to illustrate mathematical concepts or otherwise support students' mathematical thinking.

We wish to close with some words of caution. First, we encourage educators and researchers alike to critically analyze the images produced since biases in training data may yield biases in imagery (Goméz-Marchant & Hardison, 2024). We also emphasize that we are exploring pedagogical possibilities with AI. Questions remain regarding the pros and cons of actually incorporating these AI-generated images into mathematics instruction.

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DESIGNING AND RECOGNIZING FORMATIVE ASSESSMENT IN DIGITAL LEARNING ENVIRONMENTS

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This qualitative study investigates how prospective secondary mathematics teachers (PSMTs) design and recognize formative assessment in Amplify Activity Builder. Analyzing 16 designs and 50 peer reviews, we found a critical disconnect: PSMTs prioritized explicit, teacher-mediated feedback (e.g., text inputs) but largely overlooked the "epistemological feedback" inherent in dynamic tools (e.g., sliders). Results suggest PSMTs view digital tools as media substitution. We argue teacher education must explicitly train PSMTs to recognize how dynamic technological consequences function as immediate feedback.

Keywords: Formative Assessment, Preservice Teacher Education, Technology

Despite widespread recognition that formative assessment practices significantly improve student mathematical learning (Lee et al., 2020) and are essential components of effective instruction (Black & Wiliam, 2009), teacher preparation programs face persistent challenges in developing these competencies in prospective teachers (Cevikbas et al., 2024). These challenges intensify when integrating digital technologies, where the affordances and constraints of the medium fundamentally reshape how teachers can elicit, interpret, and respond to student thinking. As mathematics classrooms increasingly incorporate technology-enhanced learning environments, understanding how prospective teachers learn to design and recognize formative assessment opportunities within these contexts becomes critical for teacher education.

This study addresses this gap by examining how prospective secondary mathematics teachers (PSMTs) incorporate and recognize formative assessment practices when designing interactive mathematics activities using Amplify Activity Builder (AAB), a dynamic digital platform that combines graphing calculators, dynamic geometry tools, and real-time student response systems. Specifically, we investigate: (RQ1) What opportunities for feedback that moves learners forward do PSMTs incorporate in Activity Builder designs? and (RQ2) What opportunities do PSMTs recognize when reviewing their peers' designs?

Related Literature

Formative assessment (FA) occurs when teachers use evidence of student learning to make informed instructional decisions that move learning forward (Black & Wiliam, 2009). Research links effective FA to improved learning outcomes, engagement, and self-regulation (Lee et al., 2020; Leenknecht et al., 2021; Brookhart & Chen, 2015), making it to effective mathematics instruction and teacher education (Association of Mathematics Teacher Educators [AMTE], 2017; Conference Board of the Mathematical Sciences [CBMS], 2012). However, effective FA is not simply discrete techniques; it is a continuous, multi-step cycle requiring teachers to simultaneously gather evidence of student thinking, interpret this evidence, provide feedback, and make real-time instructional adjustments (National Council of Teachers of Mathematics [NCTM], 2014). For prospective teachers still developing pedagogical expertise, orchestrating these interconnected processes represents a significant challenge, particularly in unfamiliar technological environments.

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Digital learning environments like AAB offer both affordances and constraints for FA. AAB incorporates Desmos Graphing Calculator technology with dynamic geometry tools and virtual manipulatives, providing real-time, interactive feedback mechanisms for both students and teachers. When combined with well-designed tasks, AAB can help teachers engineer high-quality class discussions built on student thinking and create opportunities for meaningful feedback, seminal FA strategies (Wiliam & Thompson, 2008) that benefit from auditable trails of mathematical thinking (Harper & Cox, 2018).

Yet designing effective AAB lessons is not as simple as translating existing curriculum into a digital format. Effective digital design requires pedagogical transformation, not merely media substitution (Puentedura, 2013). While research documents formative assessment's importance and implementation challenges, less is known about how prospective teachers learn to design and recognize formative assessment opportunities within specific digital tools. Understanding what PSMTs recognize and overlook when designing and reviewing technology-enhanced activities can inform how teacher educators support the development of these critical competencies.

Theoretical Framework

This study draws on Wiliam and Thompson's (2008) theory of formative assessment to guide our analysis of how teachers design tasks in AAB that support FA practices. Black and Wiliam (2009) conceptualized FA not as discrete techniques, but as a coherent pedagogical approach centered on creating and capitalizing on "moments of contingency" in instruction to regulate learning processes. Specifically, FA occurs when evidence about student achievement is elicited, interpreted, and used by teachers, learners, or peers to make instructional decisions that are likely to be better, or better founded, than decisions made without such evidence.

Both versions operationalize this theory through five key aspects that operate across three agents (teacher, peer, and learner) and address three essential processes: establishing where learners are in their learning, clarifying where they are going, and determining how to bridge that gap. The five aspects are:

1. Clarifying learning intentions and criteria for success
2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding
3. Providing feedback that moves learners forward
4. Activating students as instructional resources for one another
5. Activating students as the owners of their own learning

Among the five strategies, we focus specifically on aspect 3: "Providing feedback that moves learners forward." Within digital environments like AAB, feedback can take multiple forms. It may be automated and delivered directly to students through the technology, or it may be teacher-mediated, where the platform provides information to the teacher who then decides how to respond. Understanding how PSMTs design these different types of feedback opportunities, and whether they can recognize them in peers' work, provides insight into their developing pedagogical reasoning about FA in technology-enhanced environments.

Methodology

This qualitative study involved 17 PSMTs in their first or second year of a secondary mathematics certification program at a Midwestern university, enrolled in a course on mathematical problem solving with technology taught by the first author. The initial AAB

designs (n = 16; one PSMT did not complete the assignment) were collected after PSMTs explored the technological features of AAB through pre-designed activities and prior to any formal instruction on teaching methods. Importantly, the assignment emphasized technological features and mathematical content without explicitly mentioning FA, allowing us to observe what assessment practices emerged organically.

Each PSMT reviewed approximately three randomly assigned peer AAB designs using a rubric with eight criteria tied to Wiliam and Thompson's (2008) framework, generating 50 peer reviews total. This study focuses on one criterion: "Providing Feedback that Moves Learners Forward," with four performance levels: (a) activity offers specific prompts/feedback to advance student thinking; (b) activity supports teacher feedback moments; (c) feedback opportunities lacking or unlikely to move learning forward; (d) no feedback mentioned.

Analysis proceeded in two phases. In phase one, two researchers independently coded all 16 designs for feedback opportunities. The coding process was iterative: we began with a priori categories distinguishing automated from teach-mediated feedback, then defined these through constant comparison into eight distinct types (Table 1). Initial inter-rater agreement was 82%, with disagreements resolved through consensus discussion. Primary disagreements concerned whether certain features truly created opportunities for advancing thinking versus simply confirming correctness, a distinction central to FA theory. In phase two, we systematically compared identified feedback opportunities with peer-assigned performance levels, noting where PSMTs recognized strong, weak and missing opportunities.

Table 1: Counts of Formative Assessment Practices in PSMTs' AAB

Observed Formative Assessment Practices	# of authors who used it	Recognized by reviewer as Strong	Recognized by reviewer as Weak	Recognized by reviewer as Missing	Total # of Reviews
Teacher Feedback	16	28	18	4	50
Entering Equations	5	5	5	5	15
Answers Provided	3	4	0	3	7
Draggable points	1	2	0	1	3
Slider Observations	1	1	0	2	3
Modifying Equations	4	0	1	11	12
Tested Prediction	1	0	0	3	3
Revisit and Revise	1	0	0	3	3
Peer Feedback	1	0	0	3	3

Findings

While PSMTs included various approaches (Table 1, column 1), only four were used by more than one student: Teacher Feedback, Entering Equations, Answers Provided to posed questions, and Modifying Equations. Every author included at least one opportunity for students to enter a response into a textbox for teachers to review, though the decision to offer feedback (or not) was left to the teacher in the moment. Most of these, along with all of those that provided answers, were oriented towards establishing the correctness of responses rather than allowing students to determine this for themselves.

Regarding recognition (RQ2), columns 2–4 reveal that not all opportunities were visible to PSMTs. Reviewers rated opportunities as "weak" when they recognized feedback existed but judged it unlikely to advance learning. Four types of opportunities were not visible at all: asking students to *modify equations* and examine the result on a graph, asking students to *revisit and revise* early thinking, asking students to *test predictions,* and *peer feedback* were not recognized by reviewers as opportunities for feedback to move learners forward. Though, when the author allowed students to enter their own equations before observing graphical behavior, it became more recognizable than simply modifying equations. Five authors utilized *entering equations* as a tool by which students could engage with the mathematics. Two of those went unnoticed, while three were highlighted by reviewers as a form of feedback that would move learners forward.

Other opportunities were more recognizable, including opportunities for direct teacher feedback and incorporating dynamic measurement and *draggable points*. One additional finding that we hope to touch on in the presentation that is not represented in Table 1 is that though PSMTs did not recognize many opportunities for FA outside of direct teacher feedback, they certainly recognized the lack thereof (72%).

Implications and Conclusion

Our findings reveal that PSMTs conceptualize feedback narrowly as verbal or written comments from humans, not recognizing mathematical consequences (e.g., a line shifting when a slope changes) as legitimate feedback that moves learners forward. This gap in professional vision has critical implications for teacher education.

Without explicit intervention, novice teachers default to digitizing traditional worksheets over transforming pedagogy. While PSMTs frequently included text boxes for teacher review, designs prioritized correctness over leveraging technology to prompt reflection or support conjecture testing. We introduce the term *epistemological feedback* to describe how dynamic technology provides immediate mathematical consequences that serve as feedback (e.g., a graph shifting when parameters change). Unlike correctness feedback, epistemological feedback reveals mathematical relationships, allowing learners to test conjectures and observe consequences. The invisibility of epistemological feedback in peer reviews suggests PSMTs need explicit support recognizing how dynamic environments create moments of contingency.

We recommend three pedagogical moves for teacher education. First, explicitly name and model epistemological feedback during technology explorations, consistently asking "What mathematical feedback does the tool itself provide?" Second, engage PSMTs in analyzing contrasting cases, such as comparing "digitized" worksheets with transformed tasks that leverage epistemological feedback. Third, implement design cycles where PSMTs receive guidance on whether their tasks create opportunities for students to learn from mathematical consequences.

Future research should investigate whether explicit instruction in epistemological feedback shifts PSMTs' design patterns. This limited study advances understanding of how prospective teachers develop pedagogical reasoning about technology, highlighting a critical dimension of technological pedagogical content knowledge requiring explicit attention in teacher preparation.

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ORCHESTRATING MATHEMATICAL DISCOURSE IN INTERMEDIATE CLASSROOMS: TEACHER MOVES AND STUDENT ENGAGEMENT

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Mathematical discourse improves student achievement, yet intermediate teachers face implementation challenges. This qualitative multiple case study examined how four Grade 7–8 expert teachers orchestrated discourse across diverse schools in a large urban district. Data included interviews, observations, and field notes. Findings revealed teachers employed strategic questioning, established explicit norms, and used varied grouping. Teachers with strong mathematical knowledge and experience navigated challenges more effectively (Ball et al., 2008). Participants acknowledged certain concepts resist discourse approaches, suggesting curriculum-specific considerations. This study contributes to how expert teachers adapt discourse moves to the development realities of teenagers, extending talk-moves and MKT frameworks in the intermediate grades. Findings illuminate complex teacher decision-making underlying successful discourse implementation and offer practical implications for professional development.

Keywords: Classroom Discourse, Instructional Activities and Practices, Middle School Education, Teacher Knowledge

Objectives and Research Question

Reform documents advocate for mathematics classrooms where students construct understanding through collaborative discourse (NCTM, 2014; Ontario Ministry of Education, 2020). Research demonstrates that discourse-based instruction significantly improves student achievement, particularly for students from low-income backgrounds (Chapin et al., 2003; Boaler & Staples, 2008). However, productive mathematical discourse is not easy to implement and facilitate (Ball & Forzani, 2011; Webb et al., 2019), and many teachers struggle with orchestrating meaningful student-to-student mathematical conversations (Stein et al., 2008; Leatham et al., 2015). The intermediate grades (ages 12-14) present unique challenges for discourse implementation. Adolescent learners exhibit heightened self-consciousness, increased peer influence, and developing abstract reasoning capacities (Eccles et al., 1991), making mathematical sharing in public particularly vulnerable (Marks Krpan, 2013). At this stage the social risk of being seen as wrong can outweigh the reward of contributing, so willingness to reason publicly must be deliberately cultivated through norm-setting, grouping, and questioning calibrated to emerging abstract reasoning. Despite these challenges, research on discourse practices specifically in Grade 7 and 8 mathematics classrooms remains limited (Walshaw & Anthony, 2008). This study addresses that gap by documenting how teachers recognized as expert at these grade levels adapt discourse practices to the developmental characteristics of early adolescents. It investigates: *How do experienced Grade 7 and 8 mathematics teachers orchestrate discourse to promote student learning across diverse classroom contexts?* Three sub-questions guided the inquiry: (1) What teacher moves do expert intermediate teachers employ during mathematical discourse? (2) What strategies do they use to establish learning environments conducive to discourse? (3) How does teacher knowledge shape the way they facilitate discourse?

Theoretical Framework

This study draws on social constructivist perspectives that position learning as socially mediated through language and interaction (Vygotsky, 1978; Lave & Wenger, 1991). Mathematical understanding develops through participation in communities where ideas are shared, challenged, and refined (Sfard, 2008; Cobb & Yackel, 1996). Discourse serves as the mechanism through which students externalize thinking, engage with others' reasoning, and collectively construct mathematical meaning (Lampert, 1990; Michaels & O'Connor, 2015).

The study employs Ball, Thames, and Phelps' (2008) framework of Mathematical Knowledge for Teaching (MKT), which distinguishes specialized content knowledge required for teaching from common content knowledge. This framework helps explain variation in teachers' discourse facilitation, as teachers must flexibly interpret student contributions, recognize mathematical validity in unconventional solutions, and select productive student ideas to advance collective understanding (Sleep, 2012; Stockero & Van Zoest, 2013).

Additionally, the study draws on Chapin, O'Connor, and Anderson's (2009) talk moves framework, which identifies specific teacher moves that scaffold productive discourse: wait time, revoicing, repeating, reasoning, and adding on. Another extension by O'Connor, Michaels, and Chapin (2015) emphasizes how these moves distribute cognitive authority from teacher to students while maintaining mathematical rigor. The study also draws on Liljedahl's (2020) Building Thinking Classrooms, particularly visibly random grouping and thinking tasks, which create the conditions for discourse to emerge. Talk moves and MKT then explain how teachers facilitate and sustain that discourse.

Methods

Participants and Context

Four expert intermediate mathematics teachers participated in this multiple case study, purposefully selected for their expertise with discourse-based instruction. Expertise was determined through three criteria: those with at least five years teaching mathematics in intermediate grades; those who were identified by school administrators as effective math leaders; and those who were well-versed with using discourse confirmed through pre-screening. All participants taught Grade 7 and/or 8 mathematics in a large urban district. Three held mathematics specialist additional qualifications, and all four had held school mathematics leadership roles such as lead teacher or coach. The schools they were at, served linguistically diverse communities where more than 50% of students spoke a language other than English at home. Languages include Mandarin, Cantonese, Russian, Persian, Spanish, Arabic, and Korean, among others.

Data Collection

Data collection followed case study protocols (Merriam, 2009; Stake, 1995) and included: (1) two semi-structured interviews per teacher, conducted before and after classroom observations; (2) two 45-minute classroom observation sessions per teacher, audio-recorded to capture discourse patterns; (3) researcher field notes documenting classroom environment, student interactions, and teacher moves; and (4) demographic data from school websites and publicly available statistics. Interview protocols explored teachers' beliefs about discourse, implementation strategies, challenges encountered, and decision-making processes. Observations captured naturally occurring discourse during mathematics instruction.

Data Analysis

Analysis followed Merriam's (2009) two-stage process for multiple case studies. Each case was first analyzed individually using Kvale's (2007) meaning condensation procedure, with

transcripts coded into emergent categories in NVivo. Then cross-case analysis was conducted to identify commonalities and variations across the four cases (Stake, 1995). Trustworthiness was established through triangulation across interviews, observations, and field notes, and through member checking of transcripts.

Results

Analysis revealed five major themes characterizing effective discourse orchestration in intermediate mathematics classrooms. Across the eight observed lessons, the recorded discourse was predominantly teacher-to-student and student-to-student, with teachers intervening through talk moves, asking guiding questions, and responding to student questions with another question rather than giving definitive answers. The themes below draw on both interview and observation data.

Theme 1: Establishing Foundations Before Discourse

All four teachers emphasized deliberate preparation before implementing discourse. As one teacher explained, "At the beginning of the year, I prepared the students" by explicitly teaching collaborative norms and practicing group work skills. Teachers allocated 1-2 weeks at school year's start for students to learn discourse expectations through low-stakes mathematical tasks. This preparation included modeling respectful disagreement and practicing accountable talk moves (Marks Krpan, 2013; Resnick et al., 2010). Teachers viewed this investment as essential, with another teacher noting, "It all begins with the first day, the first month of the school year and it is all in the classroom management."

Theme 2: Strategic Questioning to Advance Thinking

Teachers employed purposeful questioning beyond traditional Initiate-Response-Evaluate patterns (Sinclair & Coulthard, 1975). Questions focused on eliciting reasoning rather than answers, consistent with Stein et al.'s (2008) high-level cognitive demand framework. Teachers planned prompting questions strategically, with one explaining: "Some of my anticipated questions are going to be a lot higher level thinking; and so that would go towards some of my kids that could handle it. Some of my questions might be a little bit simpler depending on who I am working with." Teachers used wait time strategically (Ingram & Elliott, 2016), allowing students processing time before requiring responses. They revoiced student contributions to amplify ideas across the classroom and pressed students to clarify, elaborate, or justify mathematical claims (Franke et al., 2009). In one observed lesson, during the consolidation phase, a teacher avoided confirming or rejecting a student's answer. Instead, she restated it as a question and asked the group to decide whether it was correct. This moment illustrates how a single teacher move can maintain the mathematical challenge of the task while encouraging students to take ownership of the reasoning.

Theme 3: Responsive Grouping Strategies

Teachers utilized flexible grouping approaches adapted to instructional goals and student needs. Random grouping predominated for most tasks, implemented through playing cards or other randomization methods to ensure students worked with diverse peers. These practices align with Liljedahl's (2020) visibly random groupings. Teachers noted that regularly changing group compositions helped reduce the social anxiety, which is typical of early adolescence. However, teachers made strategic modifications for English language learners. One teacher explained, "even if it is random grouping, I will consider putting two people of the same language together, especially in grade 8 because the content is there". This decision reflected asset-based thinking (Civil, 2007), positioning home language as a resource for mathematical thinking rather than a deficit.

Theme 4: Mathematical Knowledge Enables Flexibility

Cross-case analysis revealed that teachers with deeper mathematical content knowledge navigated discourse more fluidly. These teachers recognized mathematical validity in unconventional student approaches, asked probing follow-up questions that advanced thinking, and connected student ideas to mathematical big ideas. One teacher demonstrated comfort admitting uncertainty, explaining: "you know what; I do not know the answer to that. Let us see if we can figure that out together; let us grab a textbook; let us get your phone; I will grab my computer; let us see if we can find an answer." This intellectual humility paradoxically strengthened discourse by positioning students as mathematical authorities (Engle & Conant, 2002).

Theme 5: Acknowledging Discourse Limitations

All participants agreed that discourse is not the only strategy to use to teach mathematics. While it may be very effective, it may come after some direct instruction. One participant shared during their interview that “if they [students] have had no exposure at all to those kinds of geometric constructions, I think that there has to be some direct instruction before they can just kind of talk." This reflects the view that foundational knowledge must precede discourse for certain content. These teachers also expressed that highly procedural topics like standard algorithms might be better taught using non-discourse approaches. Teachers drew on their content knowledge, teaching experience, and pedagogical expertise to decide when discourse would be the best approach to use (Hiebert & Grouws, 2007).

Discussion and Implications

This study illuminates the orchestration work underlying successful mathematical discourse in intermediate classrooms. The study's contribution is threefold. First, it foregrounds the intermediate grades, showing that effective discourse requires managing the social vulnerability of early adolescence, in addition to developing discourse-based facilitation skills. Second, it links frameworks usually treated separately, demonstrating how MKT (Ball et al., 2008), talk moves (Chapin et al., 2009), and task and grouping structures (Liljedahl, 2020) operate together in expert practice. Third, it adds nuance to our understanding of which concepts may be best taught using non-discourse-based approaches. The deliberate preparation phase echoes Boaler's (2015) "relational equity" and disrupts deficit narratives about adolescents (Esmonde, 2009). Teachers' strategic questioning moves extend Chapin et al.'s (2009) framework by showing that expert teachers deploy talk moves responsively rather than formulaically, and the link between MKT and discourse success (Ball et al., 2008) implies that professional development should target both pedagogical moves and mathematical content. The acknowledgment that certain content is not best taught through discourse initially resonates with Hiebert and Grouws' (2007) argument that posits that no single method is universally effective. Teachers' judgment about when discourse serves learning is itself sophisticated professional expertise (Sherin et al., 2011).

For professional development, findings suggest three priorities: (1) providing teachers with extensive practice facilitating discourse with adolescents; (2) strengthening teachers’ Mathematical Knowledge for Teaching, especially specialized content knowledge; and (3) developing teachers’ judgment about when discourse is the appropriate approach. Video analysis of discourse can support teachers in building adaptive expertise (Hatano & Inagaki, 1986; Marks Krpan, 2018).

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MATH & SHAME: EXPLORING THE RELATIONSHIP BETWEEN PRESERVICE TEACHERS' MATHISTORIES AND MATHEMATICS TEACHER EFFICACY

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Mathematics autobiographies of preservice elementary teachers are examined and compared with an administration of the Mathematics Teacher Efficacy Belief Instrument, in order to determine patterns between a preservice teacher's relationship with mathematics and their efficacy in teaching mathematics. Implications for teacher education are also considered.

Keywords: Preservice Teacher Education; Teacher Beliefs; Affect, Emotion, Beliefs, and Attitudes

The Problem

Since the publication of the seminal text "Adding it Up" (National Research Council, 2001), productive disposition toward mathematics was named as one of five strands of mathematical proficiency. In addition, the Association of Mathematics Teacher Educators (AMTE) named having "positive and productive dispositions toward teaching mathematics" (2017, p.6) as one of the primary responsibilities for mathematics teacher educators in preparing teachers of mathematics. But, what if a student's relationship with mathematics cannot be described as productive? What if the relationship is influenced by instances of shame experienced with mathematics? Going further, what happens when those same students arrive in an elementary mathematics methods course, positioned as future teachers of mathematics? And how does one's disposition toward mathematics influence one's efficacy in the teaching of mathematics?

The question driving this project is "What is the nature of the relationship between elementary preservice teachers' stories with mathematics and measures of those same students' beliefs about mathematics teaching efficacy?" In an attempt to approach the answer to the question, mathematics autobiographies (referred to as MatHistories) of students are examined and compared with an administration of the Mathematics Teacher Efficacy Belief Instrument (MTEBI) (Enochs et al., 2000), in order to determine patterns between a preservice teacher's relationship with mathematics and their efficacy in teaching mathematics. The goal of this examination is to identify pathways for repairing relationships with mathematics for the benefit of the students in the preservice teachers' future classrooms. What follows is an explanation of the literature that frames this study, the mode of inquiry, the findings, and a discussion of the implications of this work.

The Theoretical Framework

For the purpose of this brief report, the learning of mathematics is equated with developing a desired relationship with mathematics (Lampert, 2001) or moving more central within a community of practice/community of doers of mathematics (Lave & Wenger, 1991). When intention is not being focused on developing the relationship with mathematics, and instead the emphasis is on rote memorization of facts and the mindless execution of algorithms, an

undesired relationship with mathematics is formed. Instead of a productive relationship between mathematics and students, there is a negative relationship that is characterized by shame and anxiety.

When asked to do mathematics, many claim negative feelings of dislike and anxiety (Novak & Tassell, 2017). The formation of these feelings has been linked to several different factors (Harper & Daane, 1998). Students cite experiences with teachers as one of the major causes of negative attitudes and anxiety towards mathematics. Individuals who have experienced shame in the context of mathematics learning may avoid and/or withdraw from future activities associated with mathematics (Ashcraft, 2002). Further, these shame experiences with mathematics can lead to poor student performance as well as lasting mathematics anxiety (Wilson, 2018).

Self-efficacy as it relates to mathematics has been the subject of numerous studies over the past few decades. Scholars have looked at how it is affected by factors such as mathematics anxiety (Unlu et al., 2017), gender (Szczygiel, 2020), education level (Brady & Bowd, 2005), and mindset (Tassell et al., 2020). Mathematical teaching self-efficacy is defined as one's perceived belief in their ability to understand and teach mathematics. Recent studies have established links between teachers' mathematics self-efficacy, anxiety, and student performance (Unlu et al., 2017; Ramirez et al., 2018). One link shows that teachers with high mathematics anxiety have low self-efficacy, and vice versa (Unlu et al., 2017). Also, the higher a teacher's anxiety level around teaching mathematics, the lower the achievement of their students (Ramirez et al., 2018). Thus, relationships that preservice teachers have with mathematics are significantly associated with their mathematics teaching self-efficacy (Briley, 2012) and the need to help influence their relationship towards the positive.

The Mode of Inquiry

In this study, set in a large, remote public university in the midsouth, 51 preservice teachers from two sections of an elementary mathematics methods course provided a written account of their history with mathematics (i.e. MatHistory), as well as their responses to the MTEBI (Enochs et al., 2000).

Responses for the mathematics autobiography went through several cycles of coding (Saldaña, 2013). Holistic coding (Saldaña, 2013) was used to categorize histories by the overriding relationship with mathematics (e.g. positive, negative, and variations in between). Next, provisional coding (Saldaña, 2013) was used to identify emotions that were named in each history, as either triggers (i.e. events that led to the emotion), or pathways (i.e. outcomes related to the emotion). Next, the researchers engaged in open coding (Saldaña, 2013) of the triggers and pathways in order to categorize the mathematical experiences that led to the emotions and students' responses. Finally, stories, which contained a full set of trigger, emotion, pathway were categorized given positive or negative nature of each element with a focus on stories that turned toward a positive pathway.

In addition to the coding, the MTEBI subscores (Enochs et al., 2000), mathematics teaching outcome expectancy (MTOE), and personal mathematics teaching efficacy (PMTE) were calculated. MTOE is a measure of expectation that the teaching of mathematics can influence a student's relationship with mathematics, and PMTE is a measure of belief in one's ability to enact teaching of mathematics that can influence a student's relationship with mathematics. Finally, each preservice teacher's MTEBI scores were compared to the holistic coding of their histories with mathematics.

The Findings

The largest category from the holistic coding of histories with mathematics was "negative relationship." "Negative relationship" was defined as an overall negative history with mathematics, whereas a "repaired relationship" was defined as an overall negative history with mathematics that concluded with a positive disposition toward mathematics. When comparing these categories of stories with their respective subscores from the MTEBI, there was a noticeable difference in the PMTE subscores between the two groups. The major qualitative difference in the histories of mathematics between these two groups was a significant positive moment in the latter part of their history with mathematics for those in the "repaired relationship" category. This led to further questions and analysis about the context for the repair, and the resiliency of the repair given future struggles with mathematics or the teaching of mathematics. Preliminary investigations have shown that experiences in elementary

Analysis continues on this project to investigate questions as they emerge from previous analysis. What has emerged is the significance of the moment of repair for the histories with mathematics in the "repaired relationship" category and what it might look like to design and enact such experiences. Other studies have investigated the effects that methods courses can have on preservice teachers' mathematics teaching anxiety and self-efficacy. Vinson (2001) found that the majority of preservice teachers lowered their mathematics learning and teaching anxiety as well as boosted their mathematics teaching self-efficacy after being enrolled in a mathematics methods course that focused on instructing the preservice teachers on the best strategies for teaching their future students. A study conducted by Harper and Daane (1998) had similar findings, but also determined that preservice teachers need to be made aware of their mathematics anxiety and negative feelings toward mathematics, in order for them to develop a more positive relationship with mathematics and not transmit those same feelings onto their students.

Conclusion

The goal of this work is to continue to seek pathways for reconciling preservice teachers' relationships with mathematics given evidence that reconciliation is possible. By better understanding the perspectives that PSTs bring to the elementary mathematics methods classroom, through their mathematical narratives and their efficacy toward the teaching of mathematics, mathematics teacher educators can be better prepared to provide reconciling relationships with mathematics/mathematics teaching that can be implemented within their classroom and/or programs to support the high quality preparation of teachers of mathematics.

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ITERATIVE GOAL SETTING AS A CATALYST FOR PRESERVICE ELEMENTARY MATHEMATICS TEACHER DEVELOPMENT

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This case study explores the role that iterative goal setting across a semester-long mathematics methods course plays in supporting the development of elementary preservice teachers' mathematics identities and professional vision for teaching. Using narrative analysis framed around identity positioning and goal orientation, this report presents preliminary findings by offering an intrinsic case of one preservice teacher whose goal orientations and vision for mathematics teaching shifted over time. Implications from this study seek to inform work around elementary preservice teachers' mathematics identity and self-efficacy development by identifying specific reflective practices and structures that can help them grapple with the tensions of transitioning from mathematics learner to teacher.

Keywords: Preservice Teacher Education, Elementary School Education, Identity and Agency

Elementary preservice teachers (PSTs) often have negative experiences as mathematics learners that lead to challenges negotiating their future identity as teachers (Brown & McNamara, 2011; Kaasila et al., 2012). This directly impacts student learning opportunities, as PSTs may default to procedural instructional approaches rather than engaging students in active learning that promotes conceptual understanding and problem solving—limiting students' access to high quality instruction (Boaler, 2002a; Boaler & Staples, 2008; Hill et al., 2005). Teachers with low mathematics teacher self-efficacy are more likely to transmit feelings of math anxiety to students (Beilock et al., 2010; Smith & Fotou, 2023), leading to lower academic achievement (Aguirre et al., 2024; Boaler, 2002b). In order to change this paradigm, mathematics teacher educators are tasked with helping PSTs learn to navigate their past experiences and leverage them to inform future visions of becoming effective mathematics teachers. Reflection and goal setting in disciplinary methods courses can help PSTs reinterpret their understanding of mathematical learning and construct new narratives for themselves as mathematics teachers (Sfard & Prusak, 2005). In this preliminary report, I present an instrumental case study of one PST, selected from a larger qualitative study, to explore her progression of mathematics identity development. This study seeks to answer the question: How does iterative goal setting and reflection in an elementary mathematics methods course elucidate shifts in preservice teachers' identity development?

Theoretical Framing

Sfard and Prusak (2005) explain identity as a communicational practice where individuals define themselves through narratives that are "reifying, endorsable, and significant" (p. 16). Identity can position learners as either central or marginal figures in classroom spaces in ways that either affirm or challenge their self-concept (Aguirre et al., 2024; Anderson, 2007; Darragh, 2014; Harre et al., 2009). Similarly, teacher identity requires the ongoing negotiation of values and experiences, where individuals must navigate actual versus "expected" narratives that are or can potentially become part of one's identity (Sfard & Prusak, 2005). Elementary PSTs often enter their teacher preparation programs with negative self-concepts about their ability to learn and effectively teach mathematics (Brown & McNamara, 2011; Kaasila et al., 2012). As such,

mathematics teacher educators are often tasked with helping PSTs navigate tensions between their past mathematics identities as learners and future identities as mathematics teachers.

"Professional practice" refers to intentional, reflective approaches that cultivate teachers' instructional effectiveness, equity-minded dispositions, and professional identities (Darling-Hammond & Bransford, 2005; Grossman et al., 2009). It empowers teachers to advocate for learners, embrace multicultural perspectives, and enact transformative practices (Darling-Hammond & Oakes, 2021; Grossman et al., 2009; Gay, 2018; Zeichner, 2012). Reflection and goal setting are well documented professional practices to support PSTs' growth (Camp, 2017). Offering PSTs structured, iterative opportunities to reflect allows them to examine their beliefs, experiences, and aspirations—aiding them in connecting their past experiences to future teaching goals (Schutz et al., 2018). Articulating these aspirations supports PSTs in exploring pathways to shifting their mathematics identities as both learners and teachers. Research supports the role that effective goal setting requires participants to be aware of their progress (e.g., Latham et al., 1978), and that goals focused on learning versus performance are more likely to help individuals develop tools for problem solving and increase self-efficacy (e.g., Latham & Brown, 2006).

Methods

This study took place at a midsized, urban university in the Midwest. Participants included 22 elementary PSTs enrolled in a 15-week mathematics methods course in Spring 2024. At the beginning of the semester, participants wrote mathographies (Drake, 2006), reflecting on past mathematical events—including high, low, and turning points in their stories—before setting goals for themselves as learners and future teachers of math. PSTs engaged in class activities and teaching rehearsals paired with reflective opportunities to iteratively review and refine their goals in light of new experiences at four stages of the semester, tying goals to mathematics teaching practices (NCTM, 2014) and the InTASC teaching standards (CCSSO, 2013). This paper presents the experiences of one PST, "Abbi," as an instrumental case (Stake, 1995) selected from the larger data set to provide an in-depth illustration of how shifts in goal setting and attainment influenced Abbi's sense of confidence as a math teacher. Narrative analysis (Connelly & Clandinin, 2006; Polkinghorne, 1995) guided the examination of her mathography and three subsequent reflection assignments where she reviewed previous goals, reflected on teaching rehearsals, and revised/extended goals based on her progress. Rather than fragmenting the data into decontextualized codes for this case, the participant's narrative was analyzed holistically to preserve the coherence of her story while identifying key shifts in her thinking over time.

Findings

Past Learning Experiences Influence Future Aspirations

Abbi was selected as a PST whose relationship with math was neither completely negative or positive, rather it was experienced as a self-described roller coaster. As a young learner, Abbi found herself excelling in math and being positioned as a central figure in the classroom. She noted,

> I was always in the higher math group and I felt smart when it came to math. I specifically remember a time in fifth grade when my teacher commented on my work for long division in front of the class and it made me feel really good.

As she shifted into middle school, however, her experiences began to change. She found herself in a classroom setting where content was being modeled strictly through lecture and individual practice, and she did not feel comfortable asking the teacher for help. Abbi shared how this

experience impacted her identity, saying, “I remember having lots of tears in class because I just felt so behind and honestly dumb because I couldn’t keep up with my classmates. I was too embarrassed to ask for help.” Abbi found her positioning in the math classroom had flipped—instead of being a central figure she now found herself on the periphery and her identity in flux. In high school, Abbi explained that she finally had a mathematics teacher who was able to help her regain some of her lost confidence. She reflected,

> [My teacher] explained everything in a way I could understand. She actually took the time to help fill in the gaps in my math understanding that had been holding me back for so long. I wish I had a teacher who would’ve done that for me sooner.

Despite having a somewhat tumultuous experience with math as a learner, as Abbi set a number of learning focused goals for herself at the start of the semester. These goals focused on building a strong foundation for herself as a math teacher, as well as becoming comfortable asking questions in class, regardless of how this positioned her among her peers. She explained, “My experience with math has made me want to focus on embracing asking the ‘dumb questions’ as a math learner because it’s better to ask what feels like a dumb question than to go on confused.” In addition to setting goals for herself as a learner in class, she also wrote a similar goal to “create a safe and comfortable environment for my students to ask any question,” indicating her desire to provide for her students supports she did not always have herself as a learner. Abbi saw a direct connection between the experiences she wanted to build for herself as a learner in her methods course with the ones she wanted her future students to have in her own classroom.

Midsemester Shifts in Identity and Goal Articulation

In the first five weeks of the semester, PSTs engaged with course activities where they learned about effective mathematics teaching practices (e.g., National Council of Teachers of Mathematics, 2014; 2020), as well as the core teaching practices outlined by the Interstate Teacher Assessment and Support Consortium (InTASC, 2013). Part of this work included enacting a number of teaching rehearsals, where PSTs engaged in scaffolded, collaborative planning and received structured feedback to support their attainment of these practices, before transitioning to a field-based practicum in Week 6. As they shifted into classrooms, Abbi reflected on her progress towards her initial teaching goal. She described a continued desire for students to feel comfortable sharing their thinking with her and asking questions and maintained this as a target goal moving forward. She said, “I want to make sure I am continuing to engage students with questions and checks for understanding.” Abbi further tied this goal back to the mathematics teaching practice of “eliciting evidence of student thinking” (NCTM, 2013) as a way to help her make informed instructional decisions to support their growth as learners.

Approximately two to three weeks into the field experience, Abbi came back to her goals again, this time aligning them to the InTASC practices. She identified three target InTASC standards that she wanted to focus on: 1) using knowledge of students to make instructional decisions, 2) using knowledge of students’ diverse backgrounds to create inclusive learning environments, and 3) using relevant, evidence-based instructional strategies that allow students to apply content area concepts (InTASC, 2013). Here, Abbi continued to focus on creating an inclusive and supportive learning environment, however she also extended her goals by identifying core practices that could help her make actionable progress toward this goal. She began to recognize that gathering evidence of student thinking not only helped students to feel heard and supported, it also gave her important data she could use to tailor instruction in ways that met the needs of all learners. Further, Abbi began to transition from considering effective, evidence-based teaching practices from a theoretical set of ideas to clearly identifiable teaching

moves she could integrate into lessons to further support building a supportive learning environment. When asked why she opted to adjust her goals she explained, "I knew I was already doing it, and I wanted to get more specific with how I went about helping students."

"Becoming a Math Teacher:" End of Semester Growth and Ongoing Aspirations

At the end of the semester, Abbi returned to her original goals to reflect on progress, revisions, and adaptations she wanted to make based on new pedagogical learning and experiences. She began my sharing a desire to maintain one of her original goals, stating, "I still want to create a safe environment for my students to make mistakes… comfortable with being unsure to get where we finally understand by not being afraid to ask questions." Abbi's continued focus on ensuring her future students had what she sometimes felt she did not as a student—a safe space for asking questions when confused or uncertain—as integral to the type of teacher she wanted to become. In addition to this broader goal that supported creating an inclusive classroom environment, Abbi added a new goal based on her reflections. She shared,

> My new, maybe more important goal, is to make sure my students understand the algorithms they use and why they work. I feel I have the ability to explain math well using math language and vocabulary in a way that students can understand. I really like the idea of being a teacher who can lay a strong foundation in the conceptual understanding of math and the why behind it, allowing my students to hopefully be super successful in their math careers.

This illustrates how Abbi saw a path forward in designing instruction that supported the classroom environment she wanted to create by making content meaningful and accessible to students. She explained that the opportunities to "take stock of" her progress at different points in the semester helped to increase her own confidence and competence engaging in effective mathematics teaching in wants that added a new inflection point to her previous math narrative.

Discussion and Conclusion

Abbi began the semester as a PST whose goals were at odds with many of her experiences as a mathematics student. The preliminary findings from analysis of her experiences in her methods course suggest that her engagement in activities that included targeted reflection and iterative goal setting helped to reshape her views not only of what mathematics teaching looks like, but also her beliefs in her ability to become an effective mathematics teacher. These experiences reinforced her desire to ask questions in class, initially a goal about her own participation that eventually transitioned to her supporting students in the classroom. In her final reflection she explained that the evidence-based teaching practices she learned about through coursework helped her set and achieve actionable goals that become more targeted over time. This case study thus offers insights into the role that goal setting can play as a generative practice for supporting PSTs' mathematics teacher identity development by offering space for them to make note of incremental growth identify places where goal achievement supported their math identity development. Providing PSTs with structured, ongoing opportunities to reflect on pedagogical knowledge and teaching beliefs and how they evolve over time can support PSTs in navigating the tensions between their past experience as mathematics learners and their aspirations as future teachers. This work contributes to ongoing conversations about how teacher preparation programs can most intentionally support PST identity development to inform their future professional visions.

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PRESERVICE TEACHERS' AI-ASSISTED DIFFERENTIATION OF A MATHEMATICS TASK

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This study examines the theory-practice gap in elementary preservice teachers' (PSTs) AI-assisted differentiation of a volume-measurement task. This paper analyzed artifacts from thirteen PSTs who consented to research use. Qualitative artifact analysis focused on cognitive demand, scaffolding quality, learner assumptions, and indicators of AI assistance. Findings show that PSTs often reduced task quantity, simplified reflection, and increased teacher control for students with mathematical learning difficulties, while extending independence or complexity for advanced learners. Artifacts with stronger AI-likelihood indicators used sophisticated pedagogical language but enacted conventional simplifications that limited opportunities for mathematical reasoning. These findings suggest a declarative-procedural gap in AI-assisted mathematics lesson design and differentiated instruction.

Keywords: Preservice Teacher Education, Measurement, Instructional Activities and Practices, Technology, Special Education

Introduction

Differentiation is often framed as a means of expanding equitable access to learning, yet it can reproduce inequity when teachers lower the cognitive demand of mathematical tasks for students with learning difficulties. This concern is important in preservice teacher education, where PSTs learn to adapt tasks while developing beliefs about who is capable of mathematical reasoning. Effective differentiation should vary routes of access while preserving meaningful engagement with important mathematical ideas (Tomlinson, 2001). Similarly, maintaining cognitive demand is central because students need opportunities to reason, justify, and make connections, rather than simply complete simplified procedures (Smith & Stein, 1998).

Generative AI adds a new layer to this teacher education challenge. AI tools can provide research-based language about scaffolding, learner variability, and cognitive load, but such language may not translate into task designs that preserve rigorous mathematical activity (Kasneci et al., 2023; National Council of Teachers of Mathematics [NCTM], 2024). This study examines PSTs' written adaptations of a volume-measurement task for students with mathematical learning difficulties and for mathematically advanced or gifted students. The study asks: (1) How do PSTs modify a mathematics task when differentiating for different learner profiles? (2) To what extent do these modifications maintain or reduce cognitive demand? (3) How do AI-likelihood indicators relate to the enacted quality of PSTs' designs?

Theoretical Framework

This study draws on two theoretical perspectives to examine PSTs' differentiation practices. First, we apply Smith and Stein's (1998) framework of cognitive demand in mathematical tasks, which distinguishes between maintaining rigorous learning objectives while varying pathways to access (Tomlinson, 2001) and reducing expectations for students with learning difficulties. Research consistently shows that teachers struggle to implement high-quality differentiation, often lowering cognitive demand rather than providing strategic scaffolding (Boaler, 2015; Charalambous & Pitta-Pantazi, 2016)—a challenge particularly acute for PSTs, who may hold

deficit-based beliefs about student capability (Aguirre et al., 2013; Turner et al., 2012). In this study, cognitive demand refers to whether PSTs preserved opportunities for students to predict, measure, compare, reason about units, and explain errors in a volume-measurement task.

Second, we employ the declarative-procedural knowledge distinction to theorize AI's role in teacher learning. AI tools may provide declarative knowledge, such as knowing that differentiation should maintain cognitive demand, without supporting procedural knowledge development, or knowing how to design such tasks. This theory-practice gap (Thurm & Barzel, 2022) helps explain why AI assistance might produce research-based language without transforming instructional design decisions, as PSTs articulate principles they cannot yet enact. This perspective also helps explain why the emergence of generative AI tools presents both opportunities and concerns for teacher education (NCTM, 2024; Villegas-Ch et al., 2025).

Methods

Nineteen PSTs enrolled in a mathematics methods course completed a differentiation task. PSTs were asked to modify a volume-measurement activity (Lee & Link, 2012) for students with mathematical learning difficulties and for mathematically advanced or gifted students. The assignment required modifications across four dimensions of differentiation: content, process, product, and environment. PSTs also provided written justifications for their instructional choices. AI tools were available as optional supports. The dataset consisted of differentiated handouts or directions and accompanying written justifications from the 13 consenting PSTs.

This study used qualitative artifact analysis in two phases. First, open coding identified recurrent design moves and rationales across the paired adaptations. Second, the codes were organized into four analytic dimensions aligned with the theoretical framework: cognitive demand, scaffolding quality, learner assumptions, and AI-likelihood indicators. *Cognitive demand* captured whether adaptations preserved the core mathematical work of prediction, measurement, comparison, unit reasoning, and error analysis. *Scaffolding quality* captured whether supports increased access without removing opportunities for student reasoning. *Learner assumptions* captured how PSTs positioned students with mathematical learning difficulties and mathematically advanced or gifted students in relation to mathematical capability, independence, and rigor. *AI-likelihood* captured textual and design features suggesting possible AI assistance, such as dense theoretical terminology, generic research claims, parallel structure, and limited connection to the specific volume task.

Findings

Analysis revealed four patterns in PSTs' differentiated task adaptations: task-level separation, scaffolding as teacher control, unequal access to reflection, and a declarative-procedural gap in artifacts with stronger AI-likelihood indicators.

Differentiation as Task-Level Separation

The dominant pattern was differentiating by amount, difficulty, and autonomy rather than preserving a common mathematical goal with varied access. Ten of the 13 artifacts reduced the number of containers for students with mathematical learning difficulties. One PST justified this reduction by noting that students might experience "cognitive overload," suggesting that support was often interpreted as reducing task quantity. In contrast, adaptations for mathematically advanced or gifted students typically maintained or increased the number, irregularity, or complexity of containers.

This pattern was not only about quantity. In several artifacts, PSTs changed the mathematical work expected of each learner group. Students with mathematical learning difficulties were

sometimes asked only to identify which container held “the most” or “the least,” whereas advanced students were asked to predict numerical volume, compute differences, convert units, graph results, or explain discrepancies between predicted and actual measurements. Thus, some students with mathematical learning difficulties were assigned different mathematical work altogether: qualitative comparison instead of numerical prediction, observation of teacher modeling instead of active problem solving, or simplified recording instead of explanation. These adaptations changed who had access to estimation, unit reasoning, and mathematical explanation.

Scaffolding as Teacher Control

A second pattern involved different assumptions about learner independence. Eleven of the 13 artifacts positioned students with mathematical learning difficulties in teacher-led groups with modeling, step-by-step directions, or close adult guidance. In contrast, advanced students were often given independent work, peer collaboration, open-ended exploration, or additional mathematical extensions. This pattern suggests that PSTs frequently associated support with teacher control and advanced learning with autonomy. One rationale explained that a teacher-led group would help because “the teacher can lead students step by step,” illustrating how support was often framed as procedural direction rather than structured access to reasoning.

Some teacher support was productive when it directed attention to mathematical structure. One PST proposed asking, “What do you notice about the height of the bottle?” “What do you notice about the width?” and “How does this affect its volume?” These prompts maintained the task’s reasoning goals while making relevant features more visible. However, many teacher-directed adaptations did not include such mathematically focused prompts; in those cases, scaffolding functioned more as procedural management than access to reasoning.

Unequal Access to Reflection and Error Analysis

A third pattern was the removal or simplification of reflection for students with mathematical learning difficulties. Nine of the 13 artifacts removed or weakened the original prompt, “What might account for any errors in your predictions?” Some replaced it with “How did you do?” emoji self-assessments, or multiple-choice explanations. One PST stated that students with learning difficulties did not need to reflect because they were not predicting actual numbers.

This change matters because error analysis was one of the richest mathematical opportunities in the original task. Explaining inaccurate predictions can support reasoning about height, width, shape, container thickness, measurement precision, and units. When this prompt was removed, students with mathematical learning difficulties had fewer opportunities to engage in metacognitive and explanatory reasoning.

AI-Likelihood and the Declarative-Procedural Gap

Artifacts with stronger AI-likelihood indicators often included sophisticated pedagogical language, referencing cognitive load, concrete-to-abstract progression, working memory, learner variability, and research-based scaffolding. However, the corresponding adaptations were often conventional: reducing quantity, simplifying vocabulary, adding familiar reference objects, or limiting open-ended explanation. This pattern reflects a declarative-procedural gap: some PSTs could state principles of effective differentiation but did not consistently translate them into designs that maintained cognitive demand.

The relationship between AI-likelihood and quality was not straightforward. Some artifacts with moderate AI-likelihood indicators showed stronger differentiation because general pedagogical ideas were adapted to the specific mathematics of volume estimation. In contrast, artifacts with stronger AI-likelihood indicators sometimes read as polished but generic, with

limited connection to prediction, measurement, comparison, and error analysis. Because AI use was not self-reported or verified, this finding should be interpreted cautiously. The central issue is not whether PSTs used AI, but whether they critically transformed available suggestions into mathematically sound designs.

Several artifacts demonstrated more productive differentiation by maintaining the core mathematical work while varying tools, prompts, participation structures, or extensions. Productive moves included using familiar benchmarks to support estimation, asking focused questions about height and width, modeling one example before releasing responsibility, and requiring students to justify predictions or identify sources of error. Together, these patterns suggest that stronger AI-likelihood was not associated with uniformly higher design quality; polished pedagogical language did not necessarily correspond to adaptations that preserved cognitive demand.

Table 1: Examples of Access-Oriented Differentiation That Preserved Cognitive Demand

Productive Move	How it Maintained Mathematical Access
Reference tools	Familiar benchmarks supported estimation while students still predicted, measured, ordered, and compared volume.
Focused prompts	Questions about height, width, shape, and container properties directed attention to relevant structure without supplying answers.
Graduated participation	Modeling one example, completing one collaboratively, and then releasing students preserved the task while reducing uncertainty.
Metacognitive extensions	Advanced students developed estimation strategies, compared measurement methods, explained discrepancies, or created guides.
Process documentation	Students justified predictions and identified sources of error, making reasoning visible.

Discussion and Implications

Findings suggest that PSTs' central challenge was designing adaptations that preserved rigorous mathematical activity while expanding access for different learners. Across artifacts, students with mathematical learning difficulties were often given fewer containers, more teacher control, and less access to numerical estimation, error analysis, and independent reasoning. In contrast, mathematically advanced or gifted students were more likely to receive extensions involving additional quantities, conversions, graphs, or open-ended exploration. Thus, differentiation sometimes functioned as task-level separation rather than varied access to shared mathematical goals, raising equity concerns about who is positioned to engage in mathematical reasoning (Aguirre et al., 2013; Turner et al., 2012).

The AI-likelihood findings sharpen this concern. Artifacts with stronger AI-likelihood indicators often used sophisticated language about scaffolding, cognitive load, and learner variability, yet the enacted designs frequently simplified the mathematical work. For mathematics teacher educators, these findings suggest that AI-integrated assignments should foreground critique and redesign rather than task generation alone (Kasneci et al., 2023; NCTM, 2024). PSTs should evaluate whether adaptations preserve the mathematical goal, identify where cognitive demand is reduced, and revise supports to increase access without removing reasoning. One routine is to have PSTs annotate each modification by naming the mathematical goal preserved, the barrier addressed, and the reasoning opportunity maintained, consistent with calls

to support productive struggle and use evidence of student thinking in mathematics instruction (NCTM, 2014).

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CREATING A POSITIVE LEARNING ENVIRONMENT IN A MATH COURSE FOR PRESERVICE TEACHERS

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This study examines how a mathematics instructor cultivated a positive learning environment in a synchronous online course for 23 preservice teachers. Guided by self-determination theory, growth mindset, and mathematical caring, the course integrated high-cognitive-demand tasks, collaborative problem solving, flexible policies, and motivational supports. Data sources included reflection logs (Likert-scale items and open-ended responses) and a focus group; the instructor's reflexive self-study informed result interpretation. Wilcoxon signed-rank tests indicated significant gains in students' sense of inclusion and trust, though some instructor-related ratings declined late in the semester. Qualitative findings emphasized students' appreciation for flexibility, encouragement, and growth-oriented feedback. Despite time and technological pressures, participants described the environment as supportive and motivating.

Keywords: Preservice Teacher Education

In the context of higher education, creating a positive learning environment is critical for student success and well-being. In mathematics education, especially for preservice teachers, creating a positive learning environment is foundational to student learning and well-being. The disruptions caused by the COVID-19 pandemic have further amplified ongoing challenges related to engagement, motivation, and mental health. In response to these realities, the first author made a conscious effort to foster a classroom culture that supports students both intellectually and emotionally. This study examines the specific actions he took to build that environment, the challenges he encountered, and how students perceived these efforts. By reflecting on these experiences, we aim to contribute to the broader conversation on how teacher educators can create spaces where students feel safe to take risks, persist through challenges, and grow in their mathematical thinking.

Literature Review

In mathematics education, national standards emphasize not only content knowledge but also mathematical thinking, such as NCTM's five process standards and the CCSSI's eight standards for mathematical practice. Yet, many preservice teachers continue to approach mathematics passively, often viewing it as a set of rules and procedures (Lampert, 1990; Lim, 2012). While high-cognitive-demand tasks are essential for developing conceptual understanding, their effectiveness depends on thoughtful implementation (Stein et al., 2009). The five practices for orchestrating mathematical discussions (Smith et al., 2009) offer a framework for helping students make meaningful connections among strategies and ideas.

Attending to both cognitive and affective domains is critical. Challenging tasks can lead to frustration, especially when students lack the support to persist. Self-determination theory (Ryan & Deci, 2000) highlights the importance of competence, autonomy, and relatedness in fostering intrinsic motivation. Teachers who support autonomy tend to promote greater engagement, while controlling approaches can have the opposite effect.

Caring relationships also play a central role in student learning. For Noddings (1984), caring requires one to "step out of their own personal frame of reference into the other's" (p. 24). In

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

mathematics education, Hackenberg (2005) extends this idea through the concept of mathematical caring, which balances cognitive and emotional support during problem-solving. Maloney and Matthews (2020) further describe caring in terms of empathetic, transactional, and blended typologies. In this study, the first author aimed to practice empathetic care by modeling a growth mindset and a commitment to self-improvement.

Dweck (2006) distinguishes between fixed and growth mindsets. Students with a growth mindset believe that intelligence can be developed, embrace challenges, and persist through setbacks. Research on brain plasticity supports the idea that all students can learn mathematics when given the right opportunities and strategies (Boaler, 2013). Growth mindset interventions have been shown to improve academic outcomes (Blackwell et al., 2007). In this course, the first author incorporated activities and resources from youcubed.org to help students develop a "mathematical mindset" (Boaler, 2019).

Together, these frameworks—NCTM's five process standards, self-determination theory, growth mindset, and mathematical caring—guided the instructional decisions in this study. They informed the strategies used to support student engagement, persistence, and well-being in a mathematics course for preservice teachers.

Background

This study was conducted at a public university in the southwestern United States, where over 80% of the student population identifies as Hispanic. In Spring 2021, during the COVID-19 lockdown, 24 preservice middle school teachers enrolled in a capstone mathematics course participated in the study. One student's data was excluded because he joined only the first three classes via Zoom, resulting in a final sample of 23 participants.

The course was taught synchronously online, with students attending 80-minute Zoom sessions on Tuesdays and Thursdays. Weekly homework was submitted via OneNote Class Notebook by Sunday, and quizzes were completed on Blackboard using Respondus Lockdown Browser before the next class. High-cognitive-demand tasks (see Lim, 2014, 2016) were integrated into class sessions to foster mathematical thinking and conceptual understanding, often through collaborative problem-solving in breakout rooms. The course text, *Developing Essential Understanding of Expressions, Equations & Functions* (Lloyd & Herbel-Eisenmann, 2011), supported students in connecting foundational algebraic concepts to strategies discussed in class for solving novel problems.

Methods

This practitioner study drew on multiple data sources—quantitative Likert-scale items, qualitative open-ended responses, and a post-course focus group—to examine the instructor's actions, the challenges encountered, and students' perceptions of the learning environment.

Course Design and Instructional Approach

The first author designed the course to balance cognitive rigor with affective support. Core features included:

- **High-cognitive-demand tasks:** Novel problems requiring reasoning and multiple representations; discussion-oriented debriefs connected strategies to key algebraic ideas.
- **Collaborative structures:** Guided small-group work in Zoom breakout rooms, rotating roles, and public sharing of strategies to normalize partial solutions and revision.

- **Growth mindset language and routines:** Explicit reframing of errors as data for learning; reflection prompts on persistence and strategy; resources from *youcubed.org* (Boaler, 2019).
- **Relational pedagogy and care:** Consistent modeling of respect and trust, flexible office hours, and attention to students' well-being during a period of heightened stress.

Data Collection

Over the 15-week semester, students completed ten reflection logs on: growth mindset (Weeks 2, 8, and 15), problem-solving (Weeks 3 and 12), the learning environment (Weeks 4 and 11), their perceptions of the instructor (Weeks 5 and 14), and mathematics learning (Week 6). Each log began with several Likert-scale items (1 = disagree a lot to 6 = agree a lot), followed by a few open-ended questions.

For this study, we analyzed the logs on the *learning environment* and the *instructor*. In addition, five students participated in a Zoom focus group in mid-June (about one month after the course). The session, facilitated by the second author (not involved in the course), addressed:

- How did students perceive the instructor or the learning environment?
- What actions contributed to establishing a positive learning environment?
- What challenges were encountered and addressed?

Analytic Approach

Quantitative analysis. We conducted Wilcoxon signed-rank tests to compare ratings from Week 4 to Week 11 (learning environment) and from Week 5 to Week 14 (instructor).

Qualitative analysis. We used thematic analysis to identify and organize recurring ideas in responses to two open-ended prompts:

- How would you describe the learning environment of the synchronous classes so far? (Weeks 4, 11)
- The teacher says he wants to create a positive learning environment. Has he been successful? Why or why not? (Weeks 5, 14)

Themes were compared across time points and triangulated with focus group data to enhance trustworthiness.

Instructor Reflexivity

Given the practitioner-researcher role, the first author conducted a reflexive self-study (Lim, 2026) to surface beliefs, predispositions, and rationales underlying instructional decisions. This process included 50 personal journal entries and 16 Zoom-based dialogue sessions with a spiritual friend. The reflexive analysis informed interpretation of student data, particularly around trust-building, navigating tensions (breadth vs. depth; task demand vs. readiness; pacing), shifting perceptions of students (from deficit- to strength-based), and personal/professional growth.

Results – Student Reflection Logs

Table 1 shows the average student ratings (1 = disagree a lot to 6 = agree a lot) for three statements about the learning environment. Ratings increased for all items, with statistically significant gains in students' sense of respect and trust, and their feeling of inclusion.

Table 1: Comparison of Students' Ratings on the Learning Environment: Wilcoxon Signed-Rank Test (n = 22)

Item	Week 4	Week 11	W-Statistic	p-value	Significant? ($\alpha = 0.05$)
There is a high level of respect and trust in this class.	5.09	5.41	20.5	0.0254	Yes
I feel included within the learning environment.	5.05	5.36	10.0	0.0029	Yes
I feel comfortable making mistakes in this class.	4.36	4.68	24.0	0.1367	No

Written responses echoed these results. Nearly all students described the learning environment positively in Weeks 4 and 11, using words like *great*, *good*, and *comfortable*. While S14 and S19 had some reservations, both recognized the instructor's efforts. S14 appreciated that the instructor was "trying his best considering the circumstances," especially compared to previous asynchronous courses. S19 found the synchronous format "a little tough" but still noted that students had "a positive attitude and [were] willing to learn."

Between Weeks 4 and 11, students' focus shifted. In Week 4, ten students commented on structural features such as Zoom, iClickers, and breakout rooms, but only three did so in Week 11. Conversely, twelve students in Week 11 emphasized classroom atmosphere—describing it as comfortable, fun, or supportive—compared to five in Week 4. S21 wrote, "I enjoy coming to class and learn … the professor sometimes talks to us about life and gives me motivation to not give up." Three students in Week 11, up from one in Week 4, highlighted the sense of trust and respect. S7 noted, "I always say what it is I am thinking or am having trouble with."

Table 2 presents student ratings of the instructor across eight items. While most ratings stayed high, three showed statistically significant declines: creating a welcoming community, showing interest in students' thinking, and providing support. These declines may be attributed to the stress students experience near semester's end.

Table 2: Comparison of Students' Ratings about the Instructor: Wilcoxon Signed-Rank Test (n = 20)

Item	Week 5	Week 14	W-Statistic	p-value	Significant? ($\alpha = 0.05$)
The teacher creates a welcoming and inviting classroom community.	5.75	5.45	0.0	0.0143	Yes
The teacher challenges us intellectually.	5.65	5.70	6.0	0.6547	No
The teacher is interested in the way we think about math.	5.90	5.50	0.0	0.0047	Yes
The teacher understands our struggles in learning math.	5.15	4.85	34.5	0.2372	No

The teacher provides an appropriate level of support for us to overcome our struggles.	5.45	4.90	6.0	0.0045	Yes
The teacher uses growth mindset language to motivate us to remain persistent.	5.65	5.35	9.0	0.0833	No
The teacher accepts us for who we are.	5.60	5.55	6.0	0.6547	No
The teacher cares for our well-being.	5.60	5.55	6.0	0.6547	No
The teacher creates a welcoming and inviting classroom community.	5.75	5.45	0.0	0.0143	Yes

In open-ended responses, ten students in Week 5 and six in Week 14 emphasized the instructor's supportive, respectful, and student-centered approach. They appreciated the safe environment where they could ask questions, make mistakes, and stay engaged. S1 wrote that the instructor was "always open to all answers and questions," while S23 said, "he never puts us down" for mistakes. Two students in Week 5 and six in Week 14 mentioned the instructor's emphasis on kindness, citing examples like "show us a video or share a message to be kind" (S3) and "makes us feel cared for" (S6).

Interestingly, four students in Week 5—but none in Week 14—noted the instructor's focus on smiling. S5 wrote, "he is always encouraging us to smile," and S12 added, "he greets me with a smile and by my name." Finally, four students in Week 5 and six in Week 14 described how the class supported a growth mindset. Some reflected on their own development—like S3, who said, "I am being open to learn from my mistakes and not give up." Others appreciated the instructor's encouragement, as S7 shared: "even in situations where we didn't necessarily get the correct answer, he always encouraged us to see that this was still a learning experience."

Results – Focus Group

Students' Perceptions of the Instructor

Students consistently described the instructor as supportive, student-centered, and growth-oriented. They appreciated his encouragement and the way he pushed them to think deeply:

- "He motivated me to keep trying different ways … he would let us go over each other's thoughts" (S19)
- "Push you to make sure you understand it" (S16)

Several students found the instructor personally inspiring:

- "He kind of motivates me to be a teacher like him" (S16)
- "Inspired me to become a teacher" (S2)

His availability and flexibility were also frequently mentioned:

- "Every time I emailed him, he responded the same day" (S2)
- "Going to his office hours helped a lot, he was flexible with the timing" (S5)

- "He would change his office hours when needed, when they didn't work with my schedule" (S16)

Establishing a Positive Learning Environment

Students highlighted the instructor's emphasis on growth mindset as a key factor in shaping the classroom culture:

- "Making mistakes was seen as an opportunity to learn" (S19)
- "The teacher's emphasis on a growth mindset helped me think outside the box" (S16)

They also appreciated the encouragement to explore multiple problem-solving strategies: "He allowed us to explore new ideas and approaches. If we found a new way to solve a problem, he would explore that option and explain it to the class" (S3)

Collaborative learning was another positive feature. Students enjoyed working in groups and found it both engaging and effective:

- "I really like the group work that we did" (S19)
- "We did some really entertaining activities … especially when he put us in groups" (S3)

These group activities not only supported mathematical understanding but also fostered a sense of community. Tools like iClicker helped reduce anxiety around participation: "I like when he uses the clicker because it's anonymous" (S16)

Challenges Encountered and Addressed

Despite the overall positive environment, students noted several challenges. Maintaining focus during class was difficult at times due to personal stress:

- "You're in the class, but you're physically there while your mind is somewhere else" (S3)
- "Having a bad time … because of life in general" (S1)

To address this, the instructor incorporated motivational videos and maintained a consistently encouraging tone:

- "He would start with some videos … like to be kind" (S2)
- "He gave us a lot of positive reinforcement" (S16)

Students also found the original homework timeframes challenging: "The time frame for the homework was kind of hard" (S16). The instructor responded by adjusting deadlines and offering extensions.

Finally, students expressed mixed experiences with class time allocation. While some wanted more time for group work, others appreciated the instructor's efforts to prioritize it:

- "We don't have enough time for group work sometimes" (S19)
- "He wanted to give us extra time to work within our groups to really understand" (S16)

One student also described early frustrations with the LockDown Browser: "I honestly had a really bad relationship with the quizzes at the beginning" (S3). This concern was addressed when the instructor removed the LockDown Browser requirement for later assessments.

Discussion

Findings from the reflection logs and focus group point to the value of addressing students' cognitive and emotional needs in mathematics teacher education. Students generally described the instructor as supportive, responsive, and focused on helping them persist with challenging work. Although several instructor-related ratings decreased slightly toward the end of the semester—likely influenced by typical end-of-term stress—students' written comments continued to emphasize encouragement, flexibility, and care. The shift in student reflections from structural aspects of the course to classroom atmosphere suggests that the learning environment became increasingly relational over time.

The reflexive self-study (Lim, 2026) helps explain how these conditions were cultivated. First, the instructor's attention to *trust-building*—through authenticity, acknowledging mistakes, and modeling a growth mindset—aligned with students' descriptions of feeling safe to contribute and learn from errors. Second, the self-study documented *tensions in practice* (e.g., breadth vs. depth; cognitive demand vs. readiness; pacing within constraints). Decisions to prioritize depth, allow more time for group problem solving, and occasionally lower task demand to stay within students' zones of proximal development are consistent with students' appreciation for flexibility and the emphasis on understanding over speed. Third, the instructor's shifting perceptions of students—from a deficit to a strength-based lens—cohere with students' accounts of being encouraged, listened to, and supported. Finally, the self-study's theme of *personal and professional growth* (greater awareness of emotional responses and their impact) provides a plausible rationale for the steady use of kindness routines, flexible office hours, and motivational messages that students cited as helpful.

Taken together, the student data and self-study suggest that growth mindset messaging, opportunities for multiple problem-solving strategies, and collaborative routines contributed to a space where students felt more comfortable taking risks and learning from mistakes. Group work and tools such as iClicker supported participation and a sense of community, while flexible deadlines and adjustments to assessment procedures helped alleviate stress and technological challenges. The self-study adds context for these moves, indicating that they were not ad hoc but reflected ongoing reflection on trust, care, and the realities of student experience during the pandemic.

Overall, these converging sources suggest that relational and motivational teaching practices can play a meaningful role in supporting preservice teachers' engagement in mathematics, especially in online or hybrid contexts. By modeling empathy, responsiveness, and adaptability—practices the self-study framed as intentional and reflective—the instructor created conditions that supported students' mathematical engagement as well as their confidence as future educators.

Limitations

This study reflects the experiences of a single course taught by one instructor during an atypical period (the COVID-19 pandemic). The small sample size and reliance on self-report data limit generalizability. The focus group was conducted one month after the course, which may influence participants' recall. While the reflexive self-study (Lim, 2026) helps contextualize instructional choices, it is interpretive and situated; future work could triangulate instructor reflexivity with systematic classroom observations or peer debriefing.

Conclusion

This action study illustrates how intentional instructional design—emphasizing growth mindset, autonomy supports, collaborative problem solving, and relational care—can contribute to a positive learning environment for preservice teachers in mathematics. Students reported feeling increasingly included and trusted within the class and described the instructor as supportive, even when end-of-semester pressures affected some quantitative ratings. Humanizing practices in online settings, paired with high-cognitive-demand tasks, may help preservice teachers develop confidence, perseverance, and a broader sense of what it means to learn and teach mathematics. The reflexive self-study clarifies the reasoning behind these practices and underscores the ongoing work of balancing rigor with care.

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A LONGITUDINAL STUDY OF PRESERVICE MATHEMATICS TEACHERS' IDENTITY DEVELOPMENT IN HONG KONG

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This ongoing study examines preservice mathematics teachers' longitudinal identity development across prior mathematics learning, coursework in initial teacher education, and school-based practicum. It conceptualizes PSMTs' learning to teach as a process in which identities develop, and are projected, reconfigured, and negotiated across contexts. The study follows preservice mathematics teachers throughout a one-year teacher education program in Hong Kong, using a collective case study informed by narrative inquiry and ethnographic methods. Preliminary analysis of one case illustrates how earlier mathematics and student identities were projected into teacher education and practicum, and how these identities shaped the participant's emerging teacher identity. The case suggests that reform-oriented ideas were not simply rejected but were selectively taken up and reorganized across contexts.

Keywords: Identity and Agency, Preservice Teacher Education

Initial teacher education is expected to prepare preservice mathematics teachers (PSMTs) to enact reform-oriented teaching practices (Ma & Singer-Gabella, 2011; Nicol, 1998). However, many PSMTs still teach mathematics in traditional ways after beginning their careers, and the influence of teacher education is often "washed out" in schools (Zeichner & Tabachnick, 1981).

Accounting for this "wash out" effect requires tracing PSMTs' experiences from prior schooling through teacher education and into their teaching in schools. Before teacher education, an "apprenticeship of observation" during their schooling can produce durable images of teaching that favor existing practices and hinder engagement with reform-oriented alternatives (Lortie, 1975). Within teacher education, reform ideas may be understood only superficially when they conflict with PSMTs' prior beliefs (Frykholm, 1999) or remain disconnected from practice (Zeichner, 2010). During practicum and early employment, their enactment of reform-oriented teaching is further conditioned by mentors, colleagues, limited resources, or administrative expectations (Jong, 2016; Losano et al., 2018; Steele, 2001).

A growing body of evidence suggests that PSMTs' learning to teach is a complex, longitudinal process shaped by both prior and ongoing experiences, with implications for their future careers. Yet existing research has rarely adopted a holistic view that examines PSMTs' past and present experiences and the ways these experiences inform their learning to teach mathematics. The ongoing study aims to provide a comprehensive portrait of PSMTs' learning to teach over the long term. In this report, we present a preliminary analysis of the study on one case.

The study considers three contexts in which PSMTs develop as learners and teachers of mathematics: (a) prior mathematics learning contexts (i.e., schools and colleges), (b) courses within the teacher education program, and (c) school-based practicum placements. School-based practicum can approximate work contexts for PSMTs as teachers. Across these contexts, PSMTs participate in different activities, interact with different people, and position themselves and are positioned in different ways; accordingly, the identities they construct and enact may develop along distinct trajectories. Overall, this study addresses the following research question: *How do*

PSMTs' identities develop longitudinally across the contexts of prior mathematics learning, coursework in initial teacher education, and school-based practicum?

Theoretical Perspective

As this study examines PSMTs' learning to teach mathematics across contexts, identity provides a productive lens for linking individual learning and practice with the social contexts in which they are situated (Wenger, 1998, p. 145). We define identity as *socially situated ways of acting in a specific kind of practices* (Darragh, 2016; Darragh & Radovic, 2018). This definition draws from *communities of practice* (Lave & Wenger, 1991; Wenger, 1998), which delineates learning as participation and as being able, or unable, to act socially within a context (Bishop, 2012). Accordingly, we operationalize identity by examining observed actions, narrated actions, and participants' meanings, justifications, and reflections concerning these actions in relation to particular social practices (Darragh, 2016; Radovic et al., 2018; Skott, 2019).

To integrate identity development across contexts, this study focuses on PSMTs' longitudinal development of sub-identities related to learning to teach mathematics, namely, their *mathematics identity, student identity, and teacher identity* (Ma & Singer-Gabella, 2011). Each sub-identity is developed within particular social contexts, projected into other contexts, and continuously negotiated in relation to other sub-identities within those contexts.

Methodology

Three participants were recruited from a postgraduate teacher education program at a top university in Hong Kong with a stated commitment to preparing PSMTs to enact reform-oriented teaching practices. Each cohort of the program enrolls around twelve students. The PSMTs first took courses on the university campus from September to the following February and then participated in professional practicums in March and April in local schools.

We adopted a collective case study (Stake, 2000) informed by ethnographic methods (Creswell & Guetterman, 2019; Emerson et al., 2011; Spradley, 2016) and narrative inquiry (Connelly & Clandinin, 1990; Sfard & Prusak, 2005). The data collection was organized into three phases based on the structure of the teacher education program: at the early stage of the program, we interviewed the PSMTs about their past school experiences; during the coursework stage, we sat in on their teacher education classes, observed the classes, took fieldnotes without participating, and interviewed them about the classes; during practicum, we observed their teaching in practicum schools, took fieldnotes, and interviewed them about their teaching.

Preliminary data analysis in this report is grounded in qualitative content analysis, specifically deductive category assignment or structuring (Mayring, 2015). In this approach, theoretically informed categories are applied to empirical materials through a deductive, theory-guided coding process and revised through close engagement with the data. The initial category framework for this analysis was developed from the theoretical conceptualization of identity as a 3×3 matrix consisting of three contexts and three sub-identities (e.g., student identity in teacher education courses). Data segments were coded based on the operationalization of identity described above and categorized into the matrix for further interpretation.

Results

This section presents Bob's case to illustrate how the influence of teacher education was not simply washed out but filtered and reproduced through his identities. We organize the findings around Bob's sub-identities in the three contexts. These results illustrate how the current

conceptualization of identity can contribute to a fine-grained description of PSMTs' learning to teach mathematics.

Bob was a male PSMT in the program. He finished his schooling and obtained a Bachelor of Science in the Chinese Mainland. Before entering this teacher education program, he had taught for one year at an international secondary school in Hong Kong.

Prior Mathematics Learning

Bob described the mathematics instruction he experienced in school as a practice organized through "explain[ing] a concept," "show[ing] an example," and then doing "exercises and tests." At that stage, mathematics learning for him was "grounded in foundational skills," "mastered by extensive practice," "evidenced by long-term sustainment," and "regulated by the results." At university, although he tried to continue the high-school drilling model in tertiary mathematics courses initially, he found it infeasible because of the increased volume of content. To find a more effective way of doing exercises for learning mathematics, he turned to exam-oriented online courses and AI-facilitated self-study.

This emphasis on hard work and exercises reflects Bob's student identity, which is organized around the strategic allocation of effort toward what counts in the local accountability system. He strategically gave up learning arts subjects by not attending to teaching but doing his own work, judging them to be lower priority relative to his schooling goal of earning good grades in science subjects. However, his teachers tolerated him because he continued learning and did not disrupt the class. This practice continued in his university learning. In university courses, he usually learned by himself through online resources instead of attending in-person classes.

Teacher Education Courses

In teacher education courses, Bob did not simply reject the alternative ways of doing mathematics that he encountered. Rather, course ideas were filtered through his existing mathematics identity. When discussing teaching the perfect square formula with multiple representations (e.g., the sum of the areas of squares or the sum of the products of items in the table), he described it as interesting but said that he would not put it into his own mathematical system. He also said, "I would prefer my students to use algebraic thinking when dealing with algebraic problems." This suggests that alternative mathematical representations were recognized but not incorporated into what counted for him as legitimate mathematics.

Bob's participation in coursework continued to reflect his efficiency-oriented student identity. In the teacher education program, he prioritized obtaining the teaching certificate and did not engage much in classes. He sometimes looked up at the screen and took notes or photos of slides, especially when teachers introduced methods that seemed readily applicable, including flipped classroom approaches and assessment design. He explained that when he heard "an interesting idea" or "a useful website," he would "note it down." Although the participation form changed from high school to university, the efficiency-oriented practice logic was preserved.

Bob's teacher identity was largely informed by his mathematics identities at this stage. He rejected the use of calculators in his classrooms, saying, "there is no calculator… everything is calculated on paper." He believed that using calculators would undermine students' foundational skills. When asked about inquiry-based teaching, Bob positioned it as incompatible with exam-driven contexts, because he "will not have enough time to finish the lesson." He also claimed that finishing a lesson is not only a matter of going through the textbook but also of "cover[ing] all the type[s] of practice … there are so many types of necessary practice." He remained skeptical of "innovation in teaching," viewing it as a risk to instructional quality. He said that "many theories in the program cannot fit into the real classrooms" and explicitly stated that his

way of lecturing is "better than messing up during teaching." For him, lecturing could "guarantee teaching quality" and help students develop foundational mathematical competence.

School Practicum

In practicum, Bob's mathematics identity and teacher identity were simultaneously enacted through a procedural orientation to mathematics practices. He treated mathematics practices as something to be retained, recalled, and applied through correct procedures. He once framed a task as "testing your memory" and told students, "If you forget anything, check your notes." When solving an equation involving a square root with one unknown, he identified the keyword as "separate" and then demonstrated a standard solution procedure. In tutoring a student on this task, he said that "you put the root on one side" and "you need to square both sides." At the same time, this procedural orientation was not merely mechanical. He highlighted the different procedures for the equations that requires "separate" or not. His mathematics teacher identity was enacted as a focus on students' accurate replication and flexible adaptation of procedures; he asked students to remember procedures with notes, repair inaccurate algebraic manipulations, and adapt procedures to the structure of the equation.

Bob's practicum teaching was also shaped by his student identity as a practicum student accountable to mentors and university supervisors. During the practicum, he adjusted his lessons to make his teaching more visibly interactive for his mentor and university supervisors. After being reminded by his mentor, he asked students to solve questions on the board to create more interaction but found that this slowed his progress. When a university supervisor came to observe, he drew on an inquiry-oriented approach promoted in teacher education but transformed it into a highly structured task that could be easily monitored and time-controlled in his classroom. This suggests that reform-oriented practices were not absent in practicum, but were reconfigured into visible, controllable forms whose affordances for supporting student learning were limited as he reflected.

Discussion and Conclusion

This preliminary case suggests that Bob's identities developed longitudinally through the constitution, projection, reconfiguration, and negotiation of earlier identities across contexts. His mathematics identity authorized foundational fluency, sufficient practice, procedural accuracy, and question-type coverage as legitimate mathematical practices. This identity was projected from prior mathematics learning into teacher education courses and practicum, where it shaped how he interpreted course ideas and enacted teaching. His student identity also developed across contexts, from strategic effort allocation in schooling to selective engagement in teacher education and accountability to mentors and supervisors in practicum. His teacher identity then emerged through negotiating these mathematics and student identities during teaching.

This case also suggests an alternative framing of the metaphor of "wash out." In Bob's case, teacher education ideas did not simply go away. Rather, they were selectively taken up and transformed. Reform-oriented ideas appeared in his practicum teaching, but they were reorganized around clarity, time control, progress, procedural coverage, and external evaluation. Thus, the critical issue to consider in teacher education is not only whether it has an influence in PSMTs, but also how that influence is recontextualized through PSMTs' existing identities.

Therefore, teacher education should not only present reform-oriented ideas as a set of theories or strategies that PSMTs can later implement, but also engage PSMTs in examining and reconstructing their mathematics and student identities through which they interpret teaching and cultivate them through reform-oriented ways of doing mathematics and learning to teach mathematics in teacher education classrooms.

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LEARNING TO SEE AND RESPOND: PRACTICE-BASED PREPARATION FOR EQUITABLE MATHEMATICS INTERVENTION

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Elementary students experiencing mathematics anxiety, emerging bilingual (EB) status, and/or specific learning disabilities (SLD) are frequently served through fragmented interventions. This pilot study examines a cross-disciplinary, practice-based learning sequence in which elementary mathematics preservice teachers (PSTs) and school psychology graduate students (SPSY) collaboratively design mathematics interventions in general education settings. Grounded in practice-based teacher education, MTSS and UDL, affective perspectives, and DisCrit, the study centers early number concepts, including fractions as numbers. Participants analyze case scenarios, design Tier 1 and Tier 2 supports, rehearse instructional and consultation moves, and engage in structured reflection. Analysis examines shifts in instructional coherence, integration of affective and linguistic supports, and equity-oriented reasoning.

Keywords: Equity, Inclusion, and Diversity; Number Concepts and Operations; Preservice Teacher Education; Students with Disabilities.

Introduction

Elementary students experiencing math anxiety, emerging bilingual (EB) status, and/or specific learning disabilities (SLD) are frequently served through fragmented or pull-out interventions that are disconnected from core mathematics instruction. These structures shape not only students' access to mathematical content but also their mathematical identities and classroom positioning. Within multi-tiered systems of support (MTSS), Tier 1 and Tier 2 instructional responses are intended to occur within the general education context. Yet novice teachers often report limited preparation for designing and enacting instructional responses that attend simultaneously to cognitive, affective, and linguistic dimensions of mathematics learning—particularly for students positioned at the intersection of EB status and SLD.

Research identifies early number concepts, including fractions as numbers, as critical leverage points for long-term mathematics development. However, preparation programs often emphasize identification and referral over instructional decision-making, leaving PSTs with limited experience interpreting assessment data and designing responsive instruction within core mathematics teaching.

School psychologists support teachers in interpreting assessment data, yet their work is often focused on special education processes rather than Tier 1 and Tier 2 consultation. Teachers report difficulty translating psychoeducational reports into actionable classroom practices (Burns et al., 2020). This creates an opportunity for interdisciplinary preparation that strengthens consultation and instructional coherence across roles.

This pilot study examines a cross-disciplinary, practice-based learning sequence in which PSTs and SPSY collaboratively analyze case scenarios and design mathematics interventions grounded in early number concepts within general education contexts.

Purpose of Study

The purpose of this study is to examine how participation in this structured, interdisciplinary learning sequence shapes preservice teachers' conceptualizations and enactment of mathematics

intervention in general education classrooms for students experiencing mathematics anxiety and/or specific learning disabilities, with attention to emerging bilingual learners. The study also investigates how engagement in case-based collaboration supports school psychology graduate students' development of consultation practices and instructional alignment. Finally, it explores how interdisciplinary collaboration surfaces instructional and assessment considerations that can inform the design of coursework in mathematics teacher education and school psychology preparation programs.

Research Questions

1. How do PSTs conceptualize mathematics intervention in general education context, for students with math anxiety, EB status, and/or SLD, before and after participation in a practice-based interdisciplinary learning sequence?
2. How do school psychology graduate students' reflections inform the identification of skill development needs to support teachers' implementation of math interventions in general education settings?
3. How does interdisciplinary case-based collaboration surface instructional and assessment considerations that can inform mathematics teacher education and school psychology preparation?

Theoretical Framework

This study integrates four complementary perspectives, each defined below.

Practice-Based Teacher Education

While student teaching provides authentic experience, it does not guarantee systematic engagement with specific instructional problems, particularly intervention design (Hammerness et al., 2005). Practice-based approaches position approximations of practice, rehearsals, and representations of practice as central to learning to teach (Grossman et al., 2009). Structured approximations allow candidates to engage in complex decision-making within supported contexts.

Multi-Tiered Systems of Support (MTSS) and Universal Design for Learning (UDL)

MTSS frames intervention as layered support within general education systems, while UDL emphasizes proactive design responsive to learner variability, including linguistic and affective dimensions (CAST, 2024; Center on Multi-Tiered Systems of Support, 2026). Together, these frameworks situate intervention as instructional design within tiered, inclusive systems.

Affective Dimensions of Mathematics Learning

While attitudes to mathematics, even negative attitudes, cannot be equated with mathematics anxiety alone, people who think they are "bad at mathematics" are more likely to experience anxiety when engaging in mathematical tasks (Hoffman, 2010). For the purposes of this study, we adopt Richardson and Suinn (1972) definition of mathematics anxiety as a "feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in…ordinary life and academic situations."

Research in mathematics anxiety demonstrates the impact of affect on working memory (Caviola et al., 2012), persistence and task avoidance (Ashcraft and Rudig, 2012), and the development of mathematical identity (Dowker et al., 2016). Anxiety can tax cognitive resources needed for strategy selection and flexible reasoning, particularly during tasks involving number sense and early quantitative relationships.

Importantly, mathematics anxiety should not be conceptualized solely as an intrapersonal deficit. Students' affective positioning in mathematics classrooms is shaped by instructional

histories, classroom discourse norms, and narratives about ability. Thus, affect operates within sociocultural and institutional contexts, not merely within the individual learner.

Addressing intervention without attention to affective positioning risks reinforcing deficit narratives and avoidance patterns. For example, interventions emphasizing speed, error correction, or isolated skill practice may unintentionally intensify anxiety and confirm students' beliefs that mathematical competence is fixed. In contrast, interventions that foreground conceptual understanding, strategy flexibility, and opportunities for sense-making may support both cognitive growth and more fluid mathematical identities.

Within teacher preparation, PSTs often receive limited support in recognizing the affective dimension of students' mathematical engagement or in designing responses that attend to both cognitive and emotional factors. Through the disciplinary collaboration of SPSY graduate students and elementary mathematics PSTs, this study positions affect as central to instructional decision making.

Sociocultural and Equity-Oriented Perspectives

From this perspective, EB status and dis/ability are not viewed solely as individual deficits but as positioned identities shaped within instructional systems. Attention to language, representation, and participation structures is central to equitable mathematics teaching. One lens that helps to push back on ableism and racism within education systems is DisCrit. DisCrit incorporates a dual analysis of race and ability by drawing on concepts from both Disability Studies and Critical Race Theory (Annamma et al., 2013). As noted by Annamma et al. (2013), "the legacy of historical beliefs about race and ability, which were clearly based on white supremacy, has become intertwined in complex ways that carry into the present day" (p. 2); this is true of all systems, including educational systems. DisCrit provides a lens for examining how race, language, and dis/ability are socially constructed within educational systems and shape students' positioning in mathematics (Annamma et al., 2013). Utilizing Sociocultural and equity-oriented perspectives with the DisCrit position intervention as instructional design work situated within equity-oriented general education practice.

School psychologists and teachers are in a unique position to cultivate opportunities for all students. Through holding this framework, we can encourage SPSY and PSTs to advocate for all students and disrupt systemic oppression. Additionally, by drawing on DisCrit, we examine how SPSY and PSTs reason about mathematics anxiety, EB, and SLD within MTSS structures. This framework allows us to analyze not only the interventions proposed but also how participants position students' competence, agency, and access to mathematics learning within tiered systems of support.

Together, these frameworks function in a complementary manner to guide both the design and analysis of this study. Practice-based teacher education provides the pedagogical structure for approximating intervention design; MTSS and UDL situate that work within tiered, inclusive systems; affective perspectives foreground the cognitive and identity-related consequences of intervention decisions; and DisCrit interrogates how race, language, and dis/ability shape students' positioning within those systems. In combination, these lenses allow us to examine not only what interventions candidates propose, but how they reason about access, competence, and equity within general education mathematics contexts.

Methodology

Design

This pilot study uses a qualitative-dominant mixed-method design. The intervention will be implemented in Spring 2026 across two courses: Elementary Math Methods (PSTs) and School

Psychology Practicum/Intervention (SPSY graduate students). Participants include elementary PSTs in a credential program and second-year SPSY graduate students enrolled at a public university in California.

Intervention: Practice-based Learning Sequence

The five-phase sequence includes foundational instruction on math anxiety, SLD, EB-responsive practices, and early number concepts; structured case analysis of students positioned at intersections of EB, SLD, and anxiety; collaborative design of Tier 1 and Tier 2 interventions; rehearsal of instructional and consultation moves; and structured reflection.

Data Sources and Analytic Approach

All data will be collected in Spring 2026, with analysis occurring in Summer 2026 and preliminary findings presented at PME-NA Fall 2026. Data sources include pre/post intervention plans, written case analyses, participant reflections, consultation artifacts, and rubric-based ratings of intervention quality.

Analysis will employ a qualitative-dominant approach using a hybrid deductive–inductive coding process. Initial codes will be informed by constructs from the theoretical framework (e.g., instructional coherence, integration of affective and linguistic supports, equity-oriented reasoning), while allowing for emergent themes across artifacts and interdisciplinary dialogue. Pre/post comparisons will examine shifts in how PSTs and SPSY candidates conceptualize intervention within general education contexts. Cross-case analysis will identify patterns in consultation language, alignment between assessment and instruction, and positioning of students' competence and agency. To enhance analytic trustworthiness, multiple researchers will code a subset of artifacts and engage in consensus discussions, supported by analytic memos documenting interpretive decisions.

Anticipated Contributions

This pilot study contributes to the PME-NA scholarship in four primary ways. First, it extends research on preparing PSTs to design and enact mathematics interventions within general education classrooms by foregrounding instructional responses rather than referral processes. The study examines how structured approximations of practice can support intervention reasoning grounded in core mathematical ideas.

Second, the study contributes to understanding how PSTs and SPSY graduate students reason about intersectionality in mathematics learning. By analyzing case-based decision making, the study illuminates how candidates attend to math anxiety, EB, and SLDs in relation to conceptual mathematical development.

Third, the project adds to the literature on practice-based approaches for supporting early number concepts by examining how interdisciplinary teams analyze student thinking and select evidence-based intervention strategies grounded in core mathematical ideas.

Finally, it contributes to equity-oriented teacher preparation research within PME-NA by investigating how cross-disciplinary collaboration can support more coherent, inclusive, and instructionally grounded approaches to Tier 1 and Tier 2 mathematics intervention.

As a pilot study, implementation will occur in Spring 2026. Findings from the initial implementation–including analyses of participant artifacts and interdisciplinary collaboration–will be presented at PME-NA to inform mathematics teacher educators and school psychology faculty interested in integration assessment, consultation, and instructional design within preparation programs.

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USING CASES TO PREPARE FOR EQUITY-RELATED CLASSROOM ENCOUNTERS: CREATIVE INSUBORDINATION THEMES FROM PRESERVICE ELEMENTARY TEACHERS

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Preparing preservice teachers to respond to equity-related issues in their future classrooms can be a challenge for teacher educators (e.g., Chang-Bacon, 2022). Case-based scenarios, or cases, provide rich opportunities for preservice teachers to grapple with these issues. This research uses the lens of creative insubordination (Gutiérrez, 2016) to examine preservice teachers' responses to one such case, which illustrates inequitable time distribution for math and science across school districts. Written data from six semesters of elementary math methods courses were analyzed to illustrate how creative insubordination strategies emerged in preservice teachers' responses to the case. Findings from this analysis offer insights on which strategies preservice teachers considered more readily and invite discussion on how teacher educators may support preservice teachers in considering less frequently cited strategies.

Keywords: Preservice Teacher Education; Teacher Educators; Equity, Inclusion, and Diversity

Study Purpose

The National Council of Teachers of Mathematics (NCTM) has called for a mathematics education culture of equity and access, where math teachers enact culturally responsive and equity-focused pedagogy (NCTM, 2014). In response to this call, our team has worked with local teachers to develop case-based scenarios the depict authentic challenges they have encountered related to teaching math equitably. The cases are designed to be used in math methods classes to support preservice teachers (PSTs) to think through equity concerns prior to encountering them in their own classrooms. This study focuses on the Daily Schedule case (see Figure 1), considering the research question: How do themes of creative insubordination emerge in PST responses to the Daily Schedule case?

Perspectives

Cases have been used in teacher education to support PSTs in examining critical tensions (e.g., Arellano et al., 2001). Our cases follow the traditional narrative form with details that engage the PST reader in critically considering multiple interpretations (Shulman, 1992). Concerns of authenticity and deep reflection in case design (Bencze et al., 2001) were addressed in our cases by partnering with local teachers to ensure relevant scenarios, and an in-class implementation process that invites PSTs to first read and reflect on the case independently before engaging in small group discussions and a whole-group debrief of the case.

Authoring teachers were invited to consider one or more feature of culturally grounded pedagogy (CGP) in designing their case. CGP integrates the work of multiple equity-focused scholars (Aguirre et al, 2013; Banks, 2019; Gay, 2002; Ladson-Billings, 1995, 2014; Paris, 2012) and includes the following four features: 1) leverage students' multiple strengths and competencies, 2) honor students' lived experiences, 3) dismantle power and status hierarchies, 4) diversify curriculum and challenge dominant perspectives (Dobie et al., 2024). The teachers authoring the Daily Schedule designed the case with a focus on CGP feature 4.

The final prompt in the Daily Schedule case invites PSTs to consider strategies for advocating for their students, given inequities in time distribution of subject areas in their Title I school with a high multilingual population. This call to action solicited a range of strategies that align with the ideas of creative insubordination (CI) (Gutiérrez, 2016) which emerged from mathematics teachers' experiences addressing political situations. In the analysis that follows we matched PSTs' responses to the case with each of Gutiérrez's six strategies: press for explanation, counter with evidence, use the master's tools, seek allies, turn a rational issue into a moral one, and fly under the radar. This analysis offers insights into strategies that PSTs may be more comfortable implementing and strategies that may require additional support.

You are a first-grade teacher, newly hired to a Title 1 District with a high ELL population and a heavy focus on literacy. You've learned that about 60% of the students were below benchmark at the end of last year in literacy. The month before school begins you receive your daily schedule for the upcoming school year. You notice that you are only allotted 30 minutes for science twice a week, and 45 minutes a day for math. In comparison, there are 3 hours each day allotted for literacy (including phonics, intervention time, fluency, and reading comprehension). You're talking with a friend from your teacher prep program who got a job at a school in the neighboring district (non-Title 1, majority White, low ELL population), and they tell you that they are allotted an hour and a half of math and 1 hour of science per day.

After learning about this disparity between the two districts, you decide to ask someone about your schedule. Your instructional coach is your primary support for teaching and learning across all subject areas, so you approach them to ask why you don't have more time to teach math and science. Your coach says, "That's not what these kids need." When elaborating, they added that it would be nice to have more time for math and science, but last year's reading scores indicated a need to increase the focus on literacy.

Please discuss the following questions:

1. What do you notice happening in this scenario? What initial reactions and feelings come up for you?
2. What's the value in your coach's perspective in terms of supporting student learning and engagement? What are the drawbacks of that perspective for student learning and engagement?
3. What is your own personal perspective, and what experiences have you had or ideas have you encountered to support that perspective?
4. Given the circumstances, how do you think you can best advocate for your students' math and science learning moving forward?
    a. What information would you want to gather and what research might be helpful to know about to inform your advocacy? (e.g., I would want to learn more about… It would be helpful to know if researchers have studied…)

Figure 1: Daily Schedule Case

Methods

Written response data was collected from 107 PSTs across six semesters of elementary math methods classes between 2021–2025 in a university teacher education program in the mountain west. PSTs engaged with the case by a) individually reading the case and writing responses to case prompts, b) engaging in small group discussions, c) participating in a whole class discussion, and d) completing written exit tickets reflecting on experiences engaging with the case. This analysis focuses on PSTs' written responses to case prompts, which are referenced in full at the end of the case (see Figure 1). Written responses were chosen as the unit of analysis because they reflected PSTs' initial strategies prior to input from peers or the instructor.

The primary author read through the data set in its entirety and determined that responses to prompt 4 were most aligned with strategies for CI and selected these responses for coding. Two coders coded 10% of the data for the six CI strategies to calculate inter-coder agreement, finding a high level of agreement (Krippendorff's alpha = .82) The primary author coded the remainder of the data. Altogether, 168 strategies described in PST responses were coded.

Results

Analysis showed differing frequencies in the strategies described in response to this case. *Countering with evidence* and *using the master's tools* appeared with high levels of frequency, with *flying under the radar* and *pressing for explanation* occurring least often and *seeking allies* and *turning a rational issue to a moral one* having middling frequency. Highlighted below are additional details on these patterns, along with their potential significance for preparing PSTs to advocate for students.

Frequently Cited Strategies

High frequency in the countering with evidence strategy (n = 82) was unsurprising as our case prompt solicited gathering evidence: "What information would you want to gather… to inform your advocacy?" Included in this count were any responses wherein PSTs described sharing research or classroom data that proved reduced time for mathematics would not be an effective solution for improving literacy scores. Some responses acknowledged the importance of both classroom and external data, as one PST wrote, "Due to the fact that the data coming from my classroom would be pretty short-term as far as how long the research has been being done, I would want to present studies that support my point that have shown positive results coming from a study done over several months or years."

The other most commonly described strategy from our PSTs was using the master's tools (n = 53). This code was applied when PSTs justified doing what was in students' best interest using language that might be valued by administrators or stakeholders. Overwhelmingly, PST responses under this category involved using integration as a way of getting around the scheduling issue. One PST described this as a "loophole," writing, "I would try my hardest to integrate it into my literacy, that way there are still mathematical and scientific elements in everyday lessons. Sort of as a loophole to the system."

Infrequent Strategies

Given that a press for explanation is already built into the case ("you approach [your instructional coach] to ask why you don't have more time to teach math and science"), it was unsurprising to see a lower number of responses describing pressing for explanation (n = 6), which we defined as asking an authority figure to explain the time constraints in the daily schedule. The few responses that included an additional press for explanation typically involved a general statement about the types of questions a preservice teacher would be interested in

asking about the time distribution decisions, for example, "I would want to know if they had discussed alternative solutions and what those solutions were."

We expect that flying under the radar may have been less frequently described (n = 3) by our PSTs given to the extra supervision that new teachers receive. In our coding scheme, we defined this strategy as PSTs describing acting in opposition to the provided schedule without drawing attention. This is illustrated in one PST response which read, "I may fudge a couple things to incorporate more time for science and math, but if I get questioned I will share what literacy things I am doing [during] those times."

Strategies for Future Emphasis

PST responses that described discussing concerns with colleagues were coded under seeking allies (n = 9). This definition is not an exact match with Gutiérrez's (2016) description as PST responses did not differentiate allies from other colleagues. Thus, we expanded this code to include collaboration or discussion with any colleagues. For example, one PST wrote, "I can best advocate for math and science for my students by discussing their importance just as equally as literacy to other teachers, colleagues, coaches, the principal, etc." This grouping of all colleagues together when advocating or seeking support signals an opportunity to further discuss finding critical friends and seeking support from more experienced educators with PSTs (Lord, 1994).

A second strategy warrants further attention is turning a rational issue into a moral one (n = 15). While Gutiérrez (2016) characterizes this strategy as an appeal to the heroism of stakeholders, we broadened this definition to include any PST responses that included a moral claim, even if it did not extend to advocacy with an outside party. One PST explained, "I would talk to my principal and get in contact with the school board and district about the restriction this causes for students to achieve social mobility." Other responses in this category did not necessarily extend the moral accountability to a specific stakeholder which may suggest that not all PSTs not realize the ways in which they can publicly advocate for students.

Discussion

This research highlights the valuable role that cases can play in supporting PSTs to develop their understanding of CGP and developing advocacy strategies. Written responses to case prompts offer a window into the ways in which PSTs feel more comfortable engaging in creative insubordination, as well as strategies that might benefit from more attention in the future. Thus, PST educators can use cases targeted to local concerns as a formative tool to determine where PSTs may need support in understanding their options for advocacy. To encourage these efforts, our research team is working to develop design principles that researchers and teacher educators can use in developing their own cases that prompt rich discussion and grappling with relevant teaching concerns.

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PATTERNS IN PROSPECTIVE TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE WHILE CRITIQUING AI-GENERATED GEOMETRIC JUSTIFICATIONS

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This paper reports preliminary findings related to prospective elementary teachers' (PTs) mathematical content knowledge from a larger qualitative study examining how PTs critique AI-generated geometric justifications. Twelve PTs participated in the study by submitting written critiques of five AI-generated geometric justifications. Through thematic analysis of the PTs' written critiques, we analyzed patterns in PTs' content-dependent reasoning, capturing instances in which PTs' evaluation accuracy was shaped by their understanding of geometric definitions, properties of two- and three-dimensional shapes, and the hierarchical relationships among figures. We identified three recurring patterns: conceptual gaps in distinguishing cones and pyramids, partial understanding of the relationship between kites and rhombi, and limited awareness of inclusive versus exclusive definitions.

Keywords: Elementary Prospective Teachers, Geometry, Justification, Content Knowledge

Recent advances in generative artificial intelligence (AI) have introduced new opportunities and challenges in mathematics teacher education (Son et al., 2024; Yoon et al., 2024). When AI-generated explanations are used as instructional objects, prospective teachers (PTs) are positioned not only as learners but also as evaluators of mathematical reasoning (Klawa et al., 2025; Uygun et al., 2025). This evaluative stance requires activating mathematical content knowledge, particularly in geometry, where definitions, classifications, and hierarchical relationships play a central role in justification. Building on the literature, this paper focuses answering the research question: *How does PTs' mathematical content knowledge influence their critiques of AI-generated geometric justifications?*

Conceptual Framework: Justification Rubric

In this study, we adopted Melhuish et al.'s (2020) definition of justification as using definitions or properties to support claims and explain why they hold. We specifically focused on PTs' critiques of geometric justifications where spatial thinking and logical argumentation intersect to support formal justification (Stylianides et al., 2016). We adapted Brunheira and Ponte's (2019) conceptual framework, which builds on the organization of levels of justifications (see Figure 1).

This framework describes a progression in the quality of geometric justification, focusing on how claims are supported and generalized. At the most robust level, justifications rely on established geometric ideas and explicitly connect those ideas to the claim in a way that explains why it holds beyond a single case. At intermediate levels, reasoning may draw on appropriate geometric properties but remains incomplete, either because key steps are missing or because the argument is tied to a specific example rather than extended to a broader class of figures. Less developed responses attempt to justify claims using inappropriate or loosely connected ideas, revealing breaks in logical coherence or dependence on intuition instead of warranted geometric reasoning. At the lowest level, explanations do not engage with geometric

reasoning at all, instead depending on measurements or calculations without making conceptual connections to underlying geometric structures.

Level	Description	Evidence Indicators	Points
Level 3	**Correct geometric justification**: Argument uses relevant, established geometric properties to explain *why* the claim is true. Applies to a general class of figures, often using a generic example.	- Correct use of definitions, theorems, or properties. - Clear logical sequence that works for all relevant cases. - Language indicates generality ("for all…", "whenever…").	3
Level 2	**Incomplete (Partially Correct) justification**: Uses some relevant properties but omits key steps or assumptions. Reasoning may be tied to a single figure even if it hints at generality.	- Some correct reasoning but missing one or more critical information. - May rely on a specific geometric figure or property without clarifying applicability to all cases.	2
Level 1	**Incorrect justification:** Uses irrelevant or unestablished properties to justify the claim. The conclusion may be correct by coincidence, but reasoning is flawed.	- Misuse of definitions or properties. - Logical gaps that make the justification invalid. - Relies on intuition rather than evidence or theorem.	1
Level 0	**No geometric justification**: Relies solely on measurement, numerical computation, figures without connecting to geometric properties.	- "It works because I measured it" / "the textbook says so." - No explicit geometric figures or properties mentioned or explained.	0

Figure 1: Levels of Geometric Justification (Adapted from Brunheira & Ponte, 2019)

As shown in the adapted framework, three fundamental elements of justification are emphasized when PTs used this rubric to evaluate AI-generated justifications: the use of definitions, the logical sequencing of claims, and the generalizability of the argument. The rubric guided PTs' evaluations of the AI-generated justifications and informed the initial data analysis and coding process by focusing on how PTs referenced these elements in their critiques, particularly in relation to their mathematical content knowledge.

Research Setting and Participants

This pilot study involved twelve PTs enrolled in a mathematics content course at a public university in the southeastern United States. The data come from a larger qualitative study examining how PTs critique AI-generated geometric justifications across two universities. Although 50 PTs returned signed consent forms from two institutions, this paper reports only on the pilot data from the twelve participants. All names used in this paper are pseudonyms.

Materials, Methods, Sample Task, and Data Collection

This exploratory qualitative study (Kalu & Bwalya, 2017) examined twelve PTs' written evaluations of AI-generated geometric justifications. Using the rubric (Figure 1), PTs assigned a Level 0–3 score to each AI-generated response and cited evidence to provide their evaluations by attending to definitions, geometric properties, logical sequencing, and the generalizability of the arguments presented in the AI responses. An example task is provided below (see Figure 2).

Statement: **A cone is a special type of pyramid.**

ChatGPT-generated response: Yes, a cone is like a pyramid because both have one base and come to a single point at the top, called the apex. For example, if you think about a square pyramid and a cone, they both narrow as you go up. The only difference is that a cone has a circular base instead of a polygon base. So, a cone can be seen as a pyramid with a round base.

Assigned Level:
Evidence to Assigned Level:
Suggested Improvements (if any):

Figure 2: Sample Task

In this sample task, the appropriate evaluation is a Level 1, as the AI-generated response is incorrect. A cone is not a polyhedron because it has a curved surface and a circular base, which is not a polygon. A cone does not satisfy the defining properties of a pyramid. The task design positioned PTs as critics, encouraging them to think critically, question automated reasoning, and develop confidence in explaining their ideas. Researchers/instructors of the geometry content courses developed a worksheet consisting of five such tasks, including the example shown in Figure 2. PTs completed the worksheet as a homework assignment and as part of the course requirements.

Data Analysis and Example Students' Written Critiques

The data set consisted of rubric-scored written critiques from twelve PTs, collected through a homework worksheet in which participants rated AI-generated justifications. Our analysis focused on how prospective teachers (PTs) engaged with geometric definitions, properties, and hierarchical relationships when critiquing AI-generated justifications. We conceptualized each AI-generated response and the corresponding PT critique as a *content-evaluation excerpt*, in which PTs were required to interpret the mathematical validity of a claim, assign a justification level using the rubric, and provide written evidence to support their evaluation (Figure 2). These excerpts served as analytic units for examining how PTs' mathematical content knowledge shaped their evaluative judgments. The criterion for selecting written excerpts was that they illustrated variation in how PTs attended to core geometric ideas when evaluating AI-generated reasoning.

Using a thematic analysis approach (Xu & Zammit, 2020), data analysis proceeded in four steps: (a) identifying evidence of content-dependent reasoning within PTs' written critiques, (b) using the justification rubric (Figure 1) to code how PTs referenced definitions, properties, and generalizability when evaluating AI responses, (c) comparing rubric scores with the mathematical accuracy of PTs' written justifications to identify content-related inconsistencies, and (d) inductively developing themes that captured recurring patterns in PTs' mathematical content knowledge across tasks. While additional analyses are ongoing to examine how PTs' critique practices develop over time and across instructional contexts, this paper focuses on illustrating one analytic theme from the data: emergent patterns in mathematical content knowledge as revealed through PTs' written critiques of AI-generated geometric justifications. To foreground this theme, we selected illustrative excerpts from different PTs that highlight variation in how content knowledge shaped evaluative decisions.

Emergent Content Knowledge Patterns Within PTs' Written Work

In this theme, we display initial patterns in PTs' mathematical content knowledge as evidenced through their written critiques of AI-generated geometric justifications. Our analysis focuses on how PTs engaged with core geometric ideas—such as definitions, properties, and

hierarchical relationships—when evaluating the validity of AI-produced reasoning. By examining where PTs accepted, partially rejected, or accurately critiqued AI explanations, this analysis reveals how their content knowledge shaped evaluative judgments. This theme encompasses patterns where PTs' accuracy depended on their understanding of geometric concepts. Errors stem from gaps in content knowledge rather than misinterpretation of rubric criteria. Three recurring patterns emerged within the theme: Conceptual gaps in distinguishing cones and pyramids, partial understanding of kite–rhombus relationships, and limited awareness of inclusive versus exclusive definitions of trapezoid. These patterns illuminate both developing aspects of PTs' geometric understanding (see Table 1).

Table 1: Patterns in Content Knowledge and PTs' Written Work

Patterns in Content Knowledge	Illustrative Written Work:
Conceptual Gaps in Cone-Pyramid Distinction. Reflecting PTs' difficulty distinguishing between cones and pyramids based on formal geometric definitions.	**Student 1:** *"ChatGPT's response correctly explains similarities between cones and pyramids and distinguishes the circular base as the only difference."* This excerpt illustrates a conceptual gap regarding the defining property of pyramids as polyhedral. In contrast, a PT with stronger content knowledge rejected the AI reasoning: **Student 11:** *"I would give level 1 because the response is wrong. My evidence is that a shape with a curve will never be a polygon. Since pyramids are made of flat polygonal faces, a cone cannot be a type of pyramid.*
Partial Understanding of Kite-Rhombus Relationship. Capturing variability in PTs' understanding of hierarchical relationships among quadrilaterals.	**Student 2:** *"Not every kite is a rhombus. Some only require two adjacent sides to be equal, while rhombuses require four."* Although directionally correct, this explanation omits that kites require two pairs of adjacent congruent sides. In contrast, a more precise critique stated: **Student 7:** *"ChatGPT's response level is 2, which is partially correct. Because a kite has two pairs of adjacent equal sides, while a rhombus has all four sides equal. Therefore, every rhombus is a kite, but not every kite is a rhombus."*
Limited Awareness of Inclusive vs Exclusive Definition of Trapezoid. Acknowledging the existence of inclusive versus exclusive definitions when critiquing an AI-generated statement about trapezoids.	**Student 6:** *"ChatGPT's response level is 2, which is partially correct. Looking over the definitions, I can say that the statement depends on the definition being used. For example, If trapezoids are defined as having at least one pair of parallel sides, then the claim is false because parallelograms are also trapezoids."* Only two PTs explicitly referenced definitional conventions—specifically, the inclusive and exclusive definitions of trapezoids—when critiquing the AI-generated response. All other PTs relied exclusively on the exclusive definition of a trapezoid and consequently assigned the AI response a Level 1 rating.

When critiquing AI-generated justifications involving three-dimensional figures, seven out of 12 PTs struggled to distinguish cones from pyramids using formal geometric definitions. In particular, some PTs accepted AI explanations that characterized cone as a type of pyramid with a circular base. These responses suggest reliance on surface level similarities such as the presence of a single apex in both shapes rather than attention to defining properties. PTs who demonstrated stronger content knowledge rejected this reasoning by invoking the defining characteristics of pyramids as polyhedral composed exclusively of flat polygonal faces.

Most PTs (nine out of 12) correctly noted that not all kites are rhombi; however, some explanations lacked precision in defining the requirement that kites contain two pairs of adjacent congruent sides. Students varied in their precision when articulating the hierarchical relationship

between kites and rhombi. The last pattern was rare but significant. Only two PTs explicitly acknowledged the existence of inclusive versus exclusive definitions when critiquing an AI-generated statement about trapezoids. The scarcity of this pattern suggests that most PTs treat definitions as fixed and limit their ability to critically evaluate AI responses.

Results and Conclusion

This study examined how PTs' content knowledge shaped their critiques of AI-generated geometric justifications. Analysis of written excerpts show that many PTs relied on surface-level similarities, leading to acceptance of incorrect claims, whereas those with stronger content knowledge used formal definitions and hierarchical relationships to reject flawed explanations. Overall, PTs showed emerging understanding of quadrilateral classifications. The findings suggest that AI critique tasks as a useful tool for revealing PTs' content knowledge and informing targeted instructional focus in teacher education, aligning with the previous studies (e.g., Yoon et al., 2024).

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MATHEMATICS, ME, AND MY FUTURE STUDENTS: INCOMING ELEMENTARY PRESERVICE TEACHERS' BELIEFS

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PISA results for the U.S. have consistently declined in mathematics. The 2018 PISA study found students' beliefs about malleability of their intelligence positively impacted achievement, particularly when their teachers held positive beliefs about students' abilities. Yet preservice teachers (PSTs) held negative beliefs about mathematics. To inform interventions addressing these beliefs, this interpretivist study examined incoming PSTs' beliefs about mathematics and mathematical abilities, and how these align with beliefs about their hypothetical students' abilities. Data from two qualitative surveys given to elementary PSTs in an introductory content course were analyzed using open thematic analysis. Findings indicate PSTs held both positive and negative beliefs, with past teachers influencing both. PSTs held more positive beliefs about their students' abilities than their own. Implications for research and practice are discussed.

Keywords: Preservice Teacher Education; Teacher Beliefs; Affect, Emotions, Beliefs, and Attitudes

Introduction and Background Literature

The United States continues to face challenges with persistent underperformance and a declining interest in mathematics (Boaler, 2016; Gouëdard, 2021; Schleicher, 2019), which was evident in international assessments like the Programme for International Student Assessment (PISA). Between 2018 and 2022, average PISA results in mathematics declined for the U.S, reflecting not only a gap in mathematical skills, but also a problem related to beliefs and attitudes toward mathematics. Additionally, the 2018 PISA study revealed that students' beliefs in the malleability of their intelligence had significant positive associations with their test scores across seventy two countries, including the U.S. (Schleicher, 2019; Yeager & Dweck, 2020). Moreover, these associations were stronger when students were taught by teachers who held positive beliefs about their students' abilities (Gouëdard, 2021). These findings highlight the importance of beliefs, both about mathematics and mathematical abilities, in addition to mathematical skills.

However, research revealed that students, including preservice teachers (PSTs), often hold numerous non-availing beliefs about mathematics and the nature of mathematical knowledge (e.g., Celik et al., 2018; Muis, 2004). One common belief is that mathematical knowledge is unchanging and is simply received passively from a source, such as a teacher or textbook (Celik et al., 2018). PSTs also bring a range of beliefs and misconceptions into teacher education that inhibit their ability to benefit from instruction (Lloyd, 2017; Pajares, 1993). Although interventions and course experiences in teacher training promoted desired changes in these beliefs to some extent, research found that PSTs who entered teacher education with these deeply rooted mathematical beliefs often held on to them even after coursework in some cases (Auslander et al., 2019; Chamberlin, 2013). When looked closely, most of these interventions were implemented towards the later part of teacher education programs (e.g., Auslander et al., 2019). Less attention has been given to understanding what beliefs PSTs bring into teacher education, their prior experiences that framed these beliefs, and interventions targeting them. Addressing this gap, the current study aims to understand PSTs' beliefs in an introductory

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mathematics content course to inform design of interventions that leverage PSTs' existing positive beliefs while intentionally addressing the negative ones.

Moreover, although prior studies examined mathematical beliefs of PSTs, no other study to the best of my knowledge examined PSTs' beliefs about their future students' mathematical abilities, especially in an introductory course. Understanding these beliefs is an important first step in designing interventions using PSTs' strengths to address their negative beliefs. As part of a bigger study aiming to design better interventions to foster positive beliefs of PSTs in an introductory content course, this current study sought to investigate the following questions.

RQ1: *What beliefs about mathematics and mathematical abilities do incoming elementary preservice teachers bring to their introductory mathematics content course?*

RQ2: *How do incoming elementary preservice teachers' beliefs about their own mathematical abilities align or misalign with their beliefs about mathematical abilities of their future students?*

In this study, I define beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true" and can be seen as lenses that shape one's perspective or as dispositions that guide one's actions (Philipp, 2007, pg. 259).

Theoretical Framework

Interpretivism (Stinson & Walshaw, 2017) provided the lens to understand PST beliefs. Sun's (2015) beliefs about mathematics and mathematical abilities framework provided the central mechanism for data collection. Sun's (2015) beliefs about mathematics framework consisted of three constructs namely: identification with mathematics, nature of mathematics, and mastery-oriented beliefs (or beliefs about making mistakes). This study considered two additional constructs, namely experiences that affected these beliefs (Boaler et al., 2022), and beliefs about mathematical authority (Wilson & Lloyd, 2000). Beliefs about mathematical ability can be viewed as either malleable or fixed (Dweck, 2006), across two dimensions of one's own abilities and their future students' abilities. Beliefs across the dimension of hypothetical future students' abilities are specific to PSTs and are considered in this study building on Sun's work.

Data Collection, Analysis, and Methods

Participants were twelve elementary PSTs enrolled in an introductory mathematics content course on number and operation, at a public university in the southeastern U.S. Two data collection instruments namely 'Beliefs about Mathematics' and 'Beliefs about Mathematical Ability' surveys were adapted from Sun's (2015) validated Likert scale instrument. I adapted Sun's (2015) instrument by splitting it into two shorter surveys. I reworded items into open ended surveys to collect qualitative data and added a few items to collect data from participants treating them as future teachers rather than as students. For example, I added questions asking participants about their hypothetical students, which is suited for future teachers only. I open coded survey responses until saturation of codes. Coding was conducted in iterations by revisiting and refining to determine broader themes and to improve robustness of findings.

Findings

Research Question 1

Due to space restrictions, only participant beliefs about three out of five constructs are described here. Findings indicate PSTs held both positive and negative beliefs about mathematics. Surprisingly, the theme "past teachers" was related to both positive and negative beliefs. Table 1 below highlights other themes in each category.

Table 1: Themes Related to Positive and Negative Experiences with Mathematics

Overall Positive Experiences ~ 45%	Overall Negative Experiences ~ 55%
Past teachers ~ 10%	Past teachers ~ 25%
Advanced course work ~ 40%	Pandemic during high school ~ 17%
Love for mathematics ~ 10%	Harder concepts ~ 41%
Job related to mathematics ~ 20%	Geometry ~ 17%
Good experiences with math ~ 20%	

The most concerning response was from a participant Zara, who described their previous experiences with mathematics as follows.

> Zara: I always liked math. It made me feel smart, but sometime in high school the math problems started getting harder and teachers started teaching less. Not only was this during the Covid-19 pandemic so we were in and out of school but my teachers were tired and burnt out. It became more and more difficult to grasp the concepts and I had just accepted that I was bad at math.

This participant who previously felt good at math had changed their beliefs about their mathematical abilities due to various negative experiences they had, such as harder concepts in high school, pandemic, and teachers during pandemic. Understanding such past experiences of PSTs helps design better interventions to foster their positive beliefs about mathematical abilities, which can have an impact on the next generation of students.

To dig deeper into the construct of *Identification with Math*, I asked my participants to use four to five words to describe mathematics and use pictures from the internet to describe their feelings about mathematics. The most frequently used words were "challenging" and "rewarding." These words when looked together with the pictures participants used indicated that the participants associated mathematics with persistence, effort, and making mistakes which result in the rewards. They recognized that their effort in mathematics is rewarding, which means they have the potential to emphasize the value of effort, persistence and making mistakes in their future mathematics classrooms.

Regarding participant beliefs about *Making mistakes* in mathematics, findings show that while most PSTs understand mistakes support learning, many still experience frustration, embarrassment, or discouragement about making mistakes. This was evident from participant responses, one of which is highlighted below.

> Sam: In all things, including math, mistakes help you learn. In most learning environments, it is important to avoid mistakes because it will affect your grade. Many teachers don't give students the opportunity to learn and grow, they just care if the student is right or wrong. It is important to let students make mistakes without punishment as they will learn from them.

Research Question 2

To answer RQ2, participants were first asked to describe their own mathematical abilities by asking them "*How comfortable are you with mathematics on a scale of 1–10? Describe any prior experiences that shaped these beliefs?*" Then, they were asked to describe their thoughts about a hypothetical situation, where they are teaching in a classroom, see some kids struggle with content, and when this concern was shared with a fellow teacher, they respond "*Some kids*

can never do well in math, no matter how hard they try." Participant Anna's responses are presented.

> Anna (about own abilities): I don't see myself as a math person. On a scale from 1 to 10 I'll say I'm about a 6. I've never understood math well so it takes me a lot of work to understand it. I think I have a harder time when it comes to math that deals with graphs.
>
> Anna (about hypothetical students' abilities): I would think that the fellow teacher is really ignorant because they are assuming that some kids never do well when it is very possible for all kids to do well. You just need to understand that all kids learn differently.

Similar to Anna's response highlighted above, almost all participants disagreed with the hypothetical fellow teacher's response, while only one participant agreed. This indicates PSTs held more positive beliefs about their students' abilities than their own abilities. Two sample participant responses where PSTs valued their mission as future teachers are highlighted below.

> Ava: it's our job as educators to help them [students] believe in their ability to learn.
>
> Felix: This mindset is how kids end up never doing well in math, because nobody else believed in them improving, so they could not believe in themselves.

These PSTs demonstrated strong motivation to become effective teachers, potentially inspired by their own negative mathematical experiences to intentionally avoid those for their students.

Discussion and Conclusion

Building on prior research, these findings add to the literature PSTs' beliefs about their hypothetical students, which moves the field forward in designing better interventions that build on PSTs' strengths to address their negative beliefs. As Kagan (1992) asserted, PSTs' preexisting beliefs must be made explicit and challenged for them to meaningfully examine and integrate new ideas into their belief systems. Such integration requires PSTs to reexperience mathematics in research-based ways as learners of mathematics rather than as teachers (Mizell & Cates, 2004). This is possible only when PSTs' beliefs are revealed early in teacher education.

Findings that PSTs held both positive and negative beliefs about mathematics carry implications for both research and practice. For research, this underscores the need for designing effective interventions that can help PSTs overcome academic challenges in mathematics, thus fostering their positive beliefs about mathematics during their teacher education (Young & Dyess, 2021). For practice, tension between participants recognizing mistakes as opportunities for learning yet discouraging, suggests that teacher preparation programs must intentionally foster environments that normalize productive struggle and help students experience struggle as a natural and valuable part of mathematical thinking (Young & Dyess, 2021). It is important to provide PSTs with opportunities to reexperience mathematics in a mastery-oriented way by valuing mistakes, encouraging productive struggle, and positioning learners as the mathematical authority. PSTs holding more positive beliefs about their students' abilities than their own indicate they hold high values for their job as future educators. Teacher preparation programs should build on this strength by nurturing PSTs' mathematical proficiency and encouraging them to see their own growth as essential for enabling their future students' success.

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REVEALING PRESERVICE TEACHERS' STANCES TOWARD STUDENT THINKING IN AI-SIMULATED CONFERENCES

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Responding to student thinking in the moment is a challenging instructional practice that requires genuine curiosity. This study examines how elementary preservice teachers' (PSTs') stances toward student thinking were revealed through their responses during an AI-simulated mathematics conference. Transcripts from eight PSTs were analyzed to examine the moves they enacted and the stances those moves revealed. PSTs often began by eliciting student thinking, reflecting a curiosity stance. However, when incorrect thinking surfaced, many shifted toward fix-it or completion stances, using moves such as funneling, retelling as restart, and selective probing. Although PSTs may learn how to structure a mathematics conference, sustaining a curiosity stance remains challenging. These findings show how AI-simulations can make PSTs' stances toward student thinking visible.

Keywords: Preservice Teacher Education

Introduction

In the mathematics classroom, some of the most impactful instructional decisions happen while students are at work. As teachers engage with individuals or small groups of students, they must decide how to respond to student thinking in the moment. Doing this well requires drawing out and making sense of partial or incorrect thinking and responding in ways that support students' thinking without taking it over. Munson (2018) argues that such work cannot be productive unless teachers orient themselves as genuinely curious about student thinking.

Responding to student thinking productively is challenging for teachers in general and even more so for preservice teachers (PSTs). Research shows that PSTs often make assumptions about student thinking rather than questioning it, lead students toward a specific solution path, and miss opportunities to investigate student thinking (Shaughnessy & Boerst, 2018). Additionally, they tend to focus on the procedural steps taken instead of the reasoning that supports those steps (Jacobs et al., 2010; Kilic, 2018). These challenges become even more apparent when students have incorrect thinking. In these situations, PSTs have been found to avoid engaging with incorrect thinking altogether (Sherman, 2020; Shaughnessy et al., 2021) or shift into trying to fix it (Shaughnessy & Boerst, 2018).

These tendencies are concerning because engaging with incorrect thinking has been shown to have benefits for students and teachers. Students who view incorrect thinking as opportunities for learning are more open to adjusting solution pathways and persisting through challenges (Tulis et al., 2018) and have greater motivation in the mathematics classroom (Grassinger & Dresel, 2017). For teachers, engaging with students' incorrect thinking can uncover gaps and misconceptions that can inform instructional decisions (Bray, 2013). With these benefits in mind, teacher education must better support PSTs in developing and sustaining a curiosity stance toward correct and incorrect student thinking.

Practice-based approaches to teacher education have been used to support PSTs in learning to enact complex instructional practices such as responding to student thinking (Grossman et al., 2009). Within these approaches, approximations of practice provide opportunities for PSTs to

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rehearse instructional decision-making with less risk. Approximations such as scripting, rehearsals, and simulations have been shown to support PSTs in learning to respond to student thinking more productively (e.g., Campbell & Baldinger, 2022; Webel & Conner, 2017). More recently, AI-simulations have gained attention as a promising form of approximation for developing PSTs' responsiveness to student thinking (Ke et al., 2020; Lee et al., 2025). Unlike many other approximations, AI-simulations allow PSTs to engage with student thinking as it unfolds in real time. Additionally, they create opportunities for PSTs to respond to unpredictable and incorrect student thinking practice, more closely approximating interacting with real K-5 students. Since PSTs must make instructional decisions in the moment, AI-simulations may provide a context in which PSTs' stances toward student thinking become visible.

This study examines how elementary PSTs respond to student thinking during an AI-simulated small-group, in the moment interaction while students are at work. Throughout this paper, this interaction is referred to as a mathematics conference. Specifically, this work intends to answer the following research question: How are PSTs' stances toward student thinking revealed through their responses in an AI-simulated mathematics conference?

Theoretical Frameworks

This study is grounded in two related frameworks: the Eliciting-Interpreting-Advancing (EIA) practice and Munson's (2018) conceptualization of stance.

Eliciting-Interpreting-Advancing Student Thinking

The first framework is the high-leverage teaching practice of eliciting and interpreting student thinking as conceptualized by TeachingWorks (2018). The goal of this responsive practice is to draw out students' ideas and make sense of their reasoning to ultimately inform instructional decisions. However, in the context of a mathematics conference, responding requires more than eliciting and interpreting. Teachers must also decide how and when to advance student thinking in real time.

Drawing on Munson's (2018) framework for conferring in mathematics, this work is conceptualized as three connected components: Eliciting, Interpreting, and Advancing (EIA) student thinking. Eliciting refers to uncovering what students have done and how they are making sense of the task. Interpreting involves making sense of students' thinking to determine what they understand and where they might be struggling. Advancing includes responding to students' thinking in ways that support continued sense-making. Figure 1 illustrates this conceptualization of the EIA practice as a dynamic cycle.

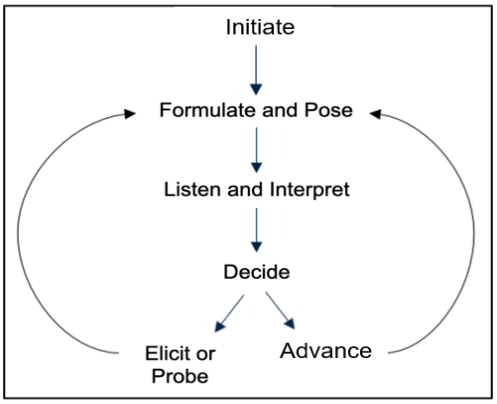


Figure 1: EIA Practice

Within this framework, teacher responses are called moves. Initiating the conversation, eliciting or probing, and advancing student thinking, each have their own set of observable moves, taken from Munson (2018). For example, initiating the conversation may involve a general eliciting move that creates space for students to share their thinking (e.g., What have you tried so far?). Eliciting further may include general or targeted follow-up moves that clarify or press on specific aspects of revealed student thinking. Advancing includes moves such as retelling, orienting students to one another, or positioning students as competent.

In this study, moves were considered productive when they supported understanding, interpreting, or advancing student thinking. Moves were considered less productive when they redirected attention away from understanding or using student thinking and toward correcting student thinking or guiding students to an answer. Therefore, in addition to Munson's (2018) moves, several less productive moves identified in prior research were included in the analysis. One example is funneling (Wood, 1998), where a teacher narrows student thinking through a series of questions that guide students toward a particular answer.

Analyzing moves alone does not fully explain how student thinking is treated during a conference. So, a second framework is used to interpret the stance reflected in those moves.

Stance

Munson's (2018) conceptualization of stance refers to a teacher's orientation toward student thinking and shapes how students' ideas are treated during a mathematics conference. Munson identifies four stances that a teacher can take during a math conference:

- Curiosity: treating student thinking as an object of inquiry
- Fix-it: treating thinking as something to correct
- Completion: orienting toward finishing the task
- Management: prioritizing behavior over thinking

Since the purpose of the EIA practice is to understand and build on student thinking, a curiosity stance supports teachers in treating this thinking as something to learn about instead of something to evaluate or correct. In contrast, fix-it, completion, and management stances focus on correcting errors, finishing tasks, and managing behavior. Considering stance alongside teacher moves helps reveal how PSTs approach and respond to student thinking during a mathematics conference.

Methods

Participants

Participants were eight graduate-level elementary PSTs enrolled in a synchronous, online mathematics methods course at a public university. All participants were working toward initial elementary teacher certification.

AI-Simulation as an Approximation of Practice

The AI-simulation was used as an approximation of practice for a small-group mathematics conference and was a required out-of-class assignment embedded within a multi-week instructional sequence focusing on the EIA practice. Prior to completing the simulation, PSTs read Munson (2018), participated in class discussions, and completed two scripting approximations focusing on the EIA practice.

The AI-simulation was conducted using a custom GPT developed through OpenAI's ChatGPT. The GPT was programmed to act like two second-grade students who were solving the following problem: *Aubree has 7 balloons. She has 5 fewer than Blair. How many balloons*

did Blair have? One AI-student (Maya) had been programmed to reason correctly about the relational structure of the problem, whereas the other AI-student (Jordan) displayed the common misconception that "5 fewer" requires subtracting 5 from 7. The GPT was instructed to maintain each student's thinking and not reveal which student was correct or incorrect. The AI-students also could reference and describe their written representations when responding.

PSTs interacted with the AI-students through a text-based chat interface and were instructed to interact with the students like they would during a small-group mathematics conference. The goal of the conference was to elicit, interpret, and advance student thinking rather than tell students how to solve the problem correctly. This was the first time that PSTs were asked to enact the EIA practice in its entirety and respond to student thinking in real time. Since PSTs could not anticipate how students would respond and were responsible for continuing the interaction through their written moves, this approximation required in the moment instructional decision making like that of a mathematics conference with actual K-5 students.

Data Analysis

The primary data source was full transcripts of each PST's interaction with the AI-simulated students. These transcripts captured PSTs' real time instructional decisions as they elicited, interpreted, and advanced student thinking while students were at work.

Data analysis occurred in three stages. First, transcripts were deductively coded using the EIA practice move codes drawn from Munson (2018) and other literature. Each teacher turn served as a unit of analysis and was coded according to the primary move enacted (e.g., general eliciting, targeted follow-up, retelling, funneling). This stage identified how PSTs responded to student thinking throughout the conference. Second, analytic memos were written for each PST's conference, examining how patterns of moves across the interaction and responses to student thinking reflected different stances toward student thinking and moments when those stances appeared to shift. Third, a cross-case analysis was completed to identify patterns in how stance was revealed and how it shifted across the eight PSTs. Themes were refined through ongoing comparisons of conferences and stance patterns across PSTs.

Results: Stances Revealed Through PSTs' Responses

This section describes how PSTs' stances toward student thinking were revealed through in the moment responses during the AI-simulated mathematics conference. Although PSTs enacted a range of teacher moves, patterns in those moves and responses to student thinking provided evidence of different stances throughout the conference.

A Curious Start

Simulations typically began with PSTs making a general eliciting move intended to learn about how students were approaching the problem. For example, Anna opened with, "What have you done so far?" Such initiations positioned students as contributors and signaled an initial curiosity stance.

These early general eliciting moves quickly surfaced the students' contrasting interpretations of the story problem, as seen in Jenna's conference:

Jenna:	Can you explain what you have drawn on your paper so far?
Maya:	Yeah! So, I drew Aubree right here, and I counted them—one, two, three, four, five, six, seven. Then I drew Blair over here, and I gave her more balloons, because it says Aubree has five less than Blair. So, I counted—seven, eight, nine, ten, eleven, twelve. Blair has twelve balloons.

Jordan: Um… I drew Aubree too with seven balloons. And then it says she has five less, so I took away five. So, um… I just put two balloons for Blair.

There were slight variations in how this curiosity was enacted. For example, Hannah entered the conversation with the AI-students by asking, "What are you guys working on?" While this question initiated the conversation, it led to Maya restating the story problem in her own words and then Jordan describing where they were in making sense of the problem. Therefore, this move does not directly help Hannah understand how each of the students was thinking. Even so, across cases, PSTs demonstrated an initial willingness to hear from students. Thus, conferences commonly began with moves that indicated a curiosity stance.

Moves that Sustained a Curiosity Stance

A few PSTs were able to sustain a curiosity stance through follow-up moves that pressed for clarification or further explanation.

General follow-up. Katie aimed to try to better understand Jordan's thinking after he said that he "started with seven and then took away five, "'cause it says five less than." She asked, "Jordan, can you show me what you did on your paper?" By doing this, Katie attempts to understand multiple aspects of Jordan's thinking, including his representation. This general follow-up move helps Katie remain curious and offered Jordan the opportunity to further explain his work and thinking.

Targeted follow-up. PSTs also sustained curiosity by questioning the AI-students about specific aspects of their explanations or representations. One example comes from Jenna, who responded to Jordan's explanation with the question, "Who has five less, Jordan?" In his initial response, Jordan used the phrase "she has five less," where the referent was left somewhat ambiguous. By asking for clarification, Jenna demonstrates close attention to Jordan's language and attempted to interpret his reasoning more precisely. In this moment, she maintains a curiosity stance by attempting to understand how Jordan was understanding the relationship between the quantities.

Grace also used a targeted follow-up move.

Maya: I think I know! I drew it right here. Aubree has seven little circles for balloons. See, Aubree has seven, and Blair has more. So, I think Blair's number has to change from that.

Jordan: I did mine different. I took seven and then did 7 – 5 = 2. So, I wrote two for Blair. But then Maya said it should. Be more, so I'm kinda confused.

Grace: Sure, that's okay, Jordan. Let's see why Maya feels like it should be more. Maya, what makes you think Blair has more?

Grace acknowledged Jordan's confusion and then asked Maya to explain her thinking. Since her question is built directly on the students' language and ideas, it reflects an effort to make sense of how each student was interpreting the problem. In this example, the targeted follow-up move was associated with a sustained curiosity stance.

Moves Associated with a Shift in Stance

Although conferences began with curiosity, PSTs tended to shift stances once Jordan's incorrect or uncertain thinking surfaced. At that point, PSTs did not press students to explain their thinking. Instead, they moved from eliciting toward advancing moves such as retelling or funneling. This pattern demonstrates a shift away from a curiosity stance to a fix-it stance where the goal is to repair incorrect thinking or a completion stance where the goal is to guide the

students through the problem toward a shared correct answer. In both cases, the goal becomes getting the right answer instead of understanding student thinking.

Limited and selective probing. Probing could have allowed PSTs to explore how Jordan was making sense of the phrase "five less." While probing did occur, it was often directed toward the student with correct thinking. One example of this occurred in Anna's conference:

Anna:	What have you done so far?
Maya:	Um, we were talking about the balloon problem. I drew two circles—one for Aubree and one for Blair—and I wrote a seven in Aubree's circle because she has seven balloons. Then I said Blair should have more than Aubree since Aubree has less. So, I was trying to figure out how many more.
Jordan:	Yeah… I wrote "7 minus 5 equals 2," 'cause it said Aubree has five less. So, I thought maybe that means Blair has two. But I'm not sure if that's right.
Anna:	Maya, I like how you drew two circles. That helps me see your thinking. So, you think Blair should have more, why?

While Anna's initial question signaled curiosity, the direction of her follow-up showed that this curiosity was selective. Instead of pressing Jordan to unpack why subtraction made sense to him, Anna directed her probing question to Maya, whose reasoning aligned with the correct interpretation of the relationship in the problem. Throughout the rest of the simulation, Jordan's thinking remained unexplored. Similar patterns were seen in other PSTs' simulated conferences.

Retelling as a restart. Another common indicator of a stance shift was retelling. If enacted productively, retelling can support student sense-making by asking to restate the problem in their own words. However, it often functioned as a reread or restart. For example, after Jordan responded to Jenna's question about who had five less, Jenna said, "[Jordan], can you read me the question again and let's start from the beginning. Maya, please don't help Jordan just yet." Here, Jenna shifted from being curious about Jordan's thinking to correcting it. She did not attempt to understand how Jordan was making sense of the problem. Instead, she used this opportunity to restart the problem. Thus, Jenna stepped out of the curiosity stance and into a fix-it stance that focused on getting the correct solution.

Additionally, by instructing Maya not to help, Jenna positioned one student as correct and the other as incorrect. At this point, the students could infer which was which. By having Jordan reread the problem, his thinking was framed as something that needed to be fixed. This shift toward a fix-it stance through rereading or restarting was seen frequently across simulations.

Funneling toward completion. Funneling moves also showed this shift. These moves essentially work to lead students to the correct solution through a series of questions. The goal then becomes different than trying to make sense of and use student thinking. Hannah, for example, asked multiple funneling questions:

- "Ok so how many fewer balloons does Aubree have than Blair?";
- "The problem says that Aubree has five less balloons than Blair. So, who has more balloons?"; and
- "Well Jordan, if the number of balloons that Aubree has is 5 less than Blair, and you said Blair has more balloons. Then how many more balloons does Blair have than Aubree?"

These questions gradually clarified the relationships between the quantities in the problem. Rather than examining how Jordan initially interpreted "five less," the funneling questions lead him toward the comparison structure of the problem, reflecting a fix-it stance being taken.

Another example comes from Anna:

Anna:	Jordan, you wrote numbers on your paper, but can you read the story to me again to tell me first how many balloons Aubrey has?
Jordan:	Um… it says Aubree has seven balloons. So, she's the one with seven.
Anna:	Yes, so let's say Aubrey is turning seven years old, so she has a bunch of balloons with helium in them all tied together with string. Can you show me what that might look like on your paper?

Anna makes two funneling moves in a row, both of which direct Jordan toward a specific solution path. The questions structure how he should proceed. This leaves Jordan little opportunity to generate or test his own reasoning. Instead of investigating Jordan's interpretation of the problem, Anna guides him toward the intended understanding through closed questioning. This pattern of guiding students to the correct answer and toward completing the problem was common across PSTs. In doing so, the conference shifts away from centering student thinking and instead guides students down a predetermined solution path.

Discussion and Conclusion

This study examined how PSTs' stances toward student thinking were revealed through their in the moment responses during an AI-simulated mathematics conference. Across simulations, PSTs frequently began with a general eliciting move that signaled curiosity and a willingness to learn about students' thinking. However, once incorrect student thinking surfaced, that curiosity was often not sustained. In these moments, PSTs tended to shift toward fix-it or completion stances, which were evident through their use of selective probing, retelling as restart, and funneling.

One interpretation of these findings is that incorrect thinking changed the nature of the interaction for PSTs. At the beginning of the conference, PSTs could focus on learning about students' thinking through general eliciting moves. However, once Jordan revealed an incorrect interpretation of "five fewer," PSTs were required to make sense of that thinking and decide how to respond in the moment. Instead of continuing to explore how Jordan was making sense of the relationship described in the problem, many PSTs moved toward helping him revise his thinking. Prior research has found that PSTs often struggle to engage with incorrect thinking (Sherman, 2020; Shaughnessy et al., 2021). The present study extends this work by showing how those challenges became visible during a real time interaction. Incorrect thinking was not ignored. Instead, it often marked the point at which PSTs shifted from trying to understand student thinking to trying to correct it. A possible explanation for this shift is that PSTs may view the role of the teacher as helping students arrive to the correct solution. When Jordan revealed incorrect thinking, many PSTs treated that thinking as something that needed to be fixed rather than investigated. This aligns with Shaughnessy and Boerst's (2018) findings that PSTs often move quickly into helping students instead of learning more about their thinking. From this perspective, the use of retelling as restart, selective probing, and funneling may not reflect a lack of attention to student thinking. Instead, these moves may stem from a belief that effective teaching means helping students get back on the correct path. If PSTs enter teacher education with experiences that position teaching as explaining, correcting, and guiding students toward solutions, then sustaining a curiosity stance when incorrect thinking is shared may be particularly challenging.

This interpretation is further supported by how PSTs distributed their attention across students. Probing was disproportionately directed toward Maya, whose thinking aligned with the intended mathematics. Jordan's thinking, on the other hand, was often left unexplored. Bray (2011) reported similar patterns, suggesting that teachers are more likely to engage with student thinking they perceive as mathematically correct. One possible explanation is that Maya's reasoning was easier for PSTs to recognize and interpret because it aligned with their own understanding of the problem. By contrast, Jordan's thinking required PSTs to make sense of reasoning that conflicted with that understanding. Thus, PSTs may have been more comfortable probing Maya's thinking than continuing to investigate Jordan's. Sustaining a curiosity stance, therefore, may require learning view to incorrect thinking as something to understand.

The findings may suggest that PSTs were learning the structure of a mathematics conference before learning to enact it productively. Most PSTs began their conferences with a general eliciting move, indicating that they recognized the importance of first learning about students' thinking. The challenge came later, when PSTs had to decide how to respond to unexpected or incorrect reasoning. Sustaining a curiosity stance requires more than knowing which move to enact next. It requires viewing student thinking as the object of focus throughout the interaction. The common use of retelling as restart and funneling suggests that many PSTs were still developing this aspect of the EIA practice. Similar patterns have been documented in prior research, where responding to students' emerging thinking is often more difficult than initiating an interaction (Jacobs et al., 2010; Kilic, 2018).

For mathematics teacher education, supporting PSTs in learning the EIA practice may require more than teaching its structure and the moves within it. Sustaining a curiosity stance appears to require learning how to engage with incorrect thinking in ways that prioritize understanding over correction. That said, teacher educators may need to create opportunities for PSTs to investigate incorrect thinking, consider possible interpretations, and practice responding. AI-simulations may provide a useful context for this work because they create opportunities for PSTs to engage with student thinking in the moment while also making their stances visible for reflection. Additionally, they allow PSTs to practice responding to incorrect thinking prior to enacting the practice with real elementary students. Future research should investigate how PSTs' stances develop across multiple approximations of practice and whether repeated opportunities to engage with incorrect student thinking support a more sustained curiosity stance.

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LA METACOGNICIÓN EN FUTUROS DOCENTES DE MATEMÁTICAS: UN ESTUDIO DE INVESTIGACIÓN-ACCIÓN Y BTC

METACOGNITION IN FUTURE MATHEMATICS TEACHERS: AN ACTION-RESEARCH AND BTC STUDY

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Este estudio examina el desarrollo de la metacognición en docentes en formación de la Licenciatura en Enseñanza y aprendizaje de las matemáticas a partir de la implementación del enfoque Building Thinking Classrooms (BTC) en sus cursos de geometría. Se adoptó una metodología de investigación-acción para la aplicación de la propuesta de investigación. Los datos analizados para los resultados incluyeron observaciones de aula, producciones matemáticas en pizarras verticales no permanentes, narrativas reflexivas y un cuestionario de conciencia metacognitiva aplicado antes y después de la intervención. Los resultados sugieren que el BTC genera condiciones estructurales que favorecen la visibilización y regulación del pensamiento matemático en la formación inicial docente.

Palabras clave: Geometría, conocimiento matemática para la enseñanza, metacognición

Introducción

La formación inicial de docentes de matemáticas enfrenta el desafío de superar prácticas centradas en la memorización de procedimientos y promover experiencias que favorezcan el desarrollo de un pensamiento matemático profundo. En el caso de la geometría—ya sea plana, del espacio o analítica—la enseñanza tradicional suele privilegiar el uso mecánico de fórmulas, lo que limita la comprensión conceptual y la reflexión sobre los propios procesos de aprendizaje.

Desde la Psicología de la Educación Matemática, la metacognición se reconoce como un elemento clave para comprender cómo los estudiantes planifican, monitorean y evalúan sus procesos cognitivos, favoreciendo el desarrollo de la autonomía intelectual. Asimismo, el modelo de niveles de Van Hiele ofrece un marco teórico pertinente para analizar el desarrollo del razonamiento geométrico, al describir una progresión en la comprensión de los conceptos y relaciones espaciales.

En este contexto, el enfoque Building Thinking Classrooms (BTC) propone reorganizar las dinámicas de aula para promover el pensamiento colectivo mediante la resolución de problemas, el trabajo colaborativo y la participación activa de los estudiantes. Sin embargo, aún es limitado el conocimiento sobre cómo este tipo de entornos favorece específicamente el desarrollo de la conciencia metacognitiva en la formación inicial docente.

El presente estudio articula tres experiencias previas desarrolladas en cursos de geometría plana y del espacio, geometría analítica y formación docente, integradas bajo el enfoque BTC. El objetivo es analizar cómo el BTC favorece el desarrollo de la conciencia metacognitiva en futuros docentes de matemáticas, en relación con su razonamiento geométrico.

Revisión de la literatura

La geometría puede entenderse, en términos generales, como el estudio de las propiedades, relaciones y medidas de las figuras en el plano y en el espacio (Real Academia Española, 2001). En el ámbito educativo, constituye una rama fundamental de las matemáticas orientada al

análisis de las estructuras, características y transformaciones de los objetos geométricos (Cordero, 2021). Más allá de su dimensión descriptiva, la geometría involucra procesos de visualización, argumentación y construcción de relaciones, lo que la convierte en un campo privilegiado para el desarrollo del razonamiento matemático.

No obstante, su enseñanza enfrenta desafíos importantes, como la alta abstracción de sus conceptos y la escasa vinculación con contextos significativos para los estudiantes, lo que suele derivar en aprendizajes fragmentados y predominantemente procedimentales (Cordero, 2021). En este sentido, diversos autores coinciden en la necesidad de diseñar tareas accesibles pero intelectualmente demandantes que favorezcan la exploración, la formulación de conjeturas y la argumentación matemática (García y López, 2008). Asimismo, el uso de herramientas tecnológicas como GeoGebra ha sido señalado como un recurso que potencia la visualización dinámica y la comprensión conceptual.

Desde una perspectiva cognitiva, el modelo de razonamiento geométrico de Van Hiele (1986) ofrece un referente central para comprender cómo los estudiantes construyen conocimiento en este dominio. Este modelo plantea una progresión de cinco niveles—visualización, análisis, clasificación, deducción y rigor—que describen formas cualitativamente distintas de interpretar los objetos geométricos. El tránsito entre niveles no depende exclusivamente de la edad, sino de las experiencias de aprendizaje diseñadas en el aula. En consecuencia, la enseñanza de la geometría requiere propuestas didácticas alineadas con el nivel de razonamiento de los estudiantes, con el fin de favorecer avances hacia formas más complejas de pensamiento geométrico.

En el presente estudio, el modelo de Van Hiele se utiliza como referente analítico para interpretar las transiciones en el razonamiento geométrico observadas durante la intervención, particularmente en relación con el paso de niveles de visualización hacia formas de análisis y razonamiento informal deductivo.

Por otra parte, la metacognición, entendida como el conocimiento y la regulación de los propios procesos cognitivos (Flavell, 1979), constituye un elemento clave en el aprendizaje matemático. Este constructo se organiza en tres procesos fundamentales: planificación, monitoreo y evaluación, los cuales intervienen directamente en la resolución de problemas y en la toma de decisiones estratégicas (Schoenfeld, 1985).

Investigaciones recientes han señalado que la metacognición no se desarrolla de manera aislada, sino que emerge en interacción con tareas matemáticas cognitivamente demandantes que implican la explicitación de estrategias, la revisión de procedimientos y la reestructuración del pensamiento (Mevarech & Kramarski, 2014; Schoenfeld, 2016). En particular, la resolución de problemas no rutinarios y la argumentación favorecen procesos de autorregulación, ya que los estudiantes deben justificar sus decisiones, identificar errores y ajustar sus estrategias (Kramarski & Heaysman, 2021).

En el contexto de la formación inicial docente, la metacognición adquiere una doble dimensión: los futuros docentes no solo regulan su propio aprendizaje matemático, sino que comienzan a desarrollar conciencia sobre cómo promover estos procesos en sus estudiantes. De este modo, la metacognición se concibe como un proceso situado, mediado por el diseño de tareas y las dinámicas de interacción en el aula.

En esta línea, el enfoque Building Thinking Classrooms (BTC), propuesto por Liljedahl (2021), plantea la creación de entornos de aprendizaje intencionalmente diseñados para promover el pensamiento matemático a través del trabajo colaborativo, la resolución de problemas no rutinarios y la participación activa de los estudiantes. Este enfoque desplaza la

enseñanza centrada en la transmisión de procedimientos hacia prácticas que favorecen la construcción colectiva del conocimiento y el desarrollo de la autonomía intelectual.

Diversas investigaciones han documentado efectos positivos asociados con la implementación del BTC, incluyendo mejoras en el rendimiento académico, el pensamiento crítico y la motivación hacia el aprendizaje de las matemáticas (Ramírez y Soto, 2025; Muir, 2025). Asimismo, este enfoque contribuye a redefinir el rol del docente como mediador del aprendizaje y diseñador de experiencias matemáticas significativas.

En el presente estudio, el BTC se asume como marco de intervención didáctica, particularmente en lo que respecta al diseño de tareas colaborativas, la organización del aula y la promoción de dinámicas que favorecen la explicitación de procesos metacognitivos. De este modo, se establece una articulación entre el desarrollo del razonamiento geométrico (Van Hiele), la regulación del aprendizaje (metacognición) y las condiciones de aula que lo posibilitan (BTC).

Metodología

El presente estudio se sustenta en el supuesto de que la metacognición se construye en interacción con tareas matemáticas cognitivamente demandantes, más que mediante la instrucción explícita de estrategias aisladas de autorregulación. Asimismo, se asume que el diseño estructural del aula influye en la emergencia de procesos metacognitivos, en la medida en que organiza las oportunidades de participación, interacción y explicitación del pensamiento. En este marco, el enfoque *Building Thinking Classrooms* (BTC) se adopta como referencia para el diseño de la intervención, al generar condiciones que favorecen el desarrollo del pensamiento matemático y metacognitivo a través de tareas no rutinarias, el trabajo colaborativo y la visibilización de los procesos de razonamiento.

La población del estudio estuvo conformada por 52 estudiantes de la Licenciatura en Enseñanza y Aprendizaje de las Matemáticas (LEAM), pertenecientes a una escuela normal pública del centro de México. Los participantes cursaban asignaturas de Geometría Plana y del Espacio y Geometría Analítica en los primeros semestres de su formación inicial docente.

La investigación se enmarca en un enfoque cualitativo con diseño de investigación-acción, orientado a comprender e interpretar los procesos de pensamiento, las interacciones en el aula y las transformaciones conceptuales que emergen durante la intervención. En este enfoque, el docente asume simultáneamente el rol de investigador, lo que implica diseñar e implementar una propuesta didáctica intencional, analizar sus efectos y reflexionar de manera sistemática sobre su propia práctica. El estudio tiene un carácter exploratorio-interpretativo, ya que busca identificar cómo se desarrollan los procesos metacognitivos en un contexto auténtico de formación inicial docente.

El proceso de investigación se desarrolló a través de ciclos de planificación, acción, observación y reflexión. En una primera fase, se realizó un diagnóstico inicial mediante la aplicación de un cuestionario de conciencia metacognitiva (pretest) y el análisis de producciones iniciales de los estudiantes, con el propósito de identificar sus formas de razonamiento geométrico, tomando como referencia el modelo de Van Hiele.

Posteriormente, se diseñó una intervención basada en tareas geométricas no rutinarias, alineadas con los principios del enfoque BTC. Estas tareas estuvieron orientadas a promover la exploración, la formulación de conjeturas, la argumentación y la reflexión sobre los propios procesos de pensamiento. De manera intencional, se incorporaron preguntas metacognitivas dirigidas a fomentar procesos de planificación, monitoreo y evaluación, tales como la selección de estrategias, la verificación de procedimientos y la valoración de resultados.

La implementación de la intervención se llevó a cabo mediante dinámicas características del

BTC, incluyendo la organización de los estudiantes en equipos de trabajo conformados de manera aleatoria, el uso de pizarras verticales para la resolución de problemas y la discusión colectiva de estrategias y soluciones. Durante este proceso, el docente promovió la explicitación del pensamiento matemático a través de preguntas abiertas y retroalimentación orientada a la reflexión, con el fin de favorecer la verbalización de procesos metacognitivos.

El análisis de la información se realizó mediante la triangulación de diversas fuentes de datos, entre las que se incluyen las producciones matemáticas elaboradas en pizarras verticales, narrativas reflexivas de los estudiantes, los resultados del cuestionario de conciencia metacognitiva aplicado en modalidad pre y post, así como observaciones y notas de campo. Para el análisis cualitativo se establecieron categorías basadas en los procesos metacognitivos de planificación, monitoreo y evaluación. De manera complementaria, el modelo de Van Hiele se utilizó como referente analítico para interpretar las transformaciones en el razonamiento geométrico observadas durante la intervención.

En cuanto a los criterios de validez, el estudio se sustentó en la credibilidad, la coherencia interna y la triangulación de datos. La inclusión de evidencias textuales provenientes de narrativas y producciones matemáticas permitió fundamentar las interpretaciones en datos empíricos. Asimismo, se reconoció el posible sesgo derivado del doble rol docente-investigador, el cual fue atendido mediante el registro sistemático de observaciones y la reflexión continua sobre el proceso de intervención.

Resultados

La implementación del enfoque Building Thinking Classrooms (BTC) en el curso de Geometría plana y del espacio permitió identificar transformaciones tanto en el razonamiento geométrico como en los procesos metacognitivos de los estudiantes. El análisis se realizó a partir del modelo de niveles de Van Hiele y de las categorías de metacognición: planificación, monitoreo y evaluación.

En una primera fase, las tareas de tipo low floor, como la identificación de figuras geométricas en el entorno y su modelación en GeoGebra 3D, favorecieron el nivel de visualización. Sin embargo, además del reconocimiento de formas, se observaron manifestaciones iniciales de conciencia metacognitiva. Por ejemplo, cuando un estudiante expresa: “Me di cuenta de que muchas construcciones en el pueblo usan cuerpos geométricos modificados”, se evidencia un proceso de monitoreo cognitivo, al reflexionar sobre cómo interpreta su entorno a partir del conocimiento matemático. En este proceso, el uso de GeoGebra funcionó como un mediador que permitió contrastar representaciones empíricas con modelos formales, favoreciendo el tránsito hacia niveles de análisis.

En tareas de mayor demanda cognitiva (high ceiling y open middle), se identificó un avance hacia el nivel de ordenación (deducción informal) en aproximadamente el 80% de los estudiantes. En la actividad sobre el baricentro, la afirmación: “El baricentro es como el punto de equilibrio porque siempre queda dentro del triángulo” muestra no solo comprensión conceptual, sino procesos de evaluación metacognitiva, al reconocer una propiedad invariante y justificarla a partir de la experiencia y la modelación.

De manera similar, en la generalización de la fórmula de la distancia, los estudiantes dejaron de aplicar procedimientos de forma mecánica y comenzaron a reconstruir su origen. Este proceso implicó planificación estratégica—al decidir cómo trasladar el razonamiento de la recta numérica al plano cartesiano—y monitoreo continuo, al verificar sus resultados ante diversas condiciones.

El análisis diagnóstico y final permitió identificar que el 100% de los estudiantes se mantuvo en el nivel de visualización (nivel 0) y evidenció dominio del nivel de análisis (nivel 1), mientras

que el 80% alcanzó el nivel de ordenación (nivel 2), construyendo generalizaciones y justificando relaciones. No se observaron evidencias correspondientes a los niveles superiores (deducción formal y rigor), lo cual resulta consistente con el momento formativo de los participantes.

Desde una perspectiva metacognitiva, el tránsito del nivel 1 al nivel 2 no solo implicó mayor complejidad conceptual, sino una transformación en la regulación del pensamiento: los estudiantes comenzaron a explicar el "por qué" de los procedimientos, y no únicamente el "cómo".

En relación con las condiciones estructurales del aula, características del enfoque BTC—como la organización aleatoria de equipos, el uso de pizarras verticales y la restricción de un solo marcador por equipo—generaron dinámicas que favorecieron la metacognición colectiva. Antes de registrar sus respuestas, los estudiantes debían consensuar estrategias, lo que promovió procesos de planificación compartida, monitoreo grupal de errores y evaluación conjunta de las soluciones.

Asimismo, las intervenciones docentes, basadas en preguntas abiertas como "*¿Funciona con números negativos?*" o "*¿puedes demostrarlo en otros casos?*," propiciaron una forma de metacognición guiada, en la que el cuestionamiento intencional sostuvo la reflexión sin anular la autonomía de los estudiantes.

Las entrevistas semiestructuradas permitieron identificar niveles más complejos de reflexión metacognitiva. Por ejemplo, la afirmación "Aprendí a construir mis propias generalizaciones utilizando mis conocimientos previos" evidencia conciencia sobre el propio proceso de aprendizaje, mientras que expresiones como "Ahora recuerdo cómo calcular la distancia porque comprendo su origen" reflejan procesos de evaluación retrospectiva. De igual forma, el reconocimiento de que "El proceso es tan importante como el resultado final" sugiere una dimensión metacognitiva asociada a la valoración del aprendizaje.

En conjunto, los resultados muestran que la metacognición no se desarrolló como un contenido explícitamente enseñado, sino como un proceso emergente derivado de las condiciones didácticas generadas por el enfoque BTC. Particularmente, el tiempo de reflexión individual favoreció la planificación inicial, mientras que el trabajo colaborativo promovió el monitoreo y la evaluación a través de la interacción y la confrontación de ideas. El uso de pizarras verticales contribuyó a hacer visible el pensamiento matemático, facilitando la revisión constante de estrategias y la construcción colectiva del conocimiento.

En síntesis, los resultados sugieren que la implementación del BTC no solo favoreció el avance en los niveles de razonamiento geométrico, sino que también promovió el desarrollo de una conciencia metacognitiva situada, vinculada tanto al aprendizaje como a la futura práctica docente.

Discusión y conclusiones

Los resultados obtenidos sugieren que la metacognición puede emerger como consecuencia del diseño estructural del aula, más que como resultado exclusivo de la instrucción explícita sobre estrategias de autorregulación. En concordancia con Schoenfeld (1985, 2016), los procesos de planificación, monitoreo y evaluación se activan cuando las tareas matemáticas exigen toma de decisiones, validación de procedimientos y argumentación, tal como se evidenció en las actividades desarrolladas durante la intervención.

En este sentido, el enfoque Building Thinking Classrooms (Liljedahl, 2021) no enseña la metacognición de forma directa; más bien, genera condiciones organizativas y normativas que favorecen la explicitación y regulación del pensamiento matemático. Elementos como la

resolución de problemas no rutinarios, la discusión colectiva y la validación de ideas contribuyen a normalizar el aprendizaje a partir del error y a desplazar la atención de la respuesta correcta hacia el proceso de razonamiento.

Desde una perspectiva sociocultural (Vygotsky, 1978), los hallazgos pueden interpretarse en términos de una regulación cognitiva que se construye inicialmente en el plano interpsicológico para posteriormente interiorizarse. Las prácticas propias del BTC—como el trabajo en grupos aleatorios, el uso de superficies verticales no permanentes y la interacción constante—favorecieron esta regulación compartida. Este resultado coincide con los planteamientos de Mevarech y Kramarski (2014), quienes destacan el carácter social de la metacognición en el aprendizaje matemático. En consecuencia, la metacognición observada en este estudio puede entenderse como un fenómeno situado, mediado por la interacción y las normas de participación en el aula.

Por otro lado, el tránsito entre los niveles de Van Hiele (1986) estuvo acompañado por el desarrollo progresivo de procesos metacognitivos. En particular, el paso del nivel de análisis al de ordenación implicó no solo una mayor complejidad conceptual, sino también cambios en la forma en que los estudiantes justifican, evalúan y validan relaciones geométricas. Este hallazgo es consistente con lo planteado por Clements y Battista (1992), quienes sostienen que el avance en el razonamiento geométrico requiere experiencias que promuevan la reflexión sobre las propiedades y sus interrelaciones.

Asimismo, se identificó una transición desde una concepción algorítmica de la geometría hacia una comprensión más relacional y reflexiva, en términos de Skemp (1976). La reconstrucción de fórmulas a partir de su significado, como ocurrió con la fórmula de la distancia, sugiere que los estudiantes comenzaron a privilegiar la comprensión conceptual sobre la aplicación mecánica de procedimientos.

Desde la perspectiva de la formación inicial docente, los resultados indican que la experiencia no solo impactó el aprendizaje matemático, sino también la forma en que los estudiantes conciben la enseñanza. Esto dialoga con la noción de conocimiento especializado del profesor de matemáticas (Ball et al., 2008), particularmente en su dimensión pedagógica, al evidenciar que comprender la estructura conceptual de los contenidos fortalece la capacidad para enseñarlos. En este sentido, la intervención contribuyó a la construcción de una identidad docente reflexiva, en línea con lo planteado por Zeichner (2010).

No obstante, es importante reconocer algunas limitaciones del estudio. En primer lugar, la duración de la intervención pudo haber sido insuficiente para favorecer el desarrollo de niveles más avanzados de razonamiento geométrico, lo que explicaría la ausencia de evidencias en los niveles de deducción formal y rigor. Asimismo, el doble rol del docente como investigador puede haber influido en la interpretación de los datos, a pesar de los esfuerzos por garantizar la triangulación y la sistematicidad del análisis. Finalmente, el carácter exploratorio del estudio y el contexto específico de formación inicial docente limitan la generalización de los resultados.

En conjunto, los hallazgos permiten afirmar que el diseño de entornos de aprendizaje orientados al pensamiento, como los propuestos por el BTC, no solo favorece el desarrollo del razonamiento geométrico, sino también la emergencia de procesos metacognitivos y la construcción de una perspectiva docente centrada en la comprensión y la reflexión.

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CONSTRUCTING PRESERVICE TEACHER AGENCY WHEN ANALYZING TASKS: A SELF-STUDY

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Through self-study, this research focuses on the challenges and tensions we faced as we implemented a semester-long project to support preservice teachers in selecting and analyzing high-quality math tasks. While several frameworks exist to analyze high-quality tasks, we observed that pre-service teachers continued to struggle. Grounded in social constructivism, we engaged our students in co-constructing rubric criteria for selecting a mathematics task based on their learning in our methods courses. Through four cycles of practice, reflection, and questioning, we implemented and revised our semester-long project while analyzing student work. Through this iterative process, we realized that some preservice teachers' focus was limited to a single, often superficial meaning of the written task, prompting rubric changes to include "teacher edits." Our findings point to tensions related to ambiguity, professional authority, and contextual decision-making.

Keywords: Instructional Activities and Practices, Preservice Teacher Education, Teacher Educators

Our interest in pre-service teachers' (PSTs') analysis of tasks began to take shape early in our mathematics teacher educator careers when the three authors participated in AMTE's STaR Fellowship. As we discussed our teaching, we noticed our PSTs experienced similar struggles when asked to identify high-quality mathematical tasks. As we engaged PSTs in activities aligned with best mathematics teaching practices outlined in NCTM's *Principles to Action* (2014), and the Mathematics Teaching Framework's (Smith & Stein, 1998) levels of cognitive demand, our PSTs still did not consistently grasp the concept of identifying high-quality mathematics tasks. Recognizing these common struggles among our PSTs, we embarked on an in-depth collaborative self-study. Using self-study, we engaged with two research questions as we analyzed student work samples and our own documented experiences: (a) What challenges and tensions do we face when fostering PSTs' ability to choose high-quality mathematics tasks? (b) How can we develop methods to address the challenges and tensions we identify?

Literature Review

High-quality mathematical tasks are central to quality mathematics teaching that supports the development of reasoning skills, effective mathematical discourse, and conceptual understanding (NCTM, 2014; Smith & Stein, 1998). Task selection is critical since "no other decision that teachers make has a greater impact on students' opportunity to learn" (Lappan & Friel, 1993, p. 524). Students who regularly engage with high-quality tasks demonstrate greater learning gains than those who work primarily with low-level tasks (Boaler & Staples, 2008; Stein & Lane, 1996; Tarr et al., 2008). Consequently, developing PSTs' capacity to distinguish and select high-quality tasks is a key goal of mathematics methods courses, as emphasized in the *Standards for Preparing Teachers of Mathematics* (AMTE, 2017). In addition, PST's need to develop professional agency, the capacity to make informed, intentional pedagogical decisions within and beyond curricular structures (AMTE, 2017), which requires a willingness to adapt tasks.

Despite this emphasis, research consistently documents PSTs struggle to identify or select high-quality tasks. Studies report mixed results when PSTs use established frameworks to evaluate tasks (Lee et al., 2023; Osana et al., 2006). Similar challenges appear when PSTs use the Worthwhile Task Evaluation Guide (Kartal et al., 2020) or lists of task characteristics (Nicol & Crespo, 2006). We observed challenges in our methods classes, motivating our collaborative self-study. Self-study is widely recognized as a powerful methodology for examining and questioning the complexities of teacher education practice, surfacing tensions, and refining instructional approaches (Hauge, 2021; LaBoskey, 2004; Loughran, 2004; Zeichner, 1999). Engaging colleagues introduces multiple lenses for analyzing problems, helps to reframe practices, and allows for transparent examination of problems of practice that deepen understanding and enhance pedagogy in ways that extend beyond individual reflection, as thoughts are challenged and transformed by others (Hauge, 2021; Pithouse-Morgan, 2022).

Methods

This research used a collaborative self-study methodology, centered on cycles of practice, reflection, and questioning, to interrogate our teaching practices of fostering PSTs' ability to select and to analyze high-quality mathematics tasks. Through sustained questioning and reflection, our practices intentionally evolved to better align with constructivist theories and improve students' task selection. As practitioner researchers, we systematically examined our practices throughout this self-study (Shaw & Lunt, 2018). We chose to conduct a self-study because our inquiry aligned with Vanassche and Kelchtermans' (2015) themes of exploring pedagogical impacts, articulating practice philosophy, and analyzing belief-practice gaps.

While recommended data collection and analysis approaches exist (Loughran & Northfield, 1998), self-study allows flexibility (Loughran, 2004). Data sources included PST-constructed rubrics, task analysis completed by PSTs throughout the courses, our reflections on instructional decisions, and notes from our collaborative reflective meetings. These constitute the base for our interpretations, consistent with the need for authentic artifacts and reflective practices in self-study (LaBoskey, 2004; Pinnegar & Hamilton, 2009). Data was collected across two large public research universities and a large public teaching university. Demographics, expectations, and structures varied across sites. As such, attending to these differences while identifying similarities was essential for interpreting PSTs responses (Vanassche & Kelchtermans, 2015). Our inquiry cycle had four phases: (1) reflecting on student work to identify challenges and alignment with constructivist theories; (2) revising strategies to address misalignments; (3) implementing new methods; and (4) collecting data used for reflection in the first phase of the next cycle. Within these cycles, data was analyzed through descriptive coding, dialogic interpretation, and reflective analysis (Saldaña, 2016; Nolan, 2010; Schuck & Russell, 2005; Tzur, 2001). Consistent with self-study's call for openness and vulnerability (LaBoskey, 2004; Mason, 2007), tensions emerged inductively from analysis.

Across these cycles of collaborative self-study, we identified tensions between research and practice, confidence and uncertainty, and autonomy and agency, which are common themes in teaching self-studies (Berry, 2004). These tensions helped us acknowledge complexities and evolve our practices, thereby enhancing our professional growth. The cyclic nature of this process allowed us to develop and test ideas to improve our practice for students' benefit (LaBoskey, 2004). This led to our adoption of broader perspectives and, ultimately, improved pedagogy (Loughran, 2004). Rather than generalizability, our goal, consistent with self-study, was working hypotheses about our practices (Lincoln & Guba, 1986).

Results

We began the first cycle of collaborative self-study inquiry shortly after we discovered our shared interest in PSTs' task selection. Starting with a reflection phase, we questioned each other about how we developed our PSTs' understandings of what makes a high-quality task and how to select one. We recognized our PSTs often favored traditional methods of teaching mathematics they experienced in their K-12 schools. Our PSTs often felt overwhelmed as they attempted to consider the multiple facets of mathematics tasks. One of our shared practices was to provide our students with tools for task selection, primarily Smith and Stein's (1998) Mathematics Task Framework (MTF) and Van de Walle et al.'s (2023) Worthwhile Task Evaluation Guide. We noticed our PSTs were uncomfortable and unsure when tasks met some of the criteria in the framework but not others. Additionally, as classroom authority figures, our PSTs were hesitant to make anything but positive judgments on tasks we provided.

In considering criteria for high-quality tasks, we recognized that Smith and Stein's (1998) framework focused solely on the cognitive demand of the task and did not allow for a holistic analysis. Conversely, Van de Walle et al.'s framework descriptions are often broad and integrate multiple concepts that may be new to developing PSTs within a single category, leaving PSTs feeling overwhelmed when using it. As we questioned the alignment of our practices with social constructivism, we began to contemplate what it might look like for PSTs to develop their own analytical strategies that fostered autonomy and stronger feelings of professional agency over the curriculum. Attempting to meet these goals, we each agreed to integrate a semester-long project during which PSTs would collaboratively construct a tool to analyze the quality of mathematics tasks (Long et al., 2025). That tool would take the form of a rubric, and our PSTs would be asked to reflect on their learning throughout our methods classes to collaboratively construct criteria and evidence for the rubric. We also decided to have our PSTs use the rubric to evaluate tasks they and their peers found online, to avoid presuming quality.

In our second round of reflections, as students were co-constructing rubrics to analyze the quality of tasks in each of our classes, a second challenge emerged: our PSTs sometimes responded to tasks in ways that demonstrated a perceived lack of autonomy and of our authority over them. Many PSTs did not consider how a teacher might implement a task beyond the written words. For example, one PST stated that students could not use tally marks to complete a place value task because the task did not explicitly state that paper and pencil were to be supplied. Thus, PSTs demonstrated their belief that teachers have little agency over curriculum and therefore did not consider the deeper potential of tasks. These beliefs may stem from PSTs' experiences with heavily scripted curricula in their own education, which can position teachers as implementers rather than designers of curricula (Aguirre et al., 2013).

As this pattern emerged, we drew on transactional theory (Raymond, 2021; Rosenblatt, 1978), used within the literature discipline, which posits the meaning created while reading depends on the stance of the reader. Readers may adopt an aesthetic stance, in which they read to create multiple possible interpretations of written text, or an efferent stance, in which the focus is limited to a single, often superficial meaning. As we became aware our students might adopt efferent stances, we began to consider ways to encourage a more aesthetic stance among PSTs. In response, we revised the initial framework from traditional categories of "exceeds expectations," "meets expectations," or the like, to adopt categories representing *teacher edits*. Namely, when analyzing a mathematics task against PST co-created criteria, the pre-service teachers decided if the task "meets as written," "requires teacher edits," or the task was "unable

to meet with teacher edits." This shifted PSTs' perspective as they analyzed tasks, prompting them to consider how they might meet expectations for high-quality tasks.

After implementing the revised framework, we engaged in a third round of reflections and observed growing PST confidence and ability. However, we also noticed PSTs often did not connect task decisions to the context in which the task would be used. When we asked PSTs to analyze a task, we provided context such as grade level, class description, a mathematics standard, and when the standard was developed (e.g., "you want to develop students' fluency with…"). Yet PSTs rarely considered these descriptions or adjusted their criteria based on the context. For example, some tasks support new conceptual understanding, while others foster procedural fluency, yet PSTs' analyses often favored criteria that valued one type over the other.

Discussion

We uncovered multiple tensions in our work with PSTs around task analysis. The first tension we came to recognize was our desire to develop clear expectations for what constitutes a high-quality task, while also recognizing the inherent ambiguity in teaching that PSTs must learn to navigate. This tension persists as we recognize a single tool may not capture the nuances needed to analyze the use of a task for a particular purpose or context, even when it is constructed by PSTs themselves. The second tension we uncovered related to PSTs' perceptions of themselves as novices, as we engaged them in the work of a professional educator. We asked our PSTs to analyze products we presented to them, which are typically created by an unknown person whom they presume has more experience and knowledge. As a result, our PSTs hesitated to alter or enhance tasks. By incorporating the idea of *teacher edits* to task analysis, we reframed alterations not as questioning the authority of their instructors or curriculum writers, but as a professional expectation of teachers. This change supported our PSTs as autonomous professionals capable of making curricular decisions. The final tension we uncovered was the need to recognize that while there are common criteria used to assess the quality of a mathematical task, the weight given to each criterion may depend on the purpose of the task. Each tension uncovered related to PSTs' abilities to assert agency over the tasks they were analyzing. As we became aware of each of these tensions, we were purposeful in constructing strategies for addressing these tensions that were aligned with social constructivism to better model constructivist pedagogies and to foster our PSTs' feelings of autonomy. Thus, uncovering these tensions allowed us to improve our practices and frame our PSTs as emerging educators.

Conclusion

Through collaborative self-study, we identified tensions related to ambiguity in defining task quality, PSTs' developing professional agency, and the contextual weighting of task criteria that required us not only to revise our semester-long project but also to reconsider how our practices positioned PSTs in relation to curriculum and professional decision-making. Encouraging a more aesthetic stance toward tasks (Raymond, 2021; Rosenblatt, 1978) supported PSTs in viewing curriculum as adaptable rather than fixed. This shift strengthened their confidence and more fully aligned our own instructional practices with social constructivism. Our findings suggest that evaluating high-quality tasks cannot be separated from considerations of instructional purpose, timing, and context. Although PSTs demonstrated growth in selecting and revising tasks, they continued to struggle with determining when and for what purposes particular tasks were most appropriate. Future research should further explore how teacher education can explicitly support PSTs in weighing task criteria differently depending on instructional goals and classroom contexts. Ultimately, fostering PSTs' capacity to select and analyze high-quality mathematics

tasks involves cultivating professional agency, comfort with ambiguity, and the ability to situate curricular decisions within purposeful instructional contexts.

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STUDENT RESPONSES TO QUESTIONS REFERENCING PEERS' CONTRIBUTIONS IN NUMBER TALKS

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Research on mathematics classroom discourse has emphasized both students' participation in discussion and the cognitive demand of their mathematical talk, yet these dimensions are often examined separately. Building on prior analyses of teacher question types, and our tracing of how these questions are used to provide opportunities for students to engage with each other's thinking, we integrate social network analysis with a cognitive-level discourse framework to trace students' responses to teacher questions that position them to engage with peers' mathematical contributions during Number Talks. Findings illustrate that teacher questions referencing peers' mathematical contributions can position students in distinct participation–reasoning roles, shaping both how students engage with peers' thinking and the cognitive level of discourse that emerges from those interactions.

Keywords: Classroom Discourse, Instructional Activities and Practices

Classroom discourse is widely recognized as central to students' opportunities to develop mathematical understanding, particularly when students attend to and respond to one another's mathematical thinking (Chapin et al., 2003; Lampert & Cobb, 2003; Webb et al., 2014). Research has further shown that teacher questioning plays a critical role in shaping these opportunities by structuring who participates in discussion and whose contributions become the focus of collective sense-making (Boaler & Brodie, 2004; Franke et al., 2009; Smith & Stein, 2011). Thus, learning opportunities in mathematics classrooms are not evenly distributed, but are mediated through patterns of questioning, response, and uptake during classroom interaction.

Our prior work examined teacher question types (Joswick et al., 2025a) and the peer-referencing discourse structures (Joswick et al., 2026), revealing patterns of who discusses whose mathematical contributions during Number Talks. Together, these studies described how teacher questioning structures opportunities for participation, but they did not examine students' responses within those opportunities. Thus, although this work clarifies how questioning shapes participation and whose ideas become the focus of discussion, it leaves unresolved the nature of students' reasoning in these positioned moments—and the learning such participation affords.

The present study centers students' responses within teacher-structured opportunities to respond in relation to peers' mathematical contributions and examines the cognitive level of the discourse (Weaver et al., 2005) those responses produce. By integrating attention to participation structure with analysis of cognitive demand, this work aims to more fully characterize the learning opportunities afforded through classroom discourse during Number Talks. Guided by this purpose, we ask: *How are teacher-structured opportunities to engage with peers' mathematical contributions taken up by students, and what cognitive levels of discourse emerge from those interactions?*

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Methodology

Data Collection

Our data come from 17 teachers enrolled in an online master's course designed to support adoption of classroom Number Talk practice, often new to participants. We used a tailored teacher learning cycle (Joswick et al., 2020) grounded in *Teacher Education by Design* (2014), involving cycles of learning, planning, implementation, reflection, and peer feedback. Teachers submitted three videos of successive Number Talk enactments using problems drawn from Number Talk literature (e.g., Humphreys & Parker, 2015; Parrish, 2010). We analyze the third, culminating video transcribed from these small-group Number Talks (3–5 students).

Data Coding

To trace how students interact with one another and each other's ideas during Number Talks, we coded the data to identify dialogic ties—opportunities for students to respond or build on peers' solutions, strategies and ideas—using our four-phase discourse analysis coding, illustrated in Figure 1. At Phase 1, we identified the question type (e.g., *Orient*) a teacher asked using our established framework (Joswick et al., 2025a). For Phase 2, we further analyzed each question to find those specifically asking students to consider their *peers' thinking* (specifically around their problem *solutions*, *strategies* used to solve, and mathematical *ideas*). Phase 3 allowed us to track directionality—who the teacher invited to answer, who responded, and whose thinking was discussed. The final phase of coding analyzed student responses, their cognitive level of discourse, using the cognitive-demand framework of Weaver et al. (2005), operationalized in our prior work to characterize the depth and complexity of students' mathematical reasoning within discourse (Joswick et al., 2025b). The research team coded independently before reconciling discrepancies to reach 100% agreement (Halpin, 2024).

Phase 1	**Phase 2**	**Phase 3**			**Phase 4**
Teacher question type	Whose/what thinking is asked about	Who was invited to answer	Who answered	Whose thinking was discussed	Student cognitive level of discourse
Teacher 58: Okay, [Student T58a] says 48. If you agree with [Student T58a], do "me too."			Student T58d gives the class signal for "me too."		
Orient (*Orient* is a prompt participation and can support students' communication development, emphasize elements of a strategy, or garner interest in a student's thinking (Joswick et al., 2025a)).	*peer solution*	*T58b, T58c, T58d*	*T58d*	*T58a*	*answering* (*answering* is a response that repeats an answer, signals agreement, or states a result without justification or connection to mathematical reasoning (Weaver et al., 2005)).

Figure 1: Four-Phase Discourse Coding, Illustrated by Teacher 58 and Four Students

Data Analysis

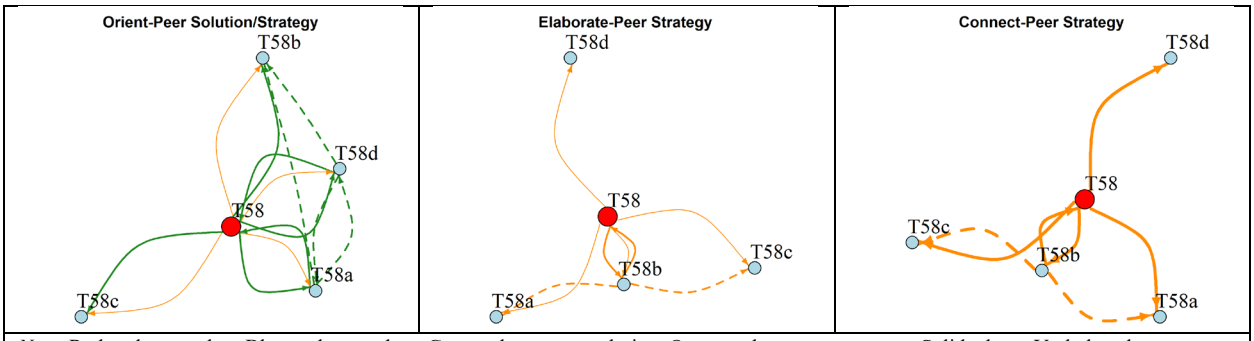
To analyze the data via Social Network Analysis (SNA) (Borgatti et al., 2022), we transformed the data into frequency matrices (i.e., layers by phase 1, 2, and 4 codes) to create a multiplex network (Magnani et al., 2021). This captures layers of student participation and cognitive discourse level with each other's thinking with directional ties (i.e., phase 3 coding), leveraging our prior SNA of student participation in collaboration networks (Sanders et al., 2025). Using R, we visualized each network with the teacher as the ego and students as alters (Perry et al., 2018). We then calculated network descriptives to examine network cohesion and teacher (i.e., ego) influence (Borgatti et al., 2022) as well as alter level descriptives—normalized eigenvector centrality (i.e., how well-connected a Number Talk participant is to other well-

connected participants).

Selected Findings

For this brief report, we highlight selected findings using illustrations from 4th year teacher, T58, and their grade 4 group of four students: T58a, T58b, T58c, and T58d. Prior to the graduate course T58 was enrolled in, they had limited experience with Number Talks. In their Number Talk, the following problems were posed successively: *2x3x8, 4x2x6,* and *6x8.* T58 was selected as an illustrative case because the Number Talk contained multiple instances of peer-referencing *Orient, Elaborate*, and *Connect* questions.

To foreground the student perspective, we present findings organized around how individual students encountered teacher-structured opportunities to engage with peer thinking embedded in *Orient*, *Elaborate*, and *Connect* questions (Joswick et al., 2025a) and the cognitive levels of discourse (Weaver et al., 2005) evident in their responses. These particular question types are analytically salient because they move classroom discourse beyond the sharing and clarification of strategies toward opportunities for students to collectively reason about numbers and build conceptual connections. Examining these moments through a four-phase analytic structure (see Figure 1) allows us to trace how positioned opportunities to engage with peers' mathematical thinking were taken up in qualitatively different ways (see Figure 2). From a student-level perspective, the analysis revealed three distinct participation–reasoning profiles: a central explanatory contributor (T58b), students positioned as idea sources without response authority (T58a, T58c), and a peripheral participant limited to agreement-level discourse (T58d).



Note. Red node = teacher, Blue node = student, Green edge = peer solution, Orange edge = peer strategy, Solid edge = Verbal exchange, Dashed edge = Connection to peer thinking, No bold = *no answer*, bold = *answering*, Moderate bold = *emerging explaining*, Heavy bold = *relating*

Figure 2. Network Visualization Across Selected Question Types

Student T58b: Central Explanatory Contributor

Student T58b emerged as the most active and oftentimes most central student across the *Orient*, *Elaborate*, and *Connect* questions regarding peer thinking. For instance, T58b had normalized eigenvector centralities of 1.00 (*Orient*), 1.00 (*Elaborate*), and 0.71 (*Connect*) indicating that T58b was well-connected to other well-connected participants (i.e., teacher and students) in the Number Talk, more well-connected than the teacher except for connect questions. The cognitive level of discourse of T58b's responses and engagement with *peers' thinking* varied systematically.

T58b solely engaged in the *emerging explaining* cognitive level of discourse (i.e., partially explaining a mathematical idea in direct response by stating a description of what the student

did) for *Elaborate* questions. When T58 then shifted to another *peer's strategy* ("Ok, what did he split?" referring to T58a), T58b again supplied the explanation, "The 2 and the 8. And then he got 16 and multiplied it by 3 then." In contrast, for *Connect* questions, T58b responded with a *relating* cognitive level of discourse (i.e., describing a connection or relationship to prior knowledge or experience). T58b's *relating* cognitive discourse response indicates how T58a's and T58b's strategies are related: "[T58a] also, uh did, um splitting it. But he split it in a different way." Despite not answering an *Orient* question, T58b's solutions were positioned centrally demonstrating influence through ideas.

Students T58a and T58c: Influence Through Ideas

In contrast to T58d (peripheral participant with *peer thinking*) and T58b (central contributor to *peer thinking*), students T58a and T58c were positioned by T58's questions as influential students through centering their solutions and strategies. In *Connect* questions, T58a had moderate normalized eigenvector centrality (0.64), indicating that despite not answering teacher questions, their *strategies* were positioned as valuable to the discussion by both T58 and T58b through questions such as, "Okay, what about [T58a]? How is [T58a]'s different or similar [from T58c]?" This particular question represents an opportunity where T58b was able to engage with T58a's strategy at a *relating* cognitive level of discourse. Similarly, in an *Elaborate* question ("Which numbers did [T58c] break?"), T58 offered up T58c's strategy for further explanation by T58b, amplifying T58c's thinking and increasing their normalized eigenvector centrality to 0.60.

Student T58d: Peripheral Participant Limited to Agreement-Level Discourse

T58d represents a student that was mostly on the periphery of the Number Talk participation. T58d is a unique case in that their only response to T58's questions centering peer thinking was for *Orient* where T58d responded to T58a's solution using the Number Talk, non-verbal hand signal for "me too" (Okay, [T58a] says 48. If you agree with [T58a], do "me too", from T58). T58d's response represents am *answering* cognitive level of discourse (i.e., short answer to a direct question from the teacher). T58d achieved their highest network position in the *Orient* network with a normalized eigenvector centrality of 0.77 while across the *Elaborate* and *Connect* questions, T58d only had connections to T58 through directed questions, but T58d did not respond to these questions or have their thinking taken up by other students.

Discussion & Conclusion

This study contributes to research on mathematics classroom discourse by integrating attention to participation structure with analysis of the cognitive level of student responses within teacher-structured opportunities to engage with peers' mathematical thinking. Our findings demonstrate how students are positioned within the group to respond to *peers' thinking*, based on their participation. As illustrated by T58, the findings highlight that students participate with *peers' thinking* in different ways and that not all students are either given or take up opportunities to engage with peers thinking in a manner that ensures higher cognitive levels of discourse. Rather, students' uptake of these positioned opportunities varies, producing differentiated access to explanation, comparison, and agreement within the same Number Talk. These patterns suggest that engagement with peers' mathematical thinking is not a uniform experience, as students occupied different participation–reasoning roles associated with different levels of cognitive discourse. Methodologically, integrating our four-phase discourse coding with multiplex social network analysis provides a sensitive lens for tracing how cognitive discourse connects through participation structures. This work adds nuance to whose ideas are being discussed, what type of questions are being asked to whom, and how different question types contribute to different cognitive levels of discourse. This approach extends prior discourse

research (e.g., Joswick et al., 2025a, 2025b) by making visible the relationship between positioned opportunity and cognitive uptake.

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ELEMENTARY TEACHERS' PERCEPTION AND INTERPRETATION OF LEARNING THEORIES IN MATHEMATICS TEACHING AND LEARNING

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Despite the significant impact of learning theories on instructional design and implementation, they remain subject to practitioners' direct examination. This study explored how elementary pre-service teachers (PSTs) interpret learning theories in the context of mathematics education. Nineteen PSTs from a graduate-level mathematics methods course selected one learning theory (among behaviorism, cognitivism, constructivism, sociocultural learning, and experiential learning) that best aligned with their experience with mathematics teaching and learning and explained their choice. Experiential learning was most aligned with PSTs' experience, and behaviorism was prioritized for designing learning environments. PSTs' selections of theories appeared to balance acknowledgment of the theories with the practicality of implementing them, foregrounding their competence and value as examiners of learning theories.

Keywords: Learning Theory, Teacher Beliefs, Teacher Knowledge, Preservice Teacher Education

Background

As education developed as a rigorous social science subject, theory and research evidence about teaching and learning have played an increasingly important role in educational decision-making (Darling-Hammond et al., 2020). Specifically, learning theories, which reveal how learning actually happens, have informed and shaped the design and implementation of instructional practices (Ertmer & Newby, 2013). Within the field of mathematics, constructivism has guided a set of reform-based practices towards student-centered instruction focused on conceptual understanding (National Council of Teachers of Mathematics [NCTM], 2014). Prior studies highlighted the positive effects of integrating theories into mathematics teacher education to deepen their understanding of how students learn specific concepts (Ivars et al., 2020), thereby further developing their professional knowledge and beliefs (Cross Francis et al., 2025; Oonk et al., 2015).

However, achieving synergy between theory and practice may be hindered by the generalized knowledge valued by the researcher community versus the practical knowledge valued by the teacher community (Buchmann, 1984). Teachers' interpretation and understanding of theory are critical for addressing the tension between theory and practice, and maintaining the fidelity of, and their enthusiasm for, designing and implementing theory-driven instruction (Allen & Wright, 2014; Lendínez Muñoz et al., 2023). Pre-service teachers (PSTs) tend to face more tensions since they are enrolled in programs with theory-heavy courses and are often simultaneously involved in field experiences (Cheng et al., 2012; Oonk et al., 2020). Therefore, it is important to understand how PSTs interpret theory and envision integrating them into their instruction.

This study explores how elementary mathematics PSTs perceive and interpret five learning theories in relation to mathematics teaching and learning. We focus on answering two research questions: (1) What learning theories do elementary PSTs perceive as most aligned with their experience of learning and teaching mathematics? (2) What experiences did PSTs draw on to support their choices and interpretation of the learning theories?

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Learning Theories Associated with Mathematics Teaching and Learning

Behaviorism is a theory of learning that foregrounds the use of positive and negative reinforcement to condition certain behaviors. In mathematics teaching and learning, this explains that students need to be conditioned to respond to specific mathematical procedures through repeated practices (Byrnes, 1992; DeLay, 1996). Later reform movements in mathematics led to a shift from behaviorism towards an emphasis on constructivism and other learning theories (e.g., cognitivism and sociocultural learning theory) (Anderson et al., 1999; NCTM, 2014). Cognitivism, deviating from behaviorism's notions of learning as a change in behavior, highlighted the importance of cognitive information processing and the role of prior knowledge and deep thinking in learning math. Constructivist-based teaching shifts the focus of learning from rote memorization of mathematical facts, as implied by behaviorism, and encourages active learning grounded in prior experiences and the student's interaction with the environment. (Bada & Olusegun, 2015). Socio-cultural learning theory argues for mathematics learning through social interactions, emphasizing scaffolding by more knowledgeable others to reach zone of proximal development (ZDP) (Walshaw, 2017). More recently, scholars have argued for the use of experiential learning, which foregrounds learning through engaging in and reflecting on experience (e.g., learning a biological concept through field experience) (Kolb et al., 2014). Experiential Learning implied building early mathematics experience by letting students use manipulatives to explore mathematical ideas (Simon, 2022).

Methods

Participants and Context

Nineteen PSTs enrolled in a graduate-level elementary mathematics method course participated in the study, and most of them are white females. The methods course is grounded in constructivist-oriented mathematics teaching practices while also exposing PSTs to other learning theories to facilitate a deep understanding of students' learning of mathematics. Meanwhile, PSTs are also engaged in field experiences and an experiential learning week within the graduate teacher education program.

Data Collection

The data source for this study included PSTs' written reflections on five learning theories (behaviorism, cognitivism, constructivism, sociocultural theory, and experiential learning) related to mathematics teaching and learning. These theories were not considered exclusive, as they overlap in ideas and may be situated at different scopes. Data were collected during the fifth week of a 15-week course. At this point, PSTs have engaged in their one-week experiential learning set of activities, read about behaviorism and socio-cultural theory, and read and discussed constructivism as a part of their coursework in the methods class. PSTs were asked to select the theory most aligned with their mathematics teaching and learning following these two prompts: (1) how mathematics learning occurs and what role the mathematics teacher plays according to this theory, and (2) the reasons they selected the theory.

Data Analysis

For research question 1, we determined the frequency with which each learning theory was mentioned by PSTs. For research question 2, we summarized the teaching and learning experiences they described as reasons for selecting each learning theory.

Findings

Research Question 1: Choice of Learning Theories

Table 1 shows the number of PSTs who thought each theory most closely echoed their

experiences or reflected high-quality mathematics teaching and learning. It is worth noting that the sum of the percentages is not 100%, as some PSTs may select multiple theories.

Table 1: Frequency of PSTs' Selection of Learning Theories

	Behaviorism	Cognitivism	Constructivism	Socio-cultural Theory	Experiential Learning
Counts (Percentage)	4 (21%)	3 (16%)	2 (11%)	1 (5%)	10 (53%)

Research Question 2: Interpretations and Reasons for Learning Theories

Behaviorism. Four PSTs indicated that they related most to behaviorism, based on their own experiences of learning and math teaching, while their interpretations presented different orientations. Emma and Jackson, interpreted behaviorism as an approach to build a positive environment in which certain behaviors (i.e., trying to solve problems rather than feeling afraid to make mistakes) were reinforced. Emma described in her reflection:

> This theory (behaviorism) argues that the likelihood of a behavior increases based on positive or negative reinforcement … Behaviorism is a way to support students' motivation and attitudes towards math, boost engagement and create consistent routines that really work for students.

By contrast, the other two PSTs chose to stick to behaviorism in their future teaching as a compromise with real-world challenges. Eleanor interpreted behaviorism as a rational choice of teachers due to the heavy workload of teachers. Evelyn described the positive results from her learning experience within the behaviorism paradigm, which encouraged her to believe that behaviorist approaches are suitable for teaching mathematics:

> I learned many of the basic principles of math through repetitive practice. We did not use base ten blocks or were rarely taught the "why" behind everything but I was still strong in math throughout school. … The (mathematical) foundations should start to just come naturally instead of taking a long time to process and decode. I believe behaviorism is the best way to achieve this.

Cognitivism. Three PSTs, Aria, Mandy, and Mia, related most to cognitivism and agreed on the importance of students' cognitive engagement when learning mathematics. Mandy and Aria framed cognitivism in a student-centered way, highlighting the role of teachers in engaging students cognitively based on their prior knowledge. However, Mia's framing was more teacher-centered, foregrounding the role of teachers in presenting the information clearly to students and managing their learning process. They all chose cognitivism based on their observations of the value of understanding students' thinking during instruction and of building cognitive connections among mathematical concepts. Mandy described her positive experience of being taught using cognitive approaches, which made her believe in the value of cognitivism:

> I had math taught to me in ways that research suggests is helpful to the learning brain. I remember being instructed to do fill-in-the-blank notes, and taking hand-written notes is supposed to put things into your memory better … As a teacher, I will try to take cognitive approaches.

Constructivism. Two PSTs, Charlotte and Amelia, chose constructivism as the closest to their learning and teaching math. These two PSTs emphasized connecting prior and current knowledge. Charlotte described constructivism as the best way for her to learn mathematics and a valuable way to build students' confidence and enjoyment in learning math through connecting with their background knowledge.

Sociocultural theory. Nora is the only PST echo the most with sociocultural theory. She appreciated the idea of the zone of proximal development and emphasized the role of teachers in providing scaffolding for students. Such appreciation derived from her early learning experiences, where her mother scaffolded her to navigate through the memorization-based mathematics teaching she struggled with.

Experiential learning. Experiential Learning was selected by ten PSTs as the most aligned with their experience of teaching and learning mathematics. They shared the belief that hands-on activities help them build their own knowledge by making mistakes and reflecting on them. Half of them referred to their own experiences of learning math and thought the best way is to "physically do something" (Abigail), "make inquiries and discoveries" (Olivia, against rote memorization), "apply them" (Lucas), and "reflect on the mistakes" (Sophia). Seven PSTs referred to their observations of the benefits of using hands-on or inquiry-based instruction, including "letting students know they are doing great, and mistakes are welcome as opportunities of learning" (Amelia), "building confidence and problem-solving skills" (Jamie), "more engaged when they can manipulate with materials and solve problems with peers" (Chole), and "students' critical thinking skills" (Ava).

Discussion and Conclusion

This study explored how elementary mathematics PSTs perceive and interpret learning theories and led to two major findings. First, this study revealed the complexity of practitioners' interpretations of learning theories and their connection to instruction, aligning with prior studies (Cheng et al., 2012). PSTs' interpretation of theory and why they foregrounded a specific theory is a thoughtful process if they balance their learning experiences, observation of teaching, and their visions for teaching mathematics. For instance, despite recognizing the drawbacks of behaviorist approaches, Eleanor still believes them to be the optimal approach with the heavy workload. These findings are more nuanced than prior studies, which only focused on PSTs' accurate interpretation and use of theory in their teaching (Oonk et al., 2015; Van Katwijk et al., 2019). These findings highlight the importance of providing PSTs with more scaffolds to unpack and reason about theory-based mathematical teaching approaches and navigate the tensions with practical situations (Lendínez Muñoz et al., 2023).

Second, this study challenged the deficit-oriented perspective of attributing the theory-practice gap to practitioners' inability to understand and apply learning science concepts (e.g., McAfee et al., 2021). PSTs were able to draw on learning theories to critically examine their learning experiences and determine which would be most applicable in a specific context. Rather than serving as the golden principle for practice, learning theories could serve as a powerful prompt for PSTs to reflect on their experiences as learners and teachers and how to address tensions between theory and practice (Cochran-Smith & Lytle, 1999). Addressing these tensions is the central task for PSTs, as it involves developing their beliefs and visions for future classroom practice (Feiman-Nemser, 2001; Oonk et al., 2020).

Implications from this study can be twofold. Research-wise, more effort should be made to incorporate practitioners' perspectives into how learning theories are explored and taught in mathematics education. Further investigation into a broader range of practitioners' perspectives

is needed, as context-wise this study is limited within a methods course. Within teacher education, the value of learning theory lies not only as the cornerstone for high-quality practice but also as a resource to build PSTs' instructional vision and to examine their own practice.

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MODELACIÓN POR FUTUROS PROFESORES EN UN CONTEXTO DE PROPORCIONALIDAD INVERSA Y ESTRÉS HÍDRICO

MODELING BY PRE-SERVICE TEACHERS IN A CONTEXT OF INVERSE PROPORTIONALITY AND WATER STRESS

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Se presentan los resultados de un estudio cualitativo cuyo objetivo fue analizar cómo futuros profesores se enfrentaron a un problema cercano a la realidad relacionado con el estrés hídrico. El estudio se sustentó en la perspectiva de modelos y modelación y contó con la participación de tres futuros docentes. Se utilizó una actividad en la que subyacía la proporcionalidad inversa, la actividad solicitaba la elaboración de una carta donde los participantes debían describir un procedimiento para estimar la cantidad de agua disponible por persona en los próximos años ante el crecimiento poblacional. Los resultados evidencian el desarrollo de dos tipos de modelos: un modelo recursivo y un modelo funcional. Durante el proceso de modelación se identificó una preocupación por el estrés hídrico, asociada a la percepción de una disponibilidad de agua cada vez más limitada.

Palabras clave: Modelación, Proporcionalidad inversa

Introducción

La proporcionalidad inversa es un tema de gran relevancia porque muchos problemas de la vida real se pueden resolver a través de ella (Ramírez & Hernández, 2017). Sin embargo, de acuerdo con varias investigaciones (Godino & Batanero, 2002; Ramírez & Hernández, 2017), la comprensión de la proporcionalidad inversa suele presentar dificultades para los estudiantes debido a que cuando se enseña este concepto frecuentemente suele limitarse a un método algorítmico y mecánico y a una manipulación de números de manera aleatoria y descontextualizada. Además, es común que se argumente que dos cantidades x e y son inversamente proporcionales porque "una cantidad aumenta, y la otra disminuye," omitiendo la condición de la constante k de proporcionalidad inversa: $x * y = k$ (Herrera-García et al., 2023).

Dado que la proporcionalidad inversa es un concepto útil para resolver problemas de la vida real, y existen dificultades de comprensión, es necesario cambiar la forma mecánica de enseñarla. Una forma es a través del uso de problemáticas cercanas a la vida real, como las denominadas MEAs (por sus siglas en inglés, Model Eliciting Activity, traducida al español como Actividad Provocadora de Modelos, de acuerdo con Vargas-Alejo et al., 2016), tal como lo propone la perspectiva de modelos y modelación (MMP, por sus siglas en inglés). Ello implica que los profesores las conozcan y aprendan a utilizarlas en el aula.

El propósito de este trabajo es presentar parte del análisis de los resultados de una investigación desarrollada con futuros profesores, basada en la MMP, la cual implica fomentar que ellos conozcan las MEAs, experimenten su resolución y reflexionen sobre el potencial de las mismas para propiciar aprendizaje. La pregunta de investigación es ¿qué modelos construyen futuros profesores al enfrentarse a una actividad sobre el estrés hídrico de su localidad en la que la proporcionalidad inversa está implícitamente involucrada?

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Marco teórico

El marco teórico que sustenta este estudio es la perspectiva de modelos y modelación (Lesh, 2010). La MMP promueve el aprendizaje matemático a través de tareas llamadas Model Eliciting Activities (MEA). Las MEAs "son similares a muchas situaciones de la vida real en las que las matemáticas son útiles" (Lesh & Doerr, 2003, p. 4). Bajo esta perspectiva, de acuerdo con Lesh y Doerr (2003), una MEA requiere la creación de modelos que pasan por procesos de prueba, revisión y refinamiento. El desarrollo de modelos implica ciclos iterativos, donde las ideas se expresan, prueban y ajustan de forma continua a través de las interacciones entre estudiantes y el docente al resolver un problema (Lesh, 2010).

El diseño de una MEA se da con base en seis principios (Lesh et al., 2000; Lesh & Doerr, 2003): principio de la realidad, principio de construcción del modelo, principio de la autoevaluación, principio de documentación del modelo, principio de prototipo simple y principio de generalización del modelo.

Metodología

La investigación fue cualitativa, tipo estudio de caso múltiple (Ponce, 2018) ya que permite analizar los patrones similares y diferenciales entre los individuos ante una misma situación problema, lo cual contribuye a contrastar los diferentes procesos de modelación que emergieron. Participaron tres futuros profesores (A1, A2 y A3) quienes se encontraban estudiando una maestría en enseñanza de las matemáticas. Los futuros profesores tenían estudios universitarios en matemáticas y estaban interesados en ser profesores del nivel medio superior y superior.

La MEA "El estrés hídrico en la ZMG" se construyó con base en los seis principios de diseño. Esta actividad está compuesta por tres partes: a) una nota periodística en la que se describe el estrés hídrico en la Zona Metropolitana de Guadalajara (ZMG), b) preguntas relacionadas con el contexto de la nota periodística que contribuyen a reflexionar sobre la problemática descrita, y c) una situación problema donde se solicita a los futuros profesores la elaboración de una carta en la que se describa un procedimiento para explicar cómo cambiaría la cantidad de agua disponible por persona si la población de la ZMG, la cual era de 5,499,680 en el año 2024, sigue aumentando. El concepto de la proporcionalidad inversa está implícitamente involucrado porque se requiere para analizar cómo en la medida que aumente la población, podría disminuir la cantidad de agua disponible ante la situación actual del estrés hídrico.

La implementación de la MEA tuvo una duración de 90 minutos (como sugiere la MMP) y consistió en las siguientes tres fases. Los futuros profesores resolvieron la MEA de manera individual, enseguida la discutieron de forma grupal y finalmente, comentaron la actividad.

Los instrumentos de recolección de datos fueron grabaciones de audio y las cartas solicitadas en la MEA. Se usaron para analizar los modelos construidos al resolver la situación problema.

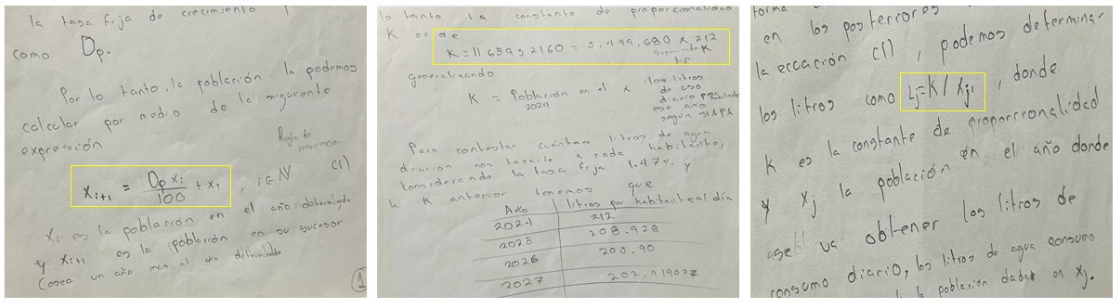
Resultados

Primera fase

Se lograron identificar dos tipos de modelos para resolver la MEA.

Modelo recursivo de proporcionalidad inversa. A partir de la tasa de crecimiento anual y la población inicial, el futuro profesor A1 propuso un procedimiento recursivo (Figura 1) de la siguiente forma: $x_{i+1} = \frac{D_p x_i}{100} + x_i$, donde D_p es la tasa de crecimiento anual, x_i es la población inicial y x_{i+1} es la población al año siguiente. Enseguida, para obtener la población de la ZMG del periodo de 2024–2038, usó fórmulas recursivas en Excel de la forma: $= (G10 * 1.47)/100 + G10$. Además, calculó la constante k de proporcionalidad inversa al multiplicar la población inicial (año 2024) por el dato de los 212 litros diarios de agua que gasta una persona en la ZMG.

Finalmente, en lápiz y papel, escribió la expresión de proporcionalidad inversa siguiente: $L_j = \frac{k}{x_j}$, donde x_j es la población anual y L_j son la cantidad de litros de agua que gasta una persona de la ZMG a lo largo de los años. Con ello predijo la cantidad de agua que le tocaría a cada persona en los siguientes años.



A B C

Figura 1: Modelo recursivo propuesto por A1

Modelos recursivo y exponencial. El futuro profesor A2 creó un modelo recursivo en el cual se observa cómo calculó la cantidad total de habitantes para el periodo 2024-2026, la cantidad de litros que le tocaría a cada habitante de la ZMG y el total de agua que se gastó en el 2024 (Figura 2). Enseguida, propuso un modelo exponencial. Finalmente, en su carta incorporó ambos y propuso al SIAPA la siguiente función para predecir la población en los siguientes años: $P(t) = 5499680(0.0147)^t + 5499680$. Sin embargo, esta expresión no concuerda con los resultados previamente obtenidos de manera recursiva.

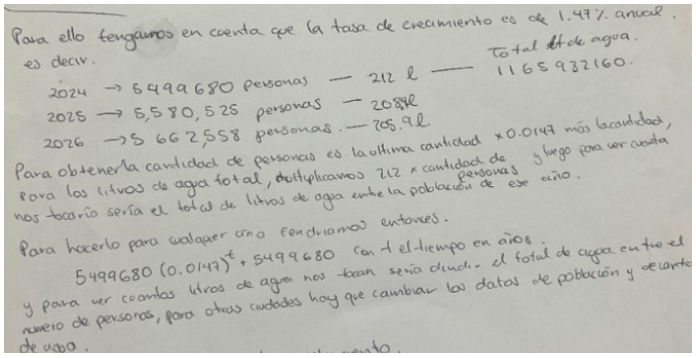


Figura 2: Modelo recursivo y exponencial propuesto por A2

Segunda fase de discusión grupal

Durante esta fase los futuros profesores explicaron sus modelos. A3, quien no construyó un modelo debido a que asumió que faltaba información, estuvo de acuerdo con el modelo creado por A1. Por su parte, A1 señaló que también era importante rescatar elementos del modelo de A2 por su aportación al entendimiento matemático del problema con base en el uso de una función exponencial.

Tercera fase

Durante esta fase, los futuros profesores describieron el uso de la proporcionalidad inversa en sus modelos. A1 indicó que el modelo recursivo (Figura 1A) lo desarrolló, primero en lápiz y papel y se apoyó en Excel para automatizar la proyección poblacional (Figura 1B). Con estos datos tabulados, regresó al lápiz y papel para calcular la constante k de proporcionalidad inversa y estructurar la fórmula de proporcionalidad inversa (Figura 1C). Ambos procedimientos le permitieron escribir su carta. A2, por su parte, señaló que no había identificado a la proporcionalidad inversa como un tema matemático subyacente en la situación problema. Ella solo se había enfocado en resolverla. Sin embargo, la había usado.

Finalmente, todos manifestaron que fue interesante resolver esta MEA por incluir, además de un conocimiento matemático, una problemática actual y, por tanto, les parecía importante llevarla al aula.

Reflexiones finales

Los resultados muestran que la proporcionalidad inversa fue utilizada en los dos tipos de modelos construidos. Ambas interpretaciones permitieron a los futuros profesores predecir el comportamiento de la distribución del agua en la ZMG, lo cual valoraron como preocupante y pertinente para su discusión en el aula con sus futuros estudiantes. Tal como lo señala Méndez et al. (2023) en investigaciones con estudiantes, se concluye que las MEAs favorecen procesos de matematización acordes con la interpretación del problema y promueven la reflexión sobre el contexto de la situación planteada. De manera particular, este estudio aporta evidencia de que los futuros docentes no solo desarrollan modelos matemáticos, sino que también reconocen el potencial interdisciplinar formativo y de impacto social del posible uso de las actividades en el aula ante los resultados obtenidos.

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LEARNING TO TEACH: PREPARING PRESERVICE TEACHERS TO FACILITATE MATHEMATICAL DISCOURSE

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Mathematical discourse is central to students' mathematical sense-making, yet preservice teachers often struggle to facilitate it effectively. This pilot study examines a three-phase discourse session embedded in a mathematics methods course, grounded in Universal Design for Learning and Mathematical Knowledge for Teaching. The session supported preservice teachers in developing discourse-focused pedagogical strategies. Pre- and post-survey results showed consistent growth, with the largest gains in revoicing, professional noticing, and supporting students' reasoning. Qualitative findings revealed shifts toward student-centered perspectives and increased awareness of discourse moves. Together, results suggest that structured engagement and practices can support preservice teachers' developing competence in facilitating meaningful mathematical discourse.

Keywords: Mathematical Discourse, Preservice Teacher Education, Universal Design for Learning, Mathematical Knowledge for Teaching

Mathematical discourse (MD) plays a vital role in supporting students' development of deeper mathematical proficiency across mathematical topics (Legesse et al., 2020; Demirci & Baki, 2023). MD is commonly conceptualized as purposeful communication among students and teachers that involves explaining reasoning, justifying solutions, and debating mathematical ideas, practices that are central to students' sense-making and conceptual development in mathematics learning. Research further indicates that classrooms characterized by sustained routines of mathematical discourse are associated with improved mathematics achievement and support students' autonomy, motivation, self-regulation, and strategic learning behaviors (Turner et al., 2003).

The National Council of Teachers of Mathematics (2014) emphasized that "effective mathematics teaching engages students in discourse to advance the mathematical learning of the whole class" (p. 29), positioning MD as central to promoting thinking in mathematics classrooms. However, research suggests that enacting this vision in practice remains challenging, particularly for preservice teachers who often struggle to identify specific teacher moves that effectively support productive MD (Lee et al., 2013; Ducharme et al., 2022; Aduko & Akayuure, 2025). This brief report describes the design and implementation of a session, situated in a mathematics methods course, aimed at supporting preservice teachers' (PSTs) understanding of their roles in promoting mathematical discourse. The report presents the design rationale, instructional activities, and initial findings related to PSTs developing competencies and growing awareness of their roles in facilitating MD.

Theoretical Framework

This study is conceptualized through the lenses of Universal Design for Learning (UDL; CAST, 2018) and Mathematical Knowledge for Teaching (MKT; Ball et al., 2008). UDL provides an equity-oriented framework for designing learning environments that offer all students multiple opportunities for equitable math learning through multiple means of engagement, representation, and action and expression. MKT further foregrounds PSTs'

developing competencies in rigorous mathematics instruction, including their ability to facilitate MD. In particular, MKT highlights the pedagogical content knowledge required to interpret, respond to, and build on students' diverse mathematical and linguistic contributions within classroom discourse. Together, these frameworks support the instructional design and teacher knowledge to shape PSTs' understandings of how to create inclusive MD spaces that value diverse ways of communicating mathematical ideas.

Methodology

Study Context and Participants

The pilot study is situated in a mathematics methods course within a satellite campus of a research university in the Southwestern U.S. By the borderland region, the campus serves the majority of the student population who self-identified as Hispanic or Latinx. Many PSTs enrolled are bilingual or bicultural, with many who were binational. The program is a post-bachelor, certificate-based program, and PSTs who are preparing to obtain teaching credentials at early grades must attend this mathematics course over the first year of coursework. Sixteen PSTs participated and granted permission to use their survey results for the purpose of the study.

Mathematical Discourse Session Design

The MD session was designed as an ongoing instructional experience intended to support PSTs' developing beliefs and pedagogical competencies related to facilitating MD. Accordingly, the design of the session was guided by three essential design principles grounded in the study's theoretical framework.

Design Principles

Promoting engaging in learning to facilitate mathematics discourse. Guided by the UDL engagement principle, the researcher designed the MD session to actively engage PSTs through a range of activities intended to promote motivation and interest. These activities included multiple ways of presenting mathematical ideas, such as the use of manipulatives and technological tools, as well as collaborative learning opportunities.

Infuse preservice teachers' mathematical knowledge for teaching. The session was designed drawing on Ball et al.'s (2008) framework of MKT, which emphasizes the mathematical knowledge, skills, and sensibilities required to enact effective mathematics instruction. Guided by this framework, the session was organized around the question: What does a PST need to know in order to support students' learning of mathematics, particularly through facilitating mathematical discourse?

Space for practice and reflection. The session was intentionally designed to provide PSTs with structured opportunities for both practice and reflection. PSTs were encouraged to reflect on their learning experiences and examine how the session influenced their beliefs and perceptions of their roles as facilitators of mathematical discourse. Such intentional structured support is critical, as research suggests that without guided opportunities to practice ambitious instructional routines, PSTs often struggle to implement them effectively in classroom settings (Bennett, 2010).

Session Implementation

The MD session was designed as an enriching instructional experience within a mathematics methods course to support PSTs' developing understanding of the importance of MD, their knowledge of pedagogical strategies, and their ability to design discourse-rich instructional activities. The implementation was organized into the following three phases.

Phase one: Establishing rationale for facilitating mathematical discourse. The first phase of the session focused on helping PSTs understand why MD is essential in elementary

mathematics classrooms. PSTs were invited to reflect on their own experiences with MD through structured prompts, positioning discourse as a pedagogical tool for reducing anxiety, increasing participation, and supporting students in expressing mathematical thinking.

Phase two: Engaging pedagogies that support mathematical discourse. The second phase represented the core of the session, during which PSTs engaged as learners in a range of pedagogical strategies designed to support meaningful MD. Multiple examples were presented and discussed across grade levels and content domains.

After engaging with pedagogical strategies for engaging students in MD, PSTs were asked to shift their attention to their roles as facilitators. This portion of the session foregrounded professional noticing of students' mathematical thinking (Jacobs et al., 2010), the use of discourse moves to interpret, and respond to student ideas (Chapin et al., 2009; Herbel-Eisenmann et al., 2013), and emphasizing on teachers' roles (Walshaw & Anthony, 2008).

After session phase: Practicing and designing mathematical discourse activities. The following phase provided PSTs with structured opportunities to design and practice facilitating their own discourse-focused instructional activities. These practice-based experiences supported PSTs in developing deeper insights into how to design and facilitate meaningful MD activities.

Instruments

The researchers developed pre- and post-survey instruments, informed by the MKT framework (Ball et al., 2008). The pre-survey consisted of 23 Likert-scale items designed to measure PSTs' competency and confidence related to facilitating mathematical discourse (e.g., revoicing, pressing for justification, anticipating student strategies, and responding to student thinking). Demographic items and questions related to prior experience were also included to provide contextual background for interpreting results. Additionally, the instrument drew on studies related to professional noticing (Jacobs et al., 2010) and research on discourse moves in mathematics classrooms (Chapin et al., 2009; Herbel- Eisenmann et al., 2013) and teachers' role (Walshaw & Anthony, 2008).

The post-survey consisted of 18 items, of which 16 multiple-choices items mirrored the pre-survey structure with two additional open-ended questions designed to gather PSTs' reflections on their experiences in the session and their evolving beliefs about their roles as teachers during discourse-oriented mathematical interactions.

The pre-survey was administered prior to implementation of the session, and the post-survey was administered following completion of the three-phase instructional session to examine shifts in PSTs' perceived competencies and beliefs. Among the 16 collected surveys, fifteen surveys were valid in supporting the initial findings presented in this study.

Data Analysis

Data sources for this pilot study included pre- and post-survey responses and PSTs' curriculum artifacts (i.e., designed mathematical discourse activities). The present study focuses on initial findings from the survey data, supplemented by descriptive narrative from artifacts and feedback of the session.

Survey data were analyzed using descriptive statistics to examine shifts in PSTs' self-reported competencies related to facilitating MD. Likert-scale responses were converted to numerical values, and item-level and overall means were calculated for both pre- and post-survey responses. Mean differences were computed to identify areas of growth across discourse-related competencies. Given the exploratory nature of this pilot implementation and the small sample size, the analysis emphasizes descriptive trends rather than inferential statistical claims. In addition to the quantitative analysis, responses to the two open-ended questions were reviewed

and analyzed using thematic analysis (Braun & Clarke, 2006). The research team generated themes capturing how PSTs articulated their understanding of discourse practices and how they considered the session shaped their emerging identities as facilitators of MD.

Initial Findings

The research team analyzed 15 valid survey results. Descriptive analysis of pre- and post-survey data revealed consistent growth across all discourse-related competencies. The overall mean increased from 3.40 to 4.27 on a five-point Likert scale, indicating a substantial shift toward agreement with statements describing competence in facilitating mathematical discourse. The largest increases were observed in items related to revoicing student ideas (+1.15), noticing mathematical thinking during discussion (+1.09), supporting students in revising their reasoning (+1.08), and facilitating peer restating (+1.08). These competencies align closely with professional noticing (Jacobs et al., 2010), implementing discourse moves (Herbel-Eisenmann et al., 2013) and core components of MKT (Ball et al., 2008), particularly teachers' ability to interpret and respond to students' thinking. While findings are exploratory due to the small sample size, the consistent upward trends suggest that structured engagement and reflection within the session may support PSTs' developing competence in facilitating meaningful MD.

Qualitative analysis revealed three emerging themes. First, PSTs described moving from viewing mathematical discourse as teacher-led explanation to understanding it as a student-centered sense-making routine. Many articulated a clearer conception of their role as facilitators who elicit, connect, and build on students' strategies. As one PST, Grace (pseudonym), noted, "Mathematical discourse isn't about getting students to talk more, it's about getting them to reveal the structures, assumptions, and strategies behind their thinking. When students articulate how they know, not just what they know." Second, responses reflected increased awareness of specific discourse moves, such as 3Rs (rephrasing, restating, and reasoning), suggesting that PSTs began to conceptualize discourse facilitation as intentional, skill-based pedagogical practice. Third, several PSTs emphasized the importance of anticipating multiple student strategies and responding productively to unexpected thinking. Together, these qualitative themes complement the quantitative findings and indicate conceptual growth in how PSTs understand and enact discourse-oriented mathematics instruction.

Discussion and Implications

The findings from this pilot study suggest that structured engagement with MD can support PSTs' developing pedagogical competencies and overall understanding of their roles as facilitators of MD. The consistent quantitative growth across all survey items indicates that discourse facilitation can be decomposed, practiced, and meaningfully developed within mathematics methods courses. The qualitative findings further suggest that growth extended beyond increased confidence to include shifts in how PSTs conceptualized MD.

For mathematics teacher educators, this study highlights the value of explicitly integrating discourse-focused design principles into mathematics methods coursework. Rather than assuming PSTs will "pick up" discourse practices through observation alone, the coherence between quantitative trends and qualitative themes provides preliminary evidence that discourse facilitation is a developable aspect of PSTs preparation. Future iterations of this work will further examine how these developing competencies translate into enacted classroom practice and how sustained engagement with discourse-oriented instruction shapes PSTs' professional identities over time.

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FROM PROCEDURES TO RELATIONAL UNDERSTANDING: SHIFTS IN PRESERVICE TEACHERS' CONCEPTUALIZATIONS OF FRACTION DIVISION

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Preservice Teachers (PSTs) need more opportunities to develop a conceptual understanding of fraction division. This exploratory study examined shifts in PSTs' fraction division definitions and contextual representations following sequenced instructions designed to support conceptual understanding. Eighteen PSTs completed pre- and post-assessments requiring written definitions and constructed word problems. We found movement from procedural and partitive descriptions toward relational, measurement-oriented definitions. However, conceptual shifts did not always translate into PSTs' word problems as they negotiated units and relationships for non-unit fractions. These findings suggest conceptual growth and representational fluency may develop unevenly and underscore the importance of intentional task sequencing and attention to multiple meanings of division in mathematics teacher education.

Keywords: Rational Numbers, Mathematical Representations, Preservice Teacher Education

Framework, Objectives, and Research Questions

Prior research has documented challenges in conceptualizing fraction division (Bowen, 2021; Copur-Gencturk, 2021), highlighting difficulties in representing fraction division through clear verbal descriptions (Lo & Luo, 2012). One area of challenge for preservice teachers (PSTs) is the concept of unitizing (i.e., considering a set of singleton units as a new unit) (Lo & Luo, 2012). Similarly, Copur-Gencturk (2021) found that even in-service teachers struggle to explain how visuals and real-world examples connect to the concept of fraction division. To address these challenges, researchers have explored how pictorial representations and hands-on activities support Preservice Teachers (PSTs) in developing conceptual understanding. For instance, Sahin et al. (2020) reported that instruction emphasizing pictorial modeling strategies positively influenced PSTs' understanding, specifically improving both accuracy and labeling. Arnal-Bailera and Arnal-Palacián (2023) found that PSTs often relied on manipulatives solely for visual support without creating meaningful contexts. They also noted that PSTs need more opportunities to learn how to appropriately select and interpret manipulatives.

Collectively, these studies highlight the potential of visual representations and the need to extend instructional time for fraction division. However, given the already rigorous and tightly structured nature of teacher preparation programs, it may not be practical to require PSTs to take additional mathematics content courses (Garner et al., 2024). Given this context, we became interested in exploring whether one intentionally designed day of instruction involving hands-on activities and contextual tasks could make a meaningful difference in supporting PSTs' understanding of fraction division. We contribute to the body of work on fraction division by exploring how specific instructional design choices can help to shift PSTs' use of algorithmic procedures toward conceptual fluency. We asked the following three research questions: (1) How did PSTs' definitions of fraction division shift after instruction? (2) How did PSTs' word problems reflect alignment with the intended symbolic division structure? (3) How did numeric complexity (unit vs. non-unit fraction divisors) influence representational coherence?

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Methods

This exploratory study involved 18 elementary and middle grades PSTs enrolled in a mathematics content course emphasizing sense-making through number theory, operations, fractions, and proportional reasoning at a southeastern regional public university. As part of the study, PSTs engaged in contextual, hands-on tasks focused on fraction division. We collected data from the pre- and post-assessments that we administered prior to and after the instruction. In those tests, we asked PSTs to explain the meaning of fraction division, write story problems for given fraction division statements, and create visual representations for given problems. In this study, we used only their definitions of fraction division and story problems as data.

We first used an inductive coding method (Saldaña, 2016), identifying patterns across all PSTs' fraction division definitions and story problems. Based on those patterns and the memos created during inductive coding, we developed separate rubrics for coding fraction definitions and story problems (Table 1). Each researcher used the rubric to code the data and then met to resolve discrepancies. Afterward, we examined differences between the pre- and post-tests to identify any shifts in PSTs' definitions and story problems, reported in the findings section.

Table 1: Coding Rubrics

Fraction Definitions	Fraction Story Problems
Missing / Irrelevant: Response is missing or does not address the task	**Not Interpretable:** Missing, incomplete, or cannot be mapped to any clear operation
Partitioning / Non-Operational: Fractions are described as "split," "cut," or "broken." Focus on parts, not division as an operation. No relationship between the two fractions	**Non-Division Context:** Fractions present, but models a different operation (e.g., addition, subtraction, multiplication, or sharing the wrong quantity)
Procedural / Operation-Aware: References division rules or algorithms (e.g., "divide," "keep-change-flip"). Shows awareness of a process. Limited connection to units or meaning	**Ambiguous Division:** Division suggested, but divisor/dividend roles unclear, units confused, or the structure does not accurately represent the problem
Conceptual / Measurement. Describes division as "how many times one fraction fits into another." Uses unit iteration, grouping, or comparison. Relational reasoning connects the dividend and the divisor.	**Explicit Fraction Division:** Clearly and coherently models the problem (measurement or partitive interpretation)

Findings

Findings are organized by the three research questions. We first report shifts in PSTs' definitions of fraction division, followed by analysis of their constructed word problems and how numeric complexity impacted their ability to generate coherent problem context.

RQ1: How did PSTs' Definitions of Fraction Division Shift After Instruction?

To examine shifts in PSTs' conceptualizations of fraction division, we categorized written definitions into four themes: Missing/Irrelevant, Partitioning/Non-Operational,

Procedural/Operation-Aware, and Conceptual/Measurement (Relational). These categories reflect increasing alignment with a measurement interpretation of division. Table 2 displays the distribution of responses across categories for pre- and post-assessments. Pre-assessment definitions frequently reflected partitioning or procedural interpretations of fraction division. Many participants described fractions as being "cut" or "split" without attending to the relational structure of division or referenced algorithms (e.g., "keep-change-flip"). Following the instructional session, a clear shift toward relational and measurement-based definitions was observed. For example, one participant initially defined fraction division as "involves keeping the first one the same, changing the sign to multiplication and flipping the second fraction," but later described it as "how many times one fraction can fit into the other," explicitly framing division as measurement. Post-assessment responses more frequently described fraction division as determining "how many times one fraction fits into another," emphasizing unit iteration, grouping, or comparison (see Table 2).

Table 2: Distribution of PST Definitions of Fraction Division (Pre/Post)

Definition Category	Pre (%)	Post (%)
Missing/Irrelevant	0%	0%
Partitioning/Non-Operational	50%	27%
Procedural/Operation-Aware	33%	17%
Conceptual/Measurement (Relational)	17%	56%
(n = 18 PSTs)		

Participants increasingly coordinated the dividend and divisor as quantities rather than relying solely on procedural language. While not all participants demonstrated fully relational reasoning, the overall distribution of responses reflected meaningful movement away from algorithmic descriptions and toward measurement interpretations. These findings suggest that the instructional sequence supported growth in conceptualizing fraction division.

RQ 2: How did PSTs' Word Problems Reflect Alignment with the Intended Symbolic Division Structure?

Although conceptual definitions shifted toward measurement interpretations, word-problem construction revealed greater variability. Pre-assessment word problems frequently modeled operations other than division despite including examples. For example, one PST wrote, "Malory has 3/4 of her book left while Caleb has 2/3. who is further along?", framing the situation as a comparison rather than a division structure. Similarly, another pre-word problem asked "I ate 3/4 of my pizza. Later I ate 2/3 of what was left. How much of the total pizza did I eat?," modeling sequential subtraction/multiplication rather than division. In some cases, division was implied but roles of the dividend and divisor were unclear, or units were not coherently coordinated.

Post-assessment responses demonstrated improved alignment for the simpler division expression (e.g., ⅓ ÷⅙). Many participants constructed measurement-based contexts that explicitly asked how many fractional units fit into a given quantity. For instance, one PST wrote, "Emma has 1/3 of a candy bar. She breaks it into small pieces. Each piece is 1/6 of a candy bar. How many small pieces did Emma make?," clearly modeling division as unit iteration. Another asked, "Joe has 1/3 yard of string. How many 1/6 yard pieces can he cut?," maintaining consistent units and coordinating the dividend and divisor appropriately.

However, alignment was less consistent when PSTs modeled division involving non-unit

fractions (¾ ÷ ⅔). These responses often reflected ambiguity in coordinating units or reverted to subtraction or comparison contexts. For example, one post-word problem asked, "We have 3/4 of a cake and Lizze wants 2/3 how much cake is left?," shifting the task toward subtraction. Another wrote, "Lily has 3/4 of chocolate and she wants to share with her friends. She gives them 2/3 each," without clarifying the referent unit or how the quantities relate structurally.

These results indicate that while participants' conceptual definitions improved, translating that understanding into coherent contextual representations required additional coordination of quantities and units. Specifically, representational coherence weakened when both the dividend and divisor were non-unit fractions, making it more difficult for students to maintain a clear problem context. To examine this pattern more closely, we compared participants' representations across the two division tasks, which differed in numeric complexity.

RQ3: How Did Numeric Complexity Influence Representational Coherence?

Taken together, these findings suggest that conceptual growth and representational fluency may develop unevenly, particularly as the symbolic demands of fraction division increase. To examine this pattern more closely, we compared participants' representations across the two division tasks, which differed in numeric complexity. A comparison suggests that increased numeric complexity amplified representational challenges. When the divisor was a unit fraction (e.g., ⅓ ÷ ⅙), participants more readily constructed measurement-based word problems that aligned with the symbolic expression. In contrast, when both the dividend and divisor were non-unit fractions (e.g., ¾ ÷ ⅔), responses displayed greater variability and misalignment.

This pattern suggests that coordinating two non-unit fractional quantities requires more sophisticated unit reasoning. Participants appeared to struggle with maintaining consistent referent units and clarifying which quantity was being measured by which. For example, one PST wrote, "She gives them 2/3 each," without specifying the referent unit or how the 3/4 quantity structured the situation, illustrating ambiguity in coordinating dividend and divisor roles. Even when relational definitions were articulated elsewhere in the assessment, representational coherence was less stable under increased numeric complexity.

Discussion and Implications

The shifts observed in PSTs' fraction definitions and visual representations suggest that how instructional tasks are sequenced and connected can influence PSTs' understanding of fraction division, even within a single, intentionally designed day of instruction. Further, our findings reinforce prior research indicating that brief interventions are insufficient for addressing some challenges associated with fraction division (Bowen, 2021; Copur-Gencturk, 2021). In our study, PSTs did not always translate conceptual definitions of fraction division into word problems, particularly in non-unit by non-unit fractions. When both fractions were non-unit, PSTs frequently attended to the numerator and denominator as separate numbers rather than reasoning about the relational structure of the division situation. PSTs' challenges with word-problem construction were also tied to interpreting the divisor in terms of the number of groups rather than the size of the unit (i.e., treating 2/3 as a count of people). Consistent with prior studies (e.g., Lo & Luo, 2012), these findings highlight the central role of unit and unitizing in fraction division and suggest that instructional approaches in teacher preparation should foreground these ideas. Moreover, these findings point to several implications for mathematics teacher education. Explicit attention to multiple meanings of division, including measurement and partitive interpretations, potentially supports PSTs' conceptual development. In addition, our results suggest that the fractions used in instructional tasks matter, and that intentional sequencing—from unit-by-unit to non-unit by non-unit fraction division—may support PSTs' ability to

coordinate units and relationships. Finally, facing these challenges firsthand offers PSTs a valuable learning opportunity, fostering intentionality and empathy in teaching fractions.

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AGENCY, AUTONOMY, AND ADAPTABILITY: TENSIONS IN THE TRANSITION TO NOVICE TEACHER

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Keywords: Identity and Agency; Cognition; Affect, Emotion, Beliefs, and Attitudes

Adaptability is a foundational professional disposition in math teaching. Teachers interpret situations (cognitive), regulate emotions (affective), and enact instructional decisions (behavioral) in real time and across personal, student, curricular, and contextual factors. Although frequently cited as a desirable (Collie & Martin, 2017), adaptability is often described as "flexibility" rather than coordinated cognitive, affective, and behavioral functioning. The *Tripartite Model of Adaptability* (Martin et al., 2013; Collie & Martin, 2016) conceptualizes adaptability as the integration of these domains, providing a lens for examining how teachers navigate professional challenges. Special attention must be paid to the transition from school-to-career (e.g., Akkermans et al., 2013; Savickas, 2005) which represents a time when adaptability becomes consequential. For novices, this transition is not merely about managing workload but about learning to act professionally within instructional constraints. In this transition, questions of autonomy (i.e., perceived freedom to act) and agency (i.e., intentional action within constraints) become central to how adaptability is experienced and enacted. Accordingly we ask: *How is adaptability described and experienced by novice secondary mathematics teachers?*

This qualitative study draws on semi-structured interviews from three novice secondary mathematics teachers following their first year of teaching. Participants, all NOYCE scholars, represented three early-career educators responding to the demands of new school settings and shifting role expectations. Analytic and reflexive memoing were also used as sources of data (Paulus et al., 2014). Analysis was iterative and theory-informed. Initial coding identified moments of cognitive, affective, and behavioral adaptability. Secondary coding examined how perceptions of autonomy and enactments of agency shaped these adaptive responses.

Findings indicate that increased autonomy in the first year does not automatically strengthen adaptability. When teachers were granted greater agency, tensions emerged: Affective tension in emotionally charged decision-making; Cognitive conflict in moral and professional dilemmas and behavioral risk-taking when considering deviations from department norms or mandated curriculum. Adaptability appeared uneven when autonomy and agency were misaligned. High autonomy paired with low confidence limited behavioral enactment of adaptability, while strong agency paired with a constrained environment heightened stress and an increased moral/ethical reasoning. When autonomy and agency were more aligned, greater coherence emerged across cognitive reframing, emotional regulation, and instructional action. The first year represents a critical, emotionally charged period in which autonomy and agency shape adaptability as decisions carry real consequences.

Induction programs must intentionally support PSTs to transition from student teaching to full professional responsibility. Support should focus on building agency in teachers through reflective decision making, addressing emotional and moral dilemmas that may emerge, and positioning the teacher as an authority of experience. Without this support, novice teachers may not exercise agency in an attempt to limit risk taking, emotional overload, or moral tensions, leading to less innovative teaching.

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ENHANCING PRE-SERVICE TEACHERS' TPACK THROUGH DYNAMIC GEOMETRY INSTRUCTION: A QUASI-EXPERIMENTAL STUDY

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Keywords: TPACK, Dynamic Geometry, GeoGebra, Preservice Teachers

Technology integration in mathematics education involves more than technology proficiency; it involves a well-integrated knowledge of content, pedagogy, and technology. In geometry, visualization and transformation-based reasoning play a crucial role. Although dynamic geometry environments provide many opportunities for exploring invariance and relational properties through manipulation, teacher preparation programs for pre-service teachers have often focused on technical skills rather than on integrated technological pedagogical content knowledge (TPACK).

The National Council of Teachers of Mathematics (NCTM, 2000) identifies technology as essential to the teaching and learning of mathematics, noting its influence on both the mathematics that is taught and how students develop understanding. Drew Polly (2014) emphasizes the integration of technology at the elementary level, noting its positive impact on teaching and learning by supporting students' exploration of tasks, enabling the construction of dynamic mathematical models, and facilitating the modelling of mathematical situations. Geometry is a fundamental component of mathematics education that supports the development of students' logical thinking, visual imagery, and spatial reasoning skills, as emphasized by Battista (2007).

This study is grounded in the Technological Pedagogical Content Knowledge (TPACK) framework (Mishra & Koehler, 2006). The framework conceptualizes teacher knowledge as an interaction among Technological Knowledge (TK), Technological Content Knowledge (TCK), Technological Pedagogical Knowledge (TPK), and Technological Pedagogical and Content Knowledge (TPACK). A quasi-experiment with a pretest and posttest design was employed to compare two classes of an elementary mathematics methods course, which were randomly assigned to treatment (GeoGebra-based teaching) and control (traditional teaching) groups. The data were collected through a modified TPACK survey (TPACK Survey; Schmidt et al., 2009), which was used to measure students' TK, TCK, TPK and TPACK. This tool was modified to fit the specific needs of this study in terms of teaching rotations and reflections with the use of GeoGebra software.

Data were gathered using the five-point Likert scale, and composite scores were generated for each domain. A total of 96 survey responses (25 control pre, 21 control post, 28 treatment pre, and 22 treatment post) were used in the TPACK analyses. The geometry assessment test was in the form of a researcher-made pre-test and post-test on conceptual and procedural knowledge regarding rotation and reflection on a coordinate plane. The two-way ANOVA analysis showed significant Group × Time interaction effects for all TPACK components.

Additionally, the results showed that there were significantly greater gains in geometry concept learning in the treatment group. The results also indicated that dynamic geometry instruction facilitates both integrated TPACK development and geometry achievement. The results imply that teacher educational institutions need to design learning activities where pre-service teachers will be able to utilize technology to support student learning and success.

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THE TENSION BETWEEN POLICY AND PEDAGOGY: A STAKEHOLDER'S ANALYSIS OF PA HB1407 AND TEACHER CERTIFICATION SPANS

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Pennsylvania House Bill 1407 represents a significant shift in the Commonwealth's approach to teacher credentialing, moving away from specialized grade spans toward a broader, binary model. Currently, Pennsylvania's three-tier system (Pre-K–4, 4–8, and 7–12) aligns with the Association of Mathematics Teacher Educators (AMTE) position that middle-level learners possess unique developmental and academic needs requiring specialized educator preparation (Hale, 2020). HB1407 proposes a consolidation into two categories: "Primary" (Pre-K–6) and "Secondary" (7–12). While the bill's sponsors argue that this change will mitigate the teacher shortage by increasing "placement flexibility" for administrators (Nelson, 2025), this shift effectively reclassifies fifth and sixth grades as elementary, potentially disregarding the specialized pedagogical content knowledge historically required for these bridge years.

To investigate the potential unintended consequences of this legislation, our research team (composed primarily of pre-service elementary and middle-level educators) conducted a stakeholder perception study. We surveyed four distinct groups (n = 70): pre-service teachers, practicing in-service teachers, school administrators, and teacher educators. By capturing a cross-section of the educational community, we sought to determine how the proposed expansion of the "primary" certification span would influence instructional quality, teacher self-efficacy, and institutional operations.

The data revealed a stark divergence in priorities between those in the classroom and those in administrative roles. Respondents who opposed the bill—predominantly pre-service and in-service teachers—expressed recurring concerns regarding a deficit in deep content knowledge for upper-elementary grades. Conversely, administrators were less likely to focus on pedagogical specialization, prioritizing the logistical benefit of staffing flexibility. Notably, our study uncovered an unanticipated impact on the research team itself; pre-service teachers pursuing Pre-K–4 certification reported significant apprehension regarding their interest and readiness to teach fifth and sixth-grade mathematics curricula. The results suggests that while HB1407 may resolve the quantity of available teachers, it may inadvertently compromise the quality of instruction and teacher retention due to a lack of specialized preparation for the middle-level learner.

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PRESERVICE TEACHERS' GROWTH THROUGH DEVELOPMENTALLY SEQUENCED MICROTEACHING

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This study examines the perceived self-growth of preservice teachers (PSTs) when experiencing two developmentally sequenced microteachings. While prior studies found the value of repeated microteaching opportunities to develop instructional skills and reflective practice (Amobi, 2005; Fernández, 2010; Remesh, 2013), our study addresses a gap by shifting the focus from the frequency of microteaching to the intentional, developmental, sequential experiences that conceptualize learning (Grossman et al., 2009; Zeichner, 2010).

The study's theoretical framework affirms that microteaching should be conceptualized as a developmentally scaffolded learning trajectory (Daro et al., 2011). This belief aligns with the developmental differences in elementary school mathematics instruction (Clements & Sarama, 2014; National Research Council, 2001; Van de Walle et al., 2019). We hypothesized that sequencing microteaching for developmental progression would support PSTs to gain a deeper understanding of teaching mathematics and to improve their confidence in teaching.

We employed a qualitative study with 12 elementary PSTs enrolled in a mathematics methods course at a midwest university. The new assignments involved two sequenced microteachings: a co-taught, concrete, K-3 mathematics lesson in mid-semester, followed by an independent, abstract, 3-6 mathematics lesson towards the end of the semester. We collected written reflective narratives from the PSTs and analyzed their preparation, facilitation, reflection, and perceived growth. In particular, we employed a thematic analysis in which we identified PSTs' perceived benefits to intentional, developmental, sequential experiences.

The key themes that emerged were: (1) developmental sequencing supported growth in confidence, (2) teaching across the grade bands deepened pedagogical content knowledge, (3) noticeable shifts occurred from procedural to conceptual instruction, (4) purposefully used manipulatives and hands-on activities, and (5) demonstrated collaboration and instructional ownership. The poster will display PSTs' in-depth reflections and excerpts on each theme.

The findings suggest developmental, sequential experiences enhanced PSTs' preparation for the complex elementary school mathematics teaching. Progressing from a co-taught, concrete lesson to an independent, abstract lesson functioned as a scaffolded professional learning experience that fostered instructional precision, conceptual growth, and professional identity development. Finally, rather than relying solely on observable teaching behaviors, our study highlights the importance of centering PSTs' reflection as a key measure of teacher growth.

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SECONDARY MATHEMATICS AND SCIENCE TEACHERS' BELIEFS OF INQUIRY-BASED VERSUS TRADITIONAL INSTRUCTION

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Inquiry-based instruction has long been a cornerstone of science (Crawford, 2014) and mathematics education reform (Cribbs et al., 2020) because it improves student learning, retention, and engagement (Crippen & Archambault, 2012; Minner et al., 2010). By emphasizing exploration and problem solving over memorization, inquiry-based approaches foster deeper conceptual understanding (Chapman, 2011; Savelsbergh et al., 2016). However, implementation remains challenging due to limited teacher preparation, misconceptions about inquiry, and practical constraints such as time and testing demands (Chin & Lin, 2013; Welch et al., 1981).

Guided by conceptual change theory (Posner et al., 1982), this study examines how pre-service teachers' (PSTs) perceptions of inquiry-based instruction changed after completing the *Inquiry Approaches to Teaching* course. This study asked: How do PSTs' perceptions of inquiry-based learning change after completing the Inquiry Approaches to Teaching course?

Participants were 20 secondary PSTs enrolled in a one-credit introductory inquiry methods course within the UTeach teacher preparation program. The course introduced inquiry-based instruction through modeling, practice, and the implementation of three inquiry-based lessons. Data was collected over one semester through online discussion posts completed before and after field experiences. PSTs responded to two prompts comparing inquiry-based and traditional instruction and describing the benefits of inquiry-based teaching. Researchers conducted iterative in vivo coding (Saldaña, 2021) using MaxQDA, collaboratively refining codes into themes through consensus.

Three primary themes emerged when examining traditional and inquiry-based teaching: (1) *Role of the Teacher*. PSTs initially described traditional instruction as teacher-centered, lecture-based, and focused on knowledge transmission. Later, they characterized inquiry-based teaching as facilitative, with teachers guiding exploration, using questioning and formative assessment, and fostering student engagement. (2) *Role of the Student.* Traditional instruction was viewed as emphasizing passive learning, note-taking, and limited collaboration. In contrast, inquiry-based learning positioned students as active participants who explored ideas, collaborated with peers, solved problems, and assumed greater ownership of learning. (3) *Role of the Learning Environment.* PSTs described traditional classrooms as fast-paced, focused on memorization, and offering few opportunities for critical thinking. Inquiry-based environments, although perceived as requiring more time and planning, were viewed as promoting hands-on learning, productive use of errors, deeper conceptual understanding, and authentic problem solving.

The findings are significant for secondary mathematics and science teacher preparation, where positive attitudes toward inquiry have historically been less prevalent than among elementary educators (Marshall et al., 2009). Early exposure to inquiry-based teaching may strengthen PSTs' confidence and willingness to implement inquiry as practicing teachers. Additionally, because much of the existing literature focuses on science education (Chinn et al., 2013; Choi & Ramsey, 2009; Fitzgerald et al., 2019; Windschitl & Thompson, 2006), this study extends the evidence base by demonstrating similar benefits for mathematics PSTs.

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DESIGNING A THEORETICALLY AND EMPIRICALLY GROUNDED SURVEY OF PRESERVICE TEACHERS' UNDERSTANDING OF INQUIRY-BASED INSTRUCTION

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For decades inquiry-based teaching has been well regarded in the mathematics education community because it has been found to enhance student learning, engagement, and retention (Crippen & Archambault, 2012; National Council of Teachers of Mathematics, 1991). Despite these benefits, few teachers implement inquiry-based teaching in their classrooms (Cribbs et al., 2020; Rönnebeck et al., 2016; Wang, 2020). One reason may be because preservice teachers (PSTs) frequently conflate inquiry-based instruction with other student-centered practices—such as active learning or collaborative work (Goldberg & Darwin, 2025). PSTs tend to ignore the epistemic and instructional features that distinguish inquiry from other pedagogical approaches (see also Kuster et al., 2019). Although researchers examined teachers' beliefs about and use of inquiry (Marshall et al., 2009) or assessed student preferences for either student- or teacher-centered instruction (Cooper et al., 2017), we still need to unpack how PSTs conceptualize inquiry-based instruction to understand if those understandings align or misalign from theoretically grounded definitions. Having a robust understanding of how PSTs conceptually understand inquiry-based instruction can enable teacher preparation programs to identify persistent misconceptions and design methods coursework that support inquiry-oriented instruction.

Considering this, our research team outlines the development of a survey designed to assess PSTs' understanding of inquiry-based instruction as guided based on a theoretically articulated framework, substantiated and refined through empirical evidence. We define inquiry-based teaching as "instruction [that] has *students* construct knowledge by discovering and interpreting information and findings for themselves" (Goldberg & Darwin, 2025, p. 4). Mathematical inquiry is further delineated as using logical reasoning, conjectures, evaluation, and exploration to determine the validity of mathematical concepts (Chinn & Clark, 2013).

In our prior work, we have published on how PSTs consider inquiry-based instruction in relation to other commonly invoked pedagogical approaches (Goldberg & Darwin, 2025). This framework allowed us to capture what literature talks about key instructional roles of involved stakeholders in the learning process, epistemic instructional goals, and classroom dynamics that characterize inquiry-based teaching. We are also leaning onto the empirical findings as informed by qualitative data drawn from seven cohorts of secondary PSTs enrolled in teacher preparation program. This study will examine data from PSTs in an introductory inquiry-based methods course (n = 155) on how they define and contextualize inquiry-based teaching after being exposed to inquiry-based instruction throughout the course. Having PSTs reflects on the idea of inquiry-based instruction highlights not only what PSTs emphasize when reasoning about inquiry, but also what is routinely absent from their explanations or how their thinking may be conflated with other pedagogical strategies.

By leveraging a theoretically articulated framework accompanied with rich empirical evidence of PST thinking, the larger goal of this project is to develop a survey instrument that

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captures how PSTs understand inquiry-based instruction, rather than whether they simply endorse it. Such work will support research and teacher education efforts that attend to conceptual change in PSTs' understanding about inquiry. At the session, we will share the conceptual grounding, empirical motivation, and design logic guiding this survey development, and invite feedback from the community on advancing measurement work related to inquiry-based mathematics instruction.

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TEACHING CASES AS A WINDOW INTO PRE-SERVICE TEACHERS' THINKING ABOUT CULTURALLY GROUNDED PEDAGOGY IN MATHEMATICS

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A central focus in mathematics teacher education is to support the varied needs of students and their communities and thoughtfully engage with issues of justice and equity (AMTE, 2017). Teacher preparation programs have an opportunity—and obligation—to prepare pre-service teachers (PSTs) to deeply consider such issues in their teaching even before they have their own classrooms. One approach we explore for doing so is the use of teaching cases, or written narratives that depict authentic, complex teaching dilemmas and require PSTs to think critically (Shulman, 1992). Such cases can support PST reflection and offer insight into PST beliefs (e.g., Abell et al., 1998; Grossman, 1992). In our research, in-service teachers design teaching cases based on their own experiences related to enacting culturally grounded pedagogy in mathematics (Dobie et al., 2024), conceptualized as (a) leveraging students' multiple competencies, (b) honoring students' lived experiences, (c) challenging status and power hierarchies, and (d) diversifying curriculum and challenging dominant perspectives (e.g., Aguirre et al., 2013; Ladson-Billings, 2014; Paris, 2012). We add the lens of teacher noticing (e.g., Sherin et al., 2011), with attention to noticing for equity (e.g., van Es et al. 2022), to investigate how cases can support PSTs to *notice* classroom, school, or societal inequities as well as student strengths; consider a range of *interpretations* for what they are noticing; and suggest possible *solutions*. In this study we ask, "How can teaching cases depicting issues related to enacting culturally grounded pedagogy in mathematics support PSTs' developing skills in noticing for equity?"

Seven cases (focused on issues such as inequitable group dynamics, assumptions about the abilities of multilingual students, and disparities in access to math between districts) were implemented 1–5 times each in math methods courses for future elementary teachers. Implementation entailed PSTs writing individual responses to case prompts, discussing the case in small groups and then as a whole class, and completing an exit ticket about their experience. Here we highlight the analysis of one case focused on inequitable group dynamics that was implemented three times. Data include written responses to three case prompts from 63 PSTs. A three-part coding scheme was designed inductively and deductively (Miles et al., 2014) to capture the details PSTs *noticed* from the case; the *interpretations* offered that aligned with the themes of culturally grounded pedagogy; and the *solutions* proposed to resolve the dilemma. Two coders coded all responses and resolved any disagreements. Findings highlight PSTs' attention to details about students and their actions; the potential for varied interpretations, alongside PSTs' tendency to consider the role of personality more than factors such as race, gender, and language; and the varied solutions offered when PSTs were prompted to consider several possible action steps. PST responses to cases offer insights into their noticing and space for teacher educators to further develop that noticing in a constrained situation on the path to building skills for noticing in more complex videos and classroom observations.

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VARIATION IN MENTOR TEACHER–TEACHER CANDIDATE CO-NOTICING ACROSS PLANNING, HUDDLES, AND DEBRIEFS

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Purpose of the Study and Theoretical Framework

Professional noticing research has largely examined teachers' attention to students' mathematical thinking during instruction (Jacobs et al., 2010). Teacher education structures practice across planning, enactment, and debrief, yet we know little about how noticing reorganizes across the three phases or how temporal conditions shape attending, interpreting, and decision-making. This study examines variation in mentor teacher and teacher candidates' attending to, interpreting, and deciding about student participation when engaging in anticipatory noticing, in-the-moment noticing, and retrospective noticing (van Es & Sherin, 2008; 2010) within the Asset-Focused Co-Noticing Tool I (ACT I). Drawing on van Es and Sherin (2008; 2010), we conceptualize noticing as a phase-sensitive. Anticipatory noticing involves projecting possible student participation prior to instruction. In-the-moment noticing involves attending to and interpreting participation during enactment while making immediate instructional decisions. Retrospective noticing involves reflecting on participation and instructional moves to inform future action. The research question is: What differences emerge in how dyads attend, interpret and respond when engaging in anticipatory, in-the-moment, and retrospective noticing while using ACT I?

Methods

Participants were two MT–TC dyads in a secondary mathematics licensure program in the Pacific Northwest during Fall 2023. Data included video recordings of planning and debrief, audio recordings of huddles, and tool artifacts. we analyze their first and second uses of ACT I. We use Jacobs et al.'s (2010) AIR framework as the coding lens for all three phases. Transcripts were segmented by phase and coded with the AIR framework: attending codes captured which aspects of student participation and mathematical work were foregrounded; interpreting codes captured how participation was explained; and responding codes distinguished short-term and long-term decisions. Cross-phase comparisons examined how anticipatory noticing was confirmed or revised during in-the-moment noticing, and how retrospective noticing reframed participation and instructional decisions.

Findings and Implications

In anticipatory noticing, dyads attended to predicted participation guided by the equity prompt they selected, interpreted that participation tentatively (e.g., who might speak or remain silent), and framed decisions as provisional, revisable hypotheses to be tested during instruction. During in-the-moment noticing, attention shifted to enacted behaviors such as silence, talk, and partner dynamics. Interpretations became immediate and behavior-linked, and decision-making narrowed to short-term tactical adjustments made under time pressure. In retrospective noticing, dyads revisited specific noticed participation moments and expanded interpretations to include

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interactional structure and task design. Decision-making shifted from tactical to strategic, articulating long-term instructional redesign for future lessons.

This study shows how mentor teacher–teacher candidate co-noticing differs across planning, enactment, and debrief. These differences can help teacher educators design planning, huddle, and debrief structures around what each phase supports.

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CARACTERIZACIÓN DE PRÁCTICAS DE PROFESORES DE MATEMÁTICAS EN FORMACIÓN INICIAL Y LA MIRADA PROFESIONAL

CHARACTERIZATION OF PRESERVICE MATHEMATICS TEACHERS' PRACTICES AND TEACHER NOTICING

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Palabras clave: Formación inicial de profesores, desarrollo profesional, conocimiento del profesor, mirada profesional

Presentación

Esta investigación tiene como objetivo caracterizar las prácticas de profesores de matemáticas en formación inicial en relación con la mirada profesional sobre el pensamiento matemático del estudiante, al orquestar recursos pedagógicos en discusiones matemáticas.

En este trabajo es posible distinguir tres aproximaciones teóricas: la Mirada Profesional (Jacobs et al., 2010), las Oportunidades Pedagógicas con Significado Matemático para la Construcción del Pensamiento Matemático (MOST) (Leatham et al., 2015). Y la Orquestación Instrumental (Drijvers, 2012; Drijvers, et al., 2010; Trouche, 2004). La primera aproximación permite dar cuenta del análisis de la competencia profesional del profesor con base en cada una de las tres habilidades de la mirada profesional: *identificar, interpretar y responder*. La segunda permite analizar los momentos de enseñanza (MOST) que ponen de manifiesto el pensamiento matemático del estudiante y así mismo ser un instrumento de análisis para el desarrollo de las tres habilidades de la mirada profesional. Y la tercera, la articulación de las dos anteriores cuando se orquestan recursos pedagógicos en discusiones matemáticas.

Metodología

Esta investigación es de estilo cualitativo y se realizó como un estudio de casos múltiple.

Los instrumentos utilizados fueron las videograbaciones como fuente central de análisis y dos entrevistas semiestructuradas, de planificación y de reflexión. El análisis de los datos es de carácter inductivo y se realiza con base en la articulación local de tres aproximaciones teóricas ya mencionadas. El análisis se llevó a cabo en tres fases. La primera fase se ejecuta alrededor del análisis mediante la teoría fundamentada (Teppo, 2015) para dar cuenta de la mirada profesional. En la segunda fase, se identifican momentos MOST con base en el instrumento analítico de esta teoría, se comparan y se clasifican. Y en la tercera fase se establecen articulaciones entre los tipos de orquestación y las respuestas del profesor al pensamiento matemático del estudiante.

Resultados

En esta investigación se obtuvieron tres grupos de resultados: el primer grupo se relacionó con las habilidades de *interpretar* y *responder* a la comprensión de los estudiantes; y el bajo nivel de desarrollo de éstas en los estudios de caso de la investigación. El segundo resultado refiere a la identificación de un número reducido de momentos que cumplen con la estructura analítica MOST. Y el tercer resultado expuso que, como tendencia, el tipo de orquestación *guía y explicación* es el más frecuente en los casos de la investigación.

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A THREE-STAGE TEACHING REHEARSAL ARC TO DEVELOP PEDAGOGICAL COMPETENCE IN PRESERVICE TEACHERS

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Elementary preservice teachers (PSTs) are often tasked with teaching mathematics using evidence-based pedagogical practices that are ambitious and include the use of high-quality mathematics tasks promoting discourse and that build conceptual understanding (Association of Mathematics Teacher Educators [AMTE], 2017; National Council of Teachers of Mathematics [NCTM], 2014). However, many elementary teacher preparation programs lack content-specific practica, particularly in mathematics, limiting PSTs' opportunities to rehearse high-leverage teaching practices (Polly et al., 2019). Without scaffolded approximations of practice, PSTs may experience a disconnect between theoretical learning and implementation in classrooms (e.g., Emerson et al., 2018). Providing coherent structures in content-focused methods courses that allow PSTs to engage in iterative planning and teaching rehearsals that are paired with targeted feedback can help them refine their implementation of these evidence-based practices.

This poster offers a framework grounded in practice-based teacher education (PBTE), which conceptualizes teacher learning through structured opportunities to engage in representations, decomposition, and approximations of practice (Grossman et al., 2009). The framework situates learning as iterative, scaffolded for PSTs to progressively assume responsibility for instructional decision making to integrates evidence-based practices in action (Figure 1).

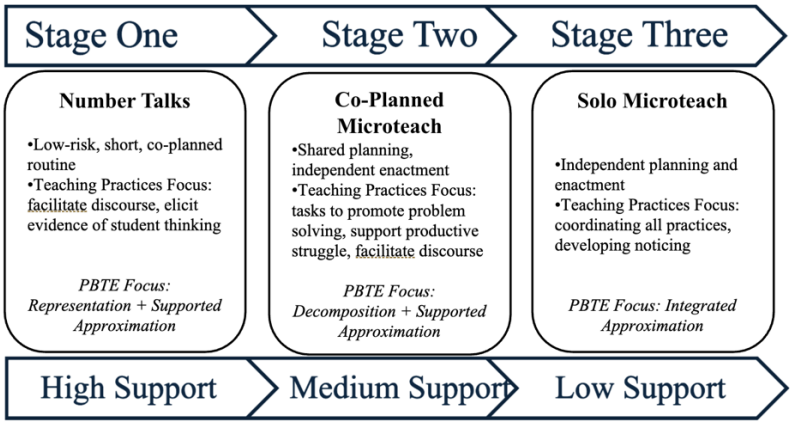


Figure 1: Three-Stage Teaching Rehearsal Arc

Implemented in an elementary math methods course at a Midwestern university, the framework was designed to address the absence of content-specific practica. Each stage of the arc focuses on high-leverage math teaching practices, gradually shifting the cognitive load of planning and implementation from shared practice to solo work. The framework offers a transferrable structure for methods courses where content-specific practica are not available. Embedding progressive rehearsal arcs in methods courses can provide PSTs with systematic opportunities to develop competence through supported approximations of practice.

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PRESERVICE AND INSERVICE TEACHERS' VIEWS OF PLANNING ACTIVITIES FOR MATHEMATICS INSTRUCTION: A COMPARATIVE ANALYIS

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Keywords: Elementary School Education, Instructional Activities and Practices, Preservice Teacher Education, Professional Development

Preservice elementary teachers often enter mathematics methods course(s) with their identity tied to their experiences as mathematics students, collegiate level courses such as college algebra, and perhaps, one or two mathematics content courses for future elementary teachers (Moler, et al., 2025). The methods course is likely one of their first experiences in which the focus shifts from their own understandings of mathematics to how ideas develop for children and pedagogical content knowledge (PCK) (e.g. Vettriselvan, et al., 2022; Lily, et al., 2024). Prior research showed differences between preservice and inservice teachers in relation to beliefs about instructional approaches and best practices for student learning (e.g. Vesga-Bravo, et al., 2022; Brugar, et al., 2023). The purpose of this research was to compare preservice and inservice elementary mathematics teachers' preferences for instructional activities.

This study answers the following research questions:

1. What activities do preservice teachers (PSTs) prefer in the development of their mathematics pedagogical strategies?
2. How do their preferences compare to inservice teachers?

Data sources were part of larger surveys that were given to inservice teachers at the end of a mathematics professional development program and elementary mathematics methods course. Forty-two inservice teachers spanning grades five and six and 35 preservice elementary teachers were asked to rank ten professional development activities from 10 (most valuable) to 1 (least valuable). The ten activities were as follows: watching videos of students solving problems, predicting student solution strategies, observing in classrooms, interviewing individual students, sorting student work, planning with other teachers, learning about problem types, videos of classroom instruction, questioning techniques, and readings. The highest ranked activities for the preservice teachers were classroom observations and planning with other teachers. These activities were also ranked high for inservice teachers (Kent, 2020; Kent & Skelton, 2024). Two activities in which there were differences in rankings between preservice teachers and inservice teachers were in the areas of viewing videos. Preservice showed a higher preference for valuing videos of students solving mathematics problems and classroom instruction than inservice teachers. Inservice teachers showed a higher preference for learning about questioning strategies from observing peer teachers thank preservice teachers.

These results could indicate that mathematics PD activities centered around peer coaching and observations of lessons in real-time should be prioritized for inservice teachers rather than video-based seminar style PD programs. Conversely, preservice teachers indicated a preference for video-based activities to learn about teaching mathematics. Future research should consider if there are specific differences between what is gleaned from video-based activities and whether improvements can be made to types of videos used for both inservice and preservice teachers. Additional research should also continue to explore advantages of observations of mathematics

lessons and development of observation instruments that support teacher learning and adaptations of best instructional practices for teachers at all levels of experience.

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FROM FAMILIAR TO UNFAMILIAR: PRE-SERVICE TEACHERS' EARLY ADAPTABILITY

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In mathematics classrooms, teachers must adapt (e.g., to student reasoning, contextual factors, and curricular pressures). This study examines how adaptability is defined and experienced during mathematics pre-service teacher (PST) education. While adaptability is frequently referenced as a desirable disposition in teacher preparation (Collie & Martin, 2017), it is often operationalized in practice as general flexibility rather than a coordinated process. As such, this study is grounded in the *Tripartite Model of Adaptability* (Martin et al., 2013; Collie & Martin, 2016), which conceptualizes adaptability as three coordinated domains: *cognitive* (i.e., interpretation, reframing, anticipatory reasoning)*, affective* (i.e., emotional regulation/ orientation)*, and behavioral* (i.e., observable instructional/relational adjustment). Our study asks: *How is adaptability described and experienced by secondary mathematics preservice teachers?*

This qualitative study draws on semi-structured interviews with three secondary mathematics PSTs following their student teaching, along with reflexive memos (Paulus et al., 2014) and field notes from NOYCE seminars that focused on adaptability. Analysis was iterative and theory-informed. Using open, *in vivo*, and axial coding (Saldaña, 2013), we generated a codebook and identified adaptability moments in personal and teaching contexts. Then we mapped these moments onto cognitive/affective/behavioral domains and conducted cross-case comparisons to examine patterns in definitions and experiences of adaptability

Table 1: PSTs' Adaptability Definitions and Personal/Classroom Experiences.

Adaptability	Dani	Erica	Shelley
Definition (emphasis)	"Shift gears" (behavioral)	"Liquid, moving with anything that comes" (cognitive)	"Stay calm and process" (affective)
Personal (familiar)	Moving from small town to large university	Responding to car failure on a road trip	Missing a train stop in another country
Classroom (unfamiliar)	Adjusting to a large high school for the first time	Weather causing change of lesson plans to move from outside to inside	Learning to manage stressful teaching moments from mentor

Findings indicate these PSTs entered preparation with domain-specific orientations toward adaptability and leveraged familiar personal experiences as interpretive anchors when navigating and adapting to unfamiliar teaching contexts. For example, behavioral orientations surfaced in proactive environmental adjustments, cognitive orientations appeared in reframing instructional disruptions, and affective orientations were evident in emotional regulation during stressful teaching moments. Importantly, PSTs described coordinating the interplay of cognitive, affective, and behavioral adaptability contributing to adaptive decision-making. These findings

suggest mathematics teacher preparation can help PSTs leverage initial strengths to cultivate cross-domain adaptability for professional decision-making.

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ELEMENTARY PRESERVICE TEACHERS' INITIAL CONCEPTIONS AND PLANS OF INTERDISCIPLINARY MATHEMATICS LESSONS: A MULTI-UNIVERSITY EXPLORATIVE STUDY

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In elementary grades, students grasp abstract mathematics and science concepts more effectively when they are taught with interdisciplinary approaches (Munier & Merle, 2009). However, interdisciplinary teaching is not an easy task for elementary teachers (Wieseman & Moscovici, 2003). More recent studies show that PSTs could be prepared better if they have opportunities to learn about interdisciplinary teaching approaches and develop necessary professional content and pedagogical knowledge during their pre-service years with attention to interdisciplinary instruction (Aslan-Tutak & Sezer, 2025; Beudels et al., 2021; Ribeirinha & Correia, 2025). Thus, the purpose of this study was to explore how PSTs across three institutions conceptualized, planned, and implemented interdisciplinary lessons.

Interdisciplinary mathematics education is defined as "conjunction of mathematics with other knowledge in problem solving and inquiry" (Williams & Roth, 2019, p. 14). Two theoretical models focus on the level of integration (Huntley, 1999) and organization of integration (Fogarty, 1991).

This study used a qualitative research design (Saldaña, 2015) to explore PSTs' initial ideas relating to interdisciplinary lesson planning. Participants were 47 pre-service teachers in four mathematics methods courses at three different universities. Responses from both the online discussions and lesson plans were collected and structurally coded using the Level of Integration and Organization of Integration frameworks (Saldaña, 2015). Implemented lessons were structurally coded during observations of implementation. Frequency tables were generated to identify pre-service teacher's initial conceptualization, integrated subjects, lesson planning, implementation, and relationships between conceptualization, lesson planning, and implementation.

Results indicated that while most PSTs conceptualized level of integration as combining both math and another subject, there were a variety of organization conceptualizations. The highest conceptualization was webbed (using theme, 28%), followed by sequential subject teaching (23%), shared content (19%), no integration (17%), and skills-only focus (13%).

Most students did not plan an integrated lesson that aligned with their initial conceptualizations, with approximately half of the lesson plans having no integrated subject. This may be due to an interpretation of story problems being an integration of literacy and mathematics. However, without a literacy focus it remains just math. For those that did integrate, art was the highest integrated subject (34%), followed by literacy (24%), social studies (21%), and science (14%), with one lesson each for PE and drama.

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When implementing the planned lessons, about two-thirds of the lessons aligned with planning. Two issues that impacted alignment were teaching inaccurate mathematics due to the PSTs misconceptions on the topic or going off on an unrelated tangent.

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AN EXPLORATORY ANALYSIS OF PROSPECTIVE ELEMENTARY TEACHERS' "TAKEAWAYS" FROM A MATHEMATICS WHOLE-CLASS DISCUSSION

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The transition from a mathematics student to a mathematics teacher requires a shift in how one perceives classroom discourse—moving from a passive focus on arriving at correct answers to an active, diagnostic focus on interpreting student thinking and leveraging misconceptions for conceptual learning. While significant research focuses on the facilitation of discussions (e.g., Smith & Stein, 2018), studies on what remains in the prospective teacher's mind once the discussion has concluded is lacking. This poster reports an analysis of prospective elementary teachers' (PTs) written responses to the prompt "What do you want to remember from this conversation about comparing fractions to a larger benchmark?"

Data were collected from 24 PTs immediately after a whole-class discussion centered on an incorrect peer contribution: *6/7 is bigger than 7/8 because sevenths are bigger than eighths.* The discussion was facilitated by an experienced mathematics teacher educator enacting the *building* teaching practice (Stockero et al., 2026). Two researchers independently applied *in vivo* coding to the 24 open-ended responses (Creswell & Poth, 2018). Through iterative comparison, codes were clustered into three primary takeaway orientations: (a) affective (16.6%), (b) mathematical (50.0%), and (c) representational (33.3%). Sub-classifications for specific mathematical takeaways will be detailed on the poster.

Table 1: Example of Each Primary Focus

Type of Primary Focus	Example
Affective	To not get overwhelmed and just acknowledge I know what I am doing. Sometimes it can be overwhelming, but I just breathe then do what is asked and I understand the problem. (PT20)
Mathematical	I had the same idea as we talked about. Instead of thinking about the larger part, think about the smaller part that is left. So this case, look at the 1/7 and 1/8 that are left. and we know that 1/7 is larger than 1/8, so seeing that 1/7 is missing from the 6/7 we know that 6/7 is smaller than the 7/8. (PT9)
Representational	I want to remember the diagram and the thinking of how we got the answer. It was helpful to see it in action and not really guess the answer. (PT21)

This study reveals that the PTs derive varied insights from the same productive whole-class discussion. While focusing on affective or representational aspects is common, these findings underscore the need for mathematics teacher educators to more deliberately guide PTs toward primary mathematical objectives. Future research could investigate how different types of scaffolding might more effectively shift PTs' attention toward the underlying mathematics of PTs' contributions or the intended pedagogical goals of the instruction.

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CREATING EQUITABLE LEARNING EXPERIENCES: DETRACKING GRADE 9 MATHEMATICS

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Research indicates that tracking worsens inequities by granting advanced opportunities to students in higher tracks while denying these benefits to those in lower tracks, thereby perpetuating existing educational disparities (Clandfield et al., 2014; James & Turner, 2017; Smaller, 2014). Several research studies reported that detracking leads to success for more students (Boaler, 2002, 2008; Boaler & Staples, 2014; Burris & Weiner, 2005; Strutchens et al., 2011). While the concept of detracking has been widely studied, a strong body of research suggests that equitable mathematics teaching practices in secondary schools requires further investigation as there is lack of or poor implementation overall (Berry & Berry, 2017; Boaler & Staples, 2008; Boston & Smith, 2009; Clandfield et al., 2014; Feiman-Newmser, 2012; Gresalfi et al., 2009; Gutierrez, 1996; Hand, 2012; Horn, 2005; Larnell et al., 2016; Leinward, 2015; NCTM, 2014; TDSB, 2018; Van Es & Sherin, 2002).

The purpose of this project was to investigate the current practices and teaching techniques of secondary school teachers that cater to a wide range of student abilities in their Grade 9 detracked mathematics course. Sixteen schools participated, with fourteen teachers interviewed from the largest school board in Canada. The interviews were transcribed and coded based on four categories: teacher beliefs, experiences in teaching the Grade 9 Mathematics course, best practices, and challenges.

Teachers reported meeting students' individual needs as a common teaching practice when teaching mixed-ability students. A key focus was assessing students' mathematical skills on an ongoing basis as well as hosting student-teacher conferences. Some teachers reported that using independent student working time was best suited to cater to the different needs in the classroom. Other teachers reported using open questions, introductory tasks that allow for multiple entry points, or low-middle-high entry level tasks in order to differentiate instruction for their students. The participants acknowledged that differentiated instruction is not a new concept, but that it has been placed at the forefront of the grade 9 mathematics course due to the restructuring of placing all (mixed ability) students in one classroom.

The classroom learning environment was also a common theme for teaching mixed-ability students in a grade 9 mathematics classroom. Teachers reported several different ways to form group work, such as naturally formed groups, working with senior-level students, hybrid groups (online and in-person mix), and the visibly random grouping approach. Teachers also spoke about the timing of introducing group work regularly at the start of the school year. Another key teaching practice is using vertical non-permanent surfaces (VNPSs), encouraging movement, openness, and collaboration throughout the classroom. The application of VNPSs was reported

by nearly all the participants in the study. Our study presents the teacher experience of detracking; however, there is a need to continue research into sustainable, equitable practices, targeting and bridging gaps of mixed abilities within the classroom.

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ADAPTABILITY AS COORDINATED DOMAIN FUNCTION: INTRODUCING THE TRIPARTITE MODEL IN SECONDARY MATHEMATICS TEACHING

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This study applies the *Tripartite Model of Adaptability* (Martin et al., 2013; Collie & Martin, 2016) to better understand how secondary mathematics teachers respond to classroom contexts during student teaching and their first year in the classroom. Rather than reducing adaptability to flexible behavior, the model conceptualizes it as a coordinated function across cognitive, affective, and behavioral domains. Demonstrating its explanatory power, the tripartite model was used to interpret pre-service teachers' (PST) experience adapting during student teaching and their first year teaching. This study is guided by two questions: (1) How is adaptability enacted across cognitive, affective, and behavioral domains in preservice and novice mathematics teachers? (2) What influences shape adaptive responses in mathematics teachers?

The Tripartite Model conceptualizes adaptability as multidimensional. *Cognitive adaptability* involves interpretation, reframing, and anticipatory reasoning as PSTs and novices draw on prior knowledge to make sense of unfolding events. *Affective adaptability* refers to emotional orientation and regulation including managing frustration, uncertainty, stress, or defensiveness to maintain responsiveness. *Behavioral adaptability* consists of observable instructional or relational adjustments such as modify pacing, questioning, tone, or task structure. Contextual understanding serves as an illustrative case of domain integration, demonstrating how knowledge-informed interpretations, emotional regulation, and aligned action operate together.

This qualitative study draws on interviews with three secondary math teachers conducted after student teaching and again after their first year in the classroom and incorporates analytic and reflexive memoing (Paulus et al., 2014). Initial coding (Saldaña, 2013) identified instances of adaptive responses and contextual information that informed their thinking, feeling, and acting. Secondary coding mapped these instances onto the tripartite domains, distinguishing between influences (e.g. prior experiences, perspective taking) and enactments (cognitive, affective, and behavioral shifts evident in teachers' in-the-moment descriptions and actions).

Findings show that adaptability in mathematics teachers emerged through coordinated cognitive, affective, and behavioral processes that deepened over time. Participants described not only behavioral adjustments (e.g. shifting strategy or pacing), but also cognitive reframing and affective regulation that appeared to underlie those shifts. We distinguish between influences (e.g. prior experiences, teacher visioning, mentorship, autonomy) and enactments, such as reinterpreting student behavior through contextual knowledge (cognitive), regulating toward empathy rather than reactive (affective), and adjusting instruction (behavioral), with cognitive reappraisal often enabling affective and behavioral change.

The Tripartite Model positions adaptability as coordinated domain functioning rather than isolated behavioral flexibility. For teacher education, this suggests that programs must: (1) Develop contextual knowledge alongside emotional regulation skills, (2) Provide structured opportunities for developing these domains, and (3) Support reflective practices that strengthen cross-domain coherence. By conceptualizing adaptability as multidimensional and integrated, the

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model offers a framework for designing teacher preparation experiences that better support PSTs and novices in adapting to current and future complex classroom environments.

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MATHEMATICS TEACHER PREPARATION: INVESTIGATING CURRICULUM AND CHARACTERISTICS OF ELEMENTARY AND SECONDARY PROGRAMS

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Most teacher preparation programs in the U.S. prepare two levels of mathematics teachers: elementary and secondary. Professional organizations (e.g., Association of Mathematics Teacher Educators [AMTE], 2017; Conference Board of Mathematical Sciences [CBMS], 2001, 2012; National Council of Teachers of Mathematics, 1991, 2014) have published recommendations for the preparation of mathematics teachers at both levels. In addition, researchers have investigated many aspects of elementary and secondary mathematics teacher preparation programs (e.g., Greenberg & Walsh, 2008; Masingila et al., 2022; Max et al., 2017; Mintos et al., 2019; Newton et al., 2014; Ronau et al., 2020; Tatto et al., 2010; Walsh, 2022).

In 2017, AMTE released *Standards for Preparing Teachers of Mathematics* (*SPTM*), a national guide that articulates a vision for mathematics teacher preparation and supports the continuous improvement of teacher preparation programs (AMTE, 2017). *SPTM* sets forth what well-prepared beginning teachers of mathematics should know, be able to do, and the dispositions they should develop throughout their programs. While *SPTM* was created by a team of experts in mathematics and mathematics education and is grounded in theory, there is a need to know how current programs are aligned with these recommendations.

In this study, we utilize a national survey to investigate current characteristics and curriculum of mathematics teacher preparation programs by collecting information (e.g., syllabi, activities, readings) from instructors of mathematics content and pedagogy courses with the aim of describing the current nature and structure of these programs in the U.S. and examining their alignment to SPTM. We use the following guiding research question: *What are the characteristics and curricula of U.S. elementary and secondary mathematics teacher preparation programs?* to describe (a) program structure, (b) activities used to support pre-service teacher content and pedagogical development, and (c) program alignment with SPTM.

In addition to a descriptive reporting of current program characteristics (e.g., goals, courses, instructors), we use the four *SPTM* standards for preservice teacher knowledge, skills, and dispositions as a conceptual framework for the study: (1) Mathematics Concepts, Practices, and Curriculum; (2) Pedagogical Knowledge and Practices for Teaching Mathematics; (3) Students are Learners of Mathematics; and (4) Social Contexts of Mathematics Teaching and Learning. Each standard has three to six indicators, with 19 total indicators (e.g., know relevant mathematics content, plan for effective instruction, anticipate and attend to students' thinking about mathematics content, draw on students' mathematical strengths). We aim to determine how current mathematics teacher education programs are aligned with these indicators, not in an evaluative sense but rather to explore the broad landscape of mathematics teacher preparation.

This study explores how current elementary and secondary mathematics teacher preparation programs support and develop beginning teachers of mathematics. Findings from this study will provide evidence to stakeholders related to how teacher preparation programs are realizing an updated vision of mathematics teacher education and create a shared database of goals, courses, readings, and activities for the mathematics teacher education community.

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EMOTIONAL REGULATION IN MATHEMATICS THROUGH EMOTIONAL COACHING FOR PRE-SERVICE TEACHERS

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Emotional regulation toward mathematics constitutes a fundamental aspect of initial teacher education, as the emotional experiences of future teachers influence their learning, professional development, and future educational practice (Rita & Pérez-Tyteca, 2017). Various studies have shown that mathematics often generates both positive and negative emotions, including anxiety, insecurity, and concern, which may affect academic performance and the willingness to teach this discipline.

In response to this issue, the present study aims to promote emotional regulation toward mathematics among pre-service teachers through the implementation of an emotional coaching process. Although previous studies have addressed emotional regulation in teacher education, such as those conducted by Caballero-Carrasco et al. (2021) and Garrido-Martos et al. (2019), further research is still needed on interventions specifically focused on emotions associated with the learning and teaching of mathematics.

The study is grounded in the Cognitive Structure of Emotions Theory (Ortony et al., 1996), as well as in the concepts of emotional knowledge, emotional regulation, and emotional coaching.

The research follows a qualitative approach through a case study involving three sixth-semester pre-service teachers enrolled in the Bachelor's Degree in Mathematics Teaching and Learning in Mexico. The methodology comprises five phases: identification of emotions and development of an emotional profile, enhancement of emotional knowledge, design of emotional regulation strategies, implementation of the strategies, and evaluation of the emotional changes achieved. Data are collected through questionnaires, semi-structured interviews, narratives, drawings, reflective journals, and emotional profiles.

The expected results aim to provide a methodology for regulating negative emotions toward mathematics among pre-service teachers and to explore the potential of emotional coaching as a strategy for strengthening their emotional and professional well-being.

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PMENA-ENHANCING PRESERVICE TEACHERS' DEVELOPMENT OF WORTHWHILE TASK DESIGN THROUGH GENERATIVE AI ASSESSMENT

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Introduction and the Problem

Development, selection, and implementation of worthwhile tasks are essential for student engagement and learning of mathematics. Worthwhile tasks are cognitively demanding, promote problem solving and critical thinking, and are characteristically open, providing a low-floor high-ceiling approach (Small, 2020; Smith & Stein, 1998; Van de Walle et al., 2023). Moreover, open questions and parallel tasks align to the Universal Design for Learning (UDL) framework providing for differentiation (Small, 2020) and emphasizing the "do" of UDL Math (Lambert, 2024). Despite the value of worthwhile tasks (Stein & Smith, 1998) pre-service teachers (PSTs) tend to resort to worksheets, procedural work, and Artificial Intelligence (AI) generated problems in lesson design and implementation (Arbaugh & Brown, 2005; Lee et al., 2026; Maiorca et al., 2024). Developing and accessing worthwhile tasks proves challenging. To aid educators, Smith and Stein (1998, 2018) developed a task analysis guide for levels of cognitive demand to identify and select these tasks. In addition, Artificial Intelligence (AI) adds a new layer to this process. The research questions are: RQ1. To what extent does the creation of AI-generated rubrics enhance PSTs' development of worthwhile tasks? RQ2. What are student perceptions of creating and using AI-generated rubrics to develop worthwhile task knowledge?

Design and Data Collection

A mixed-methods design will be employed with a convenience sample of PSTs (N=52) enrolled in two sections of the Fall 2026 methods course titled, MATH 335 Teaching Mathematics in Special Education. PSTs will complete a survey (pre-assessment) regarding their perceived knowledge of worthwhile tasks after instruction and prior to rubric design. PSTs will then create or select an initial worthwhile task (Sapkota, 2022). Aligned to Liljedahl et al.'s (2007) theoretical framework, PSTs will solve the task (predictive phase) and implement the task with a small group of PSTs (trial period) to determine mathematical and pedagogical affordances (Lee et al., 2024). PSTs will use generative AI to produce a rubric (reflective phase) aligned to Smith and Stein's Task Analysis Guide (Smith & Stein, 1998; Smith & Stein, 2018). Initial task scores will be generated by the PST and generative AI utilizing this rubric. PSTs will then develop a final task (refined initial task) based on knowledge from the rubric creation and the initial task assessment (adjustment phase). Final tasks will be self- and AI-assessed using the same rubric. Growth scores will be compared between the initial and final tasks. A post-survey will be administered to garner PSTs' perceived knowledge and to collect reflections.

Data Analysis & Results

Pre- and post-survey responses and rubric scores from initial and final worthwhile tasks will be analyzed using paired t-tests. In Vivo and Descriptive coding will be used to identify emerging trends from PST reflections (Mertler & Charles, 2011; Saldaña, 2021).

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DESIGNING EQUITY-ORIENTED DATA TALKS IN ELEMENTARY PRESERVICE TEACHER EDUCATION

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Keywords: Pre-Service Teacher Education; Statistics, Probability, and Data Science

Data science is increasingly recognized as a core strand of K–12 mathematics, engaging students in posing questions, interpreting variability, and making evidence-based decisions (Bargagliotti et al., 2020; California Department of Education, 2023; Lee & Wilkerson, 2018). Guided by an equity-oriented data science pedagogy lens, this study integrates statistical inquiry, mathematical discourse for multilingual learners, and students' lived experiences as central to justice-oriented mathematics learning (Aguirre et al., 2013; Gutstein, 2006; Moschkovich, 2010). The study asks: How do elementary preservice teachers incorporate community relevance, language supports, accessible representations, and action-oriented decision-making into Data Talk designs?

This exploratory study asks: How do elementary preservice teachers incorporate community relevance, language supports, accessible representations, and action-oriented decision-making into Data Talk designs? Qualitative analysis examines planning artifacts from a single three-hour class session in an elementary mathematics methods course at a public university. Grounded in research on K–12 data science and mathematical discourse, the session framed investigations as cycles of posing questions, collecting or curating data, representing and interpreting results, and drawing conclusions in context (Bargagliotti et al., 2020). PSTs were encouraged to embed structured sentence frames (Moschkovich, 2010) and to situate investigations in students' lived experiences (Gutstein, 2006).

Twenty-four PSTs working in grade-band groups (TK–K, K–2, 3–5, 6–8, and special education contexts) of 3–4 students, designed a "Data Talk" using a planning template organized in alignment with GAISE II as Wonder, Collect/Curate, Represent, Interpret, Act. The template required a clear investigative purpose, justified representation choices, discourse supports, anticipated misconceptions, and a decision-making or action component. Data sources included planning templates, presentation slides, and peer feedback. Artifacts were analyzed thematically with attention to community relevance, language scaffolds, representation accessibility, and action orientation. PSTs consistently selected contexts grounded in students' lived experiences (e.g., recess preferences, book genres, transportation access). Most investigations included decision-making components, positioning data science as participatory rather than procedural (Gutstein, 2006). All groups embedded sentence frames such as "I notice…" and "I claim ___ because…," foregrounding discourse as central to reasoning (Moschkovich, 2010). Several groups explicitly addressed sampling fairness, reflecting emerging data science awareness consistent with GAISE II (Bargagliotti et al., 2020).

Given the bounded nature of this single-session inquiry, findings are illustrative rather than generalizable. This artifact analysis suggests that structured planning supports preservice teachers in operationalizing equity-oriented data science practices in elementary mathematics. The poster will include the framework, Wonder–Collect/Curate–Represent–Interpret–Act progression, sample PST-designed Data Talks, thematic findings, and implications for mathematics teacher education.

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INTEGRATING LITERACY AND EARLY DATA SCIENCE IN PK-3 PRESERVICE TEACHER EDUCATION

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Data science is emerging as a central strand of K–12 mathematics, beginning in the early grades with counting, sorting, classifying, and comparing quantities as foundations for later statistical reasoning (Bargagliotti et al., 2020; California Department of Education, 2023; Lee et al., 2021; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In early childhood settings, these practices build on children's developing competencies in counting and comparison (Clements & Sarama, 2009). Equity-centered mathematics education further emphasizes language and meaningful contexts in supporting young learners' sense-making (Aguirre et al., 2013). Preservice teachers (PSTs) may struggle to connect children's literature, early numeracy, and foundational data practices in developmentally appropriate ways. This study examines: How do preservice teachers design and revise literacy-based lessons that engage preschool (P), transitional kindergarten (TK), and kindergarten (K) learners in early data science?

The study was conducted in a PK–3 mathematics methods course at a public university. In one class session, PSTs worked in groups of four to select a children's book curated by the course instructor, focused on early numeracy and designed a standards-aligned lesson for P, TK, or K that incorporated counting collections, sorting or classifying, and comparing quantities. In the following session, PSTs engaged in structured peer feedback and revised their lessons. Data sources included initial plans, peer feedback, revised plans, and written reflections. Grounded in research on early mathematics and K–12 data science, the task framed early data practices as organizing, representing, and comparing quantities in context (Bargagliotti et al., 2020). Integrating literacy with mathematics situated these investigations in familiar, play-based environments while supporting discourse and reasoning (Aguirre et al., 2013). The design cycle emphasized standards alignment, purposeful questioning, and opportunities for justifying.

Artifacts were qualitatively analyzed thematically with attention to the depth of data practices, standards alignment, literacy and play integration, and revisions following feedback. Initial plans often emphasized counting but varied in explicit attention to data organization or comparison. After peer critique, many groups strengthened the data component by adding sorting structures, representations, and prompts encouraging children to compare quantities and explain their reasoning. Revisions showed clearer alignment and more intentional questioning. Reflections suggested that feedback helped PSTs distinguish between counting as a procedural activity and engaging children in data-informed comparison. Because this inquiry was limited to a two-session design cycle, findings are illustrative rather than generalizable. However, the analysis suggests that integrating children's literature with structured peer critique may support PSTs in developing more robust conceptions of early data science in PK–3 mathematics. The poster will include research grounding early childhood data science, the lesson design and feedback structure, examples of initial and revised lesson excerpts, and implications for PK–3 teacher education.

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DEVELOPING PRE-SERVICE TEACHERS' FOUNDATIONS FOR FAMILY STEM

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Research consistently shows that parent–child math interactions are crucial drivers of children's mathematical skills, motivation, and identity (Fiskerstrand, 2022; Pomerantz et al., 2025), and that the quality of these interactions matters as much as their frequency (Silinskas & Kikas, 2019). Despite this evidence, educators are rarely prepared to cultivate meaningful family engagement around mathematics. Research indicates that home–school cooperation remains underdeveloped in teacher education programs (Cabral et al., 2024), leaving pre-service teachers (PTs) feeling underprepared (Meehan & Meehan, 2018). Yet, programs that involve community-based and practice-based models (e.g., family math nights) demonstrate shifts in PTs' beliefs and skills (Bofferding et al., 2016; Suh & Calabrese, 2025). To build upon this research, we engaged pre-service teachers in designing family STEM escape rooms that foregrounded mathematical reasoning, problem solving, and collaborative sense-making among children and caregivers. Grounded in the practice-based teacher education (Grossman et al., 2009), we addressed the following: In what ways do pre-service teachers design take-home escape room kits to support family engagement in STEM-based problem solving? To answer the research question, we utilized a multiple instrumental case study (Stake, 1995). Participants were three groups of 3–4 secondary education PTs as part of a teacher education program at the University of Eastern Finland. They each developed an escape room kit for young children and their families to engage in STEM concepts and practices in their home environments in collaboration with a local school. The data sources included video recordings of their collaboration (approximately 3.5 hours of data for each group) and a 30-minute post-interview with each group informed by prior research and the video recordings. Data analysis was informed by thematic analysis (Braun & Clark, 2006), which involved iterative coding to identify patterns in PTs discourse.

First, many PTs reflected on the unique affordances of the home environment (e.g., safe place for failures to occur), recognizing it as a valuable and often overlooked space for meaningful STEM learning. This acknowledgement suggests an emerging understanding of families as active participants in children's learning and homes as sites rich with cultural and intellectual resources. Second, PTs designed the kits to position caregivers as facilitators and children as the primary investigator. By prioritizing communication over content mastery, they provided caregivers a pedagogical role supported by "hidden" information, such as safety manuals or background context. This design choice highlights an evolving awareness that family engagement is most effective when it is relational rather than instructional. Third, in the design process, PTs made a conscious effort to distance the activities from traditional classroom expectations. As such, they used escape room mechanics (e.g., trial and error loops) to shift family's perception of what mathematics and STEM more broadly can look like. Based on these initial results, the significance of this study contributes to how we might prepare PTs to not only engage families in ambitious and collaborative inquiry in their homes, but develop their foundational understanding of how to meaningfully engage caregivers in their child's learning.

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SUPPORTING MATHEMATICS NOTICING OF PRODUCTIVE STRUGGLE IN EARLY CHILDHOOD TEACHER PREPARATION

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Early mathematical knowledge is a strong predictor of later academic achievement (Claessens et al., 2009; Duncan et al., 2007), positioning early childhood as a critical period for mathematics learning. Accordingly, instruction should be active, discourse-rich, and responsive to children's emerging ideas (National Association for the Education of Young Children [NAEYC] & National Council of Teachers of Mathematics [NCTM], 2010; NCTM, 2014). Within this broader conversation, productive struggle has become a central instructional goal. Classrooms that foster productive struggle provide opportunities for students to grapple with ideas not immediately apparent (Hiebert & Grouws, 2007; Huinker & Bill, 2017). Such struggle may surface through informal language, partial representations, social negotiation, or apparent frustration, and is not always articulated through mathematics claims (Bjorklund et al., 2020). For pre-service teachers (PSTs), this raises questions about how to define, notice, and support productive struggle in young children's mathematics. Research suggests that PSTs often struggle to interpret students' mathematics thinking in real time and may intervene prematurely when students appear stuck (Jacobs et al., 2010; Lee & Ginsburg, 2009; Warshauer, 2015). Teacher preparation must, therefore, find ways to approximate practice, creating opportunities for PSTs to notice, interpret, and support productive struggle (Grossman et al., 2009; McDonald et al., 2013).

Drawing on video case analysis (Borko et al., 2008; Sherin & van Es, 2009), we designed a video case study for elementary PSTs focused on noticing and supporting productive struggle in a K-2 context. After completing the task and discussing anticipated points of struggle, PSTs were introduced to the video context and reviewed selected clips and transcripts. In small groups, PSTs responded to a series of open-ended prompts focused on identifying moments of productive struggle and considering how to support or extend students' thinking. From this data, we conducted a qualitative thematic analysis of PSTs' written responses (Braun & Clarke, 2006) around how they anticipated struggle, recognized productive struggle, interpreted the mathematics underlying students' difficulties, and proposed instructional responses.

Preliminary analysis shows that PSTs demonstrated strengths in noticing and valuing productive struggle. Many PSTs were able to anticipate aspects of the task that might create difficulty for children. PSTs also recognized that moments of confusion, disagreement, or hesitation could be mathematically productive, noting the importance of allowing time to grapple, ask questions, and test ideas. However, PSTs' noticing was not always math-specific. In some cases, PSTs cited students as "confused" or "stuck" without articulating the mathematical ideas underlying the particular struggle. Instead, they focused on affect or participation. This suggests that while PSTs may value productive struggle, they need scaffolded support to identify what students struggle with mathematically. This underscores the importance of structured

approximations of practice that help PSTs move beyond general commitments to productive struggle toward more precise mathematical noticing in early childhood settings.

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PROFESSIONAL DEVELOPMENT AND IN-SERVICE TEACHER EDUCATION

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

AN EXPLORATORY STUDY ON OPEN PROBLEM SITUATIONS TO PROMOTE MODELING COMPETENCIES IN UNIVERSITY TEACHERS

UN ESTUDIO EXPLORATORIO SOBRE TAREAS PARA PROMOVER COMPETENCIAS DE MODELIZACIÓN EN PROFESORES UNIVERSITARIOS

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This exploratory study analyzes the work of two university mathematics teachers as they engage with an open problem situation (open-ended mathematical modeling task) focused on systems of linear equations (SLE) in the context of engineering education. The theoretical framework arises from integrating Borromeo Ferri's mathematical modeling cycle with the modeling competencies proposed by Maaß. Based on a literature review, guidelines for task design are established and a pilot implementation with two teachers is conducted, revealing a partial realization of the mathematical modeling cycle. The pilot implementation highlights the potential of incorporating open problem situations and identifies possible refinements to the proposed task.

Keywords: Professional Development, Modeling, Algebra and Algebraic Thinking

Introduction

At both the national and international levels, there is a growing interest in the design, implementation, and analysis of initial and continuing teacher education initiatives, as reflected in proposals by international organizations and in research reports found in the specialized literature.

In Mexico, it has been reported that there is a wide diversity of initiatives in the initial preparation of in-service mathematics teachers, especially at the higher education level (Beneitone et al., 2007; Camarena, 2004; Dolores, 2014; Fonseca, 2011). For this reason, it is important to promote professional development opportunities that strengthen teachers' mathematical knowledge and, above all, their didactical knowledge through reflection on their teaching practice.

Purpose of the Study

This study analyzes the work carried out by university mathematics teachers as they solve an open problem situation (open-ended task) designed from a mathematical modeling perspective, focused on the development of models based on systems of linear equations (SLE). The results of this exploratory study provide empirical evidence of the strategies developed to solve the proposed open problem situation, as well as the possible difficulties encountered in doing so.

Theoretical Framework

Mathematical modeling is considered a process in which a problem from a real-world context is translated into the context of mathematics in such a way that it can be solved and then interpreted back in relation to reality. The literature has proposed different cycles that make it possible to characterize this process; for this work, the mathematical modeling cycle from a cognitive perspective proposed by Borromeo Ferri (2018) was selected.

This cycle (Figure 1), consisting of six cognitive processes, begins with a problem from reality, from which the individual generates a mental representation of the situation through task

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

understanding. Subsequently, simplification and structuring of the task make it possible to create a real model of the situation. Then, a mathematical model is constructed from the real model, on which mathematical work is carried out to obtain mathematical results. Based on the interpretation of these mathematical results, real-world results are obtained, which must be validated by comparing those results with the mental representation of the situation.

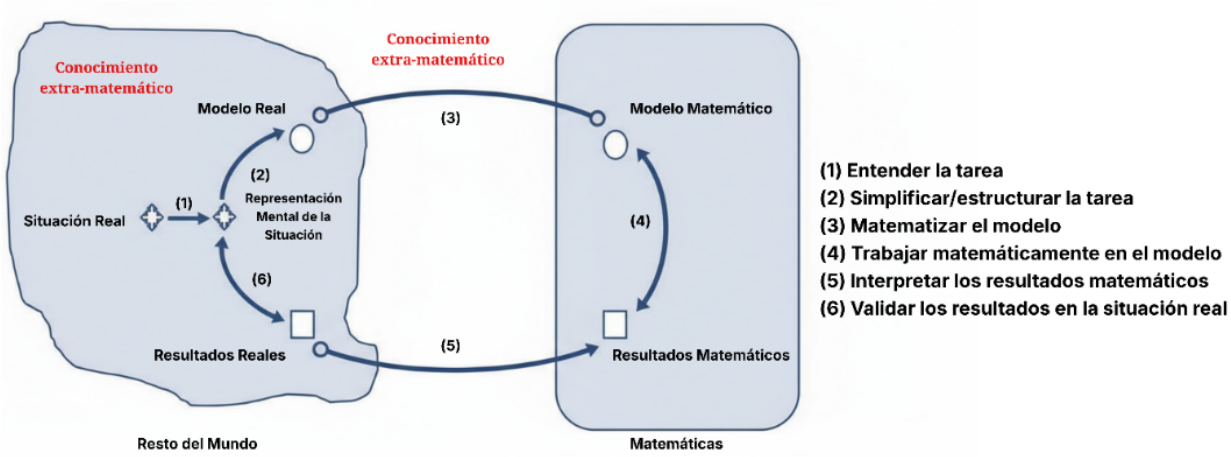


Figure 1: Mathematical Modeling Cycle from a Cognitive Perspective

The integration of the cognitive processes in the cycle presented by Borromeo Ferri (2018) with the modeling competencies proposed by Maaß (2006), as developed by Ramírez-Montes and collaborators (2021), is taken up in Table 1. Based on the authors' proposal, it is possible to delimit the types of practices or tasks that teachers should be able to develop in each of the cognitive processes of the cycle; therefore, these will form part of their didactic-mathematical knowledge and competencies.

Table 1: Alignment of Modeling Competences with the Cognitive Processes of the Mathematical Modeling Cycle

Modeling Competence	Competence to…	Cognitive Processes
Understand the real problem and develop a model based on reality	Make assumptions about the problem and simplify the situation; recognize the quantities that influence the situation, identify key variables; construct relationships among variables; search for available information and distinguish between relevant and irrelevant information.	(1) Understand the task (2) Simplify / Structure the task
Set up a mathematical model from the real model	Mathematize the relevant quantities and their relationships: simplify relevant quantities and their relationships if necessary and reduce their number and complexity; choose appropriate	(3) Mathematize the model

	mathematical notation and represent situations graphically.	
Solve mathematical questions within this mathematical model	Use heuristic strategies such as breaking the problem into subproblems, establishing relationships with similar or analogous problems, reformulating the problem, visualizing it in a different way, varying the quantities or available data, etc.; use mathematical knowledge to solve the problem.	(4) Work mathematically on the model
Interpret mathematical results in a real situation	Interpret mathematical results in extra-mathematical contexts; generalize solutions developed for a particular situation; communicate solutions using appropriate mathematical language and/or discuss them.	(5) Interpret mathematical results
Validate the solution	Critically review and reflect on the solutions found; revisit parts of the model or repeat the modeling process if the solutions do not fit the situation; consider other ways of solving the problem or developing alternative solutions; question the model as a whole.	(6) Validate the results in the real situation

Methodology

This study adopts an exploratory qualitative approach, consisting of a literature review and the design and piloting of an open problem situation based on the mathematical modeling cycle. These components are organized as follows:

1. For the literature review, a keyword-based search was conducted using search engines, with categories including: a) teacher professional development, b) mathematical modeling, c) engineering education, d) linear algebra, and e) systems of linear equations. Through this review, an overview of the current state of the central issues addressed in the study was established.
2. Based on the findings from the literature, it was determined which characteristics the proposed open problem situation should possess, as well as the selection of the context and the formulation of the open problem situation that would foster teachers' mathematical work in solving it.
3. Subsequently, the expected tasks to be carried out by teachers when solving the proposed open problem situation were identified, in connection with modeling competencies and the cognitive processes of Borromeo's mathematical modeling cycle. This specification of expected tasks provides a frame of reference for analyzing teachers' written work in the pilot study.
4. Then, an exploratory activity (pilot study) was conducted with two university-level mathematics teachers during two sessions of approximately 120 minutes each. Information was collected through an active observation process, in which the observer acted as facilitator of the activity using the technique of jointly guided work through appropriate questioning.

5. Finally, a descriptive analysis was carried out of the tasks performed by the teachers by contrasting them with the expected tasks, which made it possible to identify the strategies they adopted and the difficulties they faced while solving the open problem situations.

Results

The main findings for each of the aforementioned stages are presented below.

Literature Review

Regarding the difficulties in learning SLE, it has been found that, in school contexts, instruction tends to focus primarily on the study and solution of square systems with a unique solution, favoring an intramathematical approach and solution methods restricted to algebraic representations. This hinders learning by disconnecting algebraic meaning (equations) from geometric meaning (lines/planes), fostering misinterpretations of results and the perception of SLE solutions solely as a single intersection point (Oktaç, 2018; Trigueros et al., 2007).

Additionally, mathematical modeling has been identified as one of the prevailing trends in the literature of the discipline, both in the design of teaching proposals and in teacher education and professional development, especially with respect to mathematics education for future engineering professionals. With regard to the design of instructional proposals for mathematical training in engineering, several studies (Pizarro, 2020; Rodríguez-Gallegos & Quiroz-Rivera, 2016; Vázquez et al., 2016) agree that its incorporation, together with problems representative of students' future professional practice, fosters a deeper and more contextualized understanding of mathematical concepts. As part of professional development activities, emphasis is placed on its integration with authentic situations and appropriate contexts, the design of experiences that involve mathematical validation, and reflection on one's own learning processes (Cruz et al., 2020; Dede & Taşpinar, 2021; Ledezma et al., 2024).

The literature review carried out makes it possible to identify a scenario in which teaching strategies are undergoing transformation, driven by the incorporation of mathematical modeling as a suitable approach, particularly for the study of SLE in engineering programs. In this context, there is clear evidence of the need to update and diversify didactic knowledge and strategies through the integration of contextualized problems, the use of multiple representation registers, and the strengthening of competencies that connect real engineering contexts with disciplinary content in Linear Algebra, such as SLE.

Design of the Open Problem Situation

Drawing on the information collected during the literature review, a set of guidelines was proposed to inform decision-making in the design process, so that it would:

1. Promote the incorporation of authentic, open problem situations whose resolution requires the use of models involving various cases of SLE with real coefficients, primarily focusing on rectangular systems with infinitely many solutions.
2. Incorporate contexts suitable for engineering education that enable participants to move through each of the phases or cognitive processes of the mathematical modeling cycle, as well as to give meaning to and validate the solution of an SLE while solving the proposed problem situations.
3. Encourage the use of digital technology for the solution, validation, and interpretation of the proposed situations and the solutions obtained.

For the design of the open problem situation, the task proposed in Pochulu (2018) regarding the study of a one-liter milk container was adapted. In this case, the focus is on engaging a

modeling pathway in the context of redesigning the dimensions of a container based on SLE, formulated as follows:

> Tetra Pak® designs and produces carton packaging products for the food industry. One of its main products is the Tetra Brik®, made of layers of cardboard, polyethylene plastic, and aluminum in the shape of a parallelepiped. Considering the containers used for storing milk with a capacity of 1 liter, analyze and argue whether modifying the dimensions of the container, while keeping the amount of material and its shape fixed, changes its capacity. In addition, analyze the advantages and/or disadvantages of designing containers with different dimensions. Base your analysis and argumentation on a mathematical strategy.

For work with the open problem situation, participants are provided with real milk containers and a measuring tape. Based on the given situation and materials, it is expected that the planned tasks (Table 2) will be activated as participants follow a modeling pathway.

Table 2: Identification of Expected Tasks for Each Modeling Competence and the Associated Cognitive Processes of the Mathematical Modeling Cycle

Modeling Competence	Cognitive Processes	Expected Tasks
Understand the real problem and develop a model based on reality	(1) Understand the task (2) Simplify/ Structure the task	• Measure the real container provided as concrete material. • Construct a graphical representation (unfolded net of the material) of the real model using the actual dimensions of the container. • Analyze particular cases for different values of the container dimensions. • Identify quantities that vary (dimensions, volume) and those that remain invariant (area). • Identify the relationships among the different dimensions that make up the container. • Construct a graphical representation (unfolded net of the material) with a generalized description of the container dimensions.
Set up a mathematical model from the real model	(3) Mathematize the model	• Define unknowns (variables) that represent each of the container's dimensions. • Construct a mathematical model based on SLE, taking the measurements of the material as a constraint of the problem.
Solve mathematical questions within this mathematical model	(4) Work mathematically on the model	• Select an appropriate solution method for the proposed SLE. • Identify the solution set of the system.

Interpret mathematical results in a real situation	(5) Interpret mathematical results	• Identify constraints on the solution set based on the context of the problem. • Determine the range of admissible values for the free variable(s) in the solution set of the system.
Validate the solution	(6) Validate the results in the real situation	• Identify additional constraints for decision-making (minimum required volume, ergonomic or logistical conditions of the product). • Argue for the advantages and/or disadvantages of choosing different dimensions for the container.

Exploratory Activity (Pilot Study)

The exploratory activity was carried out over two working sessions with the participating teachers, who were guided through the use of questions that were carefully formulated on the basis of their contributions. In general terms, the participants engaged in a partial realization of the mathematical modeling cycle, developing tasks related to the first three cognitive processes in Borromeo Ferri's cycle, as shown in Table 3.

Table 3: Tasks Performed by the Teachers in the Pilot Study for Each Modeling Competence and the Associated Cognitive Processes in the Mathematical Modeling Cycle

Modeling Competence	Cognitive Processes	Tasks Performed
Understand the real problem and develop a model based on reality	(1) Understand the task (2) Simplify/ Structure the task	• Measure the real container provided as concrete material. • Construct a graphical representation (prism) of the real model using the actual dimensions of the container. • Calculate the volume of the container using the real measurements ($V = 1072.5\ cm^3$). • Analyze particular cases for different values of the container dimensions. • Identify quantities that vary (dimensions, volume) and those that remain invariant (area). • Identify the relationships among the different dimensions that make up the container. • Construct a graphical representation (prism) with a generalized description of the container dimensions.

Set up a mathematical model from the real model	(3) Mathematize the model	• Define unknowns (variables) that represent each of the container's dimensions. • Construct a mathematical model to calculate the volume of the container when one of its dimensions (height) changes and the other two remain constant. • Construct a mathematical model to calculate the surface area of the container when one of its dimensions (height) changes and the other two remain constant. • Construct a mathematical model to calculate the volume of the container when all of its dimensions change. • Construct a mathematical model to calculate the surface area of the container when all of its dimensions change.

During the first session, the teachers worked on tasks related to *understanding the problem* (the open problem situation) by constructing different particular cases in which the choice of dimensions was made arbitrarily, varying the measurements taken from the milk container that had been provided. This allowed them to formulate some initial ideas regarding the surface area and volume of the container. On this basis, they concluded that some of the proposed modifications would require a greater amount of material, which they deemed inappropriate for the company as it would entail a higher cost.

Subsequently, a case was posed in which two of the dimensions (width and length) remained constant, while the third (height) varied. Assigning a literal to the varying dimension led them to formulate two equations (one for surface area and another for volume), which allowed them, by the end of the first session, to conclude that they could not find a unique value that would keep both volume and surface area constant. In addition, by manipulating the concrete material provided, they reflected on inaccuracies in the calculation of area when folds required to form the container were not taken into account.

In the second session, the work from the previous meeting was reviewed. Through questions about the tasks they had performed, the teachers were prompted to reflect on the modification of the dimensions, taking into consideration issues related to material use and ergonomic and logistical aspects of the container redesign.

Building on these reflections, a case was proposed in which all of the container's dimensions could change, and equations were established to represent the volume (restricted to 1000 cm^3) and the total area (without a specific fixed value as a constraint). These final considerations enabled the participants to identify the area of the material sheet as one of the constraints that must remain unchanged in order to establish equations that support the construction of the mathematical model.

Final Remarks

The implementation of the exploratory activity reported here made it possible to confirm that the proposal promotes tasks that support participants' progression through several cognitive processes of the mathematical modeling cycle (Borromeo Ferri, 2018) and, consequently, the development of the associated modeling competencies (Maaß, 2006). Although a complete mathematical modeling cycle was not achieved in this exploratory phase, the use of appropriate questioning focused on participants' work can guide them in solving the open problem situation, fostering movement through each of the cognitive processes.

Additionally, the use of open problem situations such as the one proposed activates a variety of mathematical strategies, which makes it a mathematically rich situation and appropriate for a modeling-oriented approach.

Finally, the pilot implementation of the open problem situation made it possible to identify potential improvements in the formulation of the problem situation, as well as in the incorporation of additional resources to enrich the activity. In particular, the analysis of concrete cases supported participants in understanding the problem, formulating assumptions, and recognizing key relationships; therefore, integrating materials to construct physical models of these cases represents a feasible enhancement.

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UN ESTUDIO EXPLORATORIO SOBRE SITUACIONES-PROBLEMAS ABIERTAS PARA PROMOVER COMPETENCIAS DE MODELIZACIÓN EN PROFESORES UNIVERSITARIOS

AN EXPLORATORY STUDY ON OPEN PROBLEM SITUATIONS TO PROMOTE MODELING COMPETENCIES IN UNIVERSITY TEACHERS

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Este estudio exploratorio analiza el trabajo de dos profesores universitarios de matemáticas al realizar una situación-problema abierta basada en la modelización matemática, enfocada en sistemas de ecuaciones lineales para la formación de ingenieros. El soporte teórico de la propuesta resulta de la integración del ciclo de modelización matemática de Borromeo Ferri con competencias de modelización propuestas por Maaß. A partir de la revisión bibliográfica, se proponen pautas para el diseño de la situación-problema y se lleva a cabo el pilotaje con dos profesores, revelando un ciclo parcial. El pilotaje de la propuesta permite identificar el potencial de la incorporación de situaciones-problemas abiertas, así como mejoras en la misma.

Palabras clave: Desarrollo profesional, Modelización, Álgebra y pensamiento algebraico

Introducción

A nivel nacional e internacional se observa un creciente interés en el diseño, implementación y análisis de acciones de formación inicial y continua del profesorado, lo cual es visible tanto en las propuestas de organismos internacionales como en los reportes de investigación que se encuentran en la literatura especializada.

En México se ha reportado que existe una gran diversidad de iniciativas en la formación inicial de los profesores de matemáticas que se encuentran en servicio, especialmente en el nivel superior (Beneitone et al., 2007; Camarena, 2004; Dolores, 2014; Fonseca, 2011). Por lo anterior, es importante la promoción de espacios para el desarrollo profesional que potencialicen el conocimiento matemático de los profesores, pero sobre todo el conocimiento en la didáctica a partir de la reflexión sobre la práctica docente.

Propósito del estudio

Este estudio analiza el trabajo desarrollado por profesores universitarios de matemáticas al resolver una situación-problema abierta, diseñada bajo una perspectiva de modelización matemática, enfocada en el desarrollo de modelos basados en los sistemas de ecuaciones lineales (SEL). Los resultados de este estudio exploratorio proporcionan evidencia empírica sobre las estrategias desarrolladas para la resolución de la situación-problema propuesta, así como las posibles dificultades que enfrentan al resolverla.

Marco teórico

La modelización matemática se considera como un proceso en el que se traduce un problema de un contexto real, hacia el contexto de las matemáticas de tal manera que pueda ser resuelto y posteriormente regresado hacia la realidad. En la literatura se han propuesto diferentes ciclos que permiten caracterizar este proceso; para este trabajo se ha considerado utilizar el ciclo para la modelización matemática desde una perspectiva cognitiva propuesto por Borromeo Ferri (2018).

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Este ciclo (Figura 1), conformado por seis procesos cognitivos, se inicia con un problema de la realidad sobre el cual el individuo genera una representación mental de la situación a partir del entendimiento de la tarea. Posteriormente, la simplificación y estructuración de la tarea, permite la creación de un modelo real de la situación. Después, se crea un modelo matemático del modelo real, sobre el cual se trabaja matemáticamente y se obtienen resultados matemáticos. Con base en la interpretación de los resultados matemáticos, se obtienen resultados reales que deberán ser validados a través de la comparación de dichos resultados reales y la representación mental de la situación.

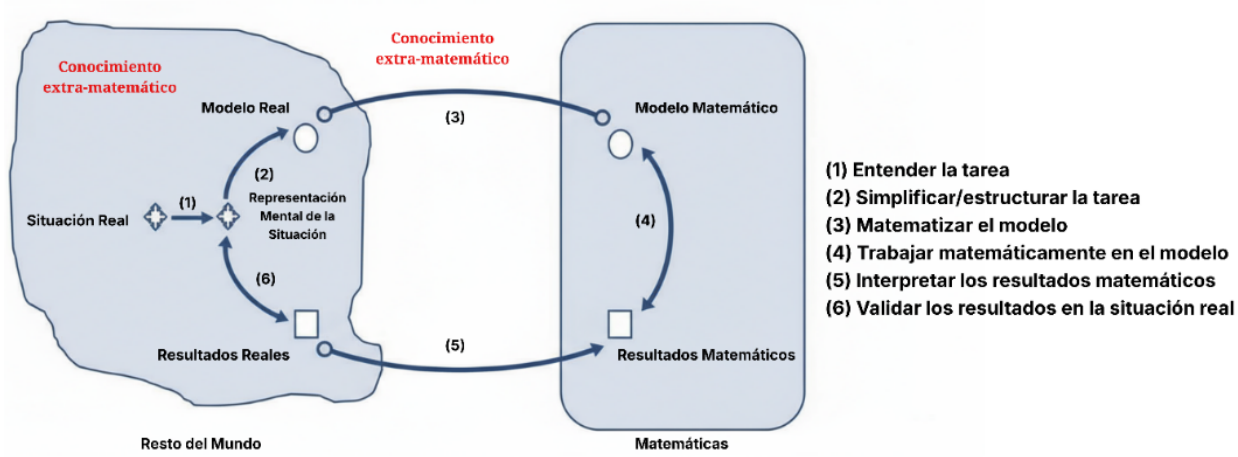


Figura 1: Ciclo de modelización desde una perspectiva cognitiva

Se retoma la integración de los procesos cognitivos del Ciclo presentado por Borromeo Ferri (2018) con las competencias de modelización que presenta Maaß (2006) realizada en el trabajo de Ramírez-Montes y colaboradores (2021) que se presenta en la Tabla 1. A partir de lo que presentan los autores, es posible delimitar el tipo de prácticas o tareas que los profesores deberán poder desarrollar en cada uno de los procesos cognitivos del ciclo, por ende, formarán parte de sus conocimientos y competencias didáctico-matemáticas.

Tabla 1: Articulación de las competencias de modelización con los procesos cognitivos del ciclo de modelización

Competencia de modelización	Competencia para…	Procesos cognitivos
Comprender el problema real y elaborar un modelo basado en la realidad	Hacer suposiciones sobre el problema y simplificar la situación; reconocer las cantidades que influyen en la situación, identificar variables clave; construir relaciones entre las variables; buscar información disponible y diferenciar entre información relevante e irrelevante	(1) Comprender la tarea (2) Simplificar/ Estructurar la tarea

Establecer un modelo matemático a partir del modelo real	Matematizar las cantidades relevantes y sus relaciones: simplificar cantidades relevantes y sus relaciones si es necesario y reducir su número y complejidad; elegir notaciones matemáticas apropiadas y representar gráficamente situaciones	(3) Matematizar el modelo
Resolver preguntas matemáticas dentro de este modelo matemático	Utilizar estrategias heurísticas como dividir el problema en subproblemas, establecer relaciones con problemas similares o análogos, reformular el problema, visualizarlo de forma diferente, variar las cantidades o los datos disponibles, etc.; emplear conocimientos matemáticos para resolver el problema	(4) Trabajar matemáticamente sobre el modelo
Interpretar resultados matemáticos en una situación real	Interpretar resultados matemáticos en contextos extramatemáticos; generalizar soluciones desarrolladas para una situación particular; comunicar soluciones usando lenguaje matemático apropiado y/o dialogar sobre ellas	(5) Interpretar resultados matemáticos
Validar la solución	Revisar críticamente y reflexionar sobre las soluciones encontradas; revisar partes del modelo o repetir el proceso de modelización si las soluciones no se ajustan a la situación; considerar otras formas de resolver el problema o desarrollar soluciones alternativas; cuestionar el modelo en general	(6) Validar los resultados en la situación real

Metodología

Se presenta un estudio cualitativo de tipo exploratorio conformado por una revisión de la literatura, así como el diseño y pilotaje de una situación-problema abierta basada en el ciclo de modelización matemática. Estas fases se organizaron de la siguiente manera:

1. Para la revisión de la literatura, se llevó a cabo una búsqueda por palabras clave en motores de búsqueda, en la que se incluyeron categorías como: a) desarrollo profesional docente, b) modelización matemática, c) formación de ingenieros, d) álgebra lineal y, e) sistemas de ecuaciones lineales. Mediante esta revisión, se ubica un panorama sobre la situación actual de las cuestiones nodales del estudio.
2. Derivado de los hallazgos en la literatura, se definieron las características que debían poseer las situaciones-problemas propuestas, así como la selección del contexto y el establecimiento de la situación que propiciara el trabajo matemático del profesorado para resolverla.
3. Posteriormente, se determinaron tareas esperadas que el profesorado desarrolle al resolver la situación-problema abierta que se propone, asociadas con las competencias de modelización y los procesos cognitivos del ciclo de modelización matemática de Borromeo. La definición de estas tareas esperadas proporciona un marco de referencia para el análisis de las producciones del profesorado en el pilotaje.

4. Luego, se llevó a cabo una actividad exploratoria (pilotaje) con dos profesores de matemáticas de nivel universitario durante dos sesiones de aproximadamente 120 minutos cada una. Se recolectó información mediante un proceso de observación activa, con la participación del observador como conductor de la actividad a través de una técnica de elaboración conjunta guiada por preguntas adecuadas.
5. Finalmente, se efectuó un análisis descriptivo de las tareas realizadas por el profesorado mediante el contraste con las tareas previstas, lo que permite establecer qué estrategias siguen y qué dificultades enfrentan durante la resolución de las situaciones-problemas abiertas.

Resultados

A continuación, se presentan los principales resultados de cada uno de los momentos descritos anteriormente.

Revisión de la literatura

Con respecto a las dificultades en el aprendizaje de los SEL, se ha encontrado que en el contexto escolar se promueve principalmente el estudio y solución de sistemas cuadrados con solución única, privilegiando un abordaje desde un contexto intramatemático y con métodos de solución que se restringen al trabajo con registros algebraicos. Esto dificulta su aprendizaje al desvincular el significado algebraico (ecuaciones) y el geométrico (líneas/planos), promover interpretaciones erróneas de los resultados y la percepción de la solución de los SEL únicamente como un punto de intersección único (Oktaç, 2018; Trigueros et al., 2007).

Adicionalmente, se ha ubicado a la modelización matemática como una de las tendencias en la literatura del campo disciplinar, tanto en el diseño de propuestas de enseñanza como respecto a la formación y desarrollo profesional de los profesores, en especial cuando se refiere a la educación matemática para futuros profesionales de la ingeniería. Con respecto al diseño de propuestas de enseñanza para la formación matemática en ingeniería, diversos estudios (Pizarro, 2020; Rodríguez-Gallegos & Quiroz-Rivera, 2016; Vázquez *et al.*, 2016) coinciden en que su incorporación, acompañada de problemas representativos del futuro quehacer profesional de los estudiantes, favorece una comprensión más profunda y contextualizada de los conceptos matemáticos. Como parte de las actividades de desarrollo profesional, se resalta su incorporación en conjunto con situaciones auténticas y contextos apropiados, el diseño de experiencias que involucren la validación matemática y la reflexión sobre los propios procesos de aprendizaje (Cruz et al., 2020; Dede y Taşpinar, 2021; Ledezma *et al.,* 2024).

La revisión de la literatura realizada permite identificar un escenario en el que las estrategias de enseñanza se encuentran en transformación, impulsadas por la incorporación de la modelización matemática como un enfoque idóneo, particularmente en el estudio de SEL dentro de los programas de formación de ingenieros. En este contexto, se evidencia la necesidad de actualizar y diversificar los conocimientos y estrategias didácticas, mediante la integración de problemas contextualizados, el uso de diversos registros de representación y el fortalecimiento de competencias que articulen contextos reales de la ingeniería con los contenidos disciplinares del Álgebra Lineal como los SEL.

Diseño de la situación-problema abierta

Mediante la información recolectada durante la revisión de la literatura, se proponen directrices que permiten la toma de decisiones para el diseño, de tal manera que este permita:

1. Promover la incorporación de situaciones-problemas auténticas y abiertas que requieran su resolución mediante modelos que consideren el uso de diversos casos de SEL $m \times n$

con coeficientes reales, atendiendo primordialmente aquellos sistemas de forma rectangular y con infinitas soluciones.
2. Incorporar contextos adecuados para la formación de ingenieros que permitan el tránsito de los participantes por cada una de las fases o procesos cognitivos del ciclo de modelización, además de la significación y validación de la solución de un SEL, al resolver las situaciones-problemas planteadas.
3. Fomentar el uso de tecnología digital para la resolución, validación e interpretación de las situaciones planteadas y las soluciones que se encuentran.

Para el diseño de la situación-problema abierta se adaptó la situación planteada en Pochulu (2018) con respecto al estudio del envase para leche con capacidad de un litro. En este caso, se enfoca en la movilización de una ruta de modelización en el contexto del rediseño de las dimensiones de un envase basada en los SEL, redactada como se muestra a continuación:

> La empresa TetraPak® diseña y produce productos de envasado de cartón para la industria alimentaria. Uno de sus principales productos es el *Tetra Brik* ® formado por capas de cartón, plástico polietileno y aluminio con forma de paralelepípedo. Considerando los envases para el almacenamiento de leche con capacidad de 1 litro, analice y argumente si al modificar las dimensiones del envase, pero manteniendo la cantidad de material y forma, se modifica su capacidad. Además, analice las ventajas y/o desventajas que tiene el diseño de envases con diferentes dimensiones. Base su análisis y argumentación en una estrategia matemática.

Para el trabajo con la situación-problema, se les proporciona a los participantes envases de leche reales y cinta métrica. A partir de la situación y el material que se les presenta, se espera que se movilicen las tareas previstas (Tabla 2) al recorrer una ruta de modelización.

Tabla 2: Determinación de tareas previstas para cada competencia de modelización y los procesos cognitivos asociados del ciclo de modelización

Competencia de modelización	Procesos cognitivos	Tareas previstas
Comprender el problema real y elaborar un modelo basado en la realidad	(1) Comprender la tarea (2) Simplificar/ Estructurar la tarea	• Medir el envase real proporcionado como material concreto. • Construir una representación gráfica (hoja de material extendida) del modelo real con las medidas concretas del envase. • Analizar casos particulares para diferentes valores de las dimensiones del envase. • Identificar cantidades que cambian (dimensiones, volumen) y aquellas que permanecen invariantes (área). • Identificar las relaciones entre las diferentes dimensiones que componen el envase. • Construir una representación gráfica (hoja de material extendida) con una generalización para las dimensiones del envase.

Establecer un modelo matemático a partir del modelo real	(3) Matematizar el modelo	• Definir incógnitas (literales) que representen cada una de las dimensiones del envase. • Construir un modelo matemático basado en los SEL, tomando en cuenta las medidas del material como restricción del problema.
Resolver preguntas matemáticas dentro de este modelo matemático	(4) Trabajar matemáticamente sobre el modelo	• Seleccionar un método de resolución adecuado para el SEL planteado. • Identificar el conjunto solución del sistema planteado.
Interpretar resultados matemáticos en una situación real	(5) Interpretar resultados matemáticos	• Identificar las restricciones del conjunto solución con base en el contexto del problema. • Determinar el rango de valores permitidos para la o las incógnitas libres en el conjunto solución del sistema.
Validar la solución	(6) Validar los resultados en la situación real	• Identificar restricciones adicionales para la toma de decisiones (volumen mínimo necesario, condiciones ergonómicas o logísticas del producto). • Argumentar ventajas y/o desventajas de la elección de diferentes dimensiones para el envase.

Actividad exploratoria (pilotaje)

La actividad exploratoria se llevó a cabo durante dos sesiones de trabajo con los profesores, guiándoles mediante el uso de preguntas convenientemente establecidas a partir de sus participaciones. De manera general, los participantes realizaron un ciclo de modelización parcial, desarrollando tareas relacionadas con los tres primeros procesos cognitivos del ciclo de Borromeo, como se muestra en la Tabla 3.

Tabla 3: Tareas realizadas por los profesores en el pilotaje para cada competencia de modelización y los procesos cognitivos asociados del ciclo de modelización

Competencia de modelización	Procesos cognitivos	Tareas realizadas
Comprender el problema real y elaborar un modelo basado en la realidad	(1) Comprender la tarea (2) Simplificar / Estructurar la tarea	• Medir el envase real proporcionado como material concreto. • Construir una representación gráfica (prisma) del modelo real con las medidas concretas del envase. • Calcular el volumen del envase considerando las medidas reales ($V = 1072.5\ cm^3$). • Analizar casos particulares para diferentes valores de las dimensiones del envase.

		• Identificar cantidades que cambian (dimensiones, volumen) y aquellas que permanecen invariantes (área). • Identificar las relaciones entre las diferentes dimensiones que componen el envase. • Construir una representación gráfica (prisma) con una generalización para las dimensiones del envase.
Establecer un modelo matemático a partir del modelo real	(3) Matematizar el modelo	• Definir incógnitas (literales) que representen cada una de las dimensiones del envase. • Construir un modelo matemático para el cálculo del volumen del envase cuando cambia una de sus dimensiones (alto) y dos permanecen constantes. • Construir un modelo matemático para el cálculo del área superficial del envase cuando cambia una de sus dimensiones (alto) y dos permanecen constantes. • Construir un modelo matemático para el cálculo del volumen del envase cuando cambian todas sus dimensiones. • Construir un modelo matemático para el cálculo del área superficial del envase cuando cambian todas sus dimensiones.

Durante la primera sesión, se realizaron tareas relacionadas con la comprensión del problema (situación-problema abierta) mediante la construcción de diferentes casos particulares en donde la elección de las dimensiones se dio de manera arbitraria, variando las medidas tomadas del envase de leche que se les proporcionó, permitiendo establecer algunas ideas con respecto al área superficial y el volumen del envase. A partir de lo anterior, se estableció que algunas de las modificaciones requerirían el uso de una cantidad mayor del material, lo cual no consideraban pertinente para la empresa al representar un mayor costo.

Posteriormente, se planteó un caso donde dos de las dimensiones (ancho y largo) permanecían constantes, mientras que otra variaba (alto). A partir de lo anterior, asignando una literal a la dimensión que varía se plantearon dos ecuaciones (una para el área superficial y otra para el volumen) lo que les permitió establecer, al final de la primera sesión, que no podían encontrar un valor único que permitiera que tanto el volumen como el área permanecieran constantes. Además, mediante la manipulación del material concreto proporcionado se reflexionó sobre imprecisiones en el cálculo del área al no considerar los dobleces que permitían formar el envase.

En la segunda sesión, se revisó el trabajo realizado en la sesión anterior. Mediante el uso de preguntas sobre las tareas realizadas se reflexionó con los profesores sobre la modificación de las dimensiones tomando en cuenta cuestiones sobre el uso del material, cuestiones ergonómicas y logísticas del rediseño del envase.

A partir de dichas reflexiones se propuso un caso en el que todas las dimensiones del envase podían cambiar, estableciendo ecuaciones que les permitían representar el volumen (restringido a 1000 cm^3) y el área total (sin encontrar un valor concreto al cual restringirse).

Estos últimos planteamientos permitieron a los participantes establecer el área de la hoja de material como una de las restricciones que debían permanecer sin cambios para el establecimiento de ecuaciones que les posibilitaran la construcción del modelo matemático.

Comentarios finales

La implementación de la actividad exploratoria reportada ha permitido constatar que la propuesta promueve tareas que facilitan el tránsito de los participantes por diversos procesos cognitivos del ciclo de modelización (Borromeo Ferri, 2018) y, por ende, el desarrollo de las competencias de modelización asociadas a ellos (Maaß, 2006). Aunque en esta exploración no se concretó un ciclo de modelización completo, el uso de preguntas adecuadas sobre el trabajo de los participantes puede guiarles en la resolución de la situación-problema abierta, propiciando el tránsito por cada uno de los procesos cognitivos.

Adicionalmente, el empleo de problemas abiertos, como el propuesto, moviliza diversas estrategias matemáticas, lo que lo convierte en una situación matemáticamente rica y adecuada para un enfoque de modelización.

Finalmente, el pilotaje de la situación-problema abierta ha permitido identificar posibles mejoras en la formulación de la situación problema, así como la incorporación de recursos adicionales para enriquecer la actividad. En particular, el análisis de los casos concretos facilitó a los participantes la comprensión del problema, la formulación de suposiciones y el reconocimiento de relaciones clave; por ello, integrar materiales para construir modelos físicos de dichos casos representa una mejora viable.

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FROM COUNTS TO CONNECTIONS: A RELATIONAL APPROACH TO TEACHER NOTICING

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Teacher noticing activities are common in teachers' professional learning programs, and research is needed to understand what they afford. Teachers' annotations of representations of practice offer a useful data source for studying noticing, yet many analyses rely on frequency-based methods that treat noticing dimensions as independent categories. Such approaches have limited capacity to capture how different aspects of noticing are coordinated within a single lesson moment. This paper explores the use of frequency-based analysis and Epistemic Network Analysis (ENA) to investigate how dimensions of teacher noticing relate to each other in the annotations of storyboard-based lesson representations. We apply and compare both approaches to illustrate their complementary analytic affordances.

Keywords: Teacher Noticing, Research Methods, Professional Development, Instructional Activities and Practices

Introduction

Teacher noticing has emerged as a central construct in research on teacher education and professional development (König et al., 2022). Lesson annotations—written comments attached to specific moments in a lesson representation—constitute a tractable data source for investigating teacher noticing, as they provide linguistically externalized and event-attached representations of what teachers attend to and their interpretations of it (Brown et al., 2024). Annotations do not capture the moment-by-moment process of noticing; rather, they represent the state of noticing at a particular instructional moment.

Noticing is widely theorized as multidimensional, with its dimensions understood to be interrelated (Rotem & Ayalon, 2022; Scheiner, 2016; van Es & Sherin, 2025). Researchers have developed analytic frameworks intended to capture its multiple facets. For example, Rotem and Ayalon (2022) distinguished cognitive, affective, and social dimensions of teacher noticing. However, conceptualizing noticing as multidimensional and interrelated has not been matched by analyzing it as such. Many studies code these dimensions separately, offering limited insights into how the dimensions of noticing are linked and coordinated. In fact, recent work suggests that how these dimensions are connected, not just how often each appears, can distinguish the noticing profiles of different professional communities. For instance, Stahnke and Gegenfurtner (2025) found that expert teachers' noticing tightly linked decision-making to student behavior and learning, whereas novices' centered on perceiving and orienting themselves within the scene. Examining this kind of relational structure, however, remains uncommon—a gap that is partly methodological: Many noticing-related studies use frequency-count approaches as independent categories, which limits the capacity to examine how they interact (Csanadi et al., 2018).

Together, these considerations raise a methodological question: How can analytic methods be aligned with conceptualizations of noticing as a multidimensional construct involving

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coordinated patterns? Addressing it requires tools that model relationships among dimensions rather than their frequencies alone. In this paper, we compare Epistemic Network Analysis (ENA) with frequency-based analysis to demonstrate how ENA can offer complementary insight into teachers' noticing through their annotations of representations of practice.

Literature Review

Methods of Analyzing Teacher Noticing

Teacher noticing research often relies on frequency-based analyses (Stahnke & Gegenfurtner, 2025). In these studies, noticing is coded into predefined categories, and shifts in category frequencies serve as primary indicators of teacher growth. For example, van Es and Sherin (2008) interpreted teachers' increased frequency of references to student mathematical thinking as evidence of growth in learning to notice. Similarly, research comparing novice and expert teachers frequently draws on counts of coded events, statements, or linguistic features to characterize their differences in noticing. For instance, Wolff et al. (2016) analyzed frequencies of teachers' eye movements and their use of mental and perceptual words to infer the cognitive richness of expert and novice teachers' perceptions of problematic classroom management. Although frequency-based analysis can describe distributions across noticing dimensions, it captures these dimensions in isolation and offers limited insight into how teachers coordinate them within a single interpretive moment (Csanadi et al., 2018).

Recent studies have attempted to address the multidimensional nature of noticing, yet most remain grounded in additive or aggregative logics. Researchers have addressed interactions among dimensions by either sacrificing analytic nuance or introducing increasingly complex categorical structures. For instance, Stahnke and Blömeke (2021) developed a nuanced framework capturing multiple subcategories of teachers' perception, interpretation, and decision-making (PID) in their focus of analysis. Although cross-coding was used to capture interactions among dimensions, the framework was ultimately collapsed into broader categories in the interest of analytic feasibility. As a result, relationships among finer-grained dimensions were examined primarily through aggregated category frequencies rather than through explicit modeling of their coordination. Other studies have sought to preserve multidimensionality by constructing combinations or composite categories to represent patterns of co-occurrence. For instance, Rotem and Ayalon (2022) coded teachers' written reflective reports across multiple cognitive, affective, and social dimensions of noticing and analyzed patterns of co-occurrence with 11 possible dimensional combinations. Similarly, to capture teachers' coordination among different aspects of noticing, Amador et al. (2025) included a dimension termed *specificity*, which was designed to reflect how teachers connected their comments to particular individuals or instructional elements, such as student thinking. While these approaches allow for some degree of interaction among dimensions, they do by multiplying categories or constructing partial combinations—increasing coding burden while capturing only a limited subset of possible relational configurations.

The Application of Epistemic Network Analysis

Epistemic Network Analysis (ENA) was originally developed to model learning as changes in the relational structure among elements of thinking and practice (Shaffer et al., 2016). It is grounded in epistemic frame theory, which holds that members of a community of practice develop shared patterns of knowledge, skills, values, and ways of reasoning that shape their participation in professional activity (Shaffer, 2006). In teacher education, ENA has been used to study changes in teachers' cognitive structures by modeling shifts in the relational organization of discourse across different stages of professional development. For example, Zhang and Hsu

(2025) applied ENA to analyze in-service teachers' coordination of epistemic frame elements across four community activities. By constructing and comparing network representations at successive time points, they traced how connections among these elements evolved as teachers engaged in collaborative reflection and lesson design. Zhang et al. (2022) similarly applied ENA to compare student teachers' negotiation processes during collaborative problem solving before and after different online learning tasks. By coding discourse into four collaborative problem-solving dimensions and modeling their co-occurrence patterns, the authors examined how relational structures differed across student groups and over time. Together, these studies suggest that ENA is well-suited for investigating and tracing the state of noticing as reflected in co-occurrence patterns among noticing dimensions—offering a relational perspective that extends beyond distributional summaries.

A handful of studies exist that apply ENA to explore the concepts related to teacher noticing, and most of them analyze teacher talk. Farrell et al. (2022) used ENA to examine how student teachers connect noticed events to pedagogical knowledge, showing differences between novices and others who exhibited more sophisticated reasoning patterns. Similarly, Stahnke and Gegenfurtner (2025) demonstrated that going beyond showing the extent to which expert and novice teachers attend to specific components, ENA can reveal expert-novice differences in their mental models of classroom management representations. Unlike synchronous talk turns, asynchronous annotations are produced without the time pressures of live interaction, freeing teachers from immediate interactional demands and potentially enabling more layered and comprehensive articulation of what they notice. These characteristics suggest that teacher noticing as exhibited in annotations is well-suited for relational analysis—that is, uncovering how noticing dimensions co-occur—rather than being treated as a set of independent, countable dimensions.

Analytic Framework

Aligned with the view that noticing should be studied multidimensionally, this paper draws on two complementary dimensions: the content teachers notice and the stance they adopt toward what they notice (van Es & Sherin, 2008, 2021). Research on representations of practice assumes that teachers can situate themselves within the instructional activity being depicted (Chieu et al., 2015). Here, stance serves as an indicator of presence—reflecting the degree of participants' involvement in the depicted experience (Tamborini & Skalski, 2012). When annotating representations of practice, participants may position themselves as active participants or as more detached observers of it (Herbst & Chazan, 2006; Schwarts et al., 2025). Building on Chieu et al.'s (2015) use of evaluation, alternativity, and reflection to capture stance, we applied and adapted this framework to our example annotations. Specifically, *evaluation* characterizes instances in which teachers position themselves outside the depicted activity to appraise the lesson as an object (e.g., *"A good approach by the teacher to collect similar ideas," T104*), *reflection* captures moments where teachers position themselves between participation and observation by interpreting underlying instructional logic or expressing uncertainty (e.g., *"I'm wondering what are the other students doing while the teacher is having these discussions with the pairs […], T101)*, and *alternativity* refers to teachers' positioning within the lesson to propose plausible alternative actions (e.g., *"Again the teacher could ask 'Are [you] sure that the triangles are congruent? Can you prove it?'," T106)*.

Our content coding framework draws on several existing frameworks (e.g., Star & Strickland, 2008; Amador et al., 2025) and is organized around the instructional triangle (Cohen et al., 2003). Accordingly, annotations tend to reflect teachers' attention to exchanges among

teachers, students, and content. In this paper, the content-related codes that have been used are *instructional practice*, *communication*, *mathematical content,* and *student learning*. We take them to mean, respectively, teachers' attention to instructional design and teachers' pedagogical moves (e.g., *"isn't the task first individual and then partnering up? I find the instruction a bit confusing," T101*), to classroom discourse and interactional features such as wording of questions/responses (e.g., *"Student 1: Since it's a competition, I guess everyone should have an equal chance to win the prize; right? Teacher: Yes, that's correct […]," T103*), to mathematical problems, representations or ideas (e.g., *"Why is C a circle center and of what circle? This is exactly what needs to be proven?", T104*), and to students' engagement, progress, and development (e.g., *"[...]I'm also wondering if students made the connection between their task and the theorem," consensual annotation*).

Methods

Description of the Example Data

The data are drawn from a professional development program called Story*Circles*. In Story*Circles*, teachers collaboratively composed and annotated a mathematics lesson in storyboard format within a multimedia environment (Herbst & Milewski, 2018). Storyboards, defined as structured visual representations of classroom events, offer a balance between the symbolic economy of written cases and the rich but potentially unwieldy detail of video (Herbst et al., 2014). Storyboards reduce incidental detail and foreground key instructional actions and discourse, making the underlying organization of classroom events more perceptible.

Our analysis draws on two lesson commentaries produced by a group of six teachers: an Individual Lesson Commentary (ILC) and a Consensual Lesson Commentary (CLC), developed while reviewing the same storyboard of the *Midpoint of the Hypotenuse* lesson created by another group. Over three weeks, teachers engaged in asynchronous individual work and weekly synchronous meetings. Asynchronous work involved individual annotation of the storyboard, while synchronous meetings provided opportunities to review annotations, discuss them collectively, and interact with peers and the facilitator. By the end of the three weeks, teachers were expected to reach consensus on the annotations to be shared with the group that had created the storyboard. The ILC was completed prior to the second synchronous meeting, with teachers independently annotating the storyboard. Annotations unrelated to stance or content (e.g., "Left a blue pin") were excluded, yielding a final dataset of 47 ILC annotations across 35 frames. The CLC was derived from ideas agreed upon during the final two synchronous meetings and represents the group's shared feedback. Across the same 35 frames, the group produced 22 consensual annotations. All annotations were done on Anotemos, an online, social, and multimodal media annotation software application (www.anotemos.com).

Coding

Each annotation was coded for stance and content using binary indicators (1 = present, 0 = absent). Because a single annotation could express more than one idea, multiple stance codes and multiple content codes could be assigned to the same annotation. This multi-label approach is consistent with prior work using annotation-based coding in related research (e.g., Amador et al., 2025). The first author and a research assistant coded the same set of annotations (n = 69 units) independently. Agreement was consistently high, with Cohen's kappa values ranging from .75 to .94 across all seven categories, indicating substantial to near-perfect reliability.

Two Methodological Approaches

We illustrate how the same corpus of annotations can be analyzed to investigate teacher noticing. We conducted both a frequency-based analysis and ENA to illustrate what information can be extracted from these two approaches.

Frequency-Based Analysis

The first approach we used here is frequency-based analysis, which is the typical method for describing and analyzing teacher noticing. For frequency-based analysis, a single annotation serves as the analytical unit. For each annotation, we counted the total number of each stance and content code and computed the proportion of annotations in which each code appeared. Because CLC was produced collectively and not attributable to individual teachers, statistical comparison between the two sets of data at the participant level is not appropriate. Instead, we report the results descriptively.

Table 1: Frequency and Proportion of Stance and Content of Noticing in the Individual Lesson Commentary (ILC) and the Consensual Lesson Commentary (CLC)

Dimensions	Codes	Frequency (Proportion)	
		ILC ($N = 47$)	CLC ($N = 22$)
Stance	Evaluation	19 (0.40)	3 (0.14)
	Reflection	17 (0.36)	3 (0.14)
	Alternativity	30 (0.64)	19 (0.86)
Content	Instructional Practice	21 (0.45)	8 (0.36)
	Communication	25 (0.53)	13 (0.59)
	Student Learning	11 (0.23)	1 (0.05)
	Mathematical Content	8 (0.17)	0 (0.00)

Table 1 shows the descriptive results of stance and content code frequencies in both ILC and CLC. In both conditions, alternativity is the most prevalent stance dimension, particularly in CLC (86%), indicating a strong emphasis on proposing alternative ways of teaching the lesson. Evaluation and reflection are more common in ILC (40% and 36%) than in CLC (both 14%). With respect to content, communication is the most frequent dimension in both ILC (53%) and CLC (59%), followed by instructional practice. In contrast, student learning and mathematical content are relatively more frequent in ILC (23% and 17%) than in CLC (5% and 0%). Overall, individual annotations exhibited a broader distribution of various stance and content dimensions, whereas collective annotations were more strongly centered on alternativity and communication.

Epistemic Network Analysis

Analytic procedure. In ENA, stanzas define the temporal window within which codes are treated as co-occurring, and units of analysis are defined as the level at which co-occurrences are aggregated into networks (Shaffer et al., 2016). In this paper, a stanza is defined as the set of annotations produced by a given teacher (unit of analysis) on a single lesson frame. This choice treats each frame as a contextually bounded lesson moment, capturing how teachers integrate stance and content codes while attending to that moment (Shaffer, 2012).

Although frequency-based analysis and ENA were based on the same coded corpus, the data were parsed differently. For frequency-based analysis, each coded annotation was treated as an independent instance, and codes were aggregated across frames. In contrast, for ENA, annotations were first grouped by stanza, and codes were analyzed as co-occurring within these

contextually bounded moments. For instance, if a teacher left three annotations on the same storyboard frame, frequency-based analysis treats them as three independent instances, whereas ENA treats them as one integrated noticing episode of that lesson moment and models the co-occurrence of noticing dimensions within it.

Then, for each unit (in our case, six individual teachers and one collective group), ENA constructs a co-occurrence matrix representing how often codes appeared together within these windows. These matrices are normalized and projected into a lower-dimensional space using singular value decomposition to generate the weighted network positions shown in the diagrams. The network is then modeled via association matrix representations (Bowman et al., 2021). Each unit thus has its own noticing network, which can be aggregated to represent shared group tendencies. In the ILC, six individual networks were generated and then averaged. This aggregation is theoretically justified: Teachers completed their annotations following an initial group meeting in which the facilitator collectively framed the task, establishing a shared interpretive context despite the annotations being produced individually and asynchronously.

What ENA reveals. Several types of information can be derived from the ENA analysis, which we illustrate using the example networks in Figure 1. First, nodes represent noticing stance and content codes, and node size indicates the relative frequency with which each code appears. Figure 1a presents the mean epistemic network for ILC, whereas Figure 1b presents the network for CLC. Both networks share the same layout, allowing for direct comparison of relational patterns. In the individual network, the most frequently represented dimensions include all three stance categories and instructional practice, with mathematical content being the least frequently annotated. In the consensual network, alternativity, communication, and instructional practice dominate the collective noticing, indicating a more focused pattern.

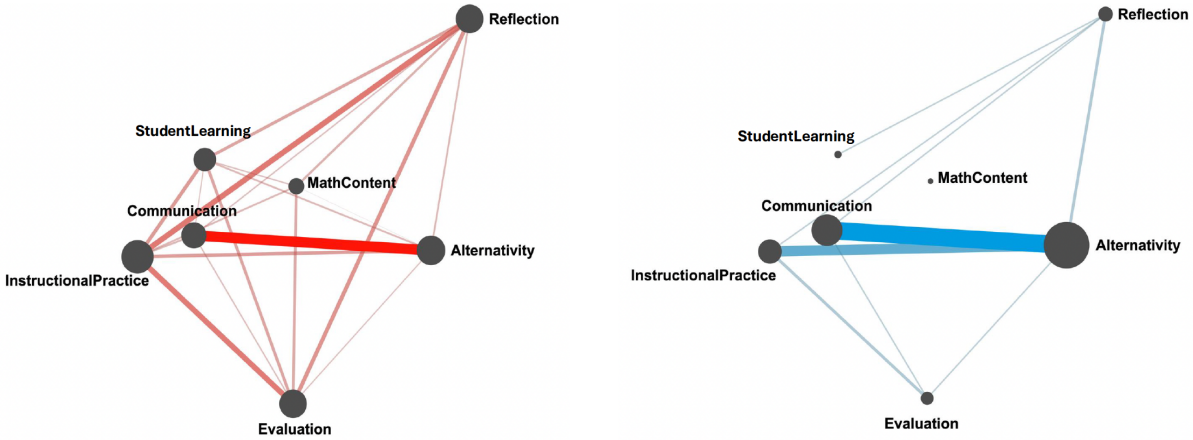


Figure 1: Mean Epistemic Networks of Teacher Annotations in (a) the Individual Lesson Commentary (ILC) and (b) the Consensual Lesson Commentary (CLC)

Second, edges represent the strength of co-occurrence between code pairs, reflecting how consistently two codes appear together within the defined stanza, averaged at the unit level. Edge thickness indicates stronger within-stanza co-occurrence, not simply higher code frequencies. In both networks, the strongest connection appears between alternativity and communication. In the individual network, most nodes are moderately integrated with others, reflecting a relatively distributed structure. Within the content and stance dimensions, relatively strong connections

appear between instructional practice and student learning, and between reflection and evaluation. In contrast, the consensual network exhibits a more centralized configuration, with co-occurrences concentrated among communication, instructional practice, and alternativity.

Finally, in addition to edge weights, nodes located closer together share common variance, even when their direct co-occurrence is relatively weak. In other words, closer nodes change in similar ways across teachers and frames. In the example, such pairs are instructional practice and communication, communication and student learning. Interpretively, this indicates that teachers who attend to instructional practice are also communication-aware, even if they mention these in separate annotations. In addition, teachers who mention students' learning also tend to attend to how ideas are communicated in the lesson.

Discussion

In this section, we compare the insights derived from frequency-based analysis with those generated by ENA to illustrate the affordances of a relational approach. Our goal is not to generalize the ENA findings but to demonstrate how ENA can help researchers rethink assumptions often embedded in frequency-based analysis.

First, the frequency results in Table 1 show a modest decrease in references to instructional practice from the individual to the collective annotation (from 45% to 36%), which might be interpreted as indicating reduced attention to teaching practice in a collective context. From this perspective, one might conclude that collective annotation contexts discourage teachers from engaging deeply with instructional design. ENA, however, complicates this interpretation by considering a second layer simultaneously. As can be interpreted from Figure 1, when annotating individually, this group of teachers primarily evaluates instructional practices (evaluation) or expresses uncertainty or interpretations about them (reflection). In collective annotation, rather than simply attending less to instructional practice, teachers shift toward proposing alternative approaches to teaching practice. Thus, ENA suggests a more nuanced interpretation of noticing of practice across contexts, which is challenging to capture solely through frequency analysis.

Beyond context-level shifts, a comparable divergence between frequency-based analysis and ENA is evident in how we interpret internal relationships between stance and content. As Martinez et al. (2017) note, frequency-based results invite additive interpretation—that is, treating highly prevalent dimensions as structurally associated. Our data challenges this logic with two instances. First, within the stance dimension, frequency analysis (Table 1) indicates that alternativity is the most prevalent category, with reflection and evaluation occurring at slightly lower but still substantial rates. Since teachers may take multiple stances, one might reasonably expect alternativity to be structurally linked with reflection or evaluation. However, ENA reveals that alternativity rarely co-occurs with either stance, whereas reflection and evaluation exhibit a stronger association, suggesting that when annotating individually, evaluative and reflective stances tend to cluster together, while alternativity occurs solely in annotations without much cumulative reasoning. Similarly, this fallacy also extends to cross-dimensional relationships. Frequency counts indicate that alternativity is the most prevalent stance, while communication and instructional practice are the most frequent content categories. On this basis, one might expect alternative suggestions concerning instructional practice or teacher-student interactions to dominate. ENA reveals, however, that individual teachers in this group more frequently direct alternative suggestions toward classroom dialogue and rarely toward instructional practices (Figure 1a). Across both contexts, the alternativity-communication linkage emerges as the strongest and most stable connection.

Finally, we briefly consider how ENA opens up new relational questions about teacher noticing. Although additional data are needed to examine these questions systematically, ENA highlights patterns that extend beyond distributional trends. For example, the combination of clustered evaluative-reflective stances and the relative isolation of alternativity of ILC suggests the need for researchers to design individual annotation tasks that better support and capture cumulative, integrative reasoning. Similarly, why do teachers consistently orient alternative suggestions toward classroom dialogue across contexts rather than toward instructional design? One possible explanation lies in the representational features of storyboard-based lessons, which render dialogue static and visually accessible and may orient teachers toward communicative aspects of instruction (Herbst et al., 2014; Herbst & Kosko, 2014). These observations illustrate how ENA enables researchers to generate relational hypotheses about teacher noticing that would be difficult to formulate on the basis of frequency analysis alone.

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AFFECTIVE, BACKGROUND, AND SCHOOL FACTORS INFLUENCING MATHEMATICS TEACHER BURNOUT

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This study examined factors influencing mathematics teacher burnout using a cross-sectional sample of 358 mathematics teachers in Oklahoma. Multiple regression analyses were conducted for each burnout dimension: exhaustion, mental distance, cognitive impairment, and emotional impairment. Mathematics teaching anxiety emerged as a consistent positive predictor across all models. Several aspects of the school environment, including staff affiliation, administrative team support, and opportunities for professional learning, were negative predictors of burnout. Selected demographics and teaching background variables were also significant, with age, loss of planning time, grade level taught, class size, and behavioral management style predicting burnout dimensions. These findings highlight the importance of mathematics-specific affective factors and the school environment for understanding mathematics teacher burnout.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs

According to Sutcher et al. (2016), teacher attrition is one of the main contributors to teacher shortages, which adversely affects districts financially (Barnes et al., 2007). Teacher burnout is correlated with teacher intentions/motivation to quit the profession (Leung & Lee, 2006; Skaalvik & Skaalvik, 2017), which makes it an important area of research. This is particularly true given the ongoing teacher shortage nationally (Learning Policy Institute, 2025a) and in the state in which the study was conducted (Oklahoma State School Board Association, 2022). Prior literature notes that a variety of factors influence burnout, such as social support and working environment (Sohail et al., 2023). However, a better understanding of a variety of factors influencing burnout is needed, particularly in areas of critical need, such as mathematics, which is experiencing a teacher shortage in 40 states (Learning Policy Institute, 2025b). It is also important to explore burnout as it relates to the concerns and perceptions in today's climate, which could greatly differ from those in previous decades as well as pre-COVID. The purpose of the current study is to examine which demographic, school, and affective factors are predictive of mathematics teachers' levels of burnout

Demographic/School Factors

Prior research provides evidence for different contributors to teachers' reported burnout. For example, Fisher (2011) found that novice teachers (less than 5 years) reported higher levels of burnout than more experienced teachers. Factors which may contribute to novice teachers leaving the profession include isolation, challenging classroom behaviors, and an overwhelming feeling of depersonalization (Hogan & White, 2021). This finding was similar to other work noting that burnout was a predictor of novice teacher turnover (Perrone et al., 2019). Conversely, Nguyen and Kremer (2022) noted that burnout seems to be cumulative, where more experienced teachers report higher levels of burnout than novice teachers. Collectively, these findings suggest burnout develops differently across career stages. Novice teachers are more susceptible to acute burnout with respect to professional demands, whereas experienced teachers exhibit cumulative burnout from prolonged exposure to institutional pressures. As a result, burnout may be represented as a career-long process rather than a standalone symptom.

Other work indicates that certain disciplines and grade level bands correlate with higher levels of teacher burnout. Specifically, secondary STEM teachers experienced higher levels of dissatisfaction and burnout in a study with over 30,000 teachers (Nguyen & Kremer, 2022). School- and classroom-level factors also influence teacher burnout. For example, higher numbers of students correlated with higher levels of burnout (Fisher, 2011). Additionally, administrative support has also been shown to be inversely related to teacher burnout (Brunsting et al., 2014). With research noting that more than 30% of teachers experience high levels of burnout (Molero Jurado et al, 2019), burnout is an urgent concern for the field given its association with teacher attrition. Consequences of high levels of burnout experienced by teachers are of particular concern in mathematics classrooms, where teacher shortages have been associated with student achievement (Siff & Song, 2025). Because mathematics functions as a gateway discipline into other areas of study, disruptions in teacher availability may have negative implications on student achievement; moreover, understanding burnout is critical not only as a means for teacher retention but also for equitable access to student educational outcomes (Arens & Morin, 2016).

Affective Factors

Self-efficacy is one of the affective factors that has repeatedly been correlated with teacher burnout (Shimony et al., 2022; Skaalvik & Skaalvik, 2007). Skaalvik and Skaalvik (2007) noted that "emotional exhaustion may result in reduced accomplishments, which again may affect self-efficacy negatively" (p. 621), indicating that burnout and efficacy may have a reciprocal relationship. Other work related to affect found a negative relationship between professional identity and teacher burnout (Li et al., 2023). This result aligns with the connection that professional identity has with teacher attrition, finding "that emotional burnout was the most salient pattern for dropout teachers" (Hong, 2010, p. 1541). Given the connection to emotion, it is not surprising that burnout and anxiety are also strongly correlated (Koutsimani et al., 2019). Extending this interpretation, professional autonomy has been proposed as a protective condition, with greater instructional-making authority associated with lower levels of "at least one component of burnout" (Farber & Bishop, 2025, p. 18). Collectively, affective and motivational factors including self-efficacy, professional identity, emotion, and anxiety appear interconnected in forming teachers' experiences of burnout rather than serving as independent influencers. By exploring background, school, and affective factors in conjunction, we can develop a better understanding of what contributes to mathematics teacher burnout.

Methods

Participants

A Qualtrics link was sent out to a listserv of 37,071 teachers in Oklahoma. There were 6,747 returned or error messages. Responses were collected from 4,136 teachers who agreed to participate in the study. After the data were cleaned and responses with large portions of missing entries were removed, the sample was reduced to 3,567, of which 358 identified themselves as mathematics teachers when given the options of math, science, or elementary teacher. It is this subsample of mathematics teachers that was the focus of the current study. When asked which grade levels taught, 31(9%) indicated K-12, 134 (37%) grades 6-9, 37 (10%) grades 6-12, and 156 (44%) grades 9-12. With regard to gender, 21% identified as Male and 78% as Female. Participants were between the ages of 20 and 71+. Three percent of the sample indicated they were of Hispanic origin. With regard to race, 85% indicated they were White, 9% Native American, 3% Black or African American, 2% Asian, 1% Multicultural or Other.

Data Collection and Analysis

Demographic and school background variables. In this study, demographic and background variables included gender, race/ethnicity, age, grade level range currently teaching, number of years of teaching experience, highest level of education, and school locale (rural, town, suburb, city, virtual). We also asked teachers to indicate where they were teaching in the area they were certified, as well as if they were currently teaching in an area that aligned with their Bachelor's or Master's degree. Finally, we asked about the typical class size, whether they had a planning period, and how often they lost their planning period due to other school activities (e.g., IEP meetings, covering another class).

Teacher burnout. Burnout was measured using the Burnout Assessment Tool (BAT; Schaufeli et al., 2020), which conceptualizes burnout as a multidimensional construct consisting of four core symptoms. Exhaustion reflects a severe loss of energy (e.g., "At work, I feel mentally exhausted") and demonstrated strong internal consistency in the present sample (α = .89). Mental Distance captures psychological detachment and aversion toward one's work and showed good internal consistency (α = .81). Emotional Impairment represents difficulties regulating emotional responses at work (e.g., "At work, I feel unable to control my emotions") and demonstrated strong internal consistency (α = .85). Cognitive Impairment reflects problems with attention, concentration, and memory and also showed strong internal consistency (α = .86). These reliability estimates are consistent with those reported in the BAT manual and support the use of the four interrelated dimensions as indicators of overall burnout.

Classroom management. Classroom management was measured using the Behavior and Instructional Management Scale (BIMS; Martin & Sass, 2010), a Likert-type instrument that assesses teachers' approaches to managing classroom behavior and instruction. The Behavior Management subscale reflects teachers' use of rules, reward systems, and responses to student misbehavior to maintain classroom order and demonstrated strong internal consistency in the present sample (α = .83). The Instructional Management subscale captures the extent to which teachers employ structured, teacher-directed instructional practices versus more student-centered and interactive approaches and showed acceptable internal consistency (α = .77). Example items include "I firmly redirect students back to the topic when they get off task" and "I nearly always use group work in my classroom." Consistent with the original scoring procedures, several Instructional Management items were reverse scored.

Rurality. For the present sample, internal consistency reliability was examined for the five RRTWLLES school-environment subscales used to represent rurality (Dorman et al., 2015). Staff Affiliation reflects the level of collegial support and sense of belonging among teachers, Administration Team Support represents the extent to which school leadership provides relational and instructional support, Mission Consensus refers to shared agreement among staff regarding school goals and direction, Opportunities for Professional Learning captures access to and facilitation of professional development despite geographic isolation, and Resource Adequacy indicates the availability of instructional, technological, and material resources. The Opportunities for Professional Learning subscale included reverse-scored items addressing barriers related to geographic isolation, whereas the remaining subscales consisted of positively worded items. Cronbach's alpha coefficients indicated strong reliability for Staff Affiliation (α = .87), Administration Team Support (α = .90), and Mission Consensus (α = .92), and acceptable reliability for Resource Adequacy (α = .74) and Opportunities for Professional Learning (α = .71). These values were largely consistent with those reported in the original validation study and suggest that the subscales demonstrated adequate to high internal consistency for use in analyses.

Professional mathematics identity. Teaching professional mathematics identity was measured using the Mathematics Teacher Identity scale (Cribbs & Utley, 2024), a Likert-type instrument that captures teachers' sense of self as mathematics educators through three subconstructs: Recognition, Interest, and Competence/Performance. Recognition reflects the extent to which individuals view themselves, and perceive others viewing them, as mathematics teachers; which demonstrated strong internal consistency in the present sample ($\alpha = .89$). Interest captures teachers' enjoyment of and desire to engage in thinking about and teaching mathematics and also demonstrated strong internal consistency ($\alpha = .91$). Competence/Performance represents beliefs about one's ability to understand and effectively teach mathematics and showed excellent internal consistency ($\alpha = .92$). The overall scale showed excellent reliability ($\alpha = .95$), consistent with the original validation study. Example items include "I can see myself as a math teacher" and "I am confident I can teach math well."

Mathematics teaching anxiety. To measure mathematics teaching anxiety, the Anxiety about Teaching Mathematics scale developed by Hadley and Dorward (2011) was used. This instrument consists of 12 Likert-type items (1 = no anxiety to 5 = extremely anxious) designed to assess teachers' anxiety related specifically to teaching mathematics, with responses summed to reflect overall levels of anxiety, where higher scores indicate greater anxiety. The items capture anxiety associated with preparing for instruction, teaching mathematical content, responding to student thinking, communicating with parents and administrators about mathematics, and anticipating student assessment outcomes. Example items include "Working out math equations on the board in front of a class of students" and "Having a surprise evaluation by an administrator during a math lesson you are teaching." For the present sample, the scale demonstrated strong internal consistency ($\alpha = .90$), which is consistent with the reliability reported in the original study.

Teaching self-efficacy toward mathematics. Mathematics teaching self-efficacy was measured using the Mathematics Teaching Efficacy Beliefs Instrument (Unfried et al, 2022). All items were rated on a 5-point Likert scale (1 = strongly disagree to 5 = strongly agree). This construct consists of two subscales: Personal Mathematics Teaching Efficacy Beliefs (PSTEB), which assesses teachers' confidence in their ability to effectively teach mathematics, and Mathematics Teaching Outcome Expectancy Beliefs (STOEB), which reflects beliefs about the extent to which effective mathematics teaching can influence student learning outcomes. Example items include "I know how to teach mathematics concepts effectively" and "When a student does better than usual in mathematics, it is often because the teacher exerted a little extra effort." For the present sample, both subscales demonstrated strong internal consistency (PSTEB $\alpha = .80$; STOEB $\alpha = .84$).

Stepwise linear regression analysis was used to create models examining how demographic variables, teaching experience, and mathematics-related constructs were predictive of teachers' levels of burnout. Four models were tested using different categories of burnout examined in the current study (Exhaustion, Mental Distance, Cognitive Impairment, and Emotional Impairment). Prior to data analysis, simple random imputation was used to create a complete dataset (Schlomer et al., 2010) as appropriate when data are determined to be missing completely at random.

Results

In order to examine which factors were predictive of mathematics teachers' levels of burnout, a stepwise regression analysis was performed for each category of burnout: Exhaustion, Mental Distance, Cognitive Impairment, and Emotional Impairment. Models were created by first

adding all variables of interest and then removing variables until only significant items remained. Variance Inflation Factor (VIF) was also used to determine if any multicollinearity existed between variables in the model, with values being lower than 2 for all variables included in the final models. Table 1 provides an overview of all four models, with empty cells indicating non-significance.

Table 1: Background, School, and Affective Factors Predicting Mathematics Teacher Burnout

Variable	Exhaustion	Mental Distance	Cognitive Impairment	Emotional Impairment
	Estimate (SE)	Estimate (SE)	Estimate (SE)	Estimate (SE)
Intercept	3.98 (.29)***	3.89 (.36)***	1.91 (.27)***	3.33 (.35)***
Background/School Factors				
Age		-0.05(.02)*		0.05 (.02)**
Grade level 6-9			0.17 (.07)*	
How often planning period is lost	0.10 (.04)**			
Typical class size			0.01 (.01)*	
Affective Factors				
Math Teaching Anxiety	0.21 (.05)***	0.20 (.05)***	0.30 (.04)***	0.42 (.05)***
Staff Affiliation	-0.19 (.07)**	-0.22 (.07)**		-0.14 (.05)*
Administration Team Support	-0.18(.05)***	-0.22 (.05)***		
Opportunities for Professional Learning		-0.17 (.07)*	-0.28 (.06)***	-0.19 (.07)**
Behavioral Classroom Management Approach		0.12 (.05)*		-0.13 (.05)*
Adjusted R^2	0.19	0.27	0.20	0.26

*p<0.05 **p<0.01 ***p<0.001

Only one demographic variable was significant for any of the models. Age was shown to be a negative predictor for mental distance (β =-.05), meaning that teachers reporting a higher age had lower levels of burnout due to mental distance. The opposite result was evident for emotional impairment, where age was a positive predictor (β =.05) in the model. Three school/teaching background variables were predictive of burnout. Losing planning time due to other school activities was positively predictive of burnout due to exhaustion (β =.10). In addition, teaching middle grades mathematics (6–9) exclusively as well as larger class sizes were both positive predictors for burnout related to cognitive impairment, (β =.17) and (β =.01), respectively.

Five constructs were found to predict one of the four categories of burnout. Mathematics teaching anxiety was a positive predictor for all four models, indicating that higher levels of anxiety predicted higher levels of burnout. The highest estimate for mathematics teaching anxiety was in the model for emotional impairment (β =.42) followed by cognitive impairment (β =.30). Several factors related to rurality were also found to be predictive of burnout, with staff affiliation negatively predicting burnout related to exhaustion (β =-.19), mental distance (β =-.22), and emotional impairment (β =-.14). Administration team support was a negative predictor for exhaustion (β =-.18) and mental distance (β =-.22). Finally, opportunities for professional learning were a negative predictor for mental distance (β =-.17), cognitive impairment (β =-.28), and emotional impairment (β =-.19). The final significant variable was behavioral management, which positively predicted mental distance (β =.12) and was a negative predictor for emotional impairment (β =-.13).

Discussion

Results provide insight into factors influencing teachers' reported levels of burnout. While age negatively predicted mental distance, indicating that older teachers report lower levels of mental distance, they also reported higher levels of emotional impairment. These results might account for the contradictory findings in prior research (Fisher, 2011), particularly if only one measure of burnout was considered. It should be noted that age had low estimates, indicating a small effect on burnout for both models where age was significant. Losing planning to other school activities and higher class sizes were both positive predictors for burnout, exhaustion, and cognitive impairment. Similar to other findings, the effect of class size on burnout was minimal (Saloviita & Pakarinen, 2021). It is important to keep in mind that even with some findings showing small estimates, these various factors also have a cumulative influence on burnout, which does not negate their significance. Higher levels of cognitive impairment were noted by 6th-9th grade teachers, which is more nuanced than prior work, noting higher levels of burnout in secondary teachers more generally (Nguyen & Kremer, 2022).

With regard to affective variables, mathematics teaching anxiety was the only significant variable across all four models for burnout. In addition, the beta values indicate moderate-level effects. This finding is consistent with other research exploring anxiety more generally (Koutsimani et al., 2019). Given the important role that anxiety has on burnout, future work might explore general teaching and individual anxiety alongside disciplinary anxiety. In addition, researchers might explore ways of reducing anxiety to help reduce burnout and subsequent attrition. Given the high rural population of the state in which the study was conducted (52%, not including rural serving schools), aspects of rurality were important to consider. Three related factors were found to be significant: staff affiliation, administrator support, and opportunities for professional learning. Staff affiliation was negatively predictive of exhaustion, mental distance, and emotional impairment. Other work notes the important role that relationships have on teacher burnout (Meredith et al., 2020), particularly the connection teachers have with other educators in mitigating intentions to leave the field (Struyve et al., 2016). Administrator team support was also negatively predictive of exhaustion and mental distance, aligned with prior related work (Brunsting et al., 2014). Other work notes that administrator dissatisfaction is the most frequently cited reason teachers provide for leaving a position in a rural setting (Ingersoll & Tran, 2023). Opportunities for professional learning were negatively predictive of mental distance, cognitive impairment, and emotional impairment. Given that teachers in rural settings often experience isolation (Shikalepo, 2020), professional learning opportunities might help mitigate these effects and subsequent burnout. This result might also connect to teacher autonomy, which has been linked to job satisfaction (Jerrim et al., 2023). The final significant variable was behavioral management approach, which was found to positively predict mental distance and negatively predict emotional impairment. This finding seems to indicate that while a stronger use of behavioral management in the classroom aids in mental distance, it has a detrimental influence on emotional impairment. Other research notes that self-efficacy toward behavioral management has a negative relationship with teacher burnout (Malinen & Savolainen, 2016). However, that study explored burnout as one measure rather than a more nuanced approach by exploring multiple aspects of burnout. The relationship between classroom management and burnout might be more complex than what has been described in prior work.

Implications

The findings suggest emotional experiences and the professional environment drive mathematics teacher burnout. The consistency between mathematics teaching anxiety and all

four categories of burnout is indicative of the important role that the factor has on burnout and the need to reduce anxiety to mitigate burnout. Given that the focus of much work related to math teaching anxiety is with elementary teachers and/or preservice teachers (Patkin & Greenstein, 2020; Szczygiel et al., 2025), more work is needed to better understand this finding as well as how to reduce mathematics teaching anxiety. It is possible that this finding might connect to a relatively large percentage of alternatively and emergency certified teachers now serving as mathematics teachers in the state. Given the findings related to opportunities for professional learning and staff affiliation, schools and districts may benefit from professional learning opportunities that promote collaborative problem solving among teachers, reinforce instructional practices, and support teachers through content-based approaches rather than only through procedural training. Protecting teachers' planning time and offering administrative support may serve as broad safeguards against teacher burnout. Additionally, a reduction in class size may preserve cognitive function while fostering autonomy and addressing teachers' emotional well-being. Examining mathematics teacher burnout requires a focus on improving working conditions while supporting teachers' affective experiences.

Limitations

Several limitations of the study are important to mention. The cross-sectional, self-report design prevents causal conclusions and may reflect teachers' perceptions rather than objective conditions. The stepwise regression approach is also sample-dependent and may omit theoretically relevant variables. Furthermore, the sample was limited to teachers in a specific context, restricting generalizability to other subjects or settings, and the survey measures do not necessarily capture the manifestation of burnout in the classroom.

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COMPARING COACH AND TEACHER PERCEPTIONS OF CONTENT-FOCUSED COACHING CYCLE CONVERSATIONS

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This study examined alignment in mathematics coaches' and teachers' perceptions of the productivity and benefits of conversations within content-focused coaching cycles. Post-conversation surveys from 36 coach–teacher dyads were analyzed across 106 online coaching conversations. Although both groups rated conversations as productive, teachers rated them more favorably than coaches. Only 14% of conversations demonstrated high benefit alignment. Certain benefits reflected this misalignment: teachers more often identified emotional support, coaches more often identified growth in teachers' content knowledge, and benefits related to planning or reflecting on lesson phases were commonly reported by only one member of the dyad. These findings suggest that coaches and teachers often interpret the same interactions differently, highlighting the importance of examining multiple perspectives in coaching research.

Keywords: Professional Development, Coaching, Teacher Knowledge, Affect and Emotion

Content-focused coaching (CFC) cycles are a common form of professional development (PD) used to strengthen mathematics instruction (West & Staub, 2003). In these cycles, coaches and teachers jointly plan a lesson, enact the lesson in the classroom, and then engage in structured reflection, a process that has been associated with improvements in both teaching and student learning (Gibbons & Cobb, 2016; Russell et al., 2020). We use the word coach to refer to individuals with part- or full-time release from teaching who provide PD to teachers in support of enhancing teaching and learning (Baker et al., 2022). Although CFC emphasizes collaboration and partnership (e.g., West & Cameron, 2013), relationships between mathematics coaches and teachers can be asymmetrical (Heritage & Clayman, 2010), given a coach and teacher have different titles and responsibilities as well as potentially differing levels of expertise (Baker et al., 2022; Fennell, 2017). This asymmetry makes it important to examine the extent to which a coach and teacher share similar perceptions of coaching interactions. Accordingly, this study examined (a) alignment in mathematics coaches' and teachers' perceptions of the productivity of their planning and debriefing conversations, (b) the degree to which their perceptions of the benefits of those conversations aligned, and (c) the perceived benefits associated with higher and lower levels of alignment.

This investigation drew upon data from a larger PD initiative in which 36 coach–teacher pairs participated in individual, online CFC cycles. The cycles were designed to support teachers' use of ambitious mathematics instruction, which refers to high-quality instructional practices associated with enhanced mathematics learning for students (National Council of

Teachers of Mathematics [NCTM], 2014). Following each planning and debriefing conversations, both members of the dyad responded separately to surveys about the conversation they had just completed. The analytic sample included 106 matched survey pairs from planning (n = 63) and debriefing (n = 43) conversations. Quantitative analyses compared coaches' and teachers' ratings of overall productivity on a Likert-scale item, while qualitative analyses examined open-ended responses describing perceived benefits of the conversations. The study addressed three research questions: (1) To what extent did coaches' and teachers' ratings of the productivity of their planning and debriefing conversations align?, (2) To what extent did coaches' and teachers' perceived benefits of their planning and debriefing conversations align?, and (3) What types of benefits do coaches and teachers report when their perceptions of conversation outcomes are aligned or misaligned?

Framing Literature

Like teacher–student relationships, coach–teacher relationships are inherently asymmetrical (Heritage & Clayman, 2010). In addition to often possessing greater expertise in mathematics content and pedagogical content knowledge (Fennell, 2017), coaches and teachers occupy distinct roles within their professional partnership. Within these roles, the coach is explicitly responsible for supporting a teacher's professional growth (Baker et al., 2022), whereas the teacher is not responsible for the coach's learning. In this sense, mathematics coaches are accountable for teacher learning in ways that parallel teachers' accountability for student learning (West & Cameron, 2013). Although both members of the coach-teacher dyad participate in coaching conversations with the shared goal of improving instruction and student learning, they enter these interactions with distinct responsibilities. Within CFC cycles, teachers are positioned to examine and refine their practice through collaborative planning and reflection on instruction (Callard et al., 2022; West & Staub, 2003). Coaches, in contrast, are responsible for facilitating these conversations in ways that support teachers to: produce concrete instructional products (e.g., viable lesson plans), develop new insights about mathematical content and pedagogy (Jakopovic, 2021; Stein et al., 2022; West & Cameron, 2013), and translate instructional experiences into professional knowledge that can inform sustained changes in practice (Desimone & Pak, 2017; West & Staub, 2003). Given these different roles and responsibilities, coaches and teachers may have different interpretations of the same coaching interaction. Divergent interpretations may shape how coaches and teachers perceive the outcomes of coaching interactions, influencing their ability to agree on instructional next steps following a coaching cycle (Kochmanski & Cobb, 2023). Thus, attending to the alignment of coaches' and teachers' interpretations of coaching productivity and outcomes is essential, just as teachers must attend to evidence of student learning to guide subsequent instruction.

Within CFC cycles, coaches must attend to multiple dimensions of support. First, coaching cycles are rarely isolated events; they are typically embedded within broader professional learning initiatives (Callard et al., 2022; West & Cameron, 2013). In such contexts, coaches support teachers in interpreting and enacting pedagogical ideas introduced through other PD experiences within their own classroom settings and are therefore accountable for teachers' learning in relation to the goals of the larger initiative. For example, when supporting ambitious mathematics teaching (Yurekli & Stein, 2025), coaches might help teachers set instructional goals, select and design tasks, anticipate student thinking, launch lessons, facilitate exploration, and orchestrate summary discussions, positioning both coaches and teachers as jointly responsible for learning to enact new instructional practices (Gibbons & Cobb, 2017; Horn &

Kane, 2015; Lampert et al., 2013). At the same time, coaches must also provide responsive, individualized support that extends beyond the focal goals of the initiative. This may include assisting with mathematical content, helping teachers navigate curricular or instructional resources, or offering emotional support, recognizing that instructional change is often demanding and uncertain (McGugan et al., 2023). In this way, coaches are also accountable for helping teachers address context-specific and personal challenges that fall outside the boundaries of targeted pedagogical reforms.

Against the backdrop of the complexity inherent to coaches providing teachers with customized support with CFC cycles, it is possible that coaches and teachers might report similar benefits from a shared planning or debriefing conversation. High alignment in the perceived benefits of a conversation would suggest that the coach has an accurate understanding of how the teacher interpreted and experienced the support, thereby enabling the coach to identify productive next steps for continued professional growth. However, such alignment may be difficult to reach for two possible reasons. First, coaches typically do not work toward a fixed set of standards for teacher learning and must instead negotiate the focus of conversations while remaining responsive to teacher-identified needs (Kochmanski & Cobb, 2023). As a result, a single coaching conversation may generate a wide range of possible benefits, both related to and beyond focal goals of larger initiatives motivating the coaching, complicating efforts to identify and interpret learning. Moreover, because the coach–teacher relationship is asymmetrical, coaches and teachers may attend to and value different aspects of the interaction based on their distinct roles. Consequently, low alignment in perceived benefits may reflect either a misinterpretation of learning (e.g., the coach infers one benefit while the teacher experiences another) or accurate recognition of different benefits that are shaped by each participant's position within the dyad.

Methods

Participants

Participants were 36 middle-grades teachers (grades 4-10) from a single state in the Pacific Northwest, with teaching experience ranging from two to 21 years. Each teacher was paired with one of 11 coaches from various geographic regions resulting in 36 coach-teacher pairings. Coaches were selected for their expertise in coaching ambitious mathematics teaching practices, with experience ranging from two to 15 years.

Project Context

The study took place within a state-funded PD program in which two cohorts of teachers each participated for one academic year. All PD activities were conducted online and consisted of three types of synchronous learning experiences: six course sessions; three video clubs; and two individualized, video-supported CFC cycles. The six course sessions introduced teachers to ambitious instructional practices (e.g., Smith & Stein, 2018). During the video clubs, teachers collaboratively analyzed classroom video of project staff enacting practices introduced in the course sessions within local schools. All course sessions and video clubs were two hours in length and held via Zoom.

Each teacher also completed two online, video-supported coaching cycles, for a total of 72 cycles. The 72 cycles serve as the primary focus of this study. In the cycles, coaches provided individualized support as teachers enacted the practices introduced in the course and video clubs. Consistent with a CFC model (Callard et al., 2022), each cycle included a planning conversation, classroom enactment that was video recorded, and a debriefing conversation. During planning

conversations, coach–teacher pairs met on Zoom to examine mathematical content, establish student learning goals, and refine lesson activities based on anticipated student thinking. Teachers then recorded their lessons using Swivl technology and shared the videos with their coaches. Both coaches and teachers independently reviewed the recordings and added annotations highlighting salient moments using the Swivl platform. In the debriefing meetings, pairs reconvened on Zoom to reflect on the lesson using evidence from the video and annotations. The first coaching cycle occurred after teachers completed four course sessions and two video clubs, and the second took place after the remaining two course sessions and final video club (see Figure 1).

<table>
<tr><th>Oct</th><th>Nov</th><th colspan="2">Dec</th><th>Jan</th><th>Feb</th><th>March</th><th>April</th></tr>
<tr><td>Course Session 1
Course Session 2
Video Club 1</td><td>Course Session 3
Course Session 4
Video Club 2</td><td colspan="3">Coaching Cycle 1</td><td>Course Session 5
Course Session 6
Video Club 3</td><td colspan="2">Coaching Cycle 2</td></tr>
</table>

Figure 1: Overview of project PD activities

Data Collected

We collected data on the planning and debriefing conversations from 72 coaching cycles. We used post-conversation surveys, immediately completed individually by both teachers and coaches after a conversation. The surveys were intended to clarify coaches' and teachers' perceptions of conversation productivity and the benefits of the conversation with respect to the teacher's professional growth. To accomplish this goal, we used two types of survey items. First, participants responded to a Likert-scale item—*To what extent was your coaching conversation productive?*—rated on a 1–9 scale. Second, the survey included two open-ended items asking participants to describe the benefits the teacher experienced because of participating in the coaching conversation.

In total, we analyzed and compared coach and teacher survey responses from 106 conversations (63 planning and 43 debriefing). Conversations were included in the analysis only when both members of the dyad completed the survey for the same interaction. The smaller number of debriefing conversations reflects the unexpected loss of project funding, which occurred while Cohort 2 was completing the second coaching cycle. As a result, post-conversation surveys were not collected for the debriefing conversations in that cycle for Cohort 2.

Data Analysis

To address RQ1, we conducted a paired-samples *t* test to examine differences between teachers' and coaches' ratings of conversation productivity.

As an initial step in addressing RQ2 and RQ3, we developed a coding scheme (see Table 1) to analyze the open-ended survey responses regarding coach and teacher perceptions regarding the benefits of the coaching interactions in terms of supporting the teachers' professional growth. The two open-response items from each survey were combined and treated as a single unit of analysis. We organized the coding scheme into two overarching categories: perceived benefits directly related to course content (six specific codes) and perceived benefits beyond the course content (five specific codes). Two coders independently coded all responses, with surveys randomized and participant roles blinded to reduce potential bias. Each code was assigned a

value of 1 if the benefit was present in the full response and 0 if it was absent. Coders then met to discuss and resolve any discrepancies.

To specifically address RQ2, which examined the extent to which the perceived benefits coaches and teachers reported were aligned, we shifted the unit of analysis to the level of the individual coaching conversation. For each conversation, we compared the codes assigned to a coach's and teacher's response and classified each coded benefit as either *paired* (i.e., the same benefit was coded for both members of the dyad) or *unpaired* (i.e., the benefit was coded for only one member of the dyad). Perceived benefits that were absent from both responses were not considered paired. Unpaired benefits were further categorized as either "coach only" or "teacher only." Using the number of paired and unpaired benefits, we classified each conversation's level of alignment using the following mutually exclusive criteria: *High*, when the number of paired benefits exceeded the number of unpaired benefits; *Moderate*, when at least one paired benefit was present *and* the difference between paired and unpaired benefits was zero, one, or two; and *Low*, when no paired benefits were present *or* when the number of unpaired benefits exceeded paired benefits by more than two. The research team collaboratively established these criteria after extensively reviewing coded conversations to identify distinctions that aligned with our perceptions of conversational alignment and meaningfully differentiated the conversations.

To address RQ3, we conducted an inductive analysis of coded responses to identify patterns in perceived benefits codes across differing levels of paired alignment and unpaired misalignment. We focused on benefits most frequently coded as paired or unpaired, with particular attention to those categorized as "coach only" or "teacher only."

Table 1: Coding Scheme (Abbreviated) for Open Response Survey Items

	Code	**Planning Definition** *Conversation supported the teacher to:*	**Debriefing Definition** *Conversation supported the teacher to:*
Benefits Related to Course Content	Goals	Develop or clarify the purpose of the lesson.	Reflect on the purpose of the lesson.
	Tasks	Select or modify a task; design task-based lessons.	Reflect on the effectiveness of a task or task-based lessons.
	Student Thinking	Anticipate student thinking.	Reflect on or interpret student thinking.
	Launching	Determine ways to launch a task.	Reflect on the effectiveness of task launch.
	Facilitating Exploration	Prepare to facilitate the exploration phase of a task-based lesson.	Reflect on the facilitation of the exploration phase of a task-based lesson.
	Summary Discussion	Determine how to structure and facilitate the summary discussion.	Reflect on the effectiveness of the summary discussion.
Benefits Outside of Course Content	Instructional Practice Goals	Identify or clarify instructional practice goals.	Reflect on progress toward instructional practice goals.
	Mathematics Content	Understand math content for the lesson.	Understand math content for the lesson.
	Emotional Support	Feel supported, inspired, or safe to experiment with practice.	Feel supported, inspired, or safe to experiment with practice.
	Curriculum Resource	Understand how to use an existing curriculum resource.	Reflect on the use an existing curriculum resource.
	New Resource	Learn to use a new teaching resource.	Reflect on the use of a new resource.

Findings

For RQ1, which compared coach and teacher perceptions of conversation productivity, both groups generally rated conversations as highly productive. However, the paired *t* test revealed that teachers' ratings (M = 8.40) were significantly higher than coaches' ratings (M = 6.92), $t(105) = 9.29$, $p < .001$, $d = 0.91$.

For RQ2, which examined the alignment of coaches' and teachers' perceived benefits of their planning and debriefing conversations, 15 conversations (14%) were classified as *high* alignment, 47 (44%) as *moderate* alignment, and 44 (42%) as *low* alignment. To illustrate a conversation considered to have *high* alignment, Coach Carter and Teacher Cindy shared two paired benefits (*Tasks* and *Curriculum Resources*) and one unpaired benefit (*Student Thinking* – teacher only) in their planning conversation. For example, Cindy noted that "[the coach] helped me modify a task … so that it would have a high ceiling and [be] beneficial to all of my students," while Carter similarly reported, "We spent much of our time walking through … how to change the task, which will provide a solid foundation for future task selection and modification." The unpaired benefit reflected Cindy's statement that "[my coach] made me think more like my students so that I could anticipate their responses," which Carter did not mention. Because paired benefits outnumbered unpaired benefits, this conversation was classified as *high* alignment. In contrast, Coach Laura and Teacher Luke's planning conversation was classified as *low* alignment, as it included three unpaired outcomes: *Goals* and *Mathematical Content* (coach only) and *Tasks* (teacher only). Laura reported that she helped Luke "think about the big ideas of the unit" (coded as *Mathematical Content*) and "narrow his content focus for the lesson" (coded as *Goals*). Luke, however, did not reference content or goals and instead stated, "my coach gave me ideas on what would be a good route for my lesson," which was coded as *Tasks*. Because no outcomes were paired, this conversation met the criteria for *low* alignment.

To answer RQ3, which focused on reported benefits frequently associated with alignment or misalignment, we constructed Table 2 to summarize the number of conversations that featured paired and unpaired benefits related to course content. For example, the 17 paired benefits for *Goals* (top left of Table 2) represent the number of conversations in which both the coach and the teacher identified benefits related to establishing mathematical learning goals for students. In contrast, the value of 48 in the total row for that same column indicates the number of conversations in which either member of the dyad identified *Goals* as a benefit. We constructed Table 3 in the same manner to summarize paired and unpaired benefits that were not directly related to course content.

Table 2: Conversations Featuring Benefits Related to Course Content

	Goals	Tasks	Student Thinking	Launching	Facilitating Exploration	Summary Discussion	Total
Paired	17	17	14	1	7	5	61
Coach Only	18	21	17	8	16	12	92
Teacher Only	13	19	24	10	12	11	89
Total	48	57	55	19	35	28	

Table 3: Conversations Featuring Benefits Outside of Course Content

	Instructional Practice Goals	Math Content	Emotional Support	Curriculum Support	New Resource	Total
Paired	18	4	15	5	1	43
Coach Only	18	18	10	4	3	53
Teacher Only	13	7	42	4	4	70
Total	49	29	67	13	8	

Three notable trends emerged from this analysis of paired and unpaired benefits. First, *Emotional Support* was far more likely to be identified by teachers than by coaches as a benefit of the coaching conversation. Teachers reported *Emotional Support* as a perceived benefit in 42 conversations in which coaches did not, whereas the reverse occurred in only 10 conversations, and both members of the dyad identified *Emotional Support* in just 15 conversations. This suggests that teachers were more attuned to the relational and affective benefits of their coaching interactions than their coach. We also note that *Emotional Support* was the most commonly applied code, appearing in 67 of the 106 conversations as a benefit reported by the coach, teacher, or both. Second, *Mathematical Content* was more often identified by coaches than by teachers. Coaches reported *Mathematical Content* as a benefit in 18 conversations in which teachers did not, compared with only seven conversations in which teachers reported this benefit without the coach, and just four conversations in which both members of the dyad identified it. This pattern suggests that coaches were more likely than teachers to view their conversations as supporting the development of teachers' mathematical content understanding. Third, outcomes related to specific phases of a task-based lesson (e.g., Smith & Stein, 2018)—such as *Launching, Facilitating Exploration,* and *Summary Discussion*—were frequently unpaired. For instance, although *Launching* was identified as a benefit in 19 conversations, it was paired in only one. Similarly, *Summary Discussion* appeared in 28 conversations but was paired in only five. These patterns indicate that while no benefit exhibited consistently high levels of paired alignment, benefits tied to the planning or implementation of particular lesson segments were especially likely to be recognized by only one member of the dyad, highlighting substantial divergence in how coaches and teachers interpreted the focus and impact of their planning and reflection conversations.

Discussion and Implication

This study examined the extent to which coaches' and teachers' perceptions of the productivity and benefits of CFC cycles corresponded. Three findings are particularly salient. First, although both groups rated conversations as highly productive, teachers consistently rated them more favorably than coaches. Second, relatively few conversations demonstrated high outcome alignment, with only 14% classified as highly aligned. Third, several outcomes were frequently unpaired: *Emotional Support* was typically identified by teachers but not coaches, *Mathematical Content* was more often identified by coaches but not teachers, and outcomes related to specific phases of instruction (e.g., *Launching*, *Facilitating Exploration*, *Summary Discussion*) were commonly reported by only one member of the dyad. Together, these findings suggest that coaches and teachers often interpret the same coaching interaction in meaningfully different ways.

These results contribute to mathematics coaching literature in several ways. Recent studies have begun to examine the outcomes of coaching through teachers' reports (e.g., Saclarides & Gillespie, 2024; Warnock et al., 2022) to better understand the nuanced and diverse ways one-on-one coaching responsively supports teachers as they attempt to implement new ways of teaching. Our findings extend such research by directly comparing teachers' and coaches' perspectives on their interactions. In addition, our findings related to RQ1 contrast with those of Smith and Desimone (2025), who found that coaches often rated PD activities involving coaching more favorably than teachers. This discrepancy likely reflects contextual differences that shape coaching experiences and outcomes (Hannan & Russell, 2020). In Smith and Desimone's study, coaches were embedded within schools and districts and provided a range of PD activities connected to curriculum implementation, whereas in the present study all teachers participated in a common, university-supported coaching cycle facilitated by mathematics coaches who were not employed by their districts. The direct contrast in these findings suggest differences in organizational positioning, relational history, and scope of responsibility may influence how coaches and teachers evaluate the productivity of PD interactions.

Findings related to RQ2 and RQ3 raise important questions about the implications of low alignment in perceived outcomes. From one perspective, limited alignment may be concerning, given prior research emphasizing the importance of coaches and teachers jointly establishing goals and shared foci for professional learning (e.g., Kochmanski & Cobb, 2023). If coaches and teachers leave conversations with different understandings of what was accomplished, this may signal missed opportunities to co-construct clear directions for instructional improvement. From an alternative perspective, divergent perceptions may reflect the inherently asymmetrical roles (Heritage & Clayman, 2010) within coaching relationships. Mathematics coaches, selected for their content and pedagogical expertise, may be more attuned to shifts in teachers' mathematical understanding, whereas teachers navigating instructional change may value emotional or motivational support from the same interactions (Yurekli & Stein, 2025). Thus, differences in reported outcomes may reflect role-related differences in what is noticed, valued, and remembered.

These findings have implications for both practice and research. For coaches, a practical strategy may be to conclude coaching conversations with brief, explicit reflection on perceived outcomes and next steps. Such meta-conversations could help surface differences in interpreted benefits of an interaction, support shared sensemaking about teacher learning, and inform more targeted follow-up support. Such conversations may also help teachers recognize professional learning that coaches detected but teachers may not yet articulate, while allowing coaches to better understand the affective dimensions of teachers' experiences that may otherwise remain unknown.

For research, future studies should examine how outcome perceptions relate to the substance of coaching interactions and subsequent instructional practice. One promising direction is to compare perceived benefits with learning opportunities evident in the conversations themselves. For instance, when a coach reports a benefit related to planning the launch of a lesson, but the teacher does not, researchers might investigate whether meaningful discussion about this phase of the lesson occurred. Additionally, future research could examine whether reported benefits—whether aligned or not—are associated with observable changes in instruction. Investigating how learning within conversations translates into classroom practice would strengthen claims about the significance of coaching and clarify the conditions under which alignment in perceptions matters most for teacher development.

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LEARNING TO FACILITATE CONTENT-FOCUSED DEBRIEFING DISCUSSIONS: EXPLORING CHANGES IN THE PRACTICES OF MATHEMATICS SPECIALISTS

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This study examined how 12 mathematics specialists' facilitation of debriefing conversations changed after learning to use the Debriefing Conversational Structure, a tool developed through a research–practice partnership and grounded in content-focused coaching. We analyzed debriefing conversations conducted before and after learning about the structure, along with specialists' reflections. Nearly all specialists increased their facilitation levels, with the largest gains in phases focused on analyzing student thinking, instructional decision-making, and future instruction. Specialists' reflections suggested that the structure supported increased levels of reflection by foregrounding key phases of reflective dialogue and promoting coherence. Uneven changes across participants highlighted the complexity of facilitating reflective dialogue. This study contributes a theoretically grounded conversational tool and future research.

Keywords: Professional Development, Teacher Educators, Teacher Noticing

Content-focused coaching (CFC) cycles are a widely used professional development (PD) structure mathematics specialists utilize to support teachers' professional learning (Stein et al., 2022; Witherspoon et al., 2021; Yurekli & Stein, 2025). In this study, mathematics specialists are educators partially or fully released from classroom teaching to engage in classroom-embedded collaboration with teachers to improve instruction (Baker et al., 2022; Mudzimiri et al., 2014). A typical CFC cycle consists of collaborative planning, teaching, and reflecting interactions in which a specialist and a teacher jointly design, implement, and analyze instruction. The central aim of CFC cycles is to deepen teachers' mathematical and pedagogical content knowledge, promoting responsiveness to student thinking and cultivating sustainable planning and reflective practices (Callard et al., 2022; West & Cameron, 2013).

Among the three phases of a CFC cycle, the debriefing phase plays a critical role in supporting teacher learning. Debriefing conversations provide structured opportunities for mathematics teachers to analyze evidence of student thinking, refine instructional decisions, and strengthen reflective habits that support continued professional learning beyond the specialist's presence (Saclarides, 2022; Walsh et al., 2023; West & Cameron, 2013). Despite its importance, comparatively little is known about how mathematics specialists learn to facilitate debriefing conversations within CFC cycles. Existing research has examined specialists' facilitation of planning conversations (Russell et al., 2020; Stein et al., 2022) and their development of general coaching knowledge and strategies (Baker & Knapp, 2019; Saclarides & Kane, 2024), but few studies focus specifically on how mathematics specialists learn to support teacher reflection during debriefing conversations. At the same time, coaching tools—such as frameworks, protocols, conversational guides, and routines—have become increasingly common supports for specialists' work with teachers (Baker & Knapp, 2019; Gibbons & Okun, 2022; Gillespie, Amador et al., 2025; Kochmanski & Cobb, 2023; Smith et al., 2025). However, limited empirical

research has examined how specific conversational structures influence specialists' facilitation of debriefing conversations. Together, these trends underscore the need to examine the professional learning of mathematics specialists' within the context of reflection-focused interactions.

This study examined 12 mathematics specialists participating in a two-year PD project designed to strengthen their facilitation of CFC cycles, with particular attention to debriefing conversations. Observations from the research team and project facilitators, along with participant feedback, indicated that learning to facilitate debriefing conversations was a central area of professional learning opportunity and that a project-developed Debriefing Conversational Structure was instrumental in supporting this learning. Accordingly, this study pursued two aims: (a) to examine how specialists' facilitation changed across debriefing conversations conducted before and after learning the conversational structure, and (b) to analyze specialists' reflections to characterize how they described the structure as supporting their professional learning. The following research questions guided the study: (1) In what ways did specialists' levels of facilitation within debriefing conversations change after participating in PD that included the Debriefing Conversational Structure?; (2) How did the Debriefing Conversational Structure support specialists' development in facilitating debriefing conversations?

Theoretical Perspectives

This study is theoretically grounded in scholarship on reflective thinking and the conversational interactions that support professional learning through reflection. Drawing on Dewey (1938) and Schön (1983, 1987), we conceptualize reflection as a process through which meaning is reorganized and reconstructed from experience. In educational contexts, reflective thinking involves revisiting prior teaching episodes to examine teacher and student actions and lesson design, while also engaging in forward-oriented thinking about how future instruction might be modified in response to what was observed and experienced (Davis, 2006; Rodgers, 2002; Schön, 1983; Walsh et al., 2023). This interplay between looking back on practice and intentionally designing future action supports reflection *on* and *for* action (Schön, 1983).

The quality of reflective discussion is consequential to the outcomes of this process. Productive reflection requires attention to intended and enacted instructional goals, evidence of student learning, and the factors that may have contributed to students' progress or challenges (Gillespie, Kruger et al., 2026; Olteanu, 2017). Such analysis informs the design of subsequent goals and modifications to future instruction, creating a cyclical process in which reflection on past experiences shapes future practice (Burhan-Horasanlı & Ortaçtepe Hart, 2024; Postholm, 2008). Within this cycle, the evidence teachers draw upon—and how that evidence is examined—plays a critical role in shaping responsive instructional design (Lefebvre et al., 2023).

Research on mathematics specialist learning and CFC further informs this study. Mathematics specialists require ongoing PD to support teachers' learning, particularly in developing reflective skills central to effective coaching practice (Saclarides, 2022). Studies of CFC models have shown that PD can lead to meaningful improvements in specialists' coaching practices (Stein et al., 2022), yet facilitating debriefing conversations—a central phase of CFC cycles—remains a complex and underexplored aspect of specialist learning (Callard et al., 2022; West & Cameron, 2013). Emerging research suggests that structured coaching tools can support specialists' facilitation of reflective conversations. For example, Baker and Knapp (2019) demonstrated that a decision-making protocol supported goal-oriented coaching discussions, while Russell et al. (2020) found that an inquiry-driven framework fostered richer, more precise dialogue between coaches and teachers. More broadly, scholars emphasize that productive

facilitation depends not only on *what* is discussed, but *how* discussion unfolds—particularly the extent to which facilitation promotes specificity in examining practice and connectivity across goals, evidence, and instructional decisions (Gibbons & Okun, 2022; van Es & Sherin, 2008). However, reflective dialogue is neither straightforward nor linear (Schön, 1983), which elevates the possibility that productive facilitation of debriefing conversations is a complex endeavor. Together, this body of work positions reflective debriefing conversations as a critical site for mathematics specialist learning and underscores the need to examine how conversational structures can support specialists in facilitating coherent, evidence-based reflection.

Methods

Participants and Project Context

Participants were 12 mathematics specialists who held various roles, including coaches, interventionists, and teacher-leaders (Baker et al., 2022). The study draws on video club (VC; Kruger et al., 2026) and coaching cycle components of a two-year PD program delivered online, which included: (a) six structured course sessions, (b) eight VCs, and (c) two one-on-one coaching cycles (see Amador et al., 2021). Course sessions introduced strategies for leading CFC cycles. In VCs, participants met in groups of four to review video clips of planning and debriefing conversations; Year 1 clips featured project personnel, while Year 2 clips featured participants' own coaching. During one-on-one coaching cycles, participants worked with local teachers, supported by a mentor coach who met with them before each cycle to prepare.

This sudy focused on two debriefing conversations participants conducted with teachers. Before the first coaching cycle, PD had emphasized planning conversations but not debriefing. Following the first cycle, participants attended one course session introducing the Debriefing Conversational Structure and participated in two VCs focused on debriefing. They then conducted a second coaching cycle with the same teacher. While the structure was introduced and modeled, participants had autonomy in deciding how and to what extent to use it.

Debriefing Conversational Structure

We developed the Debriefing Conversational Structure to be (a) simple and time-efficient, (b) adaptable to varied contexts, (c) grounded in reflective dialogue theory, and (d) aligned with CFC principles (Callard et al., 2022; West & Cameron, 2013). The structure provides an accessible framework translating theory into practice while allowing specialists to exercise professional judgment. Consistent with CFC guidance, the conversational structure focuses on evidence of student learning tied to instructional goals and on how these insights inform future teaching (West & Cameron, 2013) in order to deepen teachers' subject-matter understanding and pedagogical content knowledge (Ball et al., 2008; Shulman, 1987). The structure comprises four phases grounded in Schön's (1983) reflection-on-and-for-action: (1) *Review Goals*—what were we trying to accomplish; (2) *Examine Evidence Related to Goal*—what student data relate to our goals; (3) *Consider Contributing Factors*—what influenced success or challenges; (4) *Determine Implications for Future Teaching*—how can we improve instruction based on these insights.

Data Collection and Analysis

Two coaching cycles for each of the 12 participants were video recorded and transcribed verbatim. This yielded 24 debriefing conversations: 12 from Cycle 1, conducted before exposure to the Debriefing Conversational Structure, and 12 from Cycle 2, conducted after its introduction. Participants coached teachers across grade levels spanning third grade through high school. All cycles centered on lessons involving cognitively demanding tasks and discourse practices designed to productively support student struggle (e.g., Smith & Stein, 2011). To examine how the Debriefing Conversational Structure supported professional changes,

researchers analyzed two additional data sources. Participants completed end-of-year interviews in both years, responding to open-ended questions about their development and the aspects of the PD that supported it. In addition, after each VC, participants reflected on their facilitation of CFC cycles and identified project experiences that were impactful. All interviews and reflections were audio recorded and professionally transcribed.

To examine changes in specialists' facilitation, we conducted a two-step analysis. First, we deductively coded 1,351 specialist talk-turns from 24 debriefing conversations using a predefined scheme aligned with the Debriefing Conversational Structure and our theoretical framing of reflective dialogue. The scheme included 12 codes each connected to different phases of the conversational structure. Transcripts were organized in spreadsheets, identifiers removed, and multi-idea turns segmented. Teacher talk was retained for context only. Three researchers independently coded each specialist talk turn for the presence (1) or absence (0) of each code (Corbin & Strauss, 2014). Turns could receive multiple codes; non-instructional talk received none. Second, we assessed facilitation levels using a rubric (Figure 1) developed by the research team with input from project personnel who designed and implemented the PD experiences. After piloting and refining for reliability, three researchers independently scored each conversation on the four phases, providing written rationales, then reconciled discrepancies. Interrater reliability (kappa) ranged from 0.62 to 1.00, indicating substantial to perfect agreement (Landis & Koch, 1977).

	Level 1	**Level 2**	**Level 3**	**Level 4**
Phase 1: Review Goals	No mention of any goals	Reference to the math learning goals or teacher practice goals	Recurring references to the math learning goals or teacher practice goals	Recurring references to the math learning goals and teacher practice goals
Phase 2: Examine Evidence Related to Goal	No mention of student thinking	Mentions of instances of student thinking	Analysis of specific instances of student thinking connected to the goals.	Extended analysis of specific instances of student thinking connected to the goals.
Phase 3: Consider Contributing Factors	No mention of teacher actions or lesson design	Mentions of teacher action or lesson design	Analysis of how specific teacher action or lesson design features influenced student thinking	Extended analysis of how specific teacher action or lesson design features influenced student thinking
Phase 4: Determine Implications for Future Teaching	No talk of change to future practice	Limited talk of change to future teaching	Analysis of changes to future teaching connected to discussion of student thinking or contributing factors	Extended analysis of changes to future teaching connected to discussion of student thinking and contributing factors

Figure 1: Coach Reflection Rubric for Debriefing Conversations (Abbreviated for Space)

To investigate how the structure supported professional changes to higher levels of facilitation, we analyzed Year 1 and Year 2 interviews and reflections from all eight VCs. One researcher identified excerpts explicitly linking growth to the Debriefing Conversational Structure; a second confirmed relevance, yielding 46 segments. Two researchers independently paraphrased each segment and reconciled summaries before collaboratively identifying themes describing how participants perceived the structure as supporting their development.

Findings

In relation to the first research question, Table 1 summarizes level scores for both Cycle 1 and Cycle 2 debriefing conversations to show changes to coaches' level of facilitation after learning about the conversational structure. Four noteworthy trends were identified. First, 11 of the 12 specialists had phases coded at more advanced levels in at least two phases, suggesting that the conversational structure and related learning supported at least modest levels of advancement in facilitation. Second, three specialists demonstrated increased levels of facilitation categorized as *moderate improvements* (a net increase of one to two points), while eight specialists showed *substantial improvements* (a net increase of three or more points). Third, specialists' level of facilitation increased the most within Phases 2, 3, and 4, with nine to ten specialists showing gains in these areas. In contrast, only four specialists showed improvements in Phase 1, which involved incorporating goals into discussion. Finally, no specialist achieved a Level 4 across all phases in their second conversation, and only five reached a Level 4 in multiple phases. These patterns suggest that the conversational structure was not a mechanical script but rather a framework that required the specialists to develop nuanced facilitation skills.

Table 1: Rubric Levels for Debriefing Conversation 1 (CC1) and 2 (CC2)

	Phase 1			Phase 2			Phase 3			Phase 4		
Coach	CC1	CC2	+/-	CC1	CC2	+/-	CC1	CC2	+/-	CC1	CC2	+/-
Arnold	2	2	0	4	4	0	2	3	+1	3	4	+1
Bell	2	2	0	2	2	0	2	3	+1	2	3	+1
Briggs	2	3	+1	1	2	+1	1	3	+2	2	4	+2
Butler	2	3	+1	3	4	+1	1	3	+2	3	4	+1
Clark	2	2	0	1	3	+2	1	1	0	1	3	+2
Delgado	2	2	0	2	3	+1	3	4	+1	3	4	+1
Howard	2	2	0	3	4	+1	3	4	+1	3	4	+1
Kennedy	2	1	-1	1	3	+2	1	2	+1	2	3	+1
Lee	2	4	+2	1	2	+1	1	4	+3	1	4	+3
Logan	3	2	-1	3	3	0	3	2	-1	2	2	0
Rice	2	3	+1	2	3	+1	2	2	0	2	1	-1
Vargas	2	2	0	1	3	+2	1	2	+1	2	4	+2
Average	2.1	2.3	+0.3	2.0	3.0	+1.0	1.8	2.8	+1.0	2.2	3.3	+1.2

Analysis of participants' reflections from project PD activities and interviews indicated that 11 specialists explicitly identified the Debriefing Conversational Structure as a key contributor to their professional learning. Three themes emerged regarding how the structure supported their development. First, participants reported that the structure heightened their awareness of phases within reflective dialogue and clarified the role each phase plays in a debriefing conversation. Second, participants described how the structure offered a coherent map for connecting these phases in a purposeful sequence. Some specialists described how individual phases of a debriefing conversation were already active parts of their practice, but the structure helped them connect those phases to help teachers translate their teaching experience into professional learning. Finally, participants noted that the structure helped them balance intentional planning with responsiveness. Specifically, specialists shared that it enabled them to enter conversations with clear goals for how to support teacher development while remaining attentive and adaptable to teachers' ideas and needs as the dialogue unfolded.

The Case of Butler

To illustrate changes in facilitation and how the structure supported specialists' development, we present that case of Butler whose levels of facilitation increased across all four phases of the rubric. We first describe changes in Butler's facilitative behaviors between the first and second debriefing conversations to exemplify the patterns reported in Table 1 and provide context for how the Debriefing Conversational Structure contributed to those changes. We then use Butler's reflective commentary to illustrate the three themes through which specialists attributed their professional changes based on levels of facilitation to the use of the structure.

With respect to Phase 1: Review Goals, in Cycle 1, Butler made only a brief mention of goals at the start of the conversation. In Cycle 2, Butler again referenced the learning goals at the start of the conversation but also repeatedly connected questions throughout the conversation back to these goals. For example, Butler said the following in the Cycle 2 debriefing conversation, "our second goal, to go back and connect, was about students having a strategy to solve. How many students had strategies versus how many of them just missed the target altogether?"

For Phase 2: Examining Evidence Related to Goals, Butler's level score increased from 3 to 4 from Cycle 1 to 2. This increase reflected a shift from making isolated references to student thinking toward engaging in sustained interpretation of specific instances of student thinking in relation to the learning goals. In Cycle 2, Butler and the teacher engaged in the examination of student work samples for approximately 12 minutes, making interpretations about what students understood and the extent to which each student had progressed toward the goals. By contrast, in Cycle 1, Butler prompted occasional mentions of student thinking but did not encourage discussion aimed at interpreting what students did or did not understand.

Butler's largest facilitative increase occurred within Phase 3: Consider Contributing Factors. In Cycle 1, Butler moved directly from noticing isolated instances of student thinking to making suggestions for next instructional steps without discussing how the lesson design or teacher actions may have influenced students' thinking. In Cycle 2, however, Butler posed questions about lesson factors that may have influenced student learning to make explicit connections between analyzing student thinking and identifying implications for future teaching. For example, after discussing the thinking of two students who appeared to struggle with the task, Butler asked: "What do you think the holdback or the setback was for these students?*"* The teacher reflected that these students might have needed visual models. Butler then facilitated a sustained discussion prompting the teacher to consider how upcoming lessons could better utilize visual models to address these identified needs. This extended discussion about modifications to future teaching was grounded in both the analysis of contributing factors as well as specific instances of student thinking. This led to Butler being at a Level 4 in Cycle 2 for Phase 4: Determine Implications for Future Teaching, which was a two-point increase from Cycle 1.

Butler's reflections across two years of participation exemplify all three themes regarding how the Debriefing Conversational Structure supported specialists' professional learning. First, Butler reported that the structure heightened understanding of the importance of particular phases of reflective dialogue within the larger debriefing conversation. Specifically, Butler shared a new understanding about how to anchor reflective conversations in both evidence of student thinking and the goals established during planning. As Butler explained in a final interview after the second year of participation, "the [conversational structure] made it clear that I need to focus on that data…. And really encouraging teachers to have that data to show evidence of growth toward their goals."

For the second theme, Butler noted new insights about how the specific action of grounding conversations in data connected to goals created a durable foundation for productive dialogue about improving future teaching. In other words, Butler described how the Debriefing Conversational Structure supported new ideas about how to make connections between different phases of reflective discussion. For example, in a Year 1 interview, Butler reflected:

> The framework pushed coming back to those goals, and really helping the teacher move towards those goals, and which then helps them see their implication for practice with respect to those goals.

Related to the third theme, Butler described how the structure helped manage the challenge of facilitating conversations that were simultaneously planned and focused while remaining flexible and responsive to teachers' needs. Butler shared:

> I can ... really plan out what I want to accomplish and look at what the teacher's goals are and then really help drive the conversation… by focusing on data. … The [conversational structure] is a good framework for a debriefing conversation as I can feel planned and controlled while still being flexible and responsive to the teacher's needs and interests.

Discussion

This study examined changes in mathematics specialists' facilitation as they implemented the Debriefing Conversational Structure within CFC cycles. Specialists' reflections across two years were analyzed to better understand how the structure contributed to changes in debrief facilitation. Three key contributions of these findings are discussed below.

First, 11 of the 12 specialists demonstrated increased facilitation scores in their second debriefing conversation, and most identified the Debriefing Conversational Structure as central to their professional learning. This finding aligns with research showing that coaches often take up structured tools—such as protocols and routines—to guide their work with teachers (Baker & Knapp, 2019; Gibbons & Okun, 2022; Kochmanski & Cobb, 2023). Given that mathematics specialists are frequently "anointed and appointed" based on classroom expertise (Fennell et al., 2017, p. 3) yet receive limited preparation for supporting adult learning (Chval et al., 2010; Russell et al., 2020), structured supports may help them navigate the dual demands of responsiveness and intentionality. Participants' reflections illuminated three ways the structure supported changes: it heightened awareness of distinct phases of reflective dialogue, provided a coherent framework for connecting goals, evidence, contributing factors, and instructional implications, and scaffolded facilitation while preserving flexibility. Extending prior work that focuses primarily on enactment of coaching tools (e.g., Baker & Knapp, 2019), these findings foreground specialists' perspectives and suggest that conversational structures can serve as practical scaffolds for developing facilitative expertise in reflection-focused coaching contexts.

Second, increases in specialists' facilitation levels also point to expanded opportunities for teacher learning through professional discourse (e.g., Borko et al., 2008; Horn & Little, 2010; Lefstein et al., 2020) during the second debriefing conversation. The conversational structure was intentionally organized around core elements of pedagogical content expertise, including close attention to student thinking, analysis of how instructional decisions shaped that thinking, and consideration of instructional next steps grounded in those analyses. Research has consistently linked teachers' capacity to notice, interpret, and respond to student thinking with the development of instructional expertise (e.g., Bastian et al., 2024; Bragelman et al., 2024; Jacobs et al., 2024), and has identified analysis of instructional decision-making as a productive

focus for professional learning (Jacobs et al., 2010; Jakopovic, 2021). Similarly, reflecting on implications for future instruction based on prior teaching experiences supports ongoing professional learning (Kraft & Hill, 2020; Walsh et al., 2023).

Third, although most specialists demonstrated upward shifts, improvement was uneven. One specialist showed no change, and three exhibited only modest gains. Even in the second debriefing conversation, considerable room for development remained: only five specialists reached the highest facilitation level in more than one phase, and none did so across all four phases. These findings underscore the complexity of facilitating professional learning conversations (Mitchell et al., 2022). Prior research has documented similar challenges in CFC planning conversations, where coaches must balance responsiveness to teachers with sustained attention to instructional goals (Stein et al., 2022; Witherspoon et al., 2021). Far less is known, however, about how specialists develop the expertise required to facilitate reflective dialogue focused on enacted instruction (Saclarides, 2022). Debriefing conversations demand more than addressing discrete topics in sequence. Effective facilitation involves deepening specificity about classroom events (reflection-on-action; Schön, 1987), linking those insights to future instructional decisions (reflection-for-action; Schön, 1987), and maintaining coherence across phases of discussion. The continued room for increased level of facilitation observed in this study suggests that while conversational tools can support development, they are unlikely to be sufficient on their own. Specialists likely require sustained opportunities to rehearse, analyze, and refine how they press for specificity and connect ideas across reflective dialogue.

Implications

We offer two key implications for research and professional development. First, this study positions the Debriefing Conversational Structure as a promising tool for mathematics specialists engaged in CFC cycles. Developed through a research–practice partnership, the structure translates theories of productive reflective dialogue into a form aligned with CFC goals (Callard et al., 2022; Stein et al., 2022; West & Cameron, 2013; Witherspoon et al., 2021). Specialists with varied backgrounds and contexts demonstrated increased levels of facilitation, suggesting broader potential for impact. The structure—and its accompanying rubric—contribute practical and research-based resources for studying and supporting reflective conversations. Second, uneven changes raises questions about the level of explicitness coaching tools require and the PD needed to surface their purposes. Although intentionally open and flexible, this design may have limited uptake for some specialists. While some may thrive with adaptable tools, others may need more explicit modeling, rehearsal, and feedback to use them productively.

The promise of the structure must be considered alongside several limitations of the study. The sample was relatively small and focused on facilitative shifts were based on the comparison of two coaching cycles. Thus, while we encourage specialists and those who support their learning to consider using the Debriefing Conversational Structure and accompanying rubric, we also call for future research that examines the use of this tool across multiple coaching cycles, with larger samples, and with explicit attention to specialists' varied backgrounds and contexts. Future research should therefore directly examine how shifts in mathematics specialists' facilitation during reflective discussions influence the depth of conversation, teachers' engagement, and subsequent instructional decisions, to further substantiate these theoretical interpretations.

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TEACHERS' COLLECTIVE NOTICING OF STUDENT WORK: PATTERNS OF NORMATIVITY AND SERVICEABILITY IN PROFESSIONAL DEVELOPMENT

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Teachers may perceive student work in terms of its alignment with instructional norms (normativity) and its potential to advance lesson goals (serviceability), yet how these distinctions surface in professional discourse remains underexplored. This paper uses natural language processing to analyze discourse from 45 professional development meetings involving 77 in-service mathematics teachers who discussed student work artifacts while designing a geometry lesson storyboard. We find that teachers maintained a consistently positive evaluative stance toward all student work artifacts. Semantic patterns suggest that teachers often juxtaposed artifacts, using pairing as a strategy to position diverse approaches while seemingly compensating for perceived limitations in advancing the lesson goal. Talk amount trended higher for less normative artifacts—consistent with prior research—but did not reach statistical significance, likely due to small sizes at the artifact level.

Keywords: Teacher Noticing, Students' Mathematical Work, Professional Development, Natural Language Processing, Practical Rationality

Noticing Normativity and Serviceability in Students' Mathematical Thinking

Research on teacher noticing has focused on how teachers attend to, interpret, and respond to students' mathematical contributions in the course of instruction (van Es & Sherin, 2002; Sherin & van Es, 2009). In many classroom settings, students' thinking is not rendered transparent through their work alone. Instead, it is conveyed through written work, diagrams, or partial representations that require teachers to interpret what is shown in relation to the underlying mathematics (Kazemi & Franke, 2004). Professional development efforts that center student work therefore position teachers to make sense of complex, incomplete representations in relation to instructional goals (Goldsmith & Seago, 2011; Little et al., 2003).

Recent scholarship on mathematics teaching has argued that teacher noticing cannot be understood solely as a cognitive process of understanding students' ideas, but must also be situated within the demands of instructional practice (Scheiner et al., 2026). In problem-based instruction, teachers encounter multiple, partial, and sometimes competing student contributions and must discern which are usable for advancing the lesson. Herbst and colleagues (2023) propose categories of perception that structure this discernment. Normativity refers to a teacher's perception of whether student work abides by the norms of the instructional situation used to frame the problem; serviceability refers to a teacher's perception of whether student work can advance the instructional goal. These categories are analytically distinct: A given piece of student work may have features that teachers perceive as normative, serviceable, both, or neither. Researchers can identify features in student work that could trigger those perceptual categories , while teachers vary how much they attend to each.

Empirical studies applying this framework suggest that these categories are not merely theoretical distinctions but are differentially salient in teachers' attention. In analyses of teachers' annotations on student work from problem-based lessons, Schwarts and colleagues (2023) found

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

that teachers more frequently commented on non-normative features than normative ones, and more often identified serviceable aspects than non-serviceable ones. For example, in algebra lessons framed around equation-based word problems, teachers commented more frequently on non-normative representations—such as tables or graphs—than on conventional equation-based solutions. At the same time, features that could be leveraged toward instructional goals were more frequently identified than features offering little instructional traction. These asymmetries suggest that departures from norms and opportunities for instructional leverage may command disproportionate professional attention.

Related work indicates that norm departures are discursively consequential. Dimmel and Herbst (2020) found that events breaching instructional expectations elicited longer and more evaluative teacher responses than norm-conforming events. Such findings imply that variations in normativity may shape not only what teachers notice but also the extent and evaluative characteristics of their discourse. This raises a central question: How do distinctions of normativity and serviceability become visible in the language teachers use when collaboratively discussing student work?

To investigate this question, we use foundational natural language processing (NLP) techniques to analyze patterns of talk amount, evaluative language, and semantic association in teachers' collaborative discussions, examining whether these patterns vary when discussing student work that differs in features perceivable as normative and serviceable. We ask:

1. How does talk amount vary across student work samples that differ in features related to normativity and serviceability?
2. How does the density and polarity of evaluative language vary across such samples?
3. How do semantic associations surrounding student work differ in relation to these features?

Method

Participants and Context

Seventy-seven in-service mathematics teachers participated in Story*Circles*, a virtual professional development model in which teachers collaboratively design and refine lesson storyboards representing how a lesson around a novel geometry problem might unfold (Herbst & Milewski, 2018). Teachers averaged 14 years of mathematics teaching experience and 8 years teaching geometry. They participated in six groups that met from October 2024 to May 2025, totaling 45 one-hour synchronous sessions.

The lesson centered on a geometry problem involving three swimmers entering a rectangular pool from different locations and racing toward a prize. One swimmer starts at a corner of the pool, while the other two enter from points on the sides adjacent to that corner, and students are asked to determine where the prize should be placed so that all three swimmers travel the same distance. The task was designed to lead students to discover the Midpoint of the Hypotenuse Theorem—that in a right triangle, the midpoint of the hypotenuse is equidistant from its three vertices.

Story*Circles* uses lesson storyboards to represent instructional sequences through frames organized around lesson phases. These frames embed samples of student work—developed from real student data—that teachers discuss as they monitor student thinking, select and sequence work, and shape whole-class discussion. We analyzed teachers' talk in aggregate, rather than phase by phase, to examine how teachers collectively noticed the artifacts.

Thirteen pieces of student work (by students named after Greek letters) were available to

every group and were discussed across the six groups. The work differed in ways that could be characterized a priori in terms of normativity and serviceability. Normativity concerned adherence to norms of construction or proof situations (e.g., use of compass and straightedge, justification from givens). Serviceability concerned the presence of features that advanced the lesson's instructional goal (e.g., use of a right triangle and its midpoint). An a priori analysis identified where such features were present.

Data Analysis

Data sources included recorded Zoom meetings, anonymized transcripts, and storyboard screencaptures. The 45 transcript documents were segmented using interactional analysis (Jordan & Henderson, 1995); AI-generated descriptions[1] of screen captures rendered visual storyboard information in linguistic form for NLP-based analysis (O'Halloran et al., 2021). Prior to analysis, transcript text was lowercased and tokenized into word units using a regular expression pattern that identifies word boundaries (Jurafsky & Martin, 2024). To identify discussions of student work, we extracted segments referencing student names (e.g., Alpha, Zeta) in the transcript or corresponding storyboard. We then applied a lexical cue filter retaining only transcripts that referenced (a) student thinking or approaches, (b) representations or artifacts (e.g., diagram, sketch, figure), or (c) mathematical objects central to the Midpoint of the Hypotenuse task (e.g., midpoint, hypotenuse, perpendicular, bisect).

To evaluate the performance of this lexical cue filter, a stratified random sample of excerpts from both the pre-filtered and filtered corpora was independently coded by two raters for explicit orientation to student mathematical work, reasoning, or representations. Interrater reliability was high (92% agreement; Cohen's $\kappa = 0.83$). Precision—defined as $\frac{TP}{TP+FP}$, where TP denotes excerpts correctly identified as mathematical student-work talk and FP denotes incorrectly identified excerpts—increased from 43.2% to 85.4%. Filtering by lexical cues thus produced a reliable corpus for analyzing how teachers discuss students' mathematical work.

Word count. From the resulting corpus of student-work-related talk, we computed language metrics to address each research question. The first metric, talk amount, was operationalized as total word count: raw token counts, with no words removed. Word count is a foundational NLP measure that indexes variation in textual engagement across conditions (Jurafsky & Martin, 2024). This measure addresses the first research question by indexing how much discussion teachers devoted to artifacts of student work in which differences in normativity and serviceability could be perceived. Specifically, we compared word count per student work artifact across four a priori coded combinations of normativity and serviceability: MM (more normative, more serviceable; n = 4 artifacts), ML (more normative, less serviceable; n = 3), LM (less normative, more serviceable; n = 3), and LL (less normative, less serviceable; n = 3).

Sentiment analysis. The second metric, sentiment analysis, was used to quantify evaluative language in teachers' collective noticing of student work. Sentiment and subjectivity analysis in computational linguistics identifies lexical expressions of belief, evaluation, or emotion and distinguishes subjective from objective language (Liu, 2010; Wiebe & Riloff, 2005). We used the MPQA Subjectivity Lexicon (Wiebe & Riloff, 2005; Wilson et al., 2009), a dictionary-based resource that reported strong inter-rater agreement in identifying contextual polarity and subjective expressions (90% agreement; $\kappa = 0.84$; Wilson et al., 2009). Tokens in the corpus were matched to lexicon entries labeled for subjectivity strength (strong-subj, weak-subj) and polarity (positive, negative, neutral, both). Subjectivity strength indexes the intensity of

[1] These AI-generated descriptions were used to identify which artifact each frame depicted and were not included in the language metrics, which reflect teacher talk alone.

evaluative stance, whereas polarity indicates its directional orientation (Wilson et al., 2005; 2009). Each transcript excerpt served as the unit of analysis. Three measures were derived:

1. Strong-subjectivity density. $Strong\ Subj.Density = \frac{N_{strong}}{W} \times 1000$ where N_{strong} is the number of strong subjective cues and W is total word count. This indexes the frequency of strong evaluative language relative to excerpt length.
2. Strong-subjectivity ratio. $Strong\ Subj.Ratio = \frac{N_{strong}}{N_{strong} + N_{weak}}$. This indexes the proportion of strong subjective expressions rather than weak ones, indicating evaluative intensity.
3. Polarity density. $Polarity\ Density = \frac{N_{positive} - N_{negative}}{W} \times 1000$. This indexes the net balance of positive versus negative evaluative language per 1,000 words.

Semantic neighborhood analysis. The third metric, semantic neighborhood analysis, examined patterns of contextual association in teachers' discourse about student work. We analyzed the corpus using Word2Vec, a distributional semantic method that learns vector representations of words based on patterns of contextual co-occurrence. In the resulting vector space, words that appear in similar contexts are positioned near one another. For each target word (e.g., Alpha, Zeta), we identified the most similar words in this vector space, revealing case-specific patterns of contextual association. These semantic neighborhoods reflect contextual similarity rather than word frequency alone. Given the moderate corpus size, tokens were lemmatized to reduce sparsity (Jurafsky & Martin, 2024), with minimal stopword removal to preserve mathematically relevant language. Comparing neighborhoods across cases provided an empirical lens on recurring orientations in teachers' collective noticing of student work.

Results

More Talk Around Less Normative Student Work

Table 1 displays the mean word count per student-work artifact across normativity × serviceability combinations (n = 191,629 words in total). Across both serviceability categories, artifacts characterized by less normative features generated more discussion on average. The highest mean talk amount occurred for less normative and more serviceable artifacts (M = 19,824 words per artifact), whereas the lowest occurred for more normative and more serviceable artifacts (M = 11,902 words per artifact). Intermediate levels of discussion were observed for LL and ML artifacts. Because the unit of analysis was the artifact (n = 13), with only 4, 3, 3, and 3 artifacts in each condition, this analysis should be interpreted as exploratory.

A two-way analysis of variance revealed no statistically significant main effects or interaction at the artifact level. Although less normative artifacts generated more discussion on average, consistent with prior reports (Schwarts et al., 2023), this difference was not statistically significant (p = .285) and therefore cannot be treated as evidence without further investigation.

Table 1: Mean Word Count per Student-Work Artifact Across Normativity × Serviceability

	More Serviceable	Less Serviceable
More Normative	11,902 (20%)	12,262 (21%)
Less Normative	19,824 (33%)	15,922 (27%)

Positive Evaluative Stance Across Differences in Normativity and Serviceability

Teachers' evaluative stance toward student work remained consistently positive across artifacts, regardless of differences in perceived normativity or serviceability (Table 2). Polarity values were positive in all conditions, indicating that affirmative language predominated even for artifacts perceived as less aligned with instructional norms or less immediately useful for advancing the lesson. Measures of strong-subjectivity density and evaluative strength showed only modest variation across conditions. The highest concentration of strong evaluative language occurred in discussions of artifacts perceivable as both more normative and more serviceable (strong-subjectivity density = 100.5; strong-subjectivity ratio = 0.59), while artifacts perceivable as more normative and less serviceable exhibited the lowest density (88.1), though ratios remained similar overall. This stability suggests that teachers did not treat evaluation as a mechanism for sorting or dismissing student work. Instead, maintaining a positive stance functioned as a discursive resource that kept multiple contributions in play as teachers worked to interpret and position them within the developing lesson.

Table 2: Evaluative Language by Normativity × Serviceability

Artifact Features				
Normative	Serviceable	Strong Subj. Density	Strong Subj. Ratio	Polarity
More	More	100.5	0.59	48.7
Less	Less	99.0	0.57	51.5
Less	More	96.2	0.56	48.5
More	Less	88.1	0.56	45.9

Semantic Associations Indicate Pairing as a Compensatory Strategy

Semantic neighborhood analysis revealed patterns of contextual association surrounding selected student-work artifacts across normativity × serviceability combinations. To illustrate these patterns, we selected from each combination the artifact that elicited the most discussion, as these cases provided the densest linguistic data for modeling contextual associations. We describe the words most strongly associated with its name in the learned vector space.

For the more normative and more serviceable artifact, Nu's work, the most similar words included *ABC, medians, bisector, circum, geometric, definition,* and *construct* (cosine similarity .744–.499). These associations reflect discourse anchored in canonical geometric objects and construction methods, suggesting that teachers perceived Nu as aligned with normative diagrammatic conventions and as procedurally oriented toward the instructional goal. The clustering of canonical geometric terminology indicates that both normativity and serviceability were relatively efficiently recognized.

For the more normative but less serviceable artifact, Beta's work, associated words included *peer, classmate, epsilon, difference, visible, bisector, similarity*, and *zeta* (cosine similarity .595–.417). While the presence of *bisector* signals recognition of normative geometric structure, the frequent co-occurrence of *peer* and *classmate* suggests that teachers oriented toward pairing strategies—most notably pairing Beta with Epsilon—to compensate for perceived limitations of Beta's work for advancing the lesson goal.

For the less normative and less serviceable artifact, Epsilon's work, a similar pattern emerged, as the neighborhood included *sketch, beta, sharing, omicron, marking, classmate, peer,* and *continuation* (cosine similarity .574–.411). Words such as *sketch* reflect less canonical

geometric language, while co-occurrences of *beta* and *classmate* again indicate orientation toward pairing, suggesting that teachers worked to situate this artifact within a broader lesson sequence.

For the less normative but more serviceable artifact, Zeta's work, associated words included *eta, separate, hit, consistent, leg, quick, weigh,* and *favorite* (cosine similarity .708–.396, reflecting a broader spread of associations). The prominence of *eta* reflects a recurring instructional move in which teachers paired Zeta with Eta due to shared reliance on non-normative measurement strategies. Words such as *consistent* and *leg* indicate efforts to interpret the unconventional measurement-based reasoning, while *favorite* suggests that teachers nevertheless perceived productive value in the approach. Compared to Nu, whose neighborhood was dominated by established geometric terminology, Zeta's associations included more interpretive language, suggesting that additional discursive work was required to unpack its instructional potential.

These patterns highlight a distinctive feature of the Story*Circles* context and clarify how perceptions of normativity and serviceability were worked out in practice. Although the lesson storyboard initially positioned each student-work artifact individually within a frame, teachers frequently reconfigured artifacts in pairs when discussing sequencing decisions. Teachers treated normativity and serviceability not as fixed attributes of a single artifact but as qualities that could be established relationally by interpreting one piece of artifact alongside another. Pairing thus functioned as a compensatory strategy, mitigating perceived limitations in normativity or serviceability by juxtaposing complementary artifacts. This orientation toward comparing student work aligns with the findings from the sentiment findings: Despite differences in perceived normativity or serviceability, teachers maintained a generally positive stance toward all artifacts. Artifacts perceived as less aligned with norms or less immediately useful for the lesson were repositioned through comparison to render them instructionally usable.

The semantic findings also contextualize the talk amount trend. While less normative artifacts generated more extended discussion overall, their semantic neighborhoods contained more varied and interpretive language compared to the more concentrated geometric terminology associated with artifacts more readily perceived as normative and serviceable. This contrast suggests that normativity may be recognized efficiently when canonical features are present, whereas artifacts lacking those features require more discursive elaboration to articulate their value within the developing storyboard. Establishing serviceability, in turn, often involved interpreting how such work could be connected to other contributions or positioned within the lesson sequence to advance the instructional goal. In this sense, semantic neighborhood patterns illustrate how teachers' collective noticing extended beyond evaluation to include the interpretative and comparative work required to construct a coherent mathematical narrative across student contributions.

Illustrative Excerpts

To contextualize the quantitative patterns in talk amount, evaluative language, and semantic association, we present one representative artifact from each normativity × serviceability combination alongside an illustrative excerpt (Figure 1). These excerpts provide qualitative insight into how teachers interpreted and positioned student work within Story*Circles*.

Discussion of the more normative and more serviceable artifact, Nu's work, was concise and generally affirmative, suggesting limited discussion beyond recognition of both its normative features and instructional usefulness. Teachers described it as appropriate for a "middle-ish" portion of the lesson.

Discussion of the more normative but less serviceable artifact, Beta's work, revealed closer scrutiny of whether its geometric elements advanced the lesson goal. The quote signals recognition of normative structure—"draw the perpendicular bisectors of those sides"—while simultaneously questioning its instructional leverage. Consistent with the semantic associations linking Beta to references to peers and other artifacts, teachers considered pairing it with Epsilon to clarify similarities and differences and to strengthen its role in the developing storyboard.

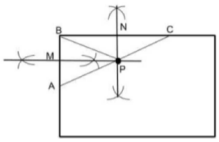
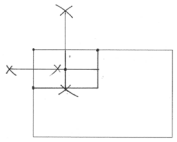
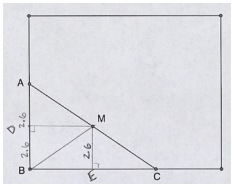
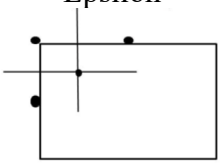
	More Serviceable	Less Serviceable
More Normative	Nu  "I think it's good… this would be, I feel like, middle-ish [of the lesson]	Beta  "Did they both say they were going to draw the perpendicular bisectors of those sides?"
Less Normative	Zeta  "This one I struggle with… when kids have ideas like this, I would almost laugh them off and be okay. But how can we really find…that's all that is an outside-the-box thinking? How could we apply it on paper? … I wouldn't even know a question to ask them to build off of their idea"	Epsilon  "I guess it's a lower barrier to entry, maybe would be a reason to include it"

Figure 1: Representative Student-Work Artifacts and Illustrative Excerpts Across Normativity × Serviceability Combinations

For the less normative and less serviceable artifact, Epsilon's work, teachers noted that it provided a "lower barrier to entry," and its sketch-like qualities prompted discussion about how it might be supported or positioned alongside other work to contribute meaningfully to the lesson. This suggests that when features of student work were perceived as less aligned with conventional norms or immediate instructional usefulness, teachers sought ways to retain and develop the artifact within the sequenced storyboard.

Discussion of the less normative but more serviceable artifact, Zeta's work, was more extended and deliberative. The excerpt highlights both the perceived unconventional nature of the reasoning as "outside-the-box thinking" and the affordances of the Story*Circles* context. Although the teacher acknowledged that in a live classroom they might "almost laugh them off", the collaborative Story*Circles* setting allowed teachers to examine how Zeta's non-normative

measurement approach might still contribute to the mathematical storyline. The teacher explicitly names struggling with this artifact; in contrast to Nu's work, whose discussion was anchored in established geometric terminology and required relatively little elaboration, Zeta's work required more interpretive work to articulate its instructional potential.

Conclusion

A central finding of this study is that teachers maintained a consistently positive evaluative stance toward student work regardless of differences in perceived normativity or serviceability. Work that was less aligned with conventional mathematical forms or less immediately useful for advancing the lesson was not dismissed; instead, teachers treated all contributions as potential resources for instruction. This consistency of evaluation suggests that the Story*Circles* context supports teachers in keeping multiple lines of student reasoning in view as they interpret student work in relation to instructional goals. This orientation aligns with a central goal of Story*Circles*: to treat student work as a shared design resource rather than an object of correction.

Semantic patterns showed how teachers carried out this work in practice. Teachers frequently paired artifacts, interpreting them in relation to one another to establish their instructional value. Pairing functioned as a compensatory strategy, addressing perceived limitations in normativity or serviceability by juxtaposing complementary approaches and repositioning contributions within the lesson's mathematical trajectory. Through this comparative work, teachers shaped a coherent mathematical storyline rather than considering artifacts in isolation.

Story*Circles* functions as a productive site for examining teachers' collective noticing, making visible how student work is collaboratively interpreted, repositioned, and leveraged toward shared instructional goals. By highlighting how teachers sustain a broadly positive orientation while engaging in comparative and interpretive work, this study underscores the role of collective noticing in transforming diverse student contributions into shared instructional resources.

Limitations and Future Directions

An exploratory analysis of talk amount showed that less normative artifacts elicited more extended discussion, but this difference was not statistically significant and therefore did not replicate prior findings. This lack of statistical significance likely reflects the small number of units of analysis at the artifact level (n = 13), which constrained inferential power. For the semantic association analysis, one artifact within each condition was selected because it elicited the most discussion and thus provided dense linguistic data for modeling contextual relationships. This artifact was treated as representative of the condition for analytic purposes, though examining additional artifacts could further test the stability of these patterns. Finally, this study relied on computational analyses of language derived from lexical features and distributional patterns. These measures provide systematic insight into discursive tendencies but do not capture speaker intention or the sequential organization of interaction; the findings, therefore, identify patterns of orientation in teacher talk rather than fully specifying teachers' reasoning processes. Future research may extend this approach by analyzing a broader range of artifacts and by integrating interactional or grammatical analyses to complement lexical and distributional measures.

Acknowledgments

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BEYOND BELIEFS: SUPPORTING REFORM DOES NOT GUARANTEE REFORM-ORIENTED PRACTICE

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Veteran teachers are often framed as resistant to reform-oriented mathematics, yet less is known about how their beliefs, knowledge, and experiences shape enactment. This Interpretative Phenomenological Analysis of seven veteran elementary teachers found that participants generally valued reform goals, but implemented them conditionally. Early procedural learning experiences shaped enduring beliefs, while uneven mathematical knowledge for teaching and teacher self-efficacy influenced instructional decisions. Systemic pressures such as testing and pacing further mediated practice. Teachers negotiated hybrid approaches that balanced reform ideals with contextual constraints. Findings suggest reform enactment is not driven by beliefs alone but by the interaction of beliefs, knowledge, efficacy, and context, underscoring the need for sustained, mathematics-focused professional learning that bridges support and practice.

Keywords: Mathematical Knowledge for Teaching, Professional Learning, Teacher Beliefs

Problem Statement

Mathematics education in the United States has long been marked by recurring tensions between traditional, procedure-focused instruction and reform-oriented approaches emphasizing conceptual understanding and problem-solving (Schoenfeld, 2007). Reform efforts, including the Common Core State Standards (CCSS), have prioritized reasoning and conceptual knowledge, yet their success depends heavily on teacher implementation (Charalambous & Philippou, 2010; Phillips & Wong, 2012). Many veteran elementary teachers learned mathematics through procedural methods and report feeling underprepared to enact reform-oriented practices such as facilitating student-centered discussion or developing conceptual understanding (Craven & Trygstad, 2020; Drake & Sherin, 2006; Swars & Chesnutt, 2016).

Research highlights the importance of teachers' beliefs, teacher self-efficacy (TSE), and mathematical knowledge for teaching (MKT) in shaping instructional decisions (Ernest, 1989; Fives & Buehl, 2012; Hill et al., 2005). However, less is known about how veteran teachers' lived experiences as mathematics learners interact with these cognitive and affective elements to influence reform enactment, particularly as they navigate ongoing reform demands. The Cognitive-Affective Model of Conceptual Change (CAMCC) (Gregoire, 2003) offers a lens for understanding how teachers interpret reform messages through prior experiences, beliefs, efficacy, and knowledge, shaping the extent to which they engage with and implement reform-oriented instruction.

Purpose and Research Questions

The purpose of this paper is to examine one analytic finding from a broader interpretative phenomenological dissertation study of veteran Florida elementary mathematics teachers' experiences with traditional and reform-oriented instruction. While the larger study explored teachers' experiences, beliefs, TSE, and MKT across their careers, this paper focuses specifically on how teachers describe their support for reform-oriented mathematics and the conditions under which that support may—or may not—translate into reform-oriented instruction. To address this purpose, the following research questions guide the study:

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

1. How do veteran elementary mathematics teachers describe their beliefs about and support for reform-oriented mathematics instruction across their experiences as learners and teachers?
2. How do beliefs, teacher self-efficacy (TSE), mathematical knowledge for teaching (MKT), and contextual factors interact to shape whether support for reform results in reform-oriented instruction?

Conceptual Framework

This study is grounded in an integrated conceptual framework (Figure 1) that integrates teacher beliefs, TSE, MKT, and the CAMCC to examine why support for reform-oriented mathematics does not always translate into reform-oriented practice. Beliefs are "messy" yet powerful drivers of teacher decision-making because they are shaped by experience and often guide behavior more strongly than knowledge alone (Ernest, 1989; Fives & Buehl, 2012; Pajares, 1992). Empirical research indicates that teachers' beliefs about the effectiveness of reform-oriented approaches are strongly associated with how they enact these practices in the classroom (Liou et al., 2019; Wilkins, 2008). At the same time, beliefs are held with varying degrees of strength and flexibility (Conner & Gomez, 2019; Green, 1971; Thompson, 1992), and teachers frequently experience mismatches between reform-oriented beliefs and enacted practice when faced with contextual demands (Grant & Kline, 2000; Raymond, 1997; Sawyer, 2017; Yates, 2006). In this study, mathematics beliefs are defined as personal judgments about mathematics formed through experience that influence instructional decision-making (Raymond, 1997; Schoen & LaVenia, 2019).

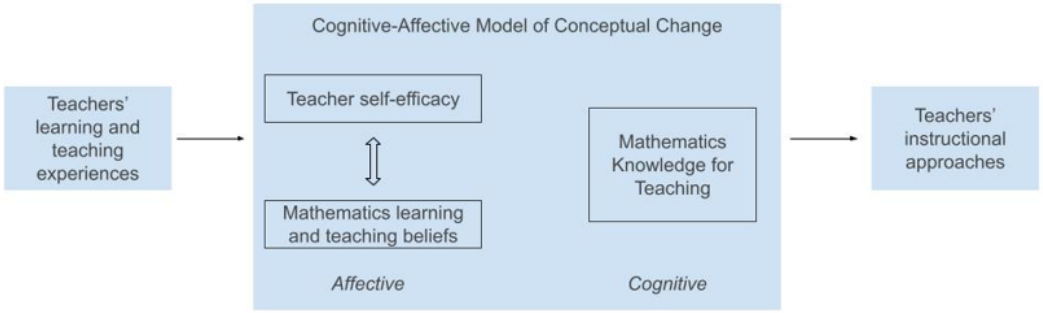


Figure 1: Conceptual Framework

To move beyond belief–practice assumptions, this framework incorporates TSE and MKT as key mechanisms that mediate enactment. TSE refers to teachers' beliefs about their capability to organize and execute instructional action (Bandura, 1994, 1997; Skaalvik & Skaalvik, 2010) and is linked to teachers' willingness to engage in reform-oriented instruction (Liou et al., 2019). MKT captures the subject matter knowledge and pedagogical content knowledge needed for teaching, including how teachers interpret content, respond to student thinking, and enact instruction (Ball et al., 2008; Shulman, 1986). Because MKT models have been critiqued for underemphasizing affective dimensions (Carrillo-Yañez et al., 2018), the CAMCC provides the organizing model for how cognitive (e.g., knowledge) and affective (e.g., beliefs, self-efficacy) processes shape teachers' responses to reform messages (Gregoire, 2003). Using this integrated lens, the study examines how veteran teachers' experiences as learners and teachers interact with beliefs, TSE, MKT, and contextual constraints to produce conditional reform enactment,

clarifying why supporting reform does not necessarily guarantee reform-oriented practice (Grant & Kline, 2000; Gregoire, 2003; Raymond, 1997).

Methodology

Statement of Positionality

This study adopted Interpretative Phenomenological Analysis (IPA) as its methodological and analysis approach, recognizing that the researcher is embedded within participants' lived experiences and must actively interpret those experiences (Neubauer et al., 2019; Smith et al., 2022). As I began the data collection, my biases, judgements, and positionality were closely monitored through reflexive journaling. I referred to my own mathematical learning and teaching background to make sense of and interpret the data, aligning with a core value in IPA. Moreover, I understood that my belief in the power of reform-oriented instruction, coupled with my understanding of the challenges that teachers face in implementing these practices, shaped how I interpreted the findings. I recognized that teaching is not binary, and the landscape of mathematics instruction is far more nuanced than a simple dichotomy between traditional and reform-oriented approaches.

Research Design

The broader study employed IPA, grounded in phenomenology, hermeneutics, and idiography (Smith et al., 2022). IPA prioritized participants' subjective experiences and meaning-making, while acknowledging the double hermeneutic, in which the researcher interprets participants' interpretations of their lived experiences (Smith et al., 2022). In keeping with IPA's idiographic focus, the study centered on a detailed, case-by-case examination of each participant's lived experiences prior to cross-case synthesis.

Sampling and Recruitment

IPA suggests that participants should be relatively homogeneous (idiographic) to ensure the data collected reflects potential areas of convergence or divergence (Smith et al., 2022). To be eligible, teachers must have been teaching elementary (kindergarten–fifth grade) mathematics at least once per week and have at least 10 years of experience in a Florida public school. Participants were recruited through purposive and snowball sampling (Tracy, 2019) using an IRB-approved digital flyer and targeted recruitment emails. Ultimately, seven teachers were selected (Table 1), a sample size considered appropriate for a robust IPA study (Smith et al., 2022). Although eligibility was set at 10 years, participants had between 20 and 35 years of teaching experience. All participants reported their gender and racial identity as female and White, respectively, and each taught mathematics anywhere from 50–65 minutes daily.

Table 1: Overview of Participants

Pseudonym	Years Taught[1]	Grade Level[1]
Rachel	27	2nd
Maria	20	3rd
Lauren	25	5th
Jill	28	1st
Janet	33	5th
Meg[2]	35	2nd, 3rd, 5th
Kate	24	1st

[1] At the time of data collection in summer after school year ended.

[2] Meg's class included one student each from grades 2, 3, 5, 9, 10, 11, and 12.

Data Collection and Analysis

Data were generated through pre-interview reflections, semi-structured interviews, and post-interview journal prompts (Figure 2). A pre-interview reflection served as stimulus material (Törrönen, 2002) to orient participants to *traditional* and *reform* terms and guide interview follow-up questions; it functioned only as preparation and was not included in the data analysis. Each participant completed a single semi-structured interview (40–75 minutes; $M = 56$), which was recorded and transcribed using MAXQDA Transcription. One day following interviews, participants completed a typed journal prompt to provide additional insight and participant control over what was shared (Smith et al., 2022). Reflexive researcher journaling in MAXQDA supported documentation of positionality, methodological decisions, and evolving interpretations (Lincoln & Guba, 1982; Smith et al., 2022; Vicary et al., 2017).

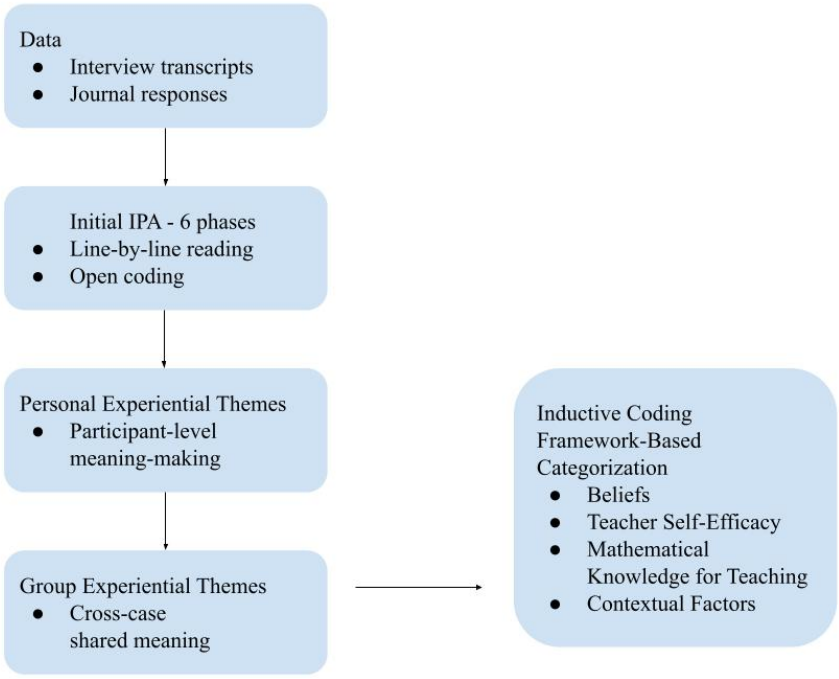


Figure 2: Two-Phase Qualitative Data Analysis

Interview transcripts and journal prompt responses were analyzed using a two-phase qualitative analytic approach (Smith et al., 2011; Figure 2). First, an IPA was conducted following the systematic, idiographic, and interpretative framework described by Smith et al. (2022), including multiple close readings, exploratory noting, generating experiential statements, developing Personal Experiential Themes (PETs) for each participant, and forming Group Experiential Themes (GETs) through cross-case analysis while attending to both convergence and divergence. Second, after GETs were generated, a framework-based analysis was conducted to examine barriers to and motivators for implementing reform-oriented mathematics instruction using the study's conceptual framework as an analytic and organizational lens. Specifically, identified barriers and motivators were categorized according to the constructs within the framework (e.g., beliefs, TSE, MKT, and contextual influences), allowing for patterns across participants to be examined while maintaining alignment with the theoretical commitments of IPA.

Findings

Research Question 1: How do Veteran Elementary Mathematics Teachers Describe Their Beliefs About and Support for Reform-Oriented Mathematics Instruction Across Their Experiences as Learners and Teachers?

Participants expressed beliefs that generally supported reform-oriented mathematics, particularly its emphasis on conceptual understanding, student-centered learning, and engagement. Reform-oriented approaches were viewed as opportunities to strengthen their beliefs aligned with student-centered learning, build confidence in instructional decision-making, and deepen MKT. For example, Lauren described reform-oriented instruction as a shift where "you're not the show," emphasizing that students should "guide their learning and their understanding," reflecting beliefs that position learners as sense makers. Participants emphasized that reform-oriented instruction helped build classroom community, encouraged learning from mistakes, and supported students in using the strategies that worked best for them. Lauren explained,

> It's important that they learn strategies that they can access. And it might not be the strategy that works for you, and it might not work for everyone in the room. But if it's something that helps them efficiently get there, I feel like that's the priority.

Participants also described how reform-oriented practices increased student motivation and engagement. Maria explained, "[Reform-oriented tasks] get [students] to buy in a little bit more and then be a little bit excited when they can make that connection between math and things that they do on a daily basis." Rachel similarly noted, "most of [my students] get excited about it [real-world problems and student-led discussions]. They're very inquisitive."

Despite these supportive beliefs, participants frequently described tensions between reform-oriented ideals and their comfort with traditional methods. Across participants, experiences with mathematics as K–12 students were overwhelmingly traditional in nature, with little to no exposure to reform-oriented instruction until later in their careers, often first encountered in teacher preparation programs or through curriculum adoption and classroom practice. Participants frequently recalled environments that emphasized memorization, speed, and performance under pressure, often with little conceptual understanding. Rachel recalled a particularly vivid source of stress when asked about what she remembered about learning mathematics:

> There was some anxiety around having to do [timed math fact tests]. Memorizing multiplication facts and having to stand up and say them all in front of the teacher. A lot of arguing [about math homework] with my parents, mainly my dad.

As teachers, all seven participants articulated difficulties letting go of the perceived control or efficiency that traditional teaching methods offered. Maria explained, "sometimes it's hard to let go of that control." Participants often framed their instructional approaches as a balance between traditional and reform-oriented instruction. Rachel described current practice as having "a nice balance," while Kate stated that she strives "to find a balance so my teaching can be memorable, but also meet the expectations given … I personally believe a hybrid would be the most effective."

These findings suggest that support for reform-oriented mathematics was not simply a matter of endorsement but of ongoing negotiation shaped by teachers' learning histories and perceived

classroom demands. While participants articulated strong alignment with the goals of conceptual understanding and student agency, their enactment was tempered by deeply rooted experiences with traditional instruction and a continued reliance on its perceived efficiency and control. Rather than rejecting reform, teachers described working to reconcile competing priorities, often adopting hybrid approaches that allowed them to maintain instructional comfort while gradually incorporating reform-oriented practices.

Research Question 2: How Do Beliefs, TSE, MKT, and Contextual Factors Interact to Shape Whether Support for Reform Results in Reform-Oriented Instruction?

Findings revealed an interrelated set of constraints operating across teachers' beliefs, TSE, MKT, and contextual conditions. Teachers' beliefs about effective mathematics instruction often interacted with their confidence and perceived MKT, jointly influencing instructional decisions. Reflecting on her limited mathematics TSE and MKT, Meg lamented that she "didn't have the confidence necessary to go forth and meet [a student] where he was," when referring to a time when she did not understand the mathematical content necessary to teach certain geometry concepts. Similarly, Janet noted that "if there is a standard that I cannot show a student how to do comfortably … that's when I will go the traditional way."

Limited access to sustained, mathematics-centered professional learning (PL) appeared to have constrained opportunities to build reform-aligned TSE and MKT. Lauren reflected, "The history of math professional development is minimal … There's so much reading professional development out there, but there's no math." Jill emphasized the potential impact of effective mathematics PL stating that "if [math PL] is well done, it is very motivating."

Externally imposed structural and policy conditions were described as constraining teachers' willingness to fully invest in reform-oriented instruction, particularly when accountability pressures made instructional time feel high stakes and outcomes uncertain. Kate explained,

> The system of testing has forced [teachers] to be failures … when you do a lot of that reform kind of math, you can't always be confident that you've wasted a whole day for something that might not pay off for a long time.

Additionally, Rachel described restrictive school norms, stating that "It was more of stay on pace and do what your team's doing … it should all look like you all are doing the same thing." Participants also identified student and classroom-related factors shaping instructional choices. Rachel noted, "Some days all they can handle is following in a book … impulsive behaviors are definitely a barrier."

Beliefs about student readiness further influenced enactment. Janet stated, "If they don't have that foundation … they're not going to understand anything that I am explaining." Kate similarly remarked, "I just believe strongly these kids need to know that four plus four is eight… you're never going to get to the fun part of math if the foundational skills are [lacking]."

At the same time, participants described motivations that supported reform enactment when conditions aligned, particularly when they took initiative in their own learning. Lauren reflected on a shift in her thinking, noting that "the way I think about math has completely changed," and that she built confidence in implementing reform-oriented practices after intentionally seeking out mathematics-focused PL on her own.

Reform-oriented instruction was most likely to occur when teachers experienced increased TSE, growth in MKT, positive student engagement, and supportive (e.g., administration, support personnel) contexts. However, even when teachers expressed strong support for reform-oriented mathematics, contextual pressures, limited MKT, and fluctuating confidence often resulted in hybrid or conditional implementation. Together, these findings indicate that support for reform-

oriented mathematics did not consistently translate into reform-oriented practice. Instead, enactment was shaped by the interaction of beliefs, MKT, TSE, and contextual constraints, resulting in conditional and context-dependent implementation.

Discussion and Implications

Sustained, Mathematics-Focused Professional Learning

Findings indicate that support for reform-oriented mathematics did not consistently translate into reform-oriented practice. Participants described valuing reform-oriented instruction while also relying on traditional methods due to limited mathematics-focused PL, fluctuating confidence, and contextual pressures. These findings point to the need for sustained, mathematics-specific PL that is embedded in teachers' daily work and grounded in classroom practice rather than one-time workshops (Shriki & Patkin, 2016).

Specifically, PL should explicitly address the gap between valuing reform principles and consistently enacting them in practice through reflection, collaboration, and opportunities to experience reform-oriented mathematics themselves as learners (Fives & Buehl, 2016; Kutaka et al., 2017; Phillipsen et al., 2019; Spillane et al., 2018). Participants' descriptions of minimal mathematics PL and reliance on self-directed learning suggest that teachers require ongoing opportunities to rehearse, refine, and reflect on reform-oriented practices in supportive environments. Without sustained, content-specific support, teachers may revert to familiar procedural methods, particularly when confidence is low or contextual pressures are high (Charalambous & Philippou, 2010; Stein et al., 2013; Swars & Chesnutt, 2016).

Targeting teacher self-efficacy and mathematical knowledge for teaching. Findings reveal that teachers' beliefs interacted with their TSE and MKT, collectively shaping whether support for reform translated into reform-oriented instruction. Participants described turning to traditional approaches when they felt uncertain about content knowledge or instructional decisions, suggesting that these elements function in a mutually reinforcing relationship in which confidence and knowledge both enable and are strengthened through reform-oriented practice (Kutaka et al., 2017; McMinn et al., 2021). PL should therefore attend to both cognitive and affective dimensions of teaching mathematics by supporting teachers' understanding of mathematical content while also strengthening confidence in implementing reform-oriented instruction (Gregoire, 2003). Opportunities to analyze student thinking, anticipate misconceptions, and explore multiple solution strategies can help teachers build MKT while also increasing confidence in facilitating conceptual and discussion-based instruction (Ball et al., 2005; Hill & Ball, 2009). When teachers experience growth in MKT alongside supportive collaboration and reflection, they are more likely to enact reform-oriented practices consistently.

Enhancing understanding of reform-oriented mathematics. Although participants expressed support for reform-oriented mathematics, findings indicated that reform was often enacted in hybrid or conditional ways. This suggests a need for PL that moves beyond introducing the visible features of reform to engaging teachers in deeper exploration of the mathematical and pedagogical principles underlying reform-oriented instruction. Analyzing student work, comparing solution strategies, and examining connections between conceptual and procedural knowledge can support teachers in developing more integrated understandings of reform goals and aligning instructional decisions with those goals (Carpenter et al., 1996; Charalambous & Philippou, 2010; Rittle-Johnson et al., 2015). Collaborative structures such as Professional Learning Communities (PLCs) and coaching can provide spaces for teachers to discuss tensions between reform ideals and classroom realities, experiment with instructional

approaches, and receive feedback in supportive environments (Kutaka et al., 2017; Shriki & Patkin, 2016).

Structural Supports and Conditions for Enactment

Participants described testing pressure, pacing expectations, limited mathematics PL, and school norms as central factors shaping whether reform-oriented instruction was enacted, constraining teachers' willingness to take instructional risks, reinforcing reliance on familiar, efficiency-driven practices even when beliefs aligned with reform goals. At the policy level, states and districts should reduce the high-stakes consequences attached to standardized test performance because such accountability pressures can reinforce traditional practices (Boehme, 2014; Lambert, 2015; Yurekli et al., 2020) and may discourage the use of instruction that best supports students' learning. Thus, school and district leaders play a critical role in providing time for collaboration, access to high quality PL, and instructional resources aligned with reform goals (Yurekli et al., 2020). PL designed around manageable, practice-proximal instructional shifts may support teachers in gradually strengthening reform-oriented instruction while navigating contextual constraints (Otten et al., 2025). When teachers experience supportive contexts, growth in MKT, and increased confidence, reform-oriented instruction is more likely to be enacted consistently rather than conditionally.

Conclusion

This study centers veteran elementary teachers' lived experiences with traditional and reform-oriented mathematics to understand how those experiences shape the conditional uptake of reform-oriented instruction. Findings reveal that teachers' support for reform was often genuine yet unevenly enacted, shaped by the reciprocal interplay of beliefs, TSE, MKT, and contextual pressures such as testing, pacing, and limited mathematics-focused PL. Collectively, the findings underscore that movement toward reform is not a simple shift in beliefs but an iterative process requiring sustained, mathematics-specific PL, alignment across policy and school structures, and conditions that allow teachers to take instructional risks without fear of negative consequences. Supporting veteran teachers in examining their learning histories, strengthening TSE and MKT, and engaging in collaborative, practice-based PL may strengthen teachers' capacity to enact reform-oriented mathematics instruction in coherent and sustainable ways.

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TALK PATTERNS IN FACILITATED TEACHER TEAMS: USING OPEN-ENDED UNIT ASSESSMENTS AS ARTIFACTS TO SUPPORT TEACHER LEARNING

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We share findings from a project in which teams of elementary teachers met regularly to discuss their students' work on district-developed, open-ended assessments and co-planned lessons incorporating information gathered from those assessments. Focusing on the patterns of talk in two fourth-grade teams over nine meetings, we analyzed shifts in the focus of conversations as participants analyzed assessments and planned lessons. We describe several patterns, including that teachers discussed features of solutions beyond their correctness and were increasingly more specific in their planning conversations. We also documented tensions in the conversations, such as concerns with the fairness of item scoring and the alignment between the assessments and curriculum, standards, and state tests.

Keywords: Instructional Leadership, Elementary School Education, Professional Development, Instructional Activities and Practices

Introduction

One empirically supported way of improving elementary mathematics teaching is engaging teachers in developing and using knowledge of student thinking in their instruction (Carpenter et al., 2000). A significant challenge, however, is embedding these kinds of learning experiences in the day-to-day work of teaching. In this project, we sought to understand how unit assessments might function as a site for such learning experiences, and how facilitated grade level team meetings might support teachers in developing and using knowledge about student thinking in their planning and teaching.

Scholarly Foundation

The study was guided by a theory which views learning in terms of "transformation of participation" (Rogoff, 1997) rather than individual competency or demonstrations of knowledge. Following Kazemi and Franke (2004), we use this approach to analyze shifts in teachers' conversations around students' mathematical work, with group discourse as the primary object of analysis. Specifically, we examine how discourse about (1) student thinking, (2) teaching, and (3) mathematics evolved over the course of the year-long project.

A robust base of literature supports the use of artifacts of student thinking to drive the development of professional knowledge in teacher teams, especially when drawn from teachers' own classrooms (e.g., Franke et al., 2001; Gibbons & Cobb, 2017; Kazemi & Franke, 2004). One type of artifact that may support such discussions is the assessments that teachers give as a part of their normal teaching responsibilities. Because tests are perceived connected to school accountability and the improvement of teaching and learning (Brown, 2004), teachers have both intrinsic and extrinsic reasons to care about how students perform on them.

Assessment results can be misinterpreted and misused in schools, as numerical scores can lead to "assumptions about students' knowledge without considering specific test items and the validity of the information that these items provide about that knowledge" (National Council of

Teachers of Mathematics [NCTM], 2014, p. 91). In many schools, assessment programs are administered electronically and are scored automatically, producing a numerical score and sometimes a ranking (3rd grade level, etc.). These scores and rankings, however, do not provide the rich examples of student thinking that are likely to provide the material to support productive teacher conversations (e.g., Shim & Thompson, 2025), but instead, can lead to identifying/sorting students for remediation (Farrell & Marsh, 2016; Gannon-Slater et al., 2017). On the other hand, assessments can be used "as a method of gathering evidence of children's mathematical thinking to inform learning and teaching" (NCTM, 2020, p. 34).

In this project, we worked closely with a district that had designed a set of assessments that focused on student strategies rather than answers. District leaders planned to facilitate meetings to support teachers' conversations and learning about student thinking through examining student performance on these assessments. The research team, in collaboration with district leaders, developed a process to study the nature of these conversations and how they shifted over time, and also to understand the potential tensions that might arise as teachers navigated various purposes and goals of assessments.

Research Question

Our research question is, "What shifts in discussions occur over the course of a school year when grade level teacher teams engage in repeated opportunities to use strategy-based assessments to analyze student thinking and plan their instruction?" Our hypotheses were that teachers would gradually move toward describing student thinking in terms of strategies rather than correctness and would connect their discussions of teaching and content to evidence regarding student thinking/strategies. Furthermore, we conjectured that they would begin to make explicit connections between evidence regarding student thinking/strategies as gleaned from the assessments and discussions of their teaching approaches. That is, they would draw on evidence about student thinking in their co-planning conversations, and they would explain patterns in their assessment data by referencing specific pedagogical choices.

Context and Methods

The study was conducted in a school district that had adopted a commercially published textbook with seven to nine units in each grade level. A team of teachers and leaders worked to adapt the unit assessments so that they would be open to a variety of strategies and created strategy rubrics for each task based on research-supported learning progressions (e.g., Battista, 2012; Carpenter et al., 2015; Empson & Levi, 2011). Each assessment had four to seven items. Table 1 provides an example of the strategy guide for one of the items on the Unit A assessment. Numerical "scores" were attached to the rubric. Though the goal of the project was to focus on using assessments to understand and support student thinking rather than provide summative evaluations, project leaders included numerical scores to enable teachers to use their characterizations of student thinking from the assessments as pieces of evidence to support their (standards-referenced) grading. The numerical scores were intended to support teachers in recognizing a progression of ideas while still allowing for flexibility in strategy.

Table 1: A Strategy Guide for the Task: *Susan says 3/5 is greater than 4/10. Do you agree or disagree with Susan? Show your thinking using number, pictures, and/or words.*

Strategy Type	Description	Examples	Score
Invalid Strategy	The student is engaging in a strategy that would not lead to a correct answer.	The student says $\frac{4}{10}$ is bigger than $\frac{3}{5}$ because 10 is more than 5.	1
Partial Fraction Reasoning	The student uses numerators or denominators to decide which is bigger but not the relationship between the two.	The student indicates that $\frac{3}{5}$ is bigger than $\frac{4}{10}$ because fifths are bigger than tenths. Although this is correct for this problem, if the numerators were different, it may not be. The student uses drawings of fraction bars or circles to compare the amounts, but draws wholes that are not the same size.	2
Direct modeling	The student represents the fractions with drawings or models and uses them to compare.	The student uses drawings of fraction bars or circles to show that $\frac{3}{5}$ is more than $\frac{4}{10}$. The diagrams have the same sized wholes.	3
Equivalent Fractions	The student generates equivalent fractions numerically to determine which is greater.	The student uses equivalent fractions to determine which one is more using numbers. $\frac{4}{10} = \frac{2}{5}$ or $\frac{3}{5} = \frac{6}{10}$, so $\frac{3}{5}$ is greater	3
Benchmark Reasoning	The student uses a benchmark such as $\frac{1}{2}$ to decide which fraction is greater.	The student claims that $\frac{3}{5}$ is more than $\frac{1}{2}$ and $\frac{4}{10}$ is less than $\frac{1}{2}$, so $\frac{3}{5}$ is greater.	3

Each school had a facilitator; two of these were school-based coaches, one was the district curriculum coordinator, and one was a researcher from a nearby university (the lead author of this report). This team of facilitators developed a set of meeting protocols with the goal of shifting teachers' instructional approach toward centering student thinking, in contrast to "data teams," which can focus on ranking students based on their ability to get correct answers (Gannon-Slater et al., 2017). Each grade level team completed a cycle of three meetings (Sessions 1, 2, and 3) for each of three units, for a total of nine meetings (see Figure 1). In Session 1, teachers examined their students' solutions to the items on the pre-assessment and worked to reach shared understanding about what counted as evidence for each category in the strategy guide and to discuss patterns in the strategies. In Session 2, teachers collectively planned a lesson with the goal of using information gathered about student thinking from the pre-assessment to guide their planning. In Session 3, teachers examined the post-assessment and were asked to compare strategy use across assessments and items and connect these patterns to their instruction.

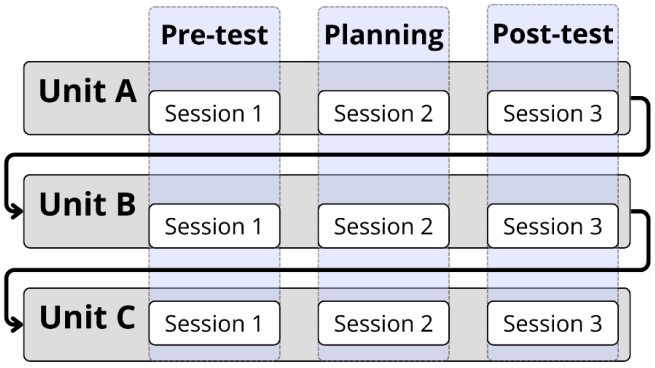


Figure 1: Depiction of the Nine Sessions with Grade-Level Team

Thirty-seven teachers in Grades 3, 4, and 5 across four schools agreed to participate in the study over the course of the 2023–2024 school year. In this paper, we focus on two teams, one at Silver Elementary and one at Bingham Elementary (pseudonyms). Both Silver and Bingham are Title I designated schools. Students eligible for free and reduced lunch constituted approximately 65% of students at Silver and 75% of students at Bingham. Both schools had recently underperformed on the state test in relation to the district as a whole, with majorities of students scoring at "below basic" or "basic" levels. The Silver team was facilitated by the district curriculum coordinator and the Bingham team was facilitated by a school-based instructional coach. The lead author attended and supported the facilitation of some meetings at each school.

We recorded all 18 teacher meetings with the groups at Silver and Bingham, which amounted to 546 minutes, of which 494 were coded. We analyzed the video data in terms of "teacher talk paths" (Murata et al., 2012), categorizing discourse as related to (1) students' mathematics thinking and (2) teaching, (3) mathematics, and (4) other. We further specified several subcodes within each of these categories (e.g., within students' mathematical thinking, we distinguished between discussions of individual student work and characterization of patterns across an assessment or across multiple assessments). Once the framework was established, we used MAXQDA software to directly code videos. We created "document portraits," in which a conversation is represented as a linear series of boxes (each an equal unit of time) that are colored according to codes. We used these portraits to examine patterns of discourse across both time and school, both in terms of percentages of time and the sequence of topics of discourse.

Findings

Session 1 (Pre-Test) Patterns

Teachers spent a substantial proportion of time (55% of the total minutes) in Session 1 interpreting and categorizing student thinking. Especially in the first meeting, conversations largely focused on how to categorize student strategies using the strategy guides, which seemed to focus teachers' attention to aspects of student thinking beyond the correct answer. For instance, they discussed cases where an incorrect answer showed some valuable understanding of the mathematics, and other cases (such as using "cross multiplication" to solve a fraction comparison problem) where conceptual understanding was difficult to ascertain. In some cases, teachers were pleasantly surprised by the understanding that students showed on the pre-assessment ("I was blown away! How they worked that out," "Wasn't that awesome?!"), suggesting that the assessments brought student thinking to the surface in ways that teachers

recognized and valued, particularly for students for whom they might not have anticipated exhibiting such thinking.

In other examples, particularly at Silver, discussions extended beyond the categories in the strategy guide, as in the case of teachers working to make sense of a student who, when trying to subtract $2\frac{2}{3}$ from $4\frac{1}{3}$, appeared to have traded a whole for 10 (thirds), as if they were trading a ten for 10 ones in a multidigit subtraction problem (see the middle part of Figure 2). The teachers were impressed by this strategy and spent several minutes discussing it without trying to assign a category from the strategy guide ("that's what I was about to say, like borrowing, you take 10, maybe she was thinking. That's kinda cool").

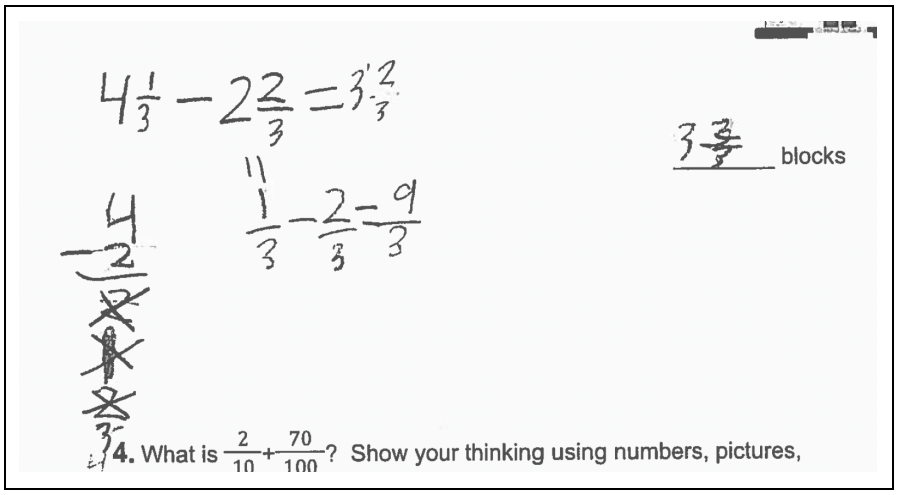


Figure 2: A Solution Discussed by Teachers at Silver Elementary

Over the year there was an increasing focus in Session 1 on patterns that the teachers noticed across tasks or solutions. Sometimes these were relatively brief, such as a teacher at Bingham noting that students who successfully completed a task tended to draw diagrams. Teachers at Silver discussed how a large number of students solved $1\frac{3}{4}$ plus $2\frac{2}{4}$ by adding the whole numbers (1 + 2) and the fractions $\left(\frac{3}{4}+\frac{2}{4}\right)$to get $3\frac{5}{4}$, while on an adjacent subtraction problem $\left(4\frac{1}{3}-2\frac{2}{3}\right)$ students unsuccessfully applied the same strategy but failed to correctly account for the need to regroup in order to subtract $\frac{2}{3}$ from $\frac{1}{3}$, as students . Sometimes these observations developed into extended conversations, such as one where teachers realized that some children were interpreting the notation "$3\frac{1}{4}$" as "3 one-fourths," that is, "$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$."

T2: [Students] were thinking that this three and one-fourth [3 ¼] was really three one-fourths. One fourth, one fourth, one fourth, instead of three *and* one fourth
T1: Yes, they weren't realizing it that it was like three and then a fourth
T3: Three wholes, yeah, and a fourth

Participants expressed surprise and appreciation for this example of children's thinking as they started to distinguish between misunderstandings about fractional quantities and misunderstandings about fraction notation.

At Bingham, conversations about patterns across assessments sometimes evolved into discussions about how students struggled with the tasks. At one point, the facilitator asked which

task the students did well on, and one teacher said, “none of them. I had about four kids, three to four kids. But no. No. And there were some I had to stop because they were in tears.” Such commentary sometimes developed into critiques of specific assessment items as being too difficult or unclear for students, which facilitators sometimes tried to redirect into discussions of how students were thinking about the ideas or implications for instruction.

Session 2 (Planning) Patterns

In Session 2, designed intended teachers to co-plan a lesson drawing on their learning from Session 1. Over the year, discourse in Session 2 shifted from generic planning to more specific planning conversations. At Bingham, in Unit A, teachers spent about 48% of the session talking about the launch-explore-summarize structure of their lesson or making what we considered to be general decisions about their lesson—such as sharing what tasks, tools, or manipulatives they were going to use. These conversations focused on the “what” of teaching, but not the why or how. Also, the “what” focused on actions and activities done by the teacher rather than anticipating what students might do.

Later in the year, however, Bingham teachers spent more time (81% in Unit B and 57% in Unit C) discussing the finer details of their lesson, hypothesizing about how children might solve the tasks and making adaptations to the task accordingly. For example, in one lesson they had planned to use the following task: *One lap around the blacktop is 2 thirds of a mile. So-and-so ran 5 laps; how much did they run?* The facilitator asked the group to “jot down what you think kiddos are gonna do.” After a brief discussion of whether children would make sense of ten-thirds as three whole miles and another third of a mile, the facilitator pressed the teachers to anticipate more strategies, referencing a common category from the assessment strategy guide: “Is there another way they would direct model?”

A student teacher who had joined the meeting shared their drawing of multiple boxes, each one subdivided into three parts. She explained, “Okay, so they would do thirds, and color in one, two, so that’s two-thirds. And then they would fill in that next one. That’s one-third. And draw another one—that's two-thirds.” Another student teacher suggested a more symbolic version of the strategy, where students would be “breaking it down. So two thirds plus two-thirds is four-thirds. And then do that another time, plus four thirds. And there’s two-thirds left. So they add four-thirds plus four-thirds plus two-thirds equals ten thirds.” The facilitator noted that children might also skip count by two-thirds, making mixed numbers as they go.

After collecting these ideas for how children might solve the task, the facilitator asked the group to consider how they might conduct the “summarize” portion of their lesson, where children would be asked to share, justify, and compare their solutions. First, they discussed the order of the solutions, and then ways to connect the solutions, e.g., “that’s a good question for them: ‘you multiplied, and you added over and over again. Why?’ And just get them to have that conversation.” This discussion was more substantive and detailed than the conversations the team was having in the beginning of the school year. Rather than just naming the task and the tools that the teacher would introduce, the group drew out anticipated strategies, in some cases using the labels from the assessment strategy guides. The facilitator encouraged the teachers to include drawn solutions (direct modeling) and suggested the practice of using different student solutions to help students make a connection between addition and multiplication.

In some Session 2 conversations, discourse shifted to the alignment between the assessments, the adopted math curriculum, and state standards. For example, when planning a lesson on decimals and fractions, one Bingham teacher talked about the need to understand the “ceiling of the standard” to clarify “what are we supposed to be getting at in this unit, and what is happening

in other units." Participants began looking up state standards and across future lessons in their curriculum to determine whether comparing decimals was an appropriate task for the lesson. This conversation continued for about six minutes, with the facilitator eventually attempting to direct conversation to the meaning of fraction/decimal notation and away from the description of the topics that need to be covered according to external documents.

Such conversations about the alignment between the assessments and standards/curricula were relatively rare in Session 2, but their presence shows the mix of concerns that teachers bring to conversations about planning. While the goal of the meetings was to focus on how students were thinking about mathematical concepts, policies around standards, mandated assessments, and curriculum occupy teachers' attention and influence their planning decisions.

Session 3 (Post-Test) Patterns

In Session 3 teachers were to discuss the results from the post-assessment in relation to the pre-assessment and their instruction during the unit. Our analysis revealed more variability in discourse foci across meetings and schools than in Sessions 1 and 2. Groups continued to spend time discussing individual pieces of student work on the assessments, but less than in Session 1 (35% of total time versus 55%). Teachers spent about 13% of their time in Session 3 discussing patterns within and across the assessments. For example, they addressed patterns from pre to post test, sometimes in very general ways, like noting that the strategies were "more efficient" or showed better understanding ("It's showing that they are understanding it a lot more than they did before. Before they didn't have any strategies, they just tried to write an answer down. Now they actually have the strategies to actually *show* you").

Groups spent about 10% of their time making explicit connections between their students' strategies on the assessment and their teaching in the previous unit. For example, they talked about the importance of using contextual problems, multiple representations, manipulatives, and explicit conceptual connections in class discussions. One Bingham teacher talked about how some of the explicit conceptual conversations in her lessons made an impact on the strategies students used on the unit test:

> I think it was really helpful to connect. … like, they did really well on 1, 2, and 4… And I think they were able to connect that as to like, *why*, and I think that helped with this one [holding up assessment], of like *why*, we talked about, giving them the rule, but then also like, here's the exception to the rule, and why we can add these together, but we have to change the amount of pieces.

In general, groups tended to see improvements from pre- to post- assessment (more children engaging in sensemaking strategies on the rubrics and fewer students using invalid strategies) and making conjectures about how their instruction created opportunities for students to develop deeper understanding.

Building on those conversations, teachers in Session 3 reflected on their own learning. For example, one teacher at Silver talked about how the project had helped her attend more to the details of student thinking.

> This kind of stuff has made me realize more that it is not about the answer. It's about like the way that they're getting to there and if we are not seeing growth necessarily in like the answer but we're seeing growth in their understanding then that's huge. So I think going into our next unit. That's kind of what I'm looking for. What are they understanding?

At Bingham, after some discussion of student growth from pretest to post test on fraction multiplication, a teacher commented, "I don't think I'll ever teach it again the 'dumb way.'"

Pressed by the facilitator to explain more, the teacher continued, "I have always taught them to convert mixed numbers to an improper fraction before you multiply them." A second teacher replied, "That's the only way I thought you could." This shows teachers expanded their knowledge of strategies through their examination of student work using the strategy guides. Other "meta" level conversations included sharing ideas for administering the assessment or critiquing the assessments.

Tensions that Surfaced During Team Meetings

In addition to the patterns described above regarding conversations in Sessions 1, 2, and 3, we also noticed tensions that arose in the meetings. While teachers sometimes expressed appreciation for how the strategy guides shifted attention away from the correctness of answers, as in the quote above, they also sometimes wrestled with this shift, possibly influenced by the fact that the assessments were being used for multiple purposes (to understand student thinking and to assign grades). At times they also critiqued the wording or structure of assessment items and worried whether the assessments were providing experiences that would help students be successful on the state test. From the designers' point of view, another tension involved the possibility that assessment categories might exacerbate teachers' perceptions of ability differences between children (Rodríguez et al., 2022). However, while there were instances where teachers characterized students in terms of deficiencies, these were often countered by facilitators directing attention to student work to identify strengths. There was some evidence that teachers attended to the possibility of reinforcing status hierarchies and sought to counter these by, for example, elevating the status of strategies like "direct modeling."

Discussion

Regarding our hypotheses, we found that teachers did start adopting the language of the strategy guides relatively quickly and made connections between the assessment results and their instruction, including by referencing the strategies in the details of their planning conversations. They expressed curiosity and appreciation for student thinking, in some cases challenging hierarchical narratives about student ability. In some cases, they explicitly described how the team meetings had started shifting their attention from answers to strategies, which helped them see development that they would not have noticed previously. These patterns show that centering grade level team meetings on unit assessments can function to support teachers' attention to and appreciation for understanding and building on student thinking (NCTM, 2014). At the same time, participants articulated some tensions as they negotiated the multiple purposes of assessment, highlighting the need to anticipate and plan for responding to such concerns.

Though not an explicit focus of our analysis, the study did suggest that the facilitator played a significant role in steering the conversations in each of the teacher teams. For example, at times the facilitator redirected focus to student thinking, or to specific aspects of the student thinking. when conversation moved to other areas or began to focus unproductively on student struggles (e.g., "a lot of my kids, they are so lost, they don't even know what to do next"). At other times, the facilitator pressed teachers mathematically, for example helping them to clarify whether a student strategy was generalizable. These kinds of moves seemed to be important for groups realizing the goals of the project, and future analyses could explore how differences in facilitation might have led to subtly different types of conversations in the teacher teams.

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BALANCING EXPERTISE AND PEER SUPPORT: EXPLORING PROFESSIONAL DEVELOPMENT FACILITATION MODELS FOR RURAL TEACHERS

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Seventy-seven rural middle-grade mathematics teachers were randomly assigned to one of two PD conditions facilitated by either Centralized Specialists (CS) or Local Teacher Leaders (LTL). While teachers in both conditions reported high levels of perceived clarity, discussion quality, and sense of community in the PD, those in the CS condition reported greater depth in discussions of instructional practices. Additionally, CS-facilitated workshops were significantly longer than LTL workshops, potentially reflecting the CSs' facilitative expertise or greater fidelity to the intended PD structure. Results from teacher surveys suggest that both models were perceived as having the potential to support meaningful collaborative discourse and the development of learning communities in rural settings, but further analysis is required to understand how workshop length and facilitator expertise impact long-term teacher growth.

Keywords: Professional Development, Middle School Education, Teacher Knowledge

Introduction

Many rural mathematics teachers receive only minimal training before they enter the classroom (Cady & Rearden, 2009). Professional development (PD) may enhance rural teachers' practices and improve their students' learning (Dille & Røkenes, 2021; Şahin et al., 2024). However, existing PD programs are often developed for teachers in non-rural contexts. Implementing these in rural settings without accounting for the uniqueness of rural contexts can fail to bring about desired shifts in instruction (Johnson & Howley, 2015). Thus, questions remain about how best to offer PD to rural teachers, many of whom work in remote areas and lack access to a professional learning community. Effective mathematics teacher PD relies heavily on collaborative partnerships (Desimone, 2009; Garet et al., 2001), particularly in adaptive models where facilitators and teachers co-construct participation and outcomes (Koellner & Jacobs, 2015).

Prior research suggests that locally-embedded facilitators have enhanced credibility, can promote teacher investment (Lindvall & Ryve, 2019), and are more naturally positioned to cultivate relational trust and community due to shared context and established relationships. External facilitators, particularly those who are not currently teaching, may face unique challenges establishing community (e.g., Anderson & Williams, 2023; Sztajn et al., 2007). However, PD led by centralized specialists, who have expertise in PD facilitation and deep knowledge of research-based instructional practices, may result in richer teacher discussions and offer greater opportunities for scalability. These differing strengths raise questions about who should facilitate PD in rural contexts and how various facilitator models impact rural teachers' perceptions and experiences.

We developed a large-scale PD project to bring rural teachers together to learn with and from colleagues, capitalizing on "existing capability" (Ball & Cohen, 1999, p. 17) in rural schools.

Our PD model was designed with two distinct facilitator conditions, in which either Centralized Specialists (CSs) or Local Teacher Leaders (LTLs) facilitated small-group, collaborative PD activities. We engaged 77 rural middle-grade mathematics teachers, who were randomly assigned to either the CS or LTL condition. Our PD focused on rural teachers' unique problems of practice, offering teachers a mechanism to address specific challenges faced while supporting the adoption of crucial instructional practices (Hiebert & Grouws, 2007; Stein et al., 2017). Both the CSs and LTLs led Problem-Solving Cycles (PSCs; Borko et al., 2015) with small teams of teachers, despite neither group having prior experience facilitating PSCs. Our goal was to prepare both groups to create a networked structure that would facilitate rural teachers' communication and collaboration, improve instruction, build professional community, reduce feelings of isolation, and potentially mitigate the elevated, expensive, and educationally deleterious effects of teacher attrition often found in rural schools (Preston et al., 2013). In this paper, we examine how the CS and LTL conditions affected rural teachers' patterns of participation in professional discourse and their perceptions of PD clarity, quality of PD discussions, and formation of authentic learning communities. Testing these two models for large-scale PD with rural mathematics teachers attends to the unique funds of knowledge and challenges in rural schools.

Theoretical Perspectives

When PD is facilitated by leaders with deep knowledge and strong expertise, PD programs are more likely to be implemented in a way that aligns with their intended purposes and designs (Heritage & Popham, 2008; Thompson & Wiliam, 2008). However, requiring advanced facilitator expertise may significantly limit a PD initiative's reach (Borko, 2004). As such, a program may instead be implemented by locally-embedded teacher leaders (Borko et al., 2021). This arrangement has several affordances and limitations. Teacher leaders are peers who are actively teaching and more familiar with the local teacher context. Their enhanced credibility can support the establishment of authentic learning communities in which teachers feel comfortable sharing and debating ideas as well as opening their practice for shared examination–PD features known to support teachers' learning (Ball & Cohen, 1999; Grossman et al., 2001; Lefstein et al., 2020; Little, 2002). However, as teacher leaders may lack experience facilitating PD, they may still be developing the knowledge and practices required to support teachers' collaborative learning, potentially affecting PD outcomes. Moreover, as they may be less familiar with mathematics education research, they may struggle to implement the PD's structures and purpose with fidelity.

In the PD program we study here, the learning of small groups of teachers was facilitated by either a CS or an LTL. The CSs were externally-located regional math specialists who had K-12 mathematics teaching experience but were not currently teaching. The CSs brought extensive experience facilitating mathematics PD across a range of implementation structures and deep knowledge of mathematics education research. Conversely, the LTLs were geographically proximate peers who were actively teaching middle grades mathematics with expertise in rural contexts. In fact, all LTLs lived within 60 miles of the teachers with whom they worked.

We anticipated that CSs would be more knowledgeable of facilitative moves and routines to support rich discourse with teachers. Also, given their expertise, we expected them to convey the purpose of the PD with clarity and enact the PD model with high fidelity. However, as external specialists, most of whom were located in an urban area and were not currently teaching, we predicted that they may find it more challenging to earn credibility with the teachers on their teams. We also expected LTLs to be able to facilitate rich teacher discourse as they were well-

positioned to establish credibility and a sense of community with the teachers, potentially resulting in teachers feeling more comfortable sharing ideas, opening up their practice, and pushing back on ideas shared by others–key indicators of generative teacher discourse in collaborative PD (Lefstein et al., 2020; Little, 2002). And yet, as LTLs have limited facilitation experience, we anticipated that they may find it more challenging to maintain discussions focused on student thinking and research-based instructional practices. This study examined the following research question: How do theorized affordances and limitations of the CS and LTL PD conditions affect teachers' a) participation in professional discourse, b) perceptions of the PD's clarity and quality of the PD discussions, and c) formation of learning communities?

Methodology

Participants

Our study involved 77 middle school mathematics teachers (Grades 6–8) from rural-serving schools in a single state in the Northwestern United States. Participants were recruited through advertisements for a PD program distributed to districts and schools across the state. Teachers could enroll individually or as part of a small team from their school or district. Teachers who enrolled individually were assigned to teams from schools located in geographically proximate areas. During enrollment, participants indicated whether they were willing to serve as an LTL. After enrollment was complete, 16 teams were formed, each consisting of two to eight teachers from nearby schools. Teams were then randomly assigned to one of two conditions based on the leadership structure: a) the CS condition, in which four university-based regional mathematics specialists each led two teams, or b) the LTL condition, in which a teacher from the team received additional training from the CSs and facilitated the group. Random assignment resulted in eight CS teams (32 teachers) and eight LTL teams (37 teachers and 8 LTLs), totaling 69 teachers in non-facilitation roles.

Project Context

The PD program spanned two and a half years and was designed as a coherent sequence of interconnected learning experiences. Core components of the program included summer institutes, asynchronous learning modules, whole group and team workshops, collaborative PSCs (Borko et al., 2015), and individual classroom studies. During the summer institutes, participants from both conditions engaged in a full day of professional learning and prepared for the collaborative work they would undertake within their teams during the academic year. Asynchronous modules and synchronous workshops oriented participants to key pedagogical concepts central to the study, specifically instructional strategies that support: Explicit Attention to Concepts (EAC) and Student Opportunities to Struggle (SOS; Hiebert & Grouws, 2007). Collaborative learning was anchored in PSCs, in which teams participated in three interconnected workshops focused on collaboratively planning, implementing, and reflecting on a lesson featuring a cognitively demanding task by analyzing classroom video. PSCs (described more fully below) constituted the primary collaborative activity in the larger PD program and the focal context for the present analysis, as they were facilitated under one of the two leadership conditions. Participants also engaged in individual classroom studies, a form of action research. Teachers selected an EAC- or SOS-related instructional strategy and designed a multi-week implementation to study the effects of using the strategy in their own classrooms. In the first year of the project, LTLs received specialized leadership training, while all teachers completed the asynchronous modules and research-related pre-participation activities. In Years 2 and 3, the PD followed a consistent annual structure including a one-day summer institute, a full PSC in the

fall, and an individual classroom study in the spring. Across all years, the PD design team held synchronous whole-group online workshops once per semester.

PSCs, the PD activity central to the present analysis, were intended to support teachers in operationalizing their learning about EAC and SOS from the asynchronous modules, workshops, and summer institutes within their own classrooms. Each PSC consisted of interconnected professional learning experiences, including one whole-group workshop at the beginning of the semester, three structured team workshops, and the implementation and video recording of a classroom lesson. In the first team workshop, small groups collaboratively solved a cognitively demanding task, selected during the summer institute for its suitability across multiple grade levels, and anticipated student strategies, models, and errors. Teachers then implemented the lesson in their classrooms and recorded the instruction, submitting their videos to their team facilitator. The facilitator selected video clips to anchor subsequent discussions in workshops 2 and 3. During the second workshop, groups viewed selected clips and engaged in a facilitated discussion focused primarily on interpreting student thinking. In the third workshop, the discussion shifted towards analyzing teachers' instructional practices in relation to observations of student thinking and lesson design. Each workshop was scheduled for 90 minutes.

The PD design team provided the CSs and LTLs with structured opportunities to understand the purpose of PSCs and how to facilitate them. CSs read Borko and colleagues' book on PSC facilitation (2015) to support their professional development and used the resource to develop facilitation materials, including sample agendas and presentation slides. Throughout the project, CSs provided ongoing support to LTLs with workshop design and facilitation through meetings, planning templates, task and video-clip selection guidance, representation modeling, and individualized team-specific assistance as needed.

Data Collected

Data collected for this inquiry include video recordings of the PSC team workshops and two quantitative surveys completed by the teachers and facilitators at the culmination of their second PSC. All data were collected in Year 3 of the project, after PD activities and research protocols were refined during Year 2.

Participation in professional discourse. Sixteen teams met three times throughout the semester, resulting in 48 PSC-related team workshops. Of the 48 workshops, 43 were successfully videorecorded. To analyze opportunities for participation in professional discourse, the length of each meeting was collected. To further explore participation patterns, we coded a random sample of two PSC meeting videos from each team. This resulted in 31 videos: 16 from CS teams and 15 from LTL teams (one LTL team faced technical issues, resulting in only one of their three workshops being recorded). This selection included 10 videos from workshop 1, 11 from workshop 2, and 10 from workshop 3.

This subset of videos was coded using the Mathematics Content and Pedagogy Professional Development Observation Protocol (MaCoP P-DOP), which the researchers developed by building upon existing literature (Banilower & Shimkus, 2004; Horizon, 2005). The MaCoP P-DOP (Amador et al., 2025) was piloted in Year 2 on videos of four randomly selected PSC workshops (two per condition), achieving nearly 80% initial rater agreement, which informed clarifications and modifications. Using the MaCoP P-DOP, video recordings are broken into three-minute segments, which are coded according to who is speaking during the segment and if one or multiple teachers contribute to the dialogue. Furthermore, the overall breadth of attendee participation was coded to note the percentage of attendees who engaged in meaningful

discourse during the meeting. Four coders coded the videos, in pairwise teams, and coders met regularly to reconcile any discrepancies.

Perceptions of PD clarity and quality of PD discussions. To gain insight into participants' perceptions of PD clarity and the quality of PD discussions, we developed the PSC Survey, which contains 18 Likert-scale items. We assessed clarity by asking participants to rate their understanding of the purpose and processes of the PSC and how prepared they felt for implementing the PSC task, on a 5-point Likert scale. We addressed the quality of the discussions occurring during PSC workshops by asking participants to rate the discussions' (a) efficiency, (b) depth, (c) focus, (d) insightfulness, and (e) equitability. For example, one survey item states: "For your PSC team discussions of student thinking, please rate your agreement with the following statement: The discussions were in-depth." All five aspects of each meeting were rated on a 5-point Likert scale. Participants completed the PSC Survey following the last of their three PSC-focused workshops in Year 3.

Perceptions of PD learning communities. The Authentic Learning Community survey (ALC) contains 17 Likert-scale items modified from existing instruments (PMRR, 2021; Sigurðardóttir, 2010) to measure participants' perceptions of the learning communities created within their teams. Items ask participants to rate their (a) willingness to make their practice public, (b) willingness to share and discuss ideas in constructive dialogue, (c) felt-sense of relevancy, and (d) felt-sense of trust within the community. Participants completed the ALC survey following the last of their three PSC-focused workshops in Year 3.

Data Analysis and Findings

Participation in Professional Discourse

A two-sided Welch's t-test found that the PSC team workshops facilitated by CSs (M = 77.00, SD = 10.23) were significantly longer than those facilitated by LTLs (M = 47.95, SD = 17.26), $t(41) = 6.14$, $p < .001$. On average, CS-facilitated workshops lasted 29.05 minutes longer than LTL-facilitated workshops, representing a very large effect size (Cohen's d = 2.03, or approximately 2 standard deviations). A difference of this magnitude indicates a highly substantial difference in how the facilitators in the two conditions managed the time of these workshops, which suggests that teachers in the CS condition may have had more opportunities to engage in professional discourse.

Overall breadth of participation was coded according to the percentage of meeting attendees who contributed to the team discussions as: Most (75-100%), Several (50-74%), Some (25-49%), and Few (0-24%). We found that the majority of discussions from both conditions were coded as having participation from at least 75% of meeting attendees: 13 (81.3%) and 9 (60%) from CS and LTL conditions, respectively. Furthermore, a Chi-squared test found no significant difference in the overall breadth of participation by condition: $X^2(1, N = 31) = 1.70$, $p = .19$.

Each three-minute video segment was coded to indicate who was speaking and whether one or multiple attendees contributed to the dialogue. For a participant's contribution to be acknowledged in the coding, their participation needed to go beyond affirming or briefly responding to closed questions. Examples of coded participation include discussing a task or student work, offering a new example or counterexample, building upon the current discussion, or summarizing what another person said. The six codes used were: N = none or limited; F = facilitator only; T = one teacher; TF = one teacher and the facilitator; MT = two or more teachers (without the facilitator); and MTF = two or more teachers and facilitator. See Figure 1 below for charts displaying discourse code counts and relative frequencies by condition.

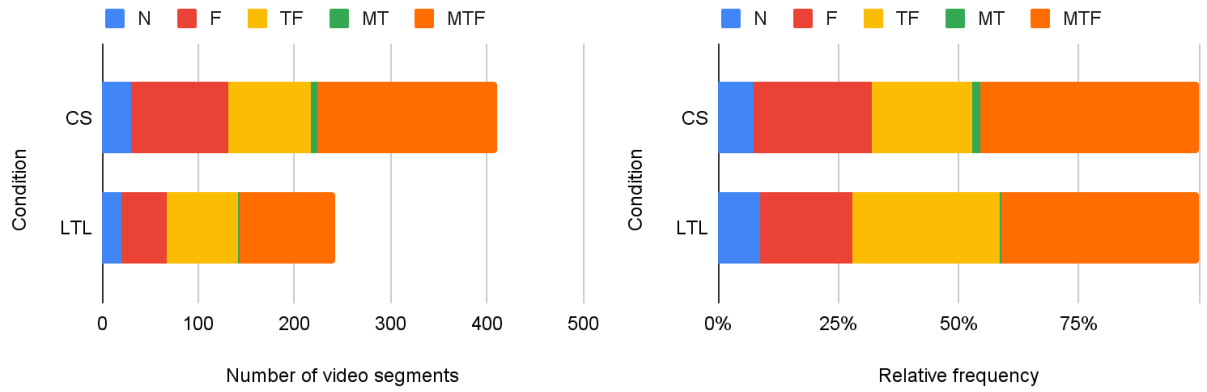


Figure 1: Left: Discourse Code Counts by Condition; Right: Relative Frequencies of Discourse Codes by Condition

Consistent with the significant differences found in meeting lengths, discourse code counts also varied notably by condition. For instance, CS-facilitated workshops included 100 segments (300 minutes) where the facilitator led the discussion alone, more than double the 47 segments (141 minutes) recorded in LTL-facilitated workshops. Similarly, CS-facilitated workshops had 186 segments (558 minutes) involving the facilitator and at least two other participants, nearly twice the 100 segments (300 minutes) found in the LTL condition. These results suggest that members of CS-facilitated teams had roughly twice as many opportunities to engage with the facilitator and participate in multi-person discussions. While these frequency differences are notable, the quality and depth of these discussions warrant further analysis. A Chi-squared test comparing the distribution of discourse codes across conditions revealed a marginally significant difference: $X^2(4, N = 446) = 10.25$, $p = .0364$.

Perceptions of PD Clarity and Quality of PD Discussions

The PSC survey was completed by 65 non-facilitating teachers (33 from CS teams, 32 from LTL teams; 94.2% response rate). Mean item responses varied from 4.22 to 4.51. When separated by condition, mean item responses for the CS and LTL teams ranged from 4.19 to 4.53 and 4.06 to 4.50, respectively. Two-sided t-tests looked for differences in responses to each Likert-scale item by condition. One significant difference was found regarding the depth of team discussions of teaching practices. CS teachers found their PSC team's discussions of teaching practices significantly more in-depth than those in LTL groups; CS group ($M = 4.41$, $SD = 0.61$) and the LTL group ($M = 4.06$, $SD = 0.74$); $t(63) = 2.16$, $p = .03$. No other significant differences were noted between groups when rating the focus, depth, insightfulness, equitability, or efficiency of PSC discussions on the math task/potential students' responses or student thinking. Moreover, no significant differences were found between groups regarding their understanding of the PSC process, purpose, or preparation.

Perceptions of PD Learning Communities

The ALC survey was completed by 63 non-facilitating teachers (31 from CS teams, 32 from LTL teams; 91.3% response rate). Mean item responses varied from 3.87 to 4.65. The highest mean score (4.65) was in response to the item, "We share examples of our math instructional practice." The lowest mean score (3.87) was in response to the item, "I feel like I can push back on an idea shared by the facilitator/team leader or another teacher." When separated by condition, mean item responses for the CS and LTL teams ranged from 3.87 to 4.68 and 3.88 to 4.66, respectively. A one-way ANOVA revealed no significant effect of condition on mean

scores across all items, $F(1, 1069) = 0.24$, $p = .624$; CS group ($M = 4.43$, $SD = 0.60$) and LTL group ($M = 4.41$, $SD = 0.66$).

Discussion

This study compared small-group PD with teachers in rural settings, facilitated by CSs and LTLs. We examined both teachers' perceptions of the PD experience and objective participation features of PD workshops. Three findings are particularly salient:

1. Teachers in both conditions reported equally high levels of connection to a collaborative community.
2. Teachers in the CS condition reported significantly more in-depth PSC discussions of instructional practices than those in the LTL condition.
3. CS-facilitated workshops were significantly longer than those facilitated by LTLs, despite similar participation patterns.

Finding #1

The absence of differences in teachers' subjective experiences of community challenges our prior assumptions about the affordances of local facilitation provided by teachers who have expertise in rural contexts and are actively teaching middle school mathematics. Despite the CSs being external experts who are no longer in the K-12 classroom, they were equally successful at creating authentic learning communities for the teachers with whom they worked. This finding suggests that skilled facilitation by CSs and/or the PSC structure may mitigate potential disadvantages inherent to the CS condition.

Finding #2

We hypothesized that teachers in CS-facilitated groups would report higher levels of clarity and discussion quality, given CS facilitators' expertise in mathematics education research and experience facilitating PD. This claim was evidenced by teachers in CS-facilitated teams reporting increased depth of discussions of instructional practices. This finding is likely due to CSs' extensive expertise in facilitating discussions of teaching practice in a variety of PD models and is a noteworthy consideration for PD designers.

However, we found similar ratings of quality markers for PD clarity and discussions of student thinking, which suggests that LTLs were also able to facilitate PD experiences that teachers perceived as clear and productive. This finding contributes to existing literature on teacher leadership and facilitation (e.g., Borko et al., 2014) by suggesting that, when provided with appropriate structures and professional learning resources, external specialists and teacher leaders can facilitate PSCs in ways that participants perceive as purposeful and productive.

Finding #3

Given CS facilitators' expertise in facilitating PD, the difference in workshop lengths across the two conditions may reflect their ability to foster sustained discussions and adherence to the intended pacing and structure of PSC workshops, which were designed to last approximately 90 minutes. Indeed, some research suggests that when teacher leaders facilitate PSC workshops, their duration can be quite short (Jarry-Shore et al., 2023). In contrast, LTLs may have adapted workshops in response to local constraints, time pressures, perceived teacher needs, or their developing understanding of PSC processes. These substantial differences in meeting length reflect well-documented tensions in facilitation research between fidelity to designed PD activities and responsive adaptation to local contexts (e.g., Borko et al., 2017; Jacobs et al., 2017). Our findings suggest that LTLs modified designed activities to be much shorter than

intended, although the specific reasons for these adaptations are beyond the scope of the present study. A further consequence of these differences in meeting length is that longer workshops may have afforded teachers in CS-facilitated groups more extensive opportunities to engage in collaborative discussions, as supported by the difference in depth of discussions of instructional practices. In this sense, longer workshops may have provided increased opportunities for teacher learning (Horn & Little, 2010). However, without direct analysis of the depth and quality of teacher discourse, meeting length alone cannot serve as a proxy for the quality of professional learning. Nevertheless, the substantially shorter workshops for LTL-led teams raise important questions about potential differences in learning opportunities across conditions. Moreover, the coexistence of mostly positive teacher perceptions with structural differences in workshop length underscores the need for PD research that integrates participant reports with fine-grained analyses of learning opportunities embedded in professional learning interactions.

Implications

Our findings have implications for PD in geographically isolated contexts, where limited access to PD and professional isolation are persistent challenges (NASEM, 2025; Şahin et al., 2024). Both PD models produced consistently positive teacher-reported experiences, suggesting that small-group collaborative PD may be effectively implemented through both locally- and centrally-facilitated models. CS-facilitated PD may offer particular advantages for scalability, access to specialized expertise, and reduced travel demands, especially in rural districts that lack the capacity to develop and sustain LTLs. At the same time, LTL-facilitated models may offer benefits in terms of sustainability and integration with local improvement efforts.

We suggest three areas for future study. First, future studies should examine how different facilitation conditions influence the substance of professional learning interactions, including the depth and alignment of discussions with the PD's intended goals. Such studies would help draw stronger links between PD design and implementation and teachers' opportunities to learn (e.g., Horn & Little, 2010). Relatedly, future research should explore whether the extended length of the CS-led PSC discussions or the adaptations observed in LTL-led groups are associated with differences in instructional improvement or student learning. Because the PSCs were a single component of a larger PD program, it remains unclear whether the length of PSC conversations impacted teachers' overall professional growth or student outcomes. Second, more work is needed to investigate the relationship between teachers' positive perceptions of PD and more objective analyses of PD interactions. For example, future studies might examine whether teachers' sense of community in LTL-led groups is driven by facilitator responsiveness to local needs (e.g., shortening workshops to honor time constraints), whereas a sense of community in CS-led groups may be fostered through more sustained, in-depth professional discussion. Examining the relationship between teachers' subjective reports and objective analyses of PD events can inform efforts to design and implement more impactful PD. Finally, the small-group PD discussions in this study were part of PSC workshops that were connected to other professional learning experiences within a larger PD program. Thus, findings regarding the two facilitation conditions may not generalize to PD that is not embedded within a broader, coherent program of professional learning. Future studies should therefore examine facilitation conditions across a wider range of PD activities and program structures.

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CO-CONSTRUCTING COACHING STANCES: DISCOURSE MOVES IN LESSON DEBRIEFS WITH TWO TEACHERS

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This study examines an elementary mathematics coach's discourse moves during one-on-one debriefs with two teachers in coaching cycles. We analyzed how directive, reflective, and facilitative stances emerged in each coach–teacher interaction. Audio recordings were examined to identify patterns in the coach's moment-to-moment discourse moves, and interviews were used to understand coach and teachers' positioning. Coaching stances varied across the two teachers and relational contexts, reflecting differences in how the coach was positioned as mentor, expert, or co-learner. Building on Saclarides et al.'s discursive move framing, we identified an emergent move acknowledging that signaled understanding without affirmation. We also distinguished between launching-general and launching-specific prompts supporting different depths of reflection. These contributions extend conceptualization of coaching discourse moves.

Keywords: Coaching, Discourse, Reflection, Positioning

Instructional coaching has continued to receive attention as a promising way for improving instructional practices of teachers and supporting students' mathematical learning (Campbell & Malkus, 2011; Russell et al., 2020). There is also growing, albeit mixed evidence of coaching effects on teaching and student learning in the mathematics education field (Russell et al., 2020). Researchers have suggested that this variability may be attributed, in part, to the difference in how coaching roles are structured, enacted and experienced across contexts (Coburn & Woulfin, 2012; Russell et al., 2020).

Researchers have emphasized that coaching effectiveness can be shaped by coaches' experiences, positions, and interactional roles, as well as teachers' positioning of coaches, prior experiences and engagement in the coaching process (Chval et al., 2010; McGatha et al., 2015; Saclarides et al., 2025). These studies suggest that coaching cannot be understood solely through fixed roles, rather coaching practices emerge through social interactions and are responsive to contextual factors. Despite this recognition, there has been limited examination of coaches' moment-to-moment discourse moves and stances.

Recent work by Saclarides et al. (2025) conceptualizes coaching stance as a discursive enactment manifest through the discourse moves coaches use to position themselves and teachers as thinking authorities during coaching conversations. Their study results identify three coaching stances: directive, reflective and facilitative. They described a directive stance as characterized by "moves in which the coach shares ideas with the teacher based on their own thinking, thus positioning the coach as the thinking authority" (p. 160), reflective by "moves in which the coach invites the teacher to share their thinking, thus positioning the teacher as the thinking authority" (p. 160) and facilitative by "moves in which the coach establishes the focus of the

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

conversation, which neither positions the coach nor the teacher as the thinking authority" (p. 160). They found that the individual coach often shifted among directive, reflective and facilitative stances, using different coaching moves accordingly. They also found that even when the coach articulated their aim to be a reflective coach, they still moved among these stances, highlighting the complex make-up of coaching practice.

While Saclarides et al. (2025) examined discourse moves and stances within a single coach-teacher dyad, our study extends this work by examining one coach working with different mathematics teachers across multiple coaching cycles. These teachers vary in grade level (K-2), years of teaching experience, the focus of the coaching support, and in how they position the coach (e.g. as expert, mentor or co-learner). This multi-teacher design captures a broader range of situational contexts and allows us to examine how a coach's discourse moves and stances vary across different teacher interactions rather than within a single teacher relational context.

Although our larger study includes six teachers, this paper presents findings from two focal teachers selected for in-depth analysis. These teachers were purposefully chosen based on how they positioned the coach, grade level taught and years of teaching experience to allow us to examine coach discourse moves in different contexts. The following research questions guide this study:

1. What discourse moves did the coach use to support teachers in reflecting on high-quality mathematics practices during one-on-one debriefs?
2. What critical elements shape how the coach enacted directive, reflective, facilitative stance and moves across one-on-one coaching debriefs with two teachers?

Theoretical Background

This study is grounded in a sociocultural perspective on learning and interaction. We particularly draw on positioning (Harre & van Langenhove, 1999) and situated learning theory (Lave & Wenger, 1991). Positioning theory does not perceive social roles as fixed, instead it posits roles as fluent "parts" that are negotiable and responsive to the nature of interaction (van Langenhove & Harre, 1999, p. 17). From this perspective, coaches and teachers position themselves and one another in roles such as learner, co-learner, expert, and mentor in response to their unfolding interactions. They also bring into the coaching experience their prior perspectives about their own and each other's positions. This lens is particularly useful for examining instructional coaching, where coaches are typically identified as experts but may seek to enact more collaborative, reflective, or facilitative stances in practice.

Situated learning theory further extends this framing by emphasizing that learning does not happen in isolation from individual or contextual needs, and social interaction (Lave & Wenger, 1991; Mangin & Dunsmore, 2013). In the context of Project AIM's (All Included in Mathematics) coaching program, this lens draws attention to how coaching interactions occur within a situated context in which teachers vary in grade level (K–2), years of teaching experience, instructional goals, and prior engagement with mathematics-focused professional learning. Together, these lenses promote attention to coaching as an interactional practice that unfolds differently across relational and situational contexts. Guided by these perspectives, the study examines a coach's discourse moves within lesson debriefs in AIM coaching cycles designed to support teachers' reflection on high-quality mathematics instruction (HQMI), and how critical interactional elements, such as positioning of coach and teacher and the use of Project AIM tools, shape these coaching discourse stances and moves.

AIM Coaching Cycle Model

The AIM coaching program has two main goals. The first goal is to enhance teachers' capacity to promote high-quality mathematics discourse. This involves supporting teachers in planning, enacting, and reflecting on lessons aimed at providing all students access and opportunities to engage in powerful mathematics and contribute to a discourse community. The second goal is to engage teachers and coaches in collaborative and self-reflective improvement of practice through a combination of individual and group coaching experiences.

In the Project AIM coaching cycle model shown in figure 1, there are two key components: individual coaching and group coaching meetings. The process begins and ends with the entire coaching group coming together, represented by the green sections (Step 1 and 6). These group meetings allow participants to learn from one another, and build a supportive community. The blue sections highlight the one-on-one coaching sessions between the AIM lead teacher and the Project AIM coach. During these individual meetings, the lead teacher has the opportunity to receive personalized support in lesson planning, and reflecting on their instruction through the examination of noteworthy moments from a video recording of the enacted lesson. The lead teacher role rotates with each cycle. Rather than asking every teacher to go through multiple coaching cycles, each teacher takes a turn as the lead teacher, typically for one or two full coaching cycles depending on the size of the coaching group (about 6 teachers). During all cycles when they are not the lead, teachers participate in the group meetings, learning from their colleagues' experiences and sharing insights. This model is intended to be a sustainable coaching arrangement that creates opportunities for both individual and collective reflections. This study focuses on the data collected in step 5 (Debrief lesson) when the coach and the lead teacher 1) reflect on video clips of noteworthy instructional moments and 2) select an instructional moment from the video they will bring to the collective lesson examination (Step 6).

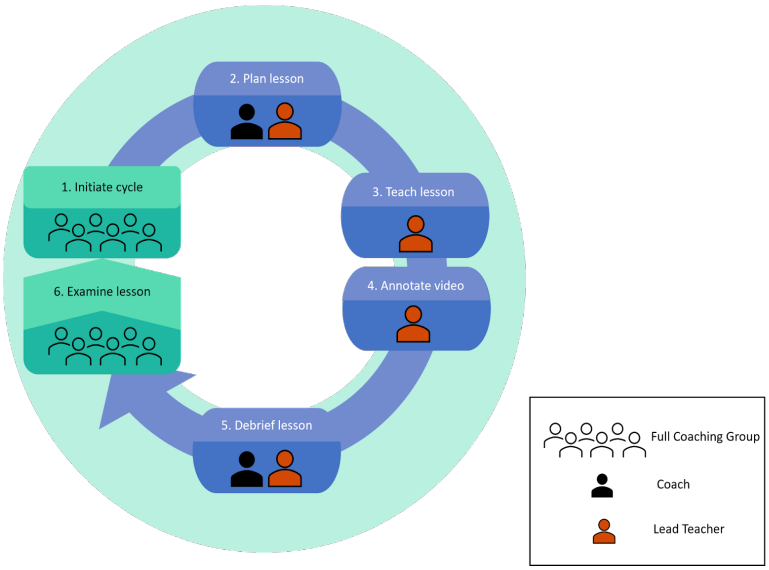


Figure 1. AIM Coaching Cycle Model

Methods

This study was conducted in a suburban elementary school in the southern United States. . This elementary school enrolls approximately 350 students and employs about 30 teachers. The school was selected to participate in Project AIM, a professional development initiative focused

on improving mathematics discourse in K–5 classrooms, due to its Title I status and its school-wide emphasis on mathematics improvement.

The coach in this study, Vanessa (a pseudonym), has over 25 years of experience teaching elementary grades (K–5) and more than six years of experience as a mathematics coach. Prior to facilitating the Project AIM coaching program during the 2024–25 academic year, Vanessa completed the Project AIM coach preparation course, an asynchronous online professional learning experience designed to prepare coaches to use AIM's coaching tools and structures. Both administrators and teachers in the school described her as having strong relationships with teachers and students. She sees herself as a reflective coach and positions herself as a co-learner and mentor within the coaching space. In the interview she stated that she believed in "letting them (teachers) know that we're all learners in this process. We don't have all the answers, but we are going to work together in order to support equitable, high-quality math discourse." The teachers in this study also viewed Vanessa as a mentor, although most did not position her as a co-learner, but instead as the mentor and expert in the coaching space.

Six K–2 teachers from the school participated in the core Project AIM professional development during the 2023–24 school year and engaged in coaching cycles during the 2024–25 school year. This study focuses on two teachers, Tara and Emily (pseudonyms), selected based on how they positioned the coach and perceived her coaching approach (directive, reflective, facilitative) in their interviews. Both Emily and Tara had established coaching relationships with Vanessa prior to the Project AIM coaching.

Emily had 6 years of experience and was teaching kindergarten at the time of the study. She was the lead teacher in the first AIM coaching cycle. She predominantly positioned the coach as a mentor, referencing these four times in her interview. Emily described Vanessa's coaching approach as reflective and facilitative and made no reference to directive coaching. The reflective positioning is evident in how Emily described the coach's interactions with her:

> She never made me feel like she was telling me the answer to anything or even suggesting anything. Rather making sure I had thought about some things that I maybe was not thinking of. So, she was perfectly supportive of that.

In the facilitative positioning, Emily gave the example of how Vanessa framed the focus of the conversation toward examining both student and teacher actions. These facilitative moves are followed by intentional reflective moves that position the teacher as the thinking authority.

Tara, a first-grade teacher with over 20 years of teaching experience was the lead teacher in the second coaching cycle. She primarily positioned the coach as an expert (mentioned 5 times) and co-learner (mentioned 3 times). She described the coach's stance as both directive and reflective, with the directive approach mentioned more frequently in her interview. For instance, she described the coach's directive approach as "you get to the point where you've got good practices. But you need someone objectively helping you tweak that." Tara also characterized Vanessa's coaching as a reflective expert:

> …When you have someone objectively looking in, she sees some pieces that I might have missed for myself, so I might have disconnected the launch to explore [phases of the lesson]. …she asked a guiding question, I'm going now, if I go back and look at it a little differently … [to the connections across phases of the lesson].

Data Collection and Analysis

This case study drew on multiple data sources to examine coaching discourse moves and shifts in these moves within lesson debriefs. Data sources included audio recorded coaching

cycle conversations, teacher and coach demographic surveys, teacher interviews, and initial and final coach interviews. All recordings were transcribed to support detailed analysis of discourse moves.

Data sources. Lesson debrief audio recordings served as the primary data sources. These recordings captured coach-teacher interactions in the moment and provided us with the opportunity to examine the coach's discourse moves and shift in stance across cycles with the teachers. The debrief data were also used to identify critical elements of interactions that shaped coaching stance and moves. Teacher interviews were conducted to understand teachers' experiences, perspectives and positioning of themselves and the coach within the coaching relationship. The interviews provided contextual information to complement the analysis of debrief meetings by offering insights into why certain discourse move patterns might emerge for individual teachers.

The coach participated in initial and final interviews to provide her perspective on the coaching process. These interviews offered insights into the coach's goals, positioning, decision-making related to discourse moves and experiences of the coaching cycle with the teachers. These interviews allowed the researchers to consider the coach's perspective alongside their outsider interpretations.

Data analysis. We engaged in a collaborative coding process (Richard & Hemphill, 2018) to analyze the lesson debrief recordings. The unit of analysis was an individual text segment (Saldana, 2025), defined as a single complete sentence representing one coach discourse move. Each segment was coded using the coaching discursive moves coding scheme (Saclarides et al., 2025, p. 160). This coding scheme includes three overarching coaching stances: directive, reflective and facilitative. Each stance then has a set of secondary-level codes to characterize the coach's particular discourse moves. Directive stance moves include *explaining*: provides an interpretation, *suggesting*: recommends an action, and *validating*: affirms the teacher's ideas or actions. Within the facilitative stance, the moves include describing: highlights noteworthy ideas or events without evaluation or interpretation, clarifying: verifies common understanding of previously discussed ideas, and framing: directs attention to the purpose of the discussion. For the reflective stance, the moves include launching: pose general prompts to invite teacher to share thinking, pressing: pose prompts inviting teachers to elaborate on thinking shared previously, and paraphrasing: restate the teacher's thinking which results in the teacher elaborating on prior statements. The moves that do not fall under these codes, coded as other moves such as "yeah, okay", procedural talk, or unfinished thoughts.

We also coded each coach discourse moves for its focus categorized as mathematics, mathematics discourse, teacher action and student action. These categories were informed by Project AIM's main reflection tool for lesson debriefs called Reflecting on Aspects of Learning Mathematics (REALM) (Sztajn et al., 2025; Yilmaz et al., 2024). REALM is a practical tool designed to support coaches and teachers in reflecting on aspects of HQMI, namely powerful mathematics and inclusive mathematics discourse. These two aspects are considered for two dimensions of teacher and student actions related to opportunity and access, and engagement and agency. For each action, REALM provides targeted reflection questions. Because AIM coaching uses the REALM tool to promote reflection, we sought to examine whether and in what ways its use connected with particular coach discourse moves.

Findings

This study examined how coach Vanessa enacted different discourse moves during debriefs to support teachers' reflection on HQMI and what interactional elements shaped the enactment

of directive, facilitative and reflective stances in meetings with the two teachers, Emily and Tara. Analysis showed that Vanessa enacted three coaching stances with both teachers. However, the distribution, patterns, and functions of the moves differed across Emily's and Tara's debriefs. We found that these differences could be tied to how each teacher positioned the coach as well as the coach's use of Project AIM tools in the coaching interactions.

Across the two debriefs, the coach used 313 moves (excluding 115 moves coded as "others"), including facilitative moves (41.4 %), followed by directive moves (40%), and reflective moves (18.6 %). Although Vanessa described her coaching stance as reflective in the interviews, analysis showed that her coaching stance emerged as dynamic and negotiated during the interaction with the two different teachers.

In Emily's debrief, Vanessa used a total of 82 discourse moves (excluding 32 coded as "other") with Emily's participation including 124 moves. Vannesa used facilitative (45.1%) and reflective (36.6%) moves predominantly. She used directive moves (18.3%) to a limited extent. (See Table 1.) This distribution indicates that Emily was mostly positioned as thinking authority to examine and reflect on her instructional practices (reflective stance moves) or invited for joint reflection and inquiry (facilitative stance moves) and the coach Vanessa rarely positioned herself as thinking authority (directive stance moves).

In Tara's debrief, Vanessa used a total of 231 discourse moves (excluding 83 moves coded as "other") while Tara's participation included 129 moves. In this case, Vannesa used directive (48.1%) and facilitative (39.8%) moves predominantly. She rarely used reflective moves (12.1%). (See Table 1.) This distribution indicates that Tara was rarely positioned as thinking authority to examine and reflect on her instructional practices while the coach was positioned as thinking authority, particularly in terms of suggesting ideas, explaining instructional moments, and validating Tara's understanding of the suggested ideas and explanations (directive stance moves).

Table 1: Distribution of Coach Discourse Moves by Coaching Stance: Emily and Tara

Coaching Stance	Coach Discourse Moves	Emily Frequency*	Tara Frequency*
Facilitative	Describing	1	22
	Clarifying	10	31
	Framing	26	39
	Total (n, %)	37, 45.1%	92, 39.8%
Directive	Explaining	4	18
	Suggesting	1	32
	Validating	10	61
	Total (n, %)	15, 18.3%	111, 48.1%
Reflective	Launching General	8	3
	Launching Specific	9	7
	Pressing	10	17
	Paraphrasing	4	1
	Total (n, %)	30, 36.6%	28, 12.1%

*"Other" discourse moves were not included in the percentage calculation.

In a facilitative move, the coach established the focus of the coaching conversation without positioning the teacher or coach thinking as authority. In this move, the coach invites joint reflection and inquiry. Seeing this move as prevalent in debriefs with both teachers aligns with

the coach's own description of her role as co-learner: "We're all learners in this process. … We are going to work together to support high quality math discourse and we all brought our mathematics knowledge to the table to have a better impact on kids." In both teachers' debriefs, Vanessa generally used the *facilitative* move of *framing* to direct attention to the purpose of the discussion for the debrief. For instance, in Tara's debrief Vanessa explained, "So the purpose of our debrief today is to reflect on the lesson that you delivered and pretty much look at the noteworthy moments in the light of REALM."

In a directive stance move, the coach shares ideas and suggestions with the teacher and positions themself as the thinking authority. As noted, directive discourse moves were the most frequently observed in debrief with Tara (n=111, 48.1%), and the least with Emily (n=15, 18.3%). In her interview, Tara described the coach playing an expert role, referring to someone with deep knowledge who could provide clear guidance and suggestions. The teacher's perception might influence the coach's approach in Tara's debrief and result in more directive coaching moves as she responded to the teacher's expectations. In Tara's debrief, the coach mainly used *directive* moves of *validating* and *suggesting*. For example, while discussing students' work on an unknown addend problem within 20, with the content goal of making connections between addition and subtraction strategies, Vanessa suggested to Tara that she extend the discussion to elicit students' reasoning, stating, "I think the extension is having that conversation with them to see, 'Tell me a little bit more about your solution methods … how did you determine 11 kids were left in the library?'"

In a reflective move, the coach positions the teacher as the thinking authority by inviting them to share and examine their own reasoning. This discourse move was observed often in the debrief with Emily (n=30, 37.0%) but relatively infrequently with Tara (n=28, 12.1%). Emily described the coach's role primarily as that of a mentor in her interview, referring to the coach as a supportive agent who did not suggest ideas but instead helped her notice and reflect. For instance, Vanessa used the *reflective* move of *pressing* to prompt Emily to examine the implications of specific instructional decisions. After discussing a lesson video moment involving an incorrect answer, Vanessa pressed Emily to reflect: "So how are mathematical mistakes used as an opportunity to rethink and examine your ideas?" In response, Emily explained how intentionally using an incorrect answer opened space for discussing multiple strategies, and validated diverse student thinking and responses. In these exchanges, reflective moves supported extended teacher talk and deeper analysis of inclusive mathematics discourse.

Vanessa also used the *reflective* move of *launching* relatively frequently with Emily. As a result of our analysis, we identified an emergent distinction within launching discourse moves: *launching general* and *launching specific*. Launching general moves are those that invite broad reflections or thinking. In contrast, launching specific moves prompt teachers to anchor their reflection in a particular instructional moment, student/teacher action, or a specific mathematical idea. Launching specific moves more frequently yielded focused and in-depth discussion. For instance, after a launching general move referencing the REALM yielded minimal uptake by Emily (e.g., a brief "Yeah" followed by a pause), Vanessa re-launched reflection with a more specific prompt: "If we were looking at the REALM, to what extent do you think each and every student had access and opportunities to engage in the powerful mathematics… versus the inclusive math discourse community?" In response, Emily engaged in more in-depth reflection and articulated how instructional choices shaped student access to and participation in classroom discourse. This contrast suggests that while *launching general* moves positioned the teacher as the thinking authority, *launching specific* moves further provided a concrete analytic frame that

supported deeper teacher reflection on HQMI. This distinction is analytically important since it captures meaningful variation within reflective moves of launching that was not differentiated in the original coding scheme. This distinction helped us notice nuances in coaching reflective moves that appear to influence the depth and richness of coaching interactions.

During analysis, we identified an additional discourse move of *acknowledging* used in both teachers' debriefs. This emergent move occurs when the coach communicates an understanding of the teacher's thinking without affirming or evaluating the idea. Although similar to validating moves, this move functions differently by signaling comprehension (e.g. Ah, got it.) rather than endorsement or affirmation of teacher ideas or action. Interactionally, this move often serves as a pivot, allowing the coach to redirect the conversation or press for further reflection without positioning the teacher's idea as correct or preferred.

Our analysis of the debriefs also revealed a pattern in how the coach used AIM tools in relation to different discourse moves. For instance, although the coach intentionally incorporated the REALM tool, which is designed to support reflection in coaching conversations, its use in discourse moves varied across the debriefs. In Emily's debrief, REALM was used in 13 discourse moves, and 10 of those moves were reflective, including five launching-specific moves, three pressing moves, one launching-general move, and one paraphrasing move. This pattern aligns with how Emily positioned the coach as a mentor and with her expectation that the coach would support her reflection. In contrast, during Tara's debrief, the coach used REALM in 11 discourse moves, but only 4 were reflective (5 were facilitative-framing, 2 were directive-explaining). This distribution aligns with Tara's overall debrief, which was dominated by directive and facilitative moves, and with her positioning of the coach as an expert whose role was to provide guidance and validation. This pattern suggests that the REALM tool did not, on its own, always promote reflective moves. Instead, how the tool was used with different moves depended on how the debrief unfolded and aligned with how the teacher positioned the coach in debrief interactions.

Discussion

Saclarides et al.'s (2025) framework for coaching stances and moves aided our interpretation of the dynamic nature of coaching interactions, as well as influences on these interactions. We investigated the work of a coach who explicitly described her intention to take on a reflective stance and act as a learning partner with teachers. We found, though, that her coaching in practice consisted of a negotiated mix of stances and discourse moves. Teachers' positioning of the coach and their expectations of coaching interactions, strongly shaped coaching practice.

Emily's positioning of the coach as a supportive mentor enabled more reflective and facilitative moves. Conversely, Tara's positioning of the coach as expert appeared to invite more directive moves. Teachers acted not as recipients of coaching determined by the coach's preferred stance, but as co-constructors of coaching practice. The affordances and purposes of tools used during coaching also played a part in shaping interactions. Our findings suggest that coaching tools are neither neutral nor fixed in supporting a particular coaching stance or set of moves. Rather, they also play a role in the co-constructed dynamics of coaching, at times prompting directive moves of explanation and suggestions, and at others inviting reflective moves, (e.g., pressing), or facilitative moves, (e.g., clarifying). Coaching practice emerged through the interaction among the coach, the teacher, and the tools in use, rather than adhering strictly to the coach's intention or a tool's design.

Our investigation further identified a distinction (Launching general vs. Launching specific) within one move from Saclarides et al.'s (2025) framework, and one additional move

(Acknowledging). Like the moves identified a priori in the framework, the prevalence and influence of these moves varied between the two lesson debrief sessions we studied, suggesting they should be considered in further study of the dynamics of, and influences on, mathematics coaching practice.

Acknowledgments

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EXAMINATION OF CONVERGENT AND DISCRIMINANT VALIDITY EVIDENCE OF TWO MATHEMATICS INSTRUCTIONAL BELIEFS INSTRUMENTS

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This study examines convergent and discriminant validity evidence of two mathematics instructional belief instruments: Lesson Vignettes and the Priorities for Mathematics Instruction (PMI) survey. Both instruments address Explicit Attention to Concepts and Student Opportunity to Struggle but operationalize the constructs differently. While the Lesson Vignettes focus on judgments about teaching in context, the PMI focuses on instructional priorities under common constraints. Instrument content and correlation analyses of 88 middle school teachers' responses revealed moderate correspondence but distinct operationalizations, suggesting these instruments capture different aspects of teacher beliefs–how teachers reason about instruction in practice and how they prioritize instructional approaches when faced with real choices–guiding researchers in their interpretation and use of belief measures in mathematics education.

Keywords: Teacher Beliefs, Measurement

Purpose

Mathematics education researchers have increasingly emphasized gathering and sharing validity evidence for research instruments (Bostic, 2019; Bostic et al., 2021; Carney et al., 2022), yet convergent and discriminant validity evidence remains rare for teacher surveys (Bostic et al., 2025). Establishing such validity evidence is especially important for belief measures, as researchers face challenges in making inferences about beliefs, since these are often tacit, context-dependent, and shaped by multiple influences (Leatham, 2006; Wilson & Cooney, 2002).

This study examines the convergent and discriminant validity of two mathematics instructional beliefs instruments: vignettes describing instruction (herein referred to as Lesson Vignettes; Stein et al., 2017), and the Priorities for Mathematics Instruction survey (PMI; Carney et al., 2021). Both instruments draw explicitly on the Hiebert and Grouws (2007) framework that focuses on two instructional practice constructs that support students' conceptual understanding: Explicit Attention to Concepts and Student Opportunities to Struggle. However, the instruments operationalize the constructs differently. The Lesson Vignettes capture teachers' instructional judgment—how they interpret and evaluate instructional moves and student thinking within specific classroom situations. The PMI captures teachers' instructional prioritizations—how they weigh and select among competing instructional priorities under realistic classroom constraints.

Because we know these instruments are highly related, we are interested in understanding the points of convergence and divergence in teachers' beliefs and how these patterns operate in relation to instructional judgments in context (Lesson Vignettes) versus instructional prioritization under constraints (PMI). Examining convergence and divergence between these two instruments can inform whether they measure overlapping or complementary aspects of teachers' beliefs, providing insight into their appropriate interpretations and uses in research. We

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answer the following research question: What convergence and divergence evidence exists between how these two instruments frame teacher beliefs?

Theoretical Framework and Related Literature

Hiebert and Grouws (2007) identified two constructs underlying instructional practices supporting students' conceptual understanding, with research evidence indicating positive effects across study design, teaching formats, and contexts. The first construct, Explicit Attention to Concepts, involves teachers and students explicitly discussing mathematical concepts and making connections among concepts, facts, and procedures, through activities such as questioning, discussing, comparing, and relating. The second construct, Student Opportunities to Struggle, involves students engaging in productive struggle with important mathematical ideas through sense-making around comprehensible problems.

These constructs represent pedagogical approaches that may be valued differentially by teachers. Explicit Attention to Concepts can be seen as a more externally-mediated approach in which the teacher ensures concepts and connections are made public and clear to students. In contrast, Student Opportunities to Struggle strategies are more concerned with experiences that engage students in developing understanding through their own sense-making. However, studies of mathematics instruction indicate many teachers focus on mastering procedural skills or limiting cognitive demand–practices at odds with Explicit Attention to Concepts and Student Opportunities to Struggle (Hill et al., 2018; Lindvall, 2025; Stein et al., 2017). Survey instruments that capture these competing instructional choices can help researchers better understand the instructional tradeoffs teachers navigate and how shifts in instructional beliefs may be facilitated.

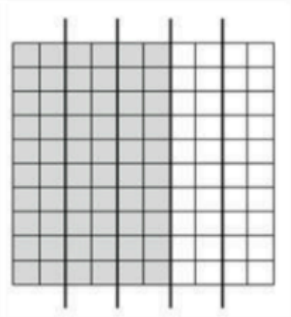
Understanding what drives instructional choices requires examining teachers' underlying beliefs. Wilson and Cooney (2002) suggest attending to what teachers value and how they assign importance to different aspects of teaching mathematics. Given that different survey instruments may operationalize and elicit beliefs about practices like Explicit Attention to Concepts and Student Opportunities to Struggle in distinct ways, comparing teachers' responses across surveys can reveal nuances in how teachers conceptualize and prioritize these instructional approaches.

Belief Measures: Explicit Attention to Concepts and Student Opportunities to Struggle

Lesson Vignettes and the PMI were designed to measure beliefs related to Explicit Attention to Concepts and Student Opportunities to Struggle instructional practices but employ different measurement approaches.

Lesson vignettes. Stein et al. (2017) posit a framework for examining interactions between high and low levels of Explicit Attention to Concepts and Student Opportunities to Struggle in context-specific classroom situations. Using the two dimensions of Explicit Attention to Concepts (EAC) and Student Opportunities to Struggle (SOS), they produced four teaching profiles, which they conceptualize as quadrants (Quadrant 1 [Q1]: high EAC/high SOS; Quadrant 2 [Q2]: high EAC/low SOS; Quadrant 3 [Q3]: low EAC/high SOS; Quadrant 4 [Q4]: low EAC/low SOS). The Lesson Vignettes comprise four lesson descriptions, each taught by a different teacher who exemplifies one of the teaching profile quadrants. See Figure 1 for a sample vignette of a lesson taught by Ms. Evans, a Quadrant 2 teacher.

Ms. Evans is finishing a unit on rational numbers (fractions, decimals, and percents). Having covered each individually, her goal for today's lesson is to help students see the relationship among these different ways of representing rational numbers. By using the same 100-grid to represent the fraction 3/5 and its decimal and percent equivalents, she wanted her students to recognize that 3/5, .6, and 60% are equivalent because they cover the same amount of area.



After passing out the task, Ms. Evans identifies the goal of the lesson and reminds students that diagrams can be a useful tool for reasoning about fractions, percents and decimals. Most students begin by dividing the 100-grid into fifths as shown here and shading three out of the five segments. When most have done so, Ms. Evans reinforces that the amount shaded is the fraction 3/5, which means 3 parts out of 5 parts, and asks if there is an equivalent fraction that could also describe the amount that is shaded. After a student volunteers that you could also write the fraction as 60/100, Ms. Evans asks the class to finish the remainder of the task. For the decimal equivalent, most students apply their knowledge that a decimal is "an amount out of 100" and identify .60 as the decimal that represents the amount that is shaded. When some students hurriedly finish by moving the decimal two places to the right and adding a "%" sign, Ms. Evans presses them to apply their knowledge of what percent means to the grid and to justify why 60% is the correct answer. She concludes the lesson with an explanation of the meaning of equivalence.

Figure 1: Vignette for Quadrant 2 Teaching by Stein et al. is licensed under CC BY 3.0 (Stein et al., 2017, Appendix A)

Items ask teachers to rate their level of agreement with statements regarding the effectiveness of the instruction depicted in each vignette on a 5-point Likert scale (e.g., “Most students will leave Ms. Evans’ class understanding the concept of equivalence”). Collectively, the measure produces four quantitative scores, one for each of the four teaching profiles, each interpreted as the extent to which a teacher’s beliefs correspond to instruction represented by the associated quadrant. The higher the quadrant score, the higher the agreement between the teacher’s beliefs and the levels of Explicit Attention to Concepts and Student Opportunities to Struggle depicted by that quadrant’s teaching profile.

Priorities for Mathematics Instruction (PMI). Carney et al. (2021) conceptualized mathematics instruction along continua reflecting competing instructional priorities. These range from practices emphasizing Explicit Attention to Concepts (EAC) and Student Opportunities to Struggle (SOS) to contrasting approaches termed Singular Methodological Focus (SMF) and Highly Scaffolded Content (HSC), respectively, resulting in the EAC–SMF and SOS–HSC belief scales. Singular Methodological Focus prioritizes focused attention on a single method or procedure to support clarity and accuracy, while Highly Scaffolded Content emphasizes explicit guidance and a gradual progression in task complexity. Importantly, we position both ends of each continuum as legitimate instructional choices shaped by classroom constraints (e.g., time, student readiness), and the PMI is designed to capture how teachers prioritize among these competing approaches.

The PMI comprises 15 items, eight related to practices on the EAC–SMF scale and seven on the SOS–HSC scale. For each item, a teacher is provided with a prompt of a belief statement (e.g., Students are more likely to succeed in math when they can…). Teachers complete the statement by selecting a position on a 6-point forced-choice slider between opposing instructional priorities, indicating both direction and intensity (see Figure 2 for a sample survey

item). Responses yield two quantitative scores, PMI EAC and PMI SOS, representing teachers' positions along the EAC–SMF and SOS–HSC continua, respectively. Importantly, scale direction does not imply that one instructional priority is inherently more desirable; instead, scores indicate the degree to which teachers' beliefs align with one emphasis relative to its contrasting counterpart.

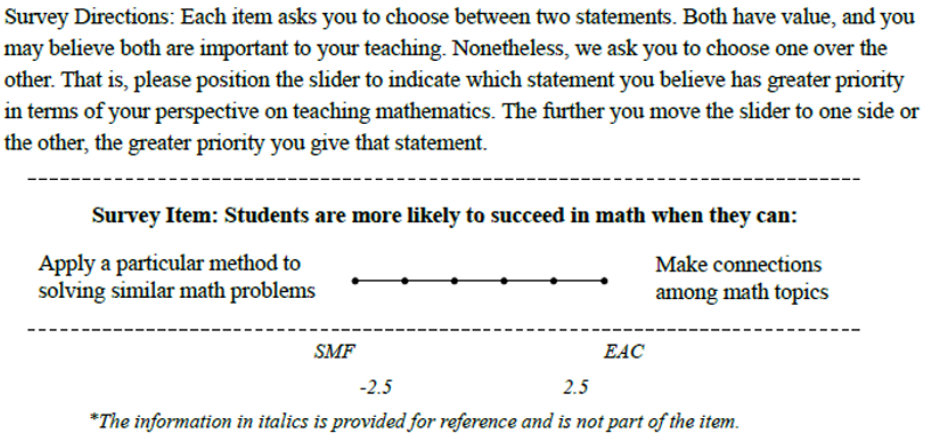


Figure 2: Example of Directions and an Item for the PMI EAC Beliefs Scale (Reprinted, with Permission, from Carney et al., 2021)

Method

Data

This study employed two distinct data sources corresponding to the qualitative and quantitative analyses. For the qualitative analysis, we examined the instruments themselves, the Lesson Vignettes and PMI survey items, to understand how each instrument operationalizes beliefs about Explicit Attention to Concepts and Student Opportunities to Struggle instructional practices. For the quantitative analysis, we used teachers' scores generated from their responses to both instruments. These scores came from 88 grades 6–8 mathematics teachers in rural-serving schools in the northwest United States. The surveys were administered at the same time at the beginning of a large research project studying professional development delivery models.

Analysis

For qualitative analysis, we used an approach of mapping the content of the Lesson Vignettes onto each item in the PMI survey (Corbin & Strauss, 2014). One researcher read each vignette twice, then determined which half of each PMI item—the EAC- or SMF-aligned response option—was better represented by the vignette content. For example, for the item shown in Figure 2, asking whether students should "apply a particular method to solving similar math problems" (SMF) or "make connections among math topics" (EAC), the Q2 Vignette was coded as EAC-aligned because the vignette described a teacher whose "goal for today's lesson is to help students see the relationship among these different ways of representing rational numbers." When vignette content did not clearly favor one response option over the other, or when neither option was represented, the item was coded as "no match or unclear." A second researcher reviewed all mappings and confirmed the results.

For quantitative evidence, we performed descriptive summaries of the response distributions of all items in the instruments, and we calculated Pearson product-moment correlations between

all instrument variables (i.e., vignette quadrant scores and PMI EAC and SOS scores) to examine convergent and discriminant validity (AERA, APA, NCME, 2014). We report the associated matrix of observed intercorrelations and compare the values to theoretically-based expectations.

Results

Content Analysis

Convergence. Table 1 shows the number of PMI items that matched instructional practices described in the Lesson Vignettes. Analysis revealed moderate correspondence between instruments. Three of four Lesson Vignettes showed clear alignment with theoretically expected PMI constructs: Q1 Vignette (High EAC/High SOS) matched 5 (63%) EAC and 5 (71%) SOS response options; Q2 Vignette (High EAC/Low SOS) matched 7 (88%) EAC response options; and Q4 Vignette (Low EAC/Low SOS) matched 7 SMF and 3 HSC response options.

Table 1: Number of PMI Items Matching Practices Described in Lesson Vignettes

PMI Construct	Lesson Vignettes (Stein et al., 2017)			
	Q1 Vignette: High EAC, High SOS	Q2 Vignette: High EAC, Low SOS	Q3 Vignette: Low EAC, High SOS	Q4 Vignette: Low EAC, Low SOS
EAC (8)	5 (63%)	7 (88%)	5 (63%)	0 (0%)
SMF (8)	2 (25%)	0 (0%)	1 (13%)	7 (88%)
SOS (7)	5 (71%)	3 (43%)	2 (29%)	0 (0%)
HSC (7)	0 (0%)	0 (0%)	0 (0%)	3 (43%)
No match or unclear	3	5	7	5

Divergence. However, substantial divergence is evident in three areas. First, Q3 Vignette (Low EAC/High SOS) showed unexpected alignment patterns: (a) matching 5 (63%) EAC response options despite representing low EAC, (b) matching only 2 (29%) SOS response options despite representing high SOS, and (c) with 7 items rated as no match or unclear. Second, HSC items showed minimal alignment across Lesson Vignettes, matching 3 (43%) response options only in Q4 Vignette and 0 (0%) in Q2 Vignette despite these vignettes representing low SOS. Third, between 3 and 7 PMI items per vignette could not be clearly mapped to vignette content.

Correlation Analysis

Correlation analysis examined the strength and direction of relationships between teachers' Lesson Vignette quadrant scores and PMI scale scores (see Table 2). Evidence of convergent validity (overlapping measurement) would include positive correlations between PMI scales and theoretically aligned vignette quadrants—specifically, PMI EAC with Q1 and Q2 (high EAC quadrants), and PMI SOS with Q1 and Q3 (high SOS quadrants)—as well as negative correlations with opposing quadrants. Evidence of discriminant validity (unique measurement) would include correlations that indicate relatedness without redundancy, unexpected patterns, and distinct relationships between the Lesson Vignettes and the PMI scales.

Table 2: Intercorrelations Among Mathematics Teaching Belief Measures

Variable	1	2	3	4	5	6
1. Q1 Score	—					
High EAC/High SOS						
2. Q2 Score	0.51	—				
High EAC/Low SOS						
3. Q3 Score	0.14	0.27	—			
Low EAC/High SOS						
4. Q4 Score	–0.22	0.11	–0.21	—		
Low EAC/Low SOS						
5. PMI EAC	0.30	0.05	0.25	–0.35	—	
6. PMI SOS	0.39	0.11	0.20	–0.46	0.54	—

Convergence. Results provided moderate evidence that the instruments measure overlapping aspects of beliefs about Explicit Attention to Concepts and Student Opportunities to Struggle. As expected, both PMI scales correlated negatively with Q4 (low EAC/low SOS): PMI EAC (r = –.35) and PMI SOS (r = –.46). PMI SOS showed expected positive correlations with high-SOS quadrants: Q1 (r = .39) and Q3 (r = .20). PMI EAC showed expected positive correlation with high-EAC quadrant Q1 (r = .30).

Divergence. In contrast, several patterns suggested that the instruments measure distinct aspects of teacher beliefs. Q2 Vignette (high EAC) showed only a weak positive correlation with PMI EAC (r = .05), while Q3 Vignette (low EAC) showed a larger positive correlation (r = .25), indicating PMI EAC response options do not strongly correlate with characterizations of high and low EAC in the vignettes. Additionally, the Q3 Vignette (high SOS) correlation with PMI SOS (r = .20), though positive and in the expected direction, was relatively weak.

Summary

Overall, content and correlation analyses revealed that the Lesson Vignettes and PMI measure overlapping yet distinct aspects of Explicit Attention to Concepts and Student Opportunities to Struggle beliefs. The clearest convergence appeared in Q1 and Q4 Vignettes, which showed expected content alignment with PMI constructs and moderate correlations in theoretically predicted directions. Some divergence involved aspects of Q2 and Q3 Vignettes. Q2 Vignette (high EAC/low SOS) showed strong EAC content alignment but weak correlation with the PMI EAC scale, suggesting the vignette depicts high EAC in ways that differ from the PMI. Q3 Vignette (low EAC/high SOS) showed unexpectedly high EAC content alignment, weak SOS content alignment, and an unexpected positive correlation with PMI EAC—patterns that were inconsistent with theoretical predictions. Together, these findings indicate the instruments operationalize Explicit Attention to Concepts and Student Opportunities to Struggle differently, capturing related but distinct aspects of teacher beliefs.

Discussion

This study examined whether two instruments measuring beliefs about Explicit Attention to Concepts and Student Opportunities to Struggle capture overlapping or unique aspects of these constructs. Content and correlation analyses revealed moderate convergence alongside meaningful divergence, indicating the instruments are related but operationalize Explicit Attention to Concepts and Student Opportunities to Struggle in distinct ways.

Integration of Results

Student Opportunities to Struggle showed the clearest divergence in Q3 Vignette, which had weak SOS content alignment despite representing high SOS, and weak correlations with PMI SOS. Explicit Attention to Concepts showed two divergent patterns: Q2 Vignette had strong EAC content alignment but weak correlations with PMI EAC, and Q3 Vignette showed unexpectedly high EAC content alignment alongside unexpected positive correlations with PMI EAC despite representing low EAC.

A key structural difference between the instruments helps explain these patterns. The Lesson Vignettes define low EAC and low SOS structurally through the quadrant design, but do not explicitly name or frame the instructional philosophy underlying those lower-scoring profiles. The Q4 Vignette is characterized by the absence of high EAC and high SOS practices rather than by the presence of a distinct competing orientation. The PMI, by contrast, explicitly defines what low EAC and low SOS represent by positioning Singular Methodological Focus and Highly Scaffolded Content as instructional orientations at the opposing ends of each continuum. A teacher who scores low on PMI EAC is prioritizing a coherent alternative approach commonly seen in instruction, centered on procedural clarity and methodological focus. Similarly, a teacher who scores low on PMI SOS is prioritizing explicit guidance and scaffolded progression rather than productive struggle. This asymmetry in operationalization may help explain why Highly Scaffolded Content items were largely absent from vignette content even in low-SOS vignettes. Rather than reflecting a limitation of either instrument, these patterns are consistent with a known structural difference in how the two instruments conceptualize the belief space: the Lesson Vignettes frame beliefs as judgments about the effectiveness of Explicit Attention to Concepts and Student Opportunities to Struggle practices, while the PMI frames them as choices among competing priorities.

The Multifaceted Nature of Beliefs

These content and correlation analyses underscore the multifaceted nature of teacher beliefs (Leatham, 2006; Philipp, 2007). The fact that teachers responded differently to the two instruments—despite both claiming to measure Explicit Attention to Concepts and Student Opportunities to Struggle—demonstrates that beliefs about these constructs cannot be fully captured by a single measurement approach. Teachers' responses differ when they evaluate contextualized lessons versus when they articulate general instructional priorities, as evidenced by the moderate but not strong correlations between instruments.

The unexpected correlation patterns reveal complexity in how teachers engage with these theoretical constructs. These patterns could stem from teachers holding context-dependent values about these practices, conceptualizing Explicit Attention to Concepts and Student Opportunities to Struggle differently than the theoretical framework suggests, or both (Leatham, 2006; Maass, 2011; Wilson & Cooney, 2002). For instance, teachers may value conceptual attention differently when evaluating a brief lesson scenario versus prioritizing it as a general instructional principle, or they may recognize Explicit Attention to Concepts in instructional contexts that extend beyond the high/low structure used to design the vignettes. Regardless of the source of

these differences, the findings demonstrate that quantification of beliefs about instructional approaches is influenced by measurement context and cannot be reduced to a simple position on theoretical dimensions. This complexity aligns with research showing beliefs are context-dependent and shaped by multiple influences (Leatham, 2006; Wilson & Cooney, 2002).

Implications for Researchers

The differential operationalization of Explicit Attention to Concepts and Student Opportunities to Struggle constructs across these instruments has important implications for researchers. First, instrument selection and comparison require careful consideration. Researchers should consider which operationalization aligns with their theoretical framework and research questions. The Lesson Vignettes may be more appropriate when researchers want to understand teachers' judgments about instructional effectiveness in specific lesson contexts, while the PMI may better capture teachers' general instructional priorities and values. Given these operational differences, findings from one instrument should not be assumed to generalize to the other, and cross-study comparisons should account for the different dimensions of beliefs captured by each instrument.

Second, score interpretation requires attention to operational context. Scores should be interpreted within the framework of how each instrument operationalizes Explicit Attention to Concepts and Student Opportunities to Struggle. A teacher who scores high on PMI SOS may not necessarily rate high-SOS vignettes as most effective, as these tap different facets of beliefs about student struggle and scaffolding. Researchers should therefore resist interpreting scores as interchangeable indicators of the same underlying belief and instead attend to what each instrument's operationalization actually captures.

Third, building a validity argument for any instrument requires accumulating multiple pieces of evidence over time—content analyses, factor structures, relationships with external variables—yet individual studies contributing this evidence are rarely published. Recent calls in the field have emphasized that this kind of iterative evidence is precisely what is needed, as it is the only way a validity argument gets built (Bharaj et al., 2025; Bostic, 2019; Bostic et al., 2021; Carney et al., 2022). Without it, researchers draw on instruments and often make interpretive claims that outpace the evidence supporting them. This study represents one such contribution to that cumulative effort, examining convergent and discriminant validity evidence for two instruments already in active use in the field. We encourage researchers to similarly document and share validity evidence at each stage of instrument development and use, so that the field can ensure that interpretations drawn from belief measures are appropriate and well-supported.

Conclusion

In conclusion, Lesson Vignettes and PMI measure potentially complementary rather than redundant aspects of teachers' beliefs about Explicit Attention to Concepts and Student Opportunities to Struggle. Both instruments provide valid but distinct information, with the Lesson Vignettes capturing beliefs through contextualized effectiveness judgments and the PMI through abstract priority statements. Rather than viewing one instrument as "better," researchers should recognize that these different operationalizations serve different purposes and capture different facets of the multifaceted construct of teacher beliefs.

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MULTIDIMENSIONAL PATTERNS OF TEACHER CHANGE IN MATHEMATICS PROFESSIONAL DEVELOPMENT

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This brief report examines patterns of teacher change during Year 1 of a statewide mathematics professional development initiative (n = 140). Mathematical Knowledge for Teaching (MKT) in fractions (Grades 3–5) and ratio and proportional reasoning (Grades 6–8) did not demonstrate statistically significant change across the year, as did collective efficacy. In contrast, mathematics teaching self-efficacy and mathematics teacher identity increased. Both grade bands performed near the national mean on the MKT assessment, although Grades 6–8 teachers consistently scored higher than Grades 3–5 teachers. The findings suggest that psychological dimensions of mathematics teaching competence and measurable changes in specialized knowledge may follow different patterns during the first year of sustained professional development, reflecting a layered process of teacher change.

Keywords: Professional Development, Mathematical Knowledge for Teaching, Teacher Beliefs, Identity and Agency

Teaching mathematics for conceptual understanding requires specialized mathematical knowledge for teaching, including the ability to represent ideas, interpret student thinking, and make responsive instructional decisions (Ball et al., 2008; Kleickmann et al., 2015). However, knowledge alone may be insufficient to support conceptual instruction. Research with elementary and prospective elementary teachers points to math anxiety and math teaching anxiety (Szczygieł et al., 2025), uneven conceptual understanding of rational number ideas (Depaepe et al., 2018), and variation in knowledge and beliefs tied to teachers' mathematical backgrounds (Holm & Kajander, 2020).

Current models of teacher competence describe professional expertise as multidimensional, integrating knowledge with affective and motivational characteristics (Blömeke & Delaney, 2012; Copur-Gencturk & Tolar, 2022). The present study is grounded in this multidimensional perspective, which conceptualizes teacher development as involving related but distinct dimensions of knowledge, beliefs, motivation, and professional identity. Within this framework, mathematical knowledge for teaching is related to, yet empirically distinct from, constructs such as mathematics teaching self-efficacy (Bandura, 1997; Wilhelm & Berebitsky, 2019), mathematics teacher identity (Cribbs et al., 2015), and other teacher beliefs (Fukaya et al., 2025). Longitudinal studies suggest that these dimensions do not necessarily develop at the same pace (Copur-Gencturk & Li, 2023).

Research on effective professional learning consistently emphasizes content focus, active learning, collaboration, modeling of practice, expert support, and sustained duration (Darling-Hammond et al., 2017). These features are intended to build teachers' knowledge and instructional confidence and, over time, may also contribute to collective efficacy within schools (Bandura, 1993, 1997; Goddard, 2002).

Taken together, the literature indicates that educating for change in mathematics instruction requires attention to knowledge and to the psychological and collective conditions that support

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

effective classroom practice. What remains less clear is how these dimensions shift during sustained professional learning. To examine how multidimensional change unfolds in practice, this brief research report focuses on Year 1 of a statewide mathematics professional development initiative and addresses two questions: (1) To what extent do teachers' Mathematical Knowledge for Teaching (MKT), mathematics teaching self-efficacy (MTSES), mathematics teacher identity (MTI), and collective efficacy (CE) change during the first year of PD? (2) Do grade bands differ in overall levels and growth from Time 1 to Time 2 across these measures?

Methods

Participants and Context

Participants were 170 educators involved in a statewide numeracy-focused professional development initiative. The broader sample included classroom teachers, instructional coaches, administrators, and district leaders; analyses for the present study focused on 140 classroom teachers participating in Grades 3–5 and Grades 6–8 strands. Because the study examined longitudinal change from Baseline (Time 1) to End-of-Year-One (Time 2), matched sample sizes varied by measure and are reported in the Results section. Teachers had an average of 10.94 years of experience (SD = 7.13) and were predominantly White (96.2%) and female (86.7%).

The PD model is a two-year, hybrid initiative designed to strengthen mathematical knowledge and instructional practice. Year 1 organized teachers into grade-band strands (Grades 3–5: fractions; Grades 6–8: ratio and proportional reasoning) and included sustained content-focused sessions, collaborative communities of practice, and coaching cycles.

Measures

Mathematics Teacher Identity (MTI; Cribbs & Utley, 2024). The 13-item, 5-point Likert instrument assesses teachers' perceptions of competence, recognition, and alignment with the role of mathematics teacher. Composite mean scores were calculated, with higher values indicating stronger mathematics teacher identity ($\alpha = 0.91$).

Mathematics Teaching Self-efficacy (MTSES; Wilhelm & Berebitsky, 2019). The 12-item, 5-point Likert scale assesses teachers' perceived capability to support student learning in mathematics. Composite mean scores were calculated, with higher values indicating stronger perceived instructional capability ($\alpha = 0.88$).

Collective Efficacy (CE; Goddard, 2002). The 12-item scale was adapted to reflect state-level beliefs within the PD context and administered on a 5-point Likert scale. Composite mean scores were calculated, with higher values indicating stronger collective efficacy beliefs ($\alpha = 0.76$).

Mathematical Knowledge for Teaching (MKT; The Regents of the University of Michigan, 2007**).** The assessment is grounded in the MKT framework distinguishing common content knowledge from knowledge specialized for teaching (Ball et al., 2008). The instrument was electronically administered, and scaled scores were provided by the publisher; item-level data were not available.

Analyses

A series of mixed-design analyses of variance (ANOVAs; Field, 2018) were conducted to examine changes over time (Baseline to Time 2) and differences by grade band for MKT, MTSES, MTI, and CE. Time served as the within-subjects factor and grade band as the between-subjects factor. Significance was evaluated at $\alpha = .05$. Effect sizes (η^2) are reported where available.

Results

Table 1: MKT Descriptive Statistics from Time 1 to Time 2

Grade Band	*n*	Time 1 Mean θ (SE)	Time 2 Mean θ (SE)
3–5	81	–0.36 (0.07)	–0.22 (0.08)
6–8	39	0.37 (0.11)	0.34 (0.12)
Total	120	–0.01 (0.06)	0.06 (0.07)

A mixed-design ANOVA revealed no significant main effect of Time, $F(1,118) = 0.68$, $p = .41$. The Time × Grade Band interaction was not significant, $F(1,118) = 1.75$, $p = .19$. A significant main effect of Grade Band was observed, $F(1,118) = 29.39$, $p < .001$, $\eta^2 = .20$.

Table 2: Descriptive Statistics from Time 1 to Time 2 for Psychological Variables

Grade Band	*n*	MTSES Time 1 Mean (SE)	MTSES Time 2 Mean (SE)	*n*	CE Time 1 Mean (SE)	CE Time 2 Mean (SE)	*n*	MTI Time 1 Mean	MTI Time 2 Mean
3–5	68	4.01 (0.05)	4.26 (0.05)	73	3.53 (0.05)	3.61 (0.05)	76	4.11 (0.05)	4.28 (0.05)
6–8	46	3.99 (0.07)	4.02 (0.07)	38	3.54 (0.07)	3.49 (0.07)	38	4.31 (0.06)	4.39 (0.06)
Total	114	4.00 (0.04)	4.14 (0.04)	111	3.54 (0.04)	3.57 (0.04)	114	4.18 (0.04)	4.31 (0.04)

Mathematics Teaching Self-Efficacy (MTSES). A significant main effect of Time was observed, $F(1,112) = 11.12$, $p = .001$, indicating an overall increase in self-efficacy from Time 1 to Time 2. A significant Time × Grade Band interaction also emerged, $F(1,112) = 6.56$, $p = .012$. Grades 3–5 teachers demonstrated a notable increase in self-efficacy (4.01 to 4.26), despite beginning the year with relatively high scores, whereas Grades 6–8 teachers remained relatively stable (3.99 to 4.02). No significant overall group difference was found, $F(1,112) = 3.21$, $p = .076$.

Collective Efficacy (CE). No significant main effect of Time was found, $F(1,109) = 0.09$, $p = .77$, indicating stability across the year. The Time × Grade Band interaction was not significant, $F(1,109) = 1.93$, $p = .17$, nor was the main effect of Grade Band, $F(1,109) = 0.46$, $p = .50$.

Mathematics Teacher Identity (MTI). A significant main effect of Time was observed, $F(1,112) = 13.96$, $p = .001$, indicating increased identification as mathematics teachers across the year. The Time × Grade Band interaction was not significant, $F(1,112) = 1.50$, $p = .223$. A small but significant main effect of Grade Band emerged, $F(1,112) = 4.33$, $p = .040$, $\eta^2 = .037$, with Grades 6–8 teachers reporting slightly higher identity overall.

Discussions and Conclusions

Year 1 results indicate that teacher development during sustained mathematics professional learning does not occur uniformly across domains. Consistent with multidimensional models of teacher competence (Blömeke & Delaney, 2012; Copur-Gencturk & Tolar, 2022), knowledge and belief-related constructs followed different patterns across the first year of the initiative.

Mathematical knowledge for teaching remained stable across the year, and a grade-band difference persisted. Importantly, both groups performed near the national mean at each time point, indicating that teachers were not entering the initiative under severe content deficits. At

the same time, Grades 6–8 teachers consistently scored higher than Grades 3–5 teachers. Because the elementary strand focused on fractions and the middle grades strand on ratio and proportional reasoning, this difference reflects patterns in content domains that are foundational for later mathematics learning (Siegler et al., 2012). The grade-band pattern also parallels broader trends in teacher preparation research, where secondary pathways typically emphasize disciplinary specialization and elementary preparation is more generalist in scope (Holm & Kajander, 2020; Kajander & Holm, 2016; Musset, 2009). Given evidence linking teachers' specialized mathematical knowledge to student achievement (Ball et al., 2008), the findings highlight the continued importance of sustained, content-focused professional learning in the elementary grades.

In contrast, mathematics teaching self-efficacy increased, particularly among Grades 3–5 teachers. Self-efficacy beliefs influence persistence and effort in challenging instructional situations (Bandura, 1997). MKT scores did not demonstrate statistically significant change while self-efficacy increased. This pattern suggests that teachers' perceived instructional capability may shift more readily than measurable changes in specialized content knowledge during early phases of professional learning. Rather than indicating inconsistency across measures, the findings may reflect the multidimensional nature of teacher development, where motivational and knowledge-based dimensions do not necessarily change on the same timeline (Copur-Gencturk & Li, 2023). Sustained professional learning may involve repeated cycles of study, enactment, and reflection as teachers continue developing across multiple domains (Desimone, 2009).

Mathematics teacher identity also increased across the year. Teachers reported stronger alignment with the role of mathematics teacher, suggesting shifts in professional self-concept that extend beyond perceived competence alone (Cribbs & Utley, 2024). The increase in mathematics teacher identity alongside stable MKT scores further supports the idea that psychological and knowledge-based dimensions of teacher competence may not develop uniformly during professional learning (Copur-Gencturk & Li, 2023).

Collective efficacy did not change during Year 1. Because collective beliefs are grounded in shared instructional experience, they may require longer periods of coordinated implementation before measurable change occurs (Goddard, 2002). The stability of collective efficacy during Year 1 further highlights that different dimensions of teacher competence and professional learning contexts may respond differently during sustained professional development.

Overall, the findings suggest that educating for change in mathematics instruction involves supporting specialized knowledge alongside the psychological dimensions that shape instructional practice. During Year 1, teachers' efficacy and identity increased, while MKT and collective efficacy did not demonstrate statistically significant change. Together, these findings highlight the layered nature of teacher development and suggest that dimensions of teacher competence may follow distinct patterns of change during sustained professional learning. Year 2 data will allow examination of whether these patterns persist as teachers engage new content strands and translate professional learning into classroom practice.

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CULTIVATING READINESS: FINDINGS FROM SOCIAL EMOTIONAL TRAINING FOR MATHEMATICS TEACHERS IN HIGH-NEED SCHOOLS

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We investigated mathematics teachers' growth and knowledge of social and emotional learning (SEL) after completing three or four years of monthly training during the academic year. The SEL trainings was led by a local nonprofit who specializes in emotional intelligence. The 11 participants were surveyed at the end of the trainings and after their first- or second-year teaching. We constructed an analytic framework using transformative SEL (tSEL) and found the most common tSEL competency area was about the teachers' agency and belonging. The teachers shared how the trainings influenced their personal lives, professional lives, and made a difference in their classrooms.

Keywords: Affect, Emotions Beliefs, and Attitudes; Instructional Activities and Practices; Communication; Preservice Teacher Education

There is a national shortage of mathematics teachers in the United States (Oyen & Scheweinle, 2020). Rural areas have a documented challenge to recruit and retain teachers (Ingersoll & Tran, 2023). We received NSF funding to support preservice mathematics teachers for the last two years of their undergraduate education and their first year or two of teaching. With the grant, the teachers received a full scholarship, monthly learning community building events, and monthly trainings on social emotional learning (SEL) by a local nonprofit, Peacemaker Resources. The preservice teachers attended the training for the last two years of their schooling and into their first years of teaching. The training focused on adverse child trauma, historical trauma, and how to incorporate SEL into their personal and professional lives.

Researchers have documented students who have adverse child experiences (ACES) and trauma affects how the brain perceives and process situations and can lead to chronic stress and limit a person's resiliency (Harter & Harter, 2022; Leung, Chan, & Ho, 2022). ACES can also result in a child struggling with memory, attention, and anxiety. However, researchers have also found that when a teacher incorporates SEL into their teaching, the students show significant improvement in their learning and academic performances (Lonhi, 2015). More recently, Baez and Baez (2023) found that incorporating cultural SEL with American Indians is critical to help them develop cultural understanding and the regulation of their emotions. Additionally, Ottmar et al. (2015) shared teachers who are trained in SEL are more able to pursue ambitious mathematics teaching.

Research Question

The following question guided our research study: How did the new mathematics teachers grow in their SEL knowledge and skills during the course of their training? In what ways, if at all, did the teachers demonstrate investment to integrate SEL in their educational settings?

Theoretical Framework

Communities of Practice (CoP) guided our framework for the structure of our five-year long grant (Wenger, 1998). Learning happens during social practices and people create their understanding of ideas through active community participation. Through active learning, the participants construct identities in relation to social engagement. The teachers who were participants in this study engaged in a learning community around SEL (CASEL, 2022) workshops monthly for three or four years. The SEL workshops also included a curriculum on historical trauma and the impact of the boarding school era on Indigenous communities, who comprise 21.7% of the population in the region where this program took place (U.S. Census Bureau, n.d.). The participants also had monthly community building activities together beyond the training and their undergraduate coursework. Example activities are axe throwing, paddling in a dragon boat, having a bonfire, snowshoeing, and a canoe trip. The CoP framework helped to guide how we organized and structured the grant project.

While the training itself was grounded in SEL, the research and analysis were guided by a transformative Social and Emotional Learning (tSEL) lens (CASEL, 2023). tSEL functioned as an interpretive framework for examining how participation in the community of practice shaped educators' engagement within high-need schools over time. Consistent with tSEL's emphasis on equity and context, the analysis attended to how identity, agency, belonging, collaborative problem solving, and curiosity emerged through mutual engagement and collective inquiry, rather than evaluating discrete outcomes. In this way, Wenger's theory of learning as participation provided the structural foundation of the study, SEL informed the content of professional learning, and tSEL guided the interpretation of how learning was applied by the practice community (Wenger, 1998; CASEL, 2022; CASEL, 2023).

Methods

As part of a five-year long NSF funded grant project on preparing preservice mathematics teachers to teach in rural high-need areas with continued support into the first year or two of teaching, we evaluated the teachers' growth and reflection on the project in relation to the five core tSEL competencies. The teachers began the project during their junior year at a public university in the rural northern Midwest. The preservice teachers were awarded full cost of attendance scholarships to finish their final two years. All the students double majored in both mathematics and mathematics education and wanted to pursue a teaching career. We had two cohorts of students. Cohort 1 (Fall 2020–Spring 2025) included eight students who we supported through their last two years of undergraduate education and through their first two years of teaching. Cohort 2 (Fall 2021–Spring 2025) included seven students who we supported through their final two years of undergraduate education and through their first year of teaching. Fourteen of the fifteen participants were teaching in rural areas with eleven of them teaching in the rural area around the university. Only eleven of the students completed the survey but because it was anonymous, we do not know who or how many were from Cohort 1 or Cohort 2.

Throughout their time in the Robert Noyce program, from upper-level undergraduate students into their first years in the classroom, the teachers participated in monthly two-hour meetings focused on integrating SEL skills for personal and professional practice. Cohort 1 was supported for the first two years of their teaching careers while Cohort 2 was supported for the first year of their teaching careers. The monthly meetings consisted of topics focused on the impact of trauma on brain development, ACEs, historical trauma, and strategies to build resilience (SEL). Trainings were conducted by a local non-profit with extensive experience training school faculty, staff and students in SEL and licensed educator.

At the final session we surveyed students on their growth and reflection. We asked the participants six questions framed within the CASEL Framework (CASEL, 2022):

1. Self-Awareness Competency - Read the skills below and reflect on your growth throughout the Noyce Program. Please select two or three SEL skills in this competency area and share examples of how you have used these skills in your classrooms or coursework.
2. Self-Management - Read the skills below and reflect on your growth throughout the Noyce Program. Please select two or three SEL skills in this competency area and share examples of how you have used these skills in your classrooms or coursework.
3. Social Awareness - Read the skills below and reflect on your growth throughout the Noyce Program. Please select two or three SEL skills in this competency area and share examples of how you have used these skills in your classrooms or coursework.
4. Relationship Skills - Read the skills below and reflect on your growth throughout the Noyce Program. Please select two or three SEL skills in this competency area and share examples of how you have used these skills in your classrooms or coursework.
5. Responsible Decision Making - Read the skills below and reflect on your growth throughout the Noyce Program. Please select two or three SEL skills in this competency area and share examples of how you have used these skills in your classrooms or coursework.
6. Any other comments regarding your SEL experience you would like to share?

We applied a transformative SEL (tSEL) analytic framework (CASEL, 2022; see Table 1) to focus on equity and context. The analysis attended to how identity, agency, belonging, collaborative problem solving, and curiosity emerged through mutual engagement and collective inquiry that will transform the educational outcomes of students in high-need schools.

Table 1: Analytic Framework

Competency Area	Definition
Identity	Refers to how adults view themselves as individuals and as part of the world around them. Having a healthy sense of identify buffers against negative or traumatic experiences and contributes to positive academic, social, and emotional outcomes.
Agency	Feeling empowered to make choices and take actions that produce a positive difference. Agency helps young people make choices about learning and career goals, overcome personal challenges, and shape the course of their lives.
Belonging	The experience of acceptance, respect, and inclusion within a group or community. Having a sense of belonging is critical to well-being, motivation, and achievement.
Collaborative Problem-Solving	The ability to build shared understanding and work together to come to solutions by pooling knowledge, skills, and efforts.
Curiosity	Leads to the pursuit of knowledge and different perspectives and contributes to attention, engagement, and learning.

After identifying the analytic framework, we first went through and documented anytime a teacher wrote about one of the competency areas independently for the first two participants. Our coding team (two researchers who participated in the SEL trainings and are mathematics education faculty, a social work faculty member, and a psychology faculty member) then checked for agreement on the first two students (18.2% of the sample). Next, we completed this process again with the rest of the responses first independently and then together. Together, we discussed our independent coding choices until we reached 100% agreement. We then tallied each time we documented a competency area during the survey.

Results and Discussion

During analysis, we coded each time the teacher documented their growth or skill as it related to CASEL's tSEL areas of focus (CASEL, 2022). Throughout analysis, we found agency was the most common competency area the teachers discussed (see Table 2).

Table 2: tSEL Competency Area Counts

Teacher	Identity	Agency	Belonging	Collaborative Problem-Solving	Curiosity
A	1	6	1	1	2
B	3	5	4	1	0
C	2	4	3	2	3
D	2	3	1	3	0
E	0	3	0	2	0
F	1	3	1	0	0
G	1	2	1	4	1
H	1	3	2	2	1
I	0	2	5	2	1
J	1	3	4	1	2
K	1	3	1	2	2
Total	13	37	23	20	12

Identity

The most common expressions around identity were about development in one's emotions. Examples of this are Teacher C shared, "One thing I've worked on is recognizing how my emotions affect my teaching and interactions with students." Emotions was a topic heavily covered during the teachers' SEL training. Teacher H explained their identity in relation to emotions by sharing, "I feel like I have gotten really good at being able to notice when I am feeling uncomfortable feeling like frustration or stress and feeling comfortable feelings like excited and fulfilled." In addition, Teacher B explained, "I realized that my negative emotions come out to play more over written communication than verbal communication" sharing how making a phone call might be better than an email when they are having negative feelings.

The discussion of self-worth was central to the theme of identity. Teacher D explained, "I know what I do is very valuable in the community." Sharing how important teaching is in a rural area. Teacher J wrote:

> When I first started at Noyce, I was very hard on myself. I would always think that I was not doing good enough. Throughout the meetings and classroom experience over the past four years, I have learned a lot about myself and how I should think.

This demonstrates Teacher J's growth in their identity, how they view themselves, and normalizing the adjustments of professional identity that accompany the first years of teaching practice.

Lastly, several teachers self-reflected in relation to teaching. An example is Teacher A explained, "I can self-reflect on what I am doing in the classroom and how it is affecting my students, my life, and everything else." Their identity around growth and the trainings was focused on their emotions, self-worth, and self-reflection.

Agency

A common subtheme within agency, the most documented competency, was teachers' growth in their emotional intelligence (EQ). Trainings focused on EQ significantly during the first two years of the program. Teacher H stated:

> I have grown the most in the emotional self-awareness skills. At the beginning of Noyce, I thought it was really difficult to name the emotions I was feeling and express my needs surrounding those emotions. Now, I employ the skills that I have learned and can recognize my emotions.

This is an example of Teacher H sharing how they have shown growth in their ability to identify emotions. Teacher C wrote:

> Early on, I found myself getting frustrated when things didn't go smoothly, but I've worked on managing those emotions, so they don't impact my teaching. Instead of letting stress take over, I've learned to take a deep breath, step back, and find a solution. When I have lessons that don't go well, I reflect on them and write down what I could do differently the next time.

This showed how Teacher C is the ability to regulate their nervous system and how they use what they learned through teaching the lesson to improve their teaching.

Several of the teachers reflected on their work life balance, which can be a struggle for many new teachers (Ingersoll & Tran, 2023). This is a topic we address during the last two years of the training as they transitioned from being a student to a new teacher. An example of this is Teacher E shared:

> I can consider how my choices will be viewed through the lens of the young people I serve and the community around them. I can take time for self-reflection on progress towards goals and the process used. I can consider how my personal and professional decisions impact the lives of others. This plays the largest role in my decision-making surrounding work-life balance.

This statement explained the teacher's ability to think not only about themselves but also how their decisions can affect others.

Belonging

We documented belonging as a competency area 23 times which made it the second most common theme we found. Many of the teachers reflected on relationships being a big part of the SEL training helped provide them skills for life and the importance of relationships in relation to teaching. Teacher C shared, "I have made some really strong relationships with students. Because of conversations I have had with them about their experiences. Being curious about

their culture has made them feel comfortable with me and in my room." This statement showed how the teacher found it important to connect with students in relation to their culture. Teacher J wrote, "I think that the Noyce program has helped me not only build very good relationships with my students, but it has helped me so much on getting students to build relationships with each other." Teacher J found the training to help them build relationships but also strategies to help others build relationships with each other.

Several teachers shared the importance of creating an environment where all students feel welcome. An example of this is Teacher C shared:

> One of my goals has been to make sure all students feel represented in my classroom. I've worked on being mindful of different learning styles and celebrating the diversity in my school community. I've also reflected more on the systemic barriers that some students face and thought about how I can better support them.

This is an example of how the teacher tried to create a belonging environment for making sure all students are welcome and the struggles some students might have. Another example is Teacher J wrote,

> Throughout the program, I have learned a lot about historical trauma and ACES. I feel that this is very important in the classroom, because different students act different ways. I also see how their lives' affect them and sometimes students are going through so much, that pushing them to do a lot of work is not helping them. Sometimes, students just need someone to talk to.

Understanding students past experiences is important and this teacher was made aware in the training how many students are bringing trauma with them to school and how to help students work through this.

Another thing shared about the Noyce program was through Teacher J, "It has been amazing experience getting to have a community of those around us that are going through the same experiences." This is an example of how important the trainings we provided the teachers with because it allowed them to have a community of belonging with them to experience the final semesters of college and their first few years of teaching.

Collaborative Problem-Solving

The competency area of collaborative problem-solving was documented 20 times. The statements made in this area were focused on the teachers working with other teachers, with their students, or with their school community as a whole. An example of this is Teacher C wrote,

> Noyce has also helped prepare me to collaborate effectively with my colleagues. Teaching isn't something you do in isolation, and I've learned just how valuable it is to share ideas, ask for feedback, and support each other. Whether it's co-planning lessons, discussing ways to better support students, or simply having a colleague to reflect with after a tough day, I've come to appreciate the power of teamwork. Through Noyce, I've had the opportunity to work with mentors and peers who challenge me to grown and push me to be a better educator. I've learned how to communicate openly, give and receive constructive feedback, and adapt to new perspectives—all skills that will help me thrive in any community.

This idea by Teacher C demonstrates how they view teaching as a group activity where many people help them work together and sharing a pool of knowledge.

Another way the students wrote about collaborative problem-solving was a focus on their students and learning from each other. Teacher G wrote,

> Working with a lot of Native students and ELL students has taught me to be intentional about the way I am teaching. I am always asking questions to learn about their perspectives and working to adapt my teaching to the needs of the students I work with.

The teachers focused on this section about how they learn from others and share their knowledge with a group of people to create shared understanding.

Curiosity

Curiosity was the least documented competency area with only twelve documented instances. The common theme with curiosity was about reflection on teaching and having room for growth. Teacher J shared, "I have learned how to reflect on myself and a teacher and a person and how I am impacting my students." This showed how curiosity promotes reflection. Another example is Student C stated,

> One of the biggest things I've learned through the Noyce Program is that growth never really stops. I've become way more reflective about my teaching—I'm always questioning what's working, what's not, and how I can better support my students. Whether it's tweaking my teaching style, bringing in different perspectives, or rethinking assessments, I've realized that being flexible and open-minded is key to meaningful learning.

This demonstrated that the training helped Teacher C be more reflective in their teacher and having a curiosity on ways to improve. The last example is Teacher G wrote,

> I am constantly looking for opportunities to grow and learn as an educator. Whether it be from formal professional development situations or informal conversations with the educators and staff members that surround me every day. I recognize that I have so much room for improvement and I am excited to continue to learn.

All of these examples document the teachers' curiosity and desire to improve their own teaching either through reflection, open-mindedness, and opportunities to learn more.

Conclusion

The identity findings related to emotional awareness and regulation are particularly significant for teachers working in rural, high-need schools. These educators must navigate complex instructional environments shaped by limited resources, high community visibility, and students experiencing poverty, trauma, and systemic inequities. Teachers who develop the capacity to recognize, manage, and reflect on their emotional responses strengthen how they view themselves both as professionals and as members of the communities they serve.

In alignment with tSEL, this healthy sense of identity helps buffer educators against the cumulative stress and secondary trauma often present in high-need educational settings while supporting positive academic, social, and emotional outcomes for students. Self-reflection thus becomes not only a pedagogical skill but a life skill that strengthens educators' sense of purpose and professional agency in meeting students' needs while sustaining effective relationships with students and colleagues in challenging contexts.

Belonging emerged as a framework for addressing cultural and social connectedness. Participants recognized the impact of personal and peer relationships in creating classroom communities, which mirrors our theoretical framework. Intentional integration of cultural representation through content and understanding of the impact of historical trauma deepened awareness of diversity and promoted an inclusive and integrative learning environment (Baez & Baez, 2023).

This emotional and professional resilience is critical given that teacher attrition is strongly connected to workplace stress, insufficient support, and burnout—factors that disproportionately affect rural and high-need settings (Ingersoll, 2001; Ingersoll & Tran, 2023). Supporting teachers' emotional development through sustained SEL and community-based professional learning therefore contributes not only to improved classroom environments but also to reducing burnout and teacher turnover, ultimately strengthening educational continuity for students in communities already facing persistent staffing challenges.

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INTRODUCING THE AUTHENTIC DATA EXPERIENCE PLANNING TOOL (ADEPT): ITERATIVE DEVELOPMENT AND TEACHER FEEDBACK

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Implementing integrated STEM instruction is challenging; data investigations are one accessible pathway to do so. This study reports on our collaborative effort to create the Authentic Data Experience Planning Tool (ADEPT), a resource that guides teachers in planning learning experiences that reflect the decision-making, problem-solving, and data literacy skills of STEM professionals. The ADEPT is a user-friendly guide anchored by established theoretical frameworks to support integration of data literacy instruction across disciplines.

Keywords: Middle School Education, High School Education, Integrated STEM/STEAM, Data Analysis and Statistics

Purpose of the Study

Our research-practice partnership (RPP) with teachers and community members from rural coastal communities in the Northeastern U.S. focused on the integration of marine science, authentic data, and technology-related competencies into STEM curricula. This study reports how we collaboratively developed a resource to guide teachers through planning a learning experience that reflects the decision-making processes, problem-solving practices, and data literacy skills that real STEM professionals use in their day-to-day work. The need for this tool arose due to the fact that all teachers wanted to address local issues that were relatable to their students. However, local issues differ greatly based on their locations along the coast. How to support all teachers with differing local civic-related topics for data investigations?

We created a planning tool that is grounded in data investigation frameworks but allows teachers across diverse STEM classrooms to focus on data investigation processes as they fit their classroom needs and individual learning objectives. We were guided by the research questions: (1) What are the essential attributes of data investigation frameworks to include for effective planning of authentic classroom experiences? and, (2) How has our planning tool been used by teachers in STEM classrooms, particularly related to data literacy and quantitative reasoning.

Theoretical Framework

Educators trying to implement integrated STEM instruction face significant challenges despite the demonstrated benefits of this practice for teachers and students (Czerniak & Johnson, 2014; Mayes, 2019; Ríordáin et al., 2016). Additionally, education research has demonstrated the value of supporting the alignment of STEM education with authentic scientific practices and competencies that are used in day-to-day work of scientists and experts (e.g., AAAS, 2011; Kelley & Knowles, 2016; Mayes et al., 2013; NGSS Lead States, 2013; Roehrig et al., 2021). Our work was grounded in the framework for data science developed by Lee and colleagues (2022), along with the positioning of data literacy at the intersection of data science, authentic

context, and quantitative reasoning proposed by Kjelvik and Schultheis (2019). Lee and colleagues (2022) divide the data investigation process into six phases: (1) Frame the problem, (2) consider and gather data, (3) process data, (4) explore and visualize data, (5) consider models, and (6) communicate and propose action. They outline key practices and processes for each phase and highlight that despite the framework appearing linear or cyclical, it should be viewed as a puzzle where each phase represents a piece as data investigations are often not linear in real life. Kjelvik and Schultheis (2019, 2020) offer accessible definitions of data literacy and authentic context and complement the Lee framework by anchoring the role authentic context can play in data investigations. These frameworks informed our project work with teachers, specifically development of the Authentic Data Experience Planning Tool (ADEPT). The ADEPT is meant to guide teachers through planning a locally relevant learning experience for their students that reflects the authentic practices used by experts and scientists when engaging in data investigations.

Methods

Study Context and Participants

Data collection and analysis followed an iterative, qualitatively driven mixed methods approach in which ongoing findings informed subsequent phases of data generation and tool refinement. Across the 2024–2025 academic year, data sources included exit slips from professional learning discussions, teacher feedback on version one of the ADEPT, and focus group discussions examining its planning and classroom implementation; these data directly informed revisions leading to version two in summer 2025. Additional fine-tuning occurred during 2025–2026 to revise targeted language and include questions related to mathematics and statistics. Consistent with a qualitatively driven mixed methods stance, data collection decisions were continually shaped by prior findings, making it difficult to specify a fixed mixed methods design *a priori*, as future data needs emerged from ongoing analysis (Hesse-Biber et al., 2016). Data triangulation included teachers' completed ADEPT digital documents, classroom observations using an ADEPT-aligned protocol, student artefacts, and teacher focus groups reflecting on ADEPT use from lesson design through implementation.

Development of the ADEPT

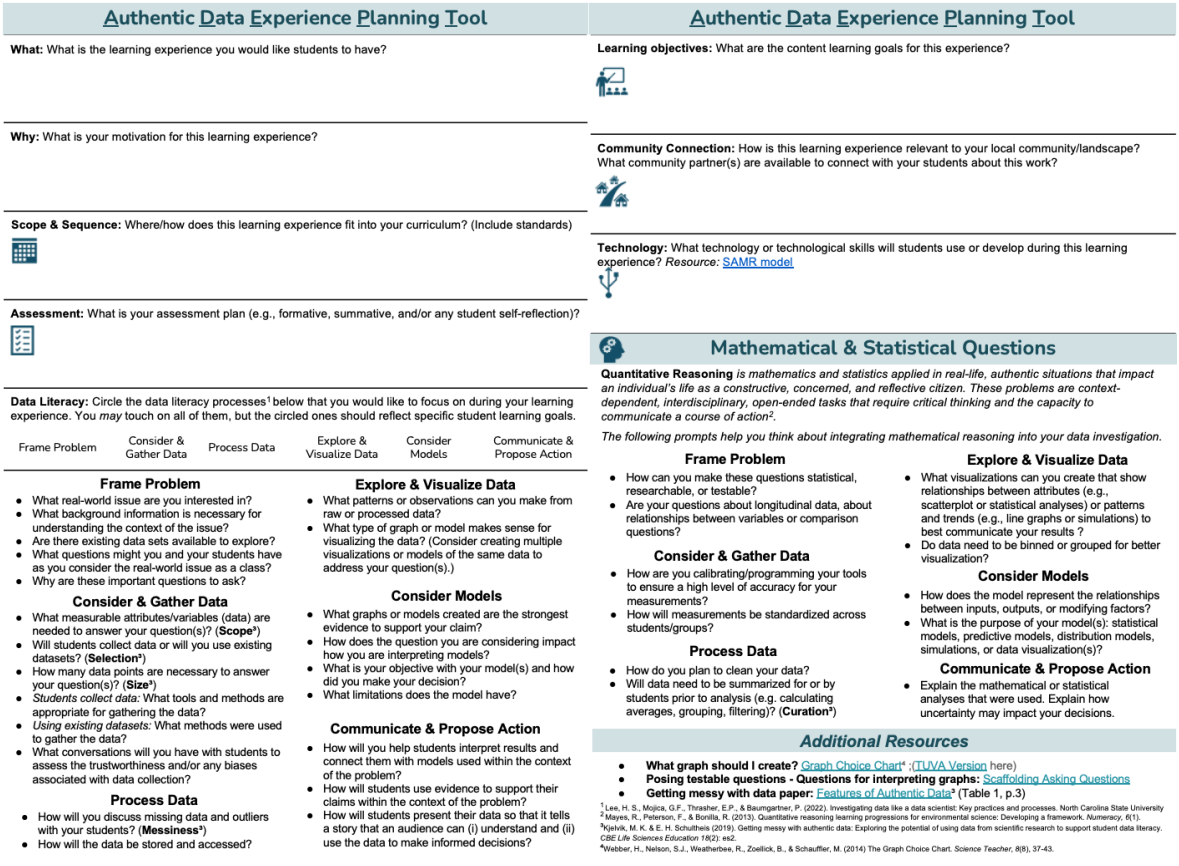
Collaborative planning and project development during Year 1 sparked several discussions about the challenges of integrating data investigations into STEM classrooms. Challenges included integration across STEM disciplines, grade levels, as well as logistical concerns of field work. The discussions highlighted the need for a useful and applicable planning tool, and in project Year 2 (2023–2024), the team collaborated with cohort 1 teachers to explore initial ideas of what such a tool should include. A key session featured Kjelvik and Schultheis as guest presenters, who shared insights from their "Data Nuggets" work. This presentation and interaction with the speakers led the team to adapt an existing Data Nugget flowchart template, guiding educators in structuring data activities. After teachers tested the template during that school year, they provided feedback, prompting the research team to refine it and simplify the structure. However, teachers later expressed concerns that our adapted template was too complex, lengthy, and not universally applicable across STEM classrooms, leading to further reflection on its usability.

In Year 3 the research team reframed its approach and the six data investigation phases from Lee et al. (2022) became the foundation for summer professional learning. Building on this framework and teachers' feedback the team developed the first version of the Authentic Data Experience Planning Tool (ADEPT) with a strong emphasis on maintaining a classroom-friendly

approach. As discussions continued, the team refined ADEPT further, leading to version 2.5, a two-page handout that streamlined the planning process. This version integrated the Lee et al. (2022) framework, along with insights from the Kjelvik and Schultheis (2019).

Results

On the first page of the ADEPT, teachers can identify major goals, scope and sequence for their learning experience, assessment ideas, and choose which data literacy processes they wish to incorporate into their learning experience. Several specific prompting questions provide more details for each data literacy process to consider while lesson planning (Figure 1). The second page of the ADEPT offers space to identify learning objectives, how teachers plan to connect their learning experience to their community, and what type of technology they plan to integrate. It also provides additional mathematical and statistical questions for each data literacy process that emphasize quantitative reasoning; these questions can be especially helpful to teachers whose teacher training was outside of mathematics.



Authentic Data Experience Planning Tool

What: What is the learning experience you would like students to have?

Why: What is your motivation for this learning experience?

Scope & Sequence: Where/how does this learning experience fit into your curriculum? (Include standards)

Assessment: What is your assessment plan (e.g., formative, summative, and/or any student self-reflection)?

Data Literacy: Circle the data literacy processes[1] below that you would like to focus on during your learning experience. You *may* touch on all of them, but the circled ones should reflect specific student learning goals.

Frame Problem | Consider & Gather Data | Process Data | Explore & Visualize Data | Consider Models | Communicate & Propose Action

Frame Problem
- What real-world issue are you interested in?
- What background information is necessary for understanding the context of the issue?
- Are there existing data sets available to explore?
- What questions might you and your students have as you consider the real-world issue as a class?
- Why are these important questions to ask?

Consider & Gather Data
- What measurable attributes/variables (data) are needed to answer your question(s)? (**Scope**[3])
- Will students collect data or will you use existing datasets? (**Selection**[3])
- How many data points are necessary to answer your question(s)? (**Size**[3])
- *Students collect data:* What tools and methods are appropriate for gathering the data?
- *Using existing datasets:* What methods were used to gather the data?
- What conversations will you have with students to assess the trustworthiness and/or any biases associated with data collection?

Process Data
- How will you discuss missing data and outliers with your students? (**Messiness**[3])
- How will the data be stored and accessed?

Explore & Visualize Data
- What patterns or observations can you make from raw or processed data?
- What type of graph or model makes sense for visualizing the data? (Consider creating multiple visualizations or models of the same data to address your question(s).)

Consider Models
- What graphs or models created are the strongest evidence to support your claim?
- How does the question you are considering impact how you are interpreting models?
- What is your objective with your model(s) and how did you make your decision?
- What limitations does the model have?

Communicate & Propose Action
- How will you help students interpret results and connect them with models used within the context of the problem?
- How will students use evidence to support their claims within the context of the problem?
- How will students present their data so that it tells a story that an audience can (i) understand and (ii) use the data to make informed decisions?

Authentic Data Experience Planning Tool

Learning objectives: What are the content learning goals for this experience?

Community Connection: How is this learning experience relevant to your local community/landscape? What community partner(s) are available to connect with your students about this work?

Technology: What technology or technological skills will students use or develop during this learning experience? *Resource:* SAMR model

Mathematical & Statistical Questions

Quantitative Reasoning *is mathematics and statistics applied in real-life, authentic situations that impact an individual's life as a constructive, concerned, and reflective citizen. These problems are context-dependent, interdisciplinary, open-ended tasks that require critical thinking and the capacity to communicate a course of action*[2].

The following prompts help you think about integrating mathematical reasoning into your data investigation.

Frame Problem
- How can you make these questions statistical, researchable, or testable?
- Are your questions about longitudinal data, about relationships between variables or comparison questions?

Consider & Gather Data
- How are you calibrating/programming your tools to ensure a high level of accuracy for your measurements?
- How will measurements be standardized across students/groups?

Process Data
- How do you plan to clean your data?
- Will data need to be summarized for or by students prior to analysis (e.g. calculating averages, grouping, filtering)? (**Curation**[3])

Explore & Visualize Data
- What visualizations can you create that show relationships between attributes (e.g., scatterplot or statistical analyses) or patterns and trends (e.g., line graphs or simulations) to best communicate your results ?
- Do data need to be binned or grouped for better visualization?

Consider Models
- How does the model represent the relationships between inputs, outputs, or modifying factors?
- What is the purpose of your model(s): statistical models, predictive models, distribution models, simulations, or data visualization(s)?

Communicate & Propose Action
- Explain the mathematical or statistical analyses that were used. Explain how uncertainty may impact your decisions.

Additional Resources
- **What graph should I create?** Graph Choice Chart[4] ;(TUVA Version here)
- **Posing testable questions - Questions for interpreting graphs:** Scaffolding Asking Questions
- **Getting messy with data paper:** Features of Authentic Data[3] (Table 1, p.3)

[1] Lee, H. S., Mojica, G.F., Thrasher, E.P., & Baumgartner, P. (2022). Investigating data like a data scientist: Key practices and processes. North Carolina State University
[2] Mayes, R., Peterson, F., & Bonilla, R. (2013). Quantitative reasoning learning progressions for environmental science: Developing a framework. *Numeracy*, 6(1).
[3] Kjelvik, M. K. & E. H. Schultheis (2019). Getting messy with authentic data: Exploring the potential of using data from scientific research to support student data literacy. *CBE Life Sciences Education* 18(2): es2.
[4] Webber, H., Nelson, S.J., Weatherbee, R., Zoellick, B., & Schauffler, M. (2014) The Graph Choice Chart. *Science Teacher*, 8(8), 37-43.

Figure 1: Authentic Data Experience Planning Tool (ADEPT v. 2.5)

Teachers identified several strengths of the ADEPT, including: (1) the prompting questions helped them focus on specific data literacy processes they might use, (2) it can be used across disciplines, and (3) it can promote community connections. For example, Phoenix is a science high school teacher who supported students in creating and programming sensors.

I really appreciate the questions offered with each of the 6 prompts in the ADEPT planner. Going through these helped me focus my learning target on the Frame the Problem prompt for this lesson. I decided to have students generate their own scientific questions that could be answered by sensors on the sonde [...] It is one of the few tools [...] that focuses on using data science with students. (Phoenix, High School Science Teacher, Spring 2025)

River is a mathematics high school teacher who used the ADEPT to focus on the phase explore and visualize data in her honors pre-calculus class.

I think the great thing about the ADEPT planner is that it helps you think about all of the parts to doing a project individually, and takes some of the pressure off of having to focus on all of them at a deep level. Instead focusing on one or two areas and doing that well is much more attainable timewise in a classroom. (River, High School Mathematics Teacher, Spring 2025)

Taylor is a veteran high school science teacher who focused on community connections.

For me, I've been in the community- I've been teaching the same school for now five years, and so it did make me reach out to local partners, more so this year, in the last probably 15 months, I guess, than I have in the past. So that part has been really good. (Taylor, High School Science Teacher, Summer 2025)

Discussion and Conclusion

Teachers highlighted their needs for a streamlined, user-friendly tool that allowed them to focus on their lesson goals while being deliberate about integrating data literacy practices. The ADEPT was developed around these key needs, in close collaboration with teachers. The ADEPT effectively translates the key components of data investigations proposed by Kjelvik and Schultheis (2019) and Lee and colleagues (2022) into a single resource for educators to integrate these frameworks into their classroom. By design, the ADEPT is flexible enough for teachers of various disciplines, grade levels, and years of experience to use. In addition, people developing professional learning opportunities can utilize the ADEPT as a planning tool and a brief assessment of participants' knowledge of data investigation phases.

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INSIGHTS ON TEACHER GOALS AND PROFESSIONAL LEARNING COMMUNITIES SUPPORTING AMBITIOUS MATHEMATICS INSTRUCTION

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This exploratory qualitative case study examined the range of ambitious mathematics instructional practices teachers selected to discuss as they participated in monthly professional learning communities. The professional learning communities were synchronous, virtual spaces designed for teachers to share excerpts of their classrooms for the purpose of reflecting on facilitating student sense making using complex tasks and productive talk moves. We report emerging findings of one professional learning community's interests and priorities and their connections to the articulated goals for improvement each teacher shared. Over time, discussion shifted from a focus on eliciting student thinking, to considering task features, and lastly to leveraging student contributions to build collective understanding of mathematical concepts.

Keywords: Professional Development, Instructional Activities and Practice, Classroom Discourse

Improving instructional quality in mathematics is essential for improving the learning opportunities outcomes for students. While rigorous curriculum materials provide access to cognitively demanding tasks, ambitious instruction also requires teachers to enact practices that position students as sense-makers and support them in developing and communicating mathematical ideas. These practices include selecting rich mathematical tasks that require varied cognitive engagement (i.e., Smith & Stein, 1998) and facilitating productive discussions that build on children's thinking (Chapin et al., 2009; Smith & Stein, 2018; Kazemi & Hintz, 2014). Together, these practices are one mechanism for fostering conceptual understanding and supporting the generalization of mathematical ideas.

Professional development often supports teachers to develop rigorous visions of practice: deepening their understanding of how mathematical concepts are developed, strengthening their pedagogical responses, and expanding their mathematical knowledge. However, these opportunities are often disembedded from the classroom, potentially limiting opportunities for teachers to translate theory into enactment. Professional learning communities can be a space for teachers to examine problems of practice that are connected to teachers' classrooms, usually with lesson artifacts, with colleagues (McLaughlin & Talbert, 2006; DuFour & Reason, 2016). Examining evidence from classrooms with peers supports reflection on the mathematical tasks, instructional choices, and classroom discussions that shape student opportunities to engage in mathematical thinking. (Horn & Little; 2010, Little, 2003).

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

This study examines a professional learning community designed to support teacher learning following a summer professional development focused on ambitious and equitable mathematics instruction. During the school year, teachers met monthly in synchronous virtual sessions to examine classroom videos, reflect on instructional decisions, and receive peer feedback. Before sharing videos, teachers identified an instructional improvement goal, which provided an entry point for understanding how conversations supported these goals. Specifically, we investigated: a) how teachers' goals for ambitious mathematics instruction aligned with practices discussed during meetings, and b) how the community's instructional focus evolved across monthly cycles of collaborative reflection. Understanding this relationship provides insight into how professional learning communities create opportunities for teachers to deepen their enactment of ambitious mathematics instruction.

Theoretical Framework

Ambitious and Equitable Instruction

Ambitious mathematics instruction emphasizes teaching practices that challenge students to engage deeply with mathematical ideas to foster a strong conceptual understanding (Franke et al., 2007; Zahner et al., 2025). In ambitious mathematics classrooms, students are positioned as sense-makers with opportunities to engage in reasoning and argumentation with their peers as they make connections to larger mathematical ideas. Enacting this vision requires teachers to make instructional decisions, including selecting cognitively demanding tasks (Stein & Smith, 1998; Youngs et al., 2022) and facilitating classroom discussions that elicit and extend student thinking (Chapin et al., 2009; Jacobs & Empson, 2016; Kazemi & Hintz, 2014; Smith & Stein, 2018).

During a summer professional development, participating teachers were introduced to two complementary instructional frameworks: *task analysis* (Stein et al., 2000), to select and maintain cognitively demanding tasks; and *productive talk* (Chapin et al., 2009; Kazemi & Hintz, 2014), to uncover and build on students' mathematical thinking. These frameworks informed the analytics lens used to identify the instructional practices teachers discussed.

Opportunities to Learn in Conversations

Drawing from the work of Horn and Kane (2015), we explored the opportunities that teachers participating in a year-long professional learning community had to learn about enacting ambitious mathematical instruction. Opportunities to learn are defined as "affordances for changing participation and practice" (Greeno & Gresalfi, 2008, p. 172) that emerge through collaborative activity. From this perspective, learning opportunities develop as teachers collectively examine instructional dilemmas, interpret classroom evidence, and negotiate productive responses.

We conceptualize the monthly professional learning community as a setting in which opportunities to learn emerged through the interaction between teachers' self-identified instructional goals and the collaborative examination of classroom practice. As teachers discussed classroom videos, they collectively attended to specific instructional practices. Examining the relationship between their stated goals and the instructional focus of the conversations provides insights into how participation in video-based professional learning communities may support teachers' enactment of ambitious instruction.

Methodology

This exploratory qualitative case study (Stake, 2005) examined the relationship between teacher' self-identified goals for instructional improvement and their discussions of practice.

Study Participants

Participants included four middle school mathematics teachers and one instructional coach engaged in their first year of a 250-hour STEM teacher leadership program, which aims to cultivate rigorous, equity-focused STEM learning cultures. The teachers reported a range of teaching experience, selecting between 3 and 21+ years. In addition, all teachers in this professional learning community had a master's in education. The instructional coach, also a middle school mathematics teacher, had experience with previous cohorts.

Professional Learning Communities

Teachers participated in six virtual professional learning communities, three in the fall semester and three in the spring. Each meeting featured classroom video excerpts, approximately 15-minutes, from two teachers in the group. Teachers were asked to record themselves facilitating classroom discussions in their mathematics classrooms. These excerpts were shared in advance via a secure platform that enabled time-stamped annotations. Each teacher also submitted a monthly reflection, stating their own goal for improvement and additional lesson context, such as the lesson goal or a description of the task. This reflection was shared with the instructional coach and members of the research team.

During each session, two teachers led a 20-minute segment, for a total of 40 minutes, of the meeting to discuss specific teaching practices related to facilitating productive discussions and maintaining cognitive demand. To facilitate this discussion, each teacher shared a classroom video with the other group members at least one week prior to their scheduled meeting. Teachers drew on annotations from their peers and instructional coach to support their discussion.

Data Collection and Analysis

Data sources for this study included five recordings of the professional learning communities and the reflection log. First, researchers reviewed professional learning community's recordings and produced descriptive and analytic memos identifying focal instructional practices (e.g., eliciting student thinking, facilitating uptake, representing ideas, task features). The team then synthesized session-level memos to characterize each meeting's collective focus.

After rich discussions across the various data sources, four project team members analyzed reflection logs and annotations from 19 classroom videos to construct analytical memos characterizing each teachers' goals for instructional improvement. The research team then met to identify patterns of alignment between individual goals and the session's collective focus, as well as shifts in instructional attention over time. These memos were shared with the research team and revised based on feedback.

Preliminary Findings

Preliminary findings address a) teachers' instructional goals for improvement; and b) the relationship between these goals and the instructional focus of the monthly discussions.

Individual Teacher Goals

Across the 19 classroom videos, articulated goals for improvement varied in specificity. The research team could not identify a goal in the reflection logs or annotations for six videos; however, goals for two of these videos emerged during the monthly discussions. Most identified goals focused on improving teachers' facilitation of student thinking, including questioning, supporting mathematical discourse. Only one teacher explicitly identified eliciting student thinking as a goal for improvement early in the year. For one spring video, a teacher's goal centered exclusively on refining their practice of working with student ideas once surfaced. Three videos reflected goals outside of the frameworks, such as integrating technology.

Progression of the Conversations

Although teachers came to each meeting with individual goals, the monthly discussions often converged around shared instructional practices. During the fall, conversations focused on eliciting student thinking through open-ended questions, increasing student participation, and examining how mathematical tasks supported productive discussions. Mid-year conversations shifted toward features of cognitively demanding tasks, including how task design influenced student engagement and a range of thinking, while maintaining attention to the intended mathematical goal. Toward the end of the year, discussions shifted toward leveraging student thinking to advance collective understanding. These discussions focused on responding to student errors and connecting individual contributions to broader mathematical concepts.

Discussion

This preliminary analysis examined the potential relationship between conversations rooted in examples of teachers enacting practices related to ambitious and equitable instruction and how individual goals for improvement supported the community's opportunities to learn. Preliminary findings suggest teachers' goals served as an entry point for collaborative inquiry, while the conversations broadened to include interconnected aspects of ambitious instruction. Although discussions generally aligned with the teachers' stated goals, the community also attended to practices that extended beyond individual priorities, such as the attention to features of tasks. These patterns suggest that opportunities to learn emerged through the interaction between teachers' goals and the collective examination of practice. Across the year, the focus shifted from eliciting student thinking, to task features, to working with student contributions. The next step for this analysis is to examine evidence from video annotations, facilitator moves, and end-of-year reflections, to understand how these cycles of reflection supported opportunities to learn and enact ambitious instruction in mathematics.

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RECOGNIZING INSTRUCTIONAL FEATURES ACROSS CONTEXTS: HOW U.S. AND NON-U.S. TEACHERS RESPOND TO THE SAME LESSON REPRESENTATIONS

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This study investigates how secondary mathematics teachers from diverse professional and cultural contexts—U.S. vs. non-U.S. in this study—recognize and attend to instructional features within problem-based but teacher-centered lesson representations. A cross-case analysis of asynchronous annotations from the teachers across five countries (Bulgaria, India, the Philippines, South Africa, and the U.S.) revealed that U.S. teachers were significantly more likely to identify the lack of student voice as a notable departure from expected practice. Conversely, many non-U.S. teachers attended to and positively evaluated the teacher's guidance in the lesson. Overall, these findings suggest that teachers from different pedagogical contexts may bring different expectations and assumptions to their interpretation of instructional materials.

Keywords: Professional Development, Culture and Asset-Based Pedagogies, Instructional Vision, Teacher Noticing

Introduction

Representations of practice—such as videos, transcripts, animations, or storyboards—have become widely used tools for eliciting teacher noticing and supporting teacher learning (e.g., Grossman et al., 2009). These representations can engage practitioners in controlled approximations of practice, temporarily reducing the complexity of real classrooms. While much research has used these tools to document what teachers notice, Herbst and colleagues have, since the early 2000s, been using representations of practice to surface the implicit assumptions and expectations that shape how teachers interpret various instructional scenarios (Herbst & Chazan, 2006; Herbst & Kosko, 2014). Other scholars have also done so later (e.g., Choy et al., 2017; Stockero & Rupnow, 2017).

One powerful method for uncovering these implicit assumptions is to design representations that deliberately deviate from expected practices. Inspired by Garfinkel's (1967) insight that assumptions often become visible only when violated, we have been exploring the potential of deploying lesson representations that depart from expectations in the context of professional development as a means to surface teachers' assumptions about desirable practice (Herbst & Chazan, 2015). These lesson representations portray practices that deviate from reform-based goals (such as those featured in National Council of Teachers of Mathematics [NCTM], 1991), featuring long stretches of teacher-centered exposition with minimal student input during the implementation of a rich mathematical task. When teachers encounter such a representation, their reactions can reveal what they take for granted as critical to the representation of good teaching (Dreher & Kuntze, 2015). Teachers tend to react to these representations more strongly than to the representations that align with their expectations, which makes those representations

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reasonable candidates for use in professional development in which designers aim to get teachers to share their ideas about what good practice should look like (Brown et al., 2024).

What a teacher finds unexpected or worth remarking upon, however, depends on the professional and cultural context in which they are operating. Brousseau's (1997) theory of the didactical contract offers a useful orienting lens: Teachers develop implicit expectations about teaching and learning through sustained participation in particular pedagogical traditions, and those expectations may differ substantially across contexts. A lesson designed to explicitly ignore aspects of U.S. reform-based standards may therefore land differently depending on the observer—what registers as a conspicuous absence to one teacher may seem unremarkable, or even appropriate, to another.

This professional divergence is frequently rooted in differing cultural scripts for teaching, as a growing body of cross-cultural empirical work shows that what teachers attend to in a shared instructional artifact varies systematically across cultural and professional contexts. Using the same teaching vignette, Dreher et al. (2021) found that German experts were concerned that students' existing thinking was insufficiently valued, whereas Taiwanese experts focused on whether students' inadequate thinking was further prompted differences they trace to culturally embedded norms of instructional quality. Similar patterns appear with practicing teachers from different countries watching the same lesson episodes: Ding et al. (2022) found that U.S. teachers foregrounded concrete representations, whereas Chinese teachers attended to concrete-to-abstract sequences and the mathematical relationships that those models could surface. Recently, Raval et al. (2025) found that Indian pre-service teachers paid closer attention to detailed mathematical discourse in a geometry-focused video, whereas Swedish pre-service teachers were more likely to notice general instructional strategies, reflecting differences in their familiarity with geometrical reasoning. While these studies mostly focused on what teachers from different countries attend to shared videos or vignettes, less is known about how teachers from different countries engage with a lesson representation that departs from pedagogical expectations.

To address this gap, the current study examines how teachers in U.S. and non-U.S.[1] contexts engage with the same lesson representation differently—one designed to deviate from U.S. reform-based recommendations (NCTM, 1991). By analyzing what teachers attend to, prioritize, or pass over, we aim to make visible the instructional assumptions that appear to vary across pedagogical traditions. Our guiding question is: *How do teachers from U.S. and non-U.S. contexts differ in what they notice and how they evaluate the instructional features of a problem-based, teacher-centered lesson representation?*

Method

Context

The study was conducted within a Story*Circles* professional development setting. In Story*Circles*, teachers collaboratively storyboard lessons by scripting, visualizing, and arguing about instructional interactions in response to realistic classroom contingencies (Herbst & Milewski, 2018). These approximations provide opportunities to rehearse ambitious teaching strategies and engage in peer-supported instructional decision-making.

[1] We use "non-US" not to imply homogeneity across these contexts, but as an initial contrast category relative to the U.S. reform tradition that structured the lesson design. Future work should examine within-group patterns across participating countries.

Design of the Lesson Representations

In Story*Circles*, teachers start by reviewing a lesson representation built around a novel problem. The representations were deliberately constructed to deviate from some expectations common in reform-based, student-centered instruction as articulated in U.S. professional teaching standards (NCTM, 1991). Specifically, the lessons were designed to follow the Worthwhile Tasks standard, which calls for introducing new content through problems, while departing from the Discourse Standard (NCTM, 1991), which calls for teachers to orchestrate classroom discourse in ways that elicit and build on student thinking (Lampert, 2001). While the lesson started with a problem whose work would usher in new knowledge, the teacher in each lesson representation solved the problem with minimal student participation, providing direct exposition rather than facilitating student exploration and discourse (see Figure 1).

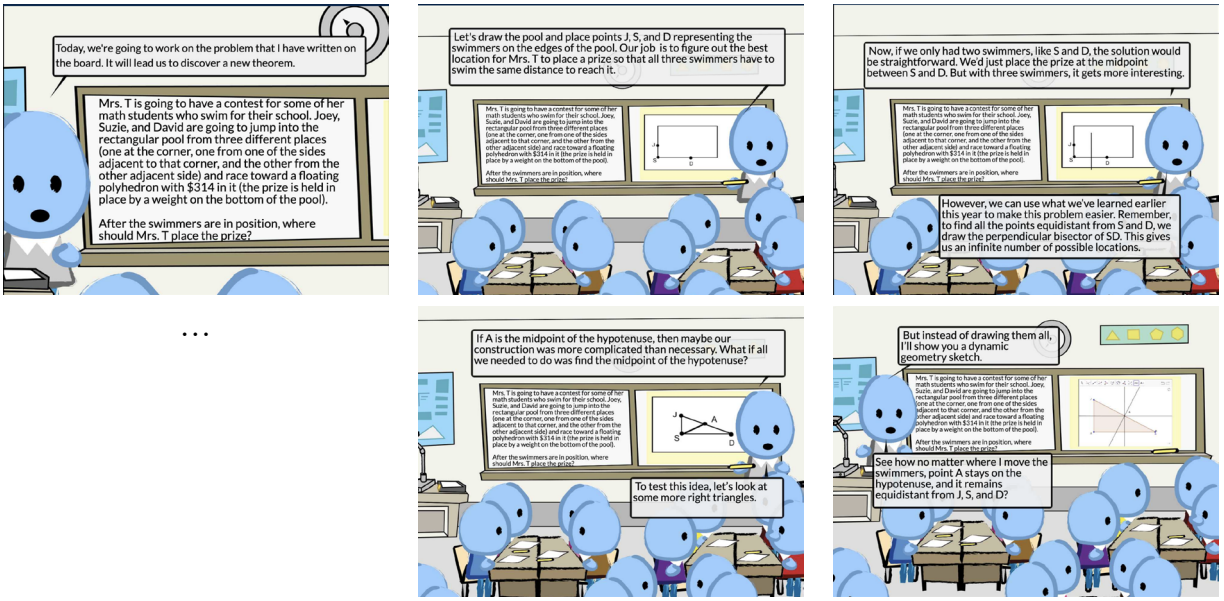
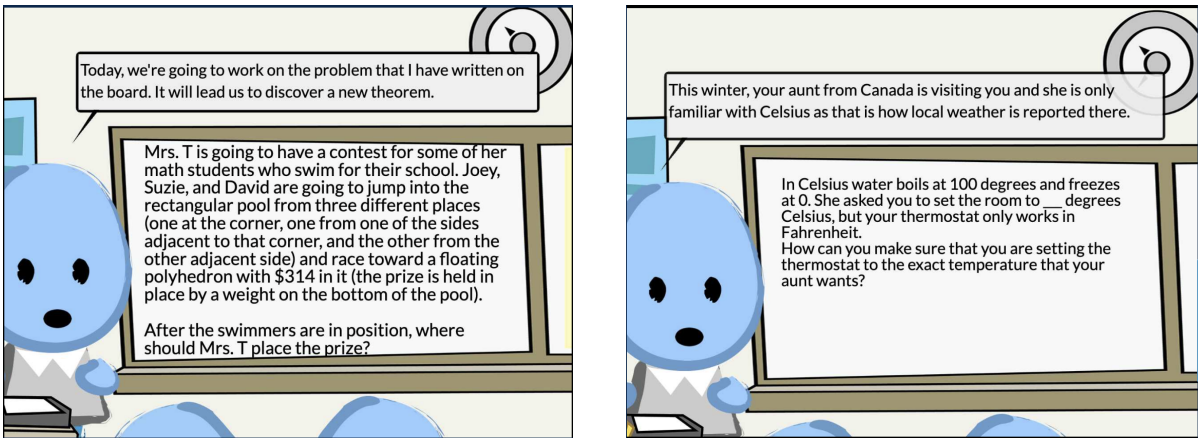


Figure 1: Example Sequence from the Geometry Lesson

This design intentionally presented instruction that deviates from expectations teachers might hold for exemplary professional development in order to acquaint teachers with the content of the lesson and elicit the desire to involve more student participation–none of which goals were conveyed to them directly. Prior to each synchronous session, participants reviewed the lesson representation asynchronously and posted comments using the *Anotemos* software embedded in Canvas. Before the module was presented, participants were informed only that the lesson would utilize problem-based instruction within a teacher-centered approach. Teachers placed written comments directly on the storyboard at moments of their own choosing. The activity was entirely open-ended, meaning that teachers were not paused at predetermined moments or asked to evaluate specific pedagogical moves. While two optional, pre-written sentence starters ("Students might be thinking…" and "The teacher should… because…") were available in the interface to assist participants in framing their thoughts, the participants could use these sentence starters at any moment of the lesson they wanted to comment on.

Participants included 64 secondary mathematics teachers from the United States (n = 36) and four non-U.S. contexts (n = 28). Of the 310 annotations in the data set, 107 were excluded

because they did not reflect teachers' spontaneous attention to the lesson. These included responses to required navigation prompts (e.g., "What key mathematical ideas do you notice in this frame?" and "Before continuing, take a moment to try the problem for yourself"); empty pin placements with no written comment; on-screen drawings; and irrelevant entries (e.g., "Hi, this is a test"). The final analytic sample consisted of 201 written annotations produced in response to two lesson representations (one in Algebra and one in Geometry), which started with novel problems (converting temperatures across two scales and placing a prize inside a pool) that ushered respective new ideas (the equation of the Line through Two Points and the Midpoint of the Hypotenuse Theorem) (see Figure 2). These annotations served as the primary unit of analysis for examining patterns of teachers' reactions to the lesson.



(a) (b)

Figure 2: The Problem Presented in (a) the Geometry Lesson and (b) the Algebra Lesson

Data Analysis

The coding scheme was developed abductively, moving between the design intentions of the lesson and the patterns we explored in teachers' annotations. We began from the structural design intent of the lesson representations: (1) Lack of student participation, (2) Teacher showing how the problem leads to the lesson goal, (3) Addressing common errors previously observed in the context of these problems. Then, rather than treating these design intentions as fixed codes, the research team went through all the annotations and traced how each appeared—how teachers attended to, commented on, or reacted to these features. This tracing did two things. It surfaced three pedagogical features aligned with the design intentions—(1) Active Student Engagement, (2) Teacher Facilitation and Eliciting Reasoning, and (3) Addressing Student Ideas and Misconceptions—and it revealed a fourth that the design had not anticipated: many teachers, particularly those in non-U.S. contexts, attended closely to the qualities of the problem itself. We therefore added (4) Problem Approachability and Quality, grounded in the data. The final coding scheme was thus settled only after this back-and-forth between design intention and data, rather than being imposed in advance or read off the annotations alone.

For each pedagogical feature, we specified the design characteristic integrated into the lessons and developed indicators for recognizing when teachers' comments showed attention to that feature (see Table 1). Annotations that addressed multiple features received multiple codes. Features were coded as Yes/No, where "Yes" indicated that a teacher's comment attended to the feature—whether by critiquing its absence or by approving of the teacher's choice—and "No"

indicated that the comment did not address the feature. This distinction separates whether teachers attended to a feature (captured by the Yes/No code) from how they evaluated it (examined in the qualitative analysis). We compared the proportions of "Yes" codes for each feature between groups using a chi-square test. Alongside this quantitative comparison, we analyzed the "Yes" annotations qualitatively to characterize how each group attended to and evaluated each feature. We also attended to patterns of "No" codes—that is, features a group did not comment on—since the absence of attention to a feature is itself informative about what each group treated as worth remarking upon.

Table 1: Coding Scheme for Identifying Noted Pedagogical Features

Pedagogical Feature	Departure from reform standard (NCTM, 1991; Lampert, 2001)	Example "Yes" Indicators	Example
Active Student Engagement	Teacher solves the problem without involving students	- Comments that "teacher is doing all the work" - Statements about students not having opportunities to work on problems	*"Instead of simply providing information about Fahrenheit; the teacher can engage students by asking about their prior knowledge. This approach encourages active participation and allows students to connect new concepts with what they already know."*
Teacher Facilitation & Eliciting Reasoning	Teacher shows all reasoning steps without student input	- Suggestions to elicit student ideas or thinking - Comments about need for questioning techniques - Proposals to have students explain their reasoning	*"Instead of directly defining a proportional relationship; students can analyze the graph's data points and make connections to concepts like the y-intercept; slope; and the corresponding equation."*
Addressing Student Ideas & Misconceptions	No student contributions; misconceptions are pre-stated by the teacher	- Comments about needing to hear student thinking - Suggestions to collect student ideas or solutions	*"Teacher is trying to break down the problem here… give students to come up with various possible points and then progress to the idea of perpendicular bisector and equidistant points*."
Problem Approachability & Quality	The problem is unfamiliar and conceptually generative	- Comments about problem difficulty or complexity	"*Initially the language of the question seems a bit vague- whether the points*

	are on other 2 corners or somewhere on the sides; and if on the adjacent sides; are they given or they have to be chosen?"

Findings

Analysis of 201 annotations (71 from U.S.-based teachers, 130 from non-U.S. teachers) revealed differences in how teachers from different national contexts attended to instructional features within the lessons. While both U.S. and non-U.S.-based groups of teachers evaluated each of the lessons, what features they chose to comment on—and how they evaluated each feature—differed in ways that may suggest different underlying assumptions about mathematics instruction. The analysis aggregates responses across lessons to focus on comparing them according to the national context where the participants came from.

Heightened Attention to Student Thinking

The most significant statistical difference appeared in the recognition of Addressing Student Ideas and Misconceptions ($p = 0.039$). U.S. teachers attended to this feature at a significantly higher rate (12.7%) than non-U.S. teachers (3.8%), with an odds ratio of 3.63, indicating U.S. teachers were approximately 3.6 times more likely to notice that student thinking was not addressed or elicited by the teacher. For these teachers, the lesson appeared inconsistent with what they expected from PD materials in that it fails to represent "exemplary" practice, as reflected in the comments such as: "*The teacher should… allow students to try this because… error analysis is a great way to assess conclusions made without evidence*." These annotations reflect an expectation—consistent with reform-based teaching standards (NCTM, 1991)—that student thinking should emerge, become visible, and drive instructional decisions. However, the lower rate of recognition among non-U.S. teachers does not indicate they failed to notice the lack of student participation but rather suggests different assumptions about when and how student thinking should surface in instruction.

Divergent Interpretations of Teacher-Centered Instruction

While both groups recognized features with teacher facilitation at a similar frequency, their qualitative evaluations revealed a divergence. When the lesson showed the teacher demonstrating solution steps with minimal student input, U.S. teachers predominantly responded to this as inconsistent with their facilitation expectations. Their annotations critiqued the teacher's "telling" approach (e.g., "*The teacher should ask what they noticed about the relationship instead of giving them the answer*"), explicitly noting that students were not given opportunities to construct their own understanding. This may suggest that these teachers read the teacher-centered approach as inconsistent with reform-based expectations for discourse-rich instruction. In contrast, a substantial number of non-U.S. teachers' annotations actively appreciated these same instructional moves. These teachers valued the teacher's guiding questions and explanations as effective pedagogical support (e.g., "*Question by teacher is good approach; where students start thinking in all possible ways*," "*Teacher gave a good clue to the students*"). Rather than viewing the teacher's guidance as problematic, these annotations framed the teacher's moves as helpful scaffolding rather than as a failure to elicit student thinking.

A parallel pattern emerged in *Active Student Engagement***,** with the groups recognizing features at a similar frequency. U.S. teachers who attended to this feature tended to frame the lack of student participation as a problem to be corrected, calling for students to do the mathematical work themselves rather than watch the teacher do it (e.g., "*Are the students*

discussing this; or is the teacher telling them?"). Non-U.S. teachers more often folded student involvement into appreciation of the teacher's moves, valuing opportunities for students to draw or explore while still crediting the teacher's guidance (e.g., "*Teacher can ask children to draw the figure themselves*"). Thus, even where the two groups noticed the same feature at similar rates, what they took the absence of student engagement to mean—a departure to be fixed versus a reasonable instructional choice to be supplemented—reflected the same divergence seen in their reading of teacher facilitation

Foregrounding the Worthwhile Task

While both groups remarked upon Problem Approachability at similar rates (8.5% vs. 11.5%), their annotations revealed different expectation priorities. Non-U.S. teachers more frequently attended to the mathematical and structural qualities of the problem itself, commenting on its "novel nature," "explorative quality," or conceptual richness (e.g., "*This problem provides opportunities for spatial reasoning, visualization and calculating midpoint*"). These annotations often appreciated the problem's mathematical generativity while simultaneously suggesting the teacher should provide additional scaffolding or clarity to support student access (e.g., "*Teacher must make it clear that points J & D are randomly selected*," "*Include specific temperature. Start with room temperature so that the student can differentiate the boiling and freezing temp*.")*.*U.S. teachers, by contrast, attended to the problem less through its mathematical generativity than through its accessibility to students. Their comments more often anticipated where the task's wording or lack of given measurements might create confusion (e.g., "*The vagueness of 'from one of the sides' could really trip up a lot of students*"), and where they appreciated the problem, they did so for its real-world relevance rather than its explorative quality (e.g., "*I love the real-world relevance of the problem*"). Thus, while both groups remarked on the problem at similar rates, non-U.S. teachers tended to foreground its conceptual richness, whereas U.S. teachers foregrounded student access and clarity.

In summary, the same lessons elicited different responses depending on teachers' pedagogical contexts. For U.S. teachers, the lessons departed from what they may have expected exemplary PD materials to model—particularly around student voice and discourse. For many non-U.S. teachers, the same instructional moves were characterized positively, with attention focused on problem quality and teacher clarity. This distinction suggests that what teachers find worth remarking upon is not universal but shaped by the instructional assumptions and professional expectations of their particular context.

Conclusion

Though the lessons were intentionally designed to depart from some expectations teachers might hold for exemplary professional development materials—particularly those aligned with U.S. reform-based teaching expectations (NCTM, 1991)—what teachers found worth remarking upon was not universal. Our findings suggest that teachers from different pedagogical contexts bring different implicit assumptions about teaching and learning to their reading of the same lesson representation. These assumptions may be shaped not only by their classroom traditions but also by what they expect from representations of instruction in a professional development setting (Schwarts et al., 2025). What U.S. teachers frequently identified as a significant instructional concern—students not actively engaging with mathematical problems was less salient to teachers from other pedagogical contexts, who attended instead to problem characteristics and mathematical precision. Yet, an important feature aligned with U.S. Standards—the use of novel problems to usher in new content—was more appreciated by non-U.S. teachers. These findings remind us that pedagogical approaches like teaching through problems

(Lampert, 2001) are not culturally neutral. What counts as good teaching, or as a departure from it, appears to be shaped by the particular instructional traditions and professional contexts in which teachers work.

Several limitations warrant consideration. First, our sample of non-U.S. teachers was relatively small (n = 28) and came from only four countries, limiting generalizability. The "non-U.S." category is heterogeneous, combining teachers from pedagogical traditions that may themselves differ substantially. Second, we analyzed written asynchronous annotations rather than synchronous verbal discussions or teachers' reflections on their own classroom practice. Teachers' written annotations may not fully capture their pedagogical reasoning; future work could examine synchronous discussions alongside annotations to better access the reasoning behind them. Third, we examined teachers' responses only by national context (U.S. vs. non-U.S.) and did not distinguish them by lesson (Algebra vs. Geometry). The lesson itself (Algebra vs. Geometry) could be another variable shaping teachers' responses. Analyzing annotations by lesson would be a useful direction for future work.

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A FRAMEWORK FOR CATEGORIZING TEACHERS' EVALUATIONS OF STUDENTS' RESPONSES TO SIGNIFICANCE TESTING PROBLEMS

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We propose a framework to characterize how in-service high school teachers evaluate students' written responses to significance-testing problems solved through an informal simulation-based approach. The framework emerged from a design research study conducted within a short professional development course involving 16 teachers. Grounded theory was used to code teachers' written analyses of student responses. Four main categories were identified: brief, evaluative, descriptive, and interpretative, each with two subcategories. These results were compared with similar frameworks in the literature to highlight the subtle but meaningful contributions we report. The framework may contribute to support research and professional development focused on fostering teachers' interpretations of student reasoning in statistics.

Keywords: Professional Development, Assessment, Data Analysis and Statistic

Research on mathematics teachers has shown a sustained interest in characterizing the knowledge required to teach effectively. Within this line of work, several theoretical frameworks of teachers' professional knowledge have been proposed (e.g., Ball et al., 2008; Fennema & Franke, 1992; Flores-Medrano et al., 2014). Among the components identified in these models, particular attention has been paid to the importance of knowing students' thinking. For instance, Fennema and Franke (1992) long ago highlighted the knowledge of learners' cognitions in mathematics, understood as knowledge of how students think, learn, and encounter difficulties with specific mathematical content. More recently, the Mathematical Knowledge for Teaching framework identifies Knowledge of Content and Students (KCS) as a central subdomain that integrates mathematical understanding with knowledge of students' reasoning, errors, and ways of thinking (Ball et al., 2008; Depaepe et al., 2013).

As statistical reasoning differs from mathematical reasoning (Wild & Pfannkuch, 1999), teachers need specific knowledge for teaching statistics (Groth, 2007). Accordingly, different frameworks have been developed to characterize Statistical Knowledge for Teaching (SKT), most of which highlight the importance of KCS (e.g., Burgess, 2006; Groth, 2013; Batur & Baki, 2024). However, little is known about how teachers interpret students' reasoning in statistics. At the high school level, we highlight the work of Nagle, Casey, and Carney (2018), who investigated teachers' noticing in a statistical task related to the line of best fit using an informal approach, and Shin (2020, 2021), whose studies involved preservice teachers analyzing video clips from a high school statistics class. In another study, Yilmaz and Yetkin (2023) analyzed preservice mathematics teachers' work during three cycles of lesson study implemented with seventh-grade students concerning bar, pie, and line graphs. Noticing is the common framework across these four studies.

On the other hand, research on hypothesis and significance testing in statistics education has focused mainly on teachers' content knowledge (e.g., Batanero et al., 2018; Matuszewski, 2018). However, we have not found studies related to teachers' KCS in this specific area.

With this perspective, we conducted a study within a short professional development course at a public upper-secondary school, working with in-service teachers, and building on our recent research on students' reasoning about significance tests and sampling distributions from an

informal, technology-supported perspective (Sánchez et al., 2024). The general question we posed was: *How do teachers evaluate students' written responses to significance-testing problems solved through an informal approach using simulated sampling distributions?*

In the process of analyzing the data to address this question, a framework emerged that organizes teachers' responses to a task intended to prompt them to analyze the students' reasoning. The aim of this article is to present this framework.

Background

Evaluation and Formative Assessment

The term *evaluation* in English is often associated with summative practices, whereas *assessment* more commonly refers to processes embedded in everyday teaching. This distinction does not exist in Spanish, where the word *evaluación* is used for both contexts, and more broadly to refer to estimating students' knowledge. Formative assessment (translated into Spanish as *evaluación formativa*) has been extensively researched as a way of examining students' work with the aim of supporting their learning and providing effective feedback. This has remained a relevant research topic within the PME-NA community, even in recent years (e.g., Confrey, 2024; Namakshi et al., 2024; Staples & Rostkowski, 2025), yet it has still been described as precarious (Confrey, 2024).

In contrast to these broad uses of assessment and evaluation, when analyzing teachers' interpretations of student thinking from a *noticing* perspective, teachers' responses are often described as "evaluative" referring to interpretations in which teachers acknowledge or judge the merits, affordances, or limitations of students' processes (e.g., Nagle et al., 2015; Yilmaz & Yetkin, 2023). This reflects a judgment-oriented use of the term *evaluation*, which foregrounds interpretation while setting aside other common aspects, such as grading or feedback. This clarification helps define the scope of our research questions, while also situating them within the literature. In this report, we use the term *evaluation* as a literal translation of the Spanish word to refer to teachers' estimating students' knowledge when analyzing student responses, not necessarily implying judgment or interpretation.

Developing Knowledge of Content and Students Through the Analysis of Student Work

As we explain below, the procedure used to elicit teachers' knowledge about students consisted of asking them to analyze students' written responses to tasks on significance testing. In practice, this prompt became a task of evaluating students' answers. Our review of the literature indicates that such evaluative work is a common practice for fostering KCS (e.g., Baldinger, 2020; Nagle et al., 2018; Namakshi et al., 2024). The written analysis of student responses offers several advantages over other professional development formats. For instance, written interactions can help teachers set aside preconceptions about students, as well as concerns related to classroom management, discipline, and curricular pressures that often dominate their attention (Crespo, 2000; Talanquer et al., 2015). Moreover, this format allows teachers to engage collaboratively in examining student thinking and to question and extend their own mathematical ideas (Crespo, 2000).

Research has identified patterns in how teachers evaluate students' written reasoning. Teachers who make brief or general inferences tend to adopt an evaluative or judgment-based orientation, often attributing students' difficulties to lack of attention, effort, or memory. In contrast, teachers who seek to understand students' thinking display a more interpretive orientation, focusing on specific features of students' reasoning (Crespo, 2000; Talanquer et al., 2015). Studies also report on the relationship between the nature of students' responses and teachers' interpretations. For example, Talanquer et al. (2015) found that selecting correct

answers can lead teachers to overlook misunderstandings present in students' justifications, whereas focusing on incorrect responses may draw attention away from productive elements in students' reasoning. Finally, even when teachers attempt to interpret student thinking, limited content knowledge can hinder their ability to make sound inferences about students' understanding (Nagle et al., 2018; Talanquer et al., 2015).

The literature identifies several factors that can strengthen teachers' capacity to evaluate student reasoning, including solving the same tasks as students before analyzing their work (Baldinger, 2020), working in small collaborative groups (Crespo, 2000; Yilmaz, 2022), and participating in professional development focused on student thinking (Jacobs et al., 2010; Yilmaz, 2022). All three features were incorporated into the present study.

We acknowledge that referring to students' written responses as students' thinking or reasoning (e.g., Namakshi et al., 2024) may be controversial, as thinking is a mental activity and the relationship between students' thinking and what they write is complex. For instance, students' verbal justifications may include additional reasoning not documented in written responses (Staples & Rostkowski, 2025). We assume, following Staples and Rostkowski (2025), that written justifications are expected to include claims, evidence, and reasoning; however, to avoid overinterpretation, we refer to students' answers simply as responses.

High School Students' and Teachers' Conceptions of Significance Testing

Knowledge of content and students in the domain of significance testing is informed by a broad body of findings from statistics education research. This literature documents recurring patterns in how students and teachers understand key ideas and the difficulties they encounter. In what follows, we highlight some results that are particularly relevant to this study.

A common tendency among students is to believe, inappropriately, that the purpose of a significance test is to confirm or prove that a hypothesis is true (or false; Batanero, 2000). Confusion is also frequently reported among three related distributions: the sample distribution, the sampling distribution, and the population distribution (Inzunza & Jiménez, 2013). It is worth noting that similar difficulties have been documented not only in students but also in both preservice and in-service high school teachers (Batanero et al., 2018; Haller & Krauss, 2002; Matuszewski, 2018; Thompson et al., 2007).

When an informal approach supported by computer simulation is used, students often struggle to coordinate two different perspectives: the real world, in which the empirical sample is collected, and the hypothetical perspective represented by the SSD, in which the null hypothesis is assumed to be true (Case & Jacobbe, 2018; Sánchez et al., 2024; Silvestre et al., 2024).

Method

The purpose of our study was to characterize how in-service high school teachers evaluate students' written responses to significance-testing problems solved through an informal approach using simulated sampling distributions (SSDs). The specific aim of this article is to present a general framework to describe and anticipate the ways in which teachers evaluate students' responses to such problems. This framework emerged from activities carried out in a professional development course. The methodological approach underlying the research from which the framework emerged was design research (Bakker, 2018).

Participants

Sixteen in-service teachers from a public upper secondary school in Mexico voluntarily participated in a professional development course offered during the inter-semester break. Of these, nine held graduate degrees (three doctoral and six master's degrees). Seven had received training to teach statistics, and five had taught the topic of hypothesis testing. Two participants

did not provide their background information. Among the 14 teachers who did provide data, the mean number of years teaching math or statistics at the upper secondary level at the time of the course was 15.6 years, and the median was 11.5 years. The authors of this article, together with a respected teacher at the same school served as instructors for the course.

Procedure

A 20-hour intersession course, organized into five four-hour sessions, was delivered to teachers remotely and synchronously. It was conducted in five sessions of four hours each and aimed to support teachers in two directions: (1) solving significance testing problems from an informal approach with the aid of computer simulation, and (2) fostering the analysis of students' reasoning by examining their responses to significance testing problems. Session 1 introduced the course, reviewed some fundamental statistical concepts, and taught participants how to generate SSDs using the software Fathom. Session 2 focused on solving significance tests problems from an informal perspective with the support of simulation. Subsequently, teachers solved three problems and compared the traditional approach with the simulation-based approach. Sessions 3, 4, and part of Session 5 were devoted to problems in which teachers evaluated students' responses to significance testing tasks, identified and explained the ways of reasoning that led students to their answers.

Instrument

Ten student responses to significance tests problems were selected for teachers to analyze. These responses were part of a data set from a previous study on the same topic conducted with students (García-Ríos, 2017). The statistical problems had been previously solved by the teachers during the first session of the course. In selecting the student responses, the researchers sought examples that displayed misconceptions or relevant features of students' reasoning, with the aim that teachers would identify them. The 10 responses were administered across the sessions as follows: three in Session 3, four in Session 4, and three in Session 5. Teachers worked in pairs during these Sessions. As an example, one of the tasks, called Situation, administered in these sessions is presented in Table 1. The students' response in this Situation illustrates the challenge of coordinating a real-world perspective with a hypothetical one.

Data Analysis Procedure

The analysis followed principles of Classic Grounded Theory (CGT), a rigorous and systematic methodological approach for generating an emergent, inductive, and abstract conceptual theory (Glaser & Strauss, 1967). A feature that distinguishes CGT is that the analysis is not based on pre-established categories or theoretical frameworks. To do so, the process began with coding through a constant comparison of the data. With additional procedures, such as saturation and theoretical coding, categories, their properties, and their mutual relationships were identified, as well as the emergence of a central category (Birks & Mills, 2015).

In our case, the first author coded all the responses provided by pairs of teachers across the different situations. Each code was subsequently discussed with the second author to clarify its meaning, merge related codes, and define the names and characteristics of each category. This process was conducted twice throughout the research to support category saturation.

We also highlight that two literature reviews were conducted. The first was an a priori review, which helped contextualize and define the research problem. This review focused on significance testing from an informal approach and on teachers' knowledge of significance testing. The second was an a posteriori literature review. After completing the coding process and developing the categories, we conducted a literature review related to the main category

(evaluation) in order to frame and deepen our results and extend the discussion. In this a posteriori review, we found noticing frameworks to be particularly relevant to our study.

Table 1: Example Task Situation

Context	High school students with no prior background in statistical inference generated SSDs in Fathom and used them to solve a problem. The following responses represent intuitive student conceptions. For each response, answer the question: How did the students reason to reach their conclusion?
Problem 1	It is believed that half of the population that drinks cola prefers Pepsi and the other half prefers Coke. Coca-Cola claims that the majority (more than 50%) of cola drinkers prefer Coke over Pepsi. To support this claim, the company surveyed 60 randomly selected people who regularly consume cola. Out of the 60 participants, 35 preferred Coca-Cola. Is the sample information sufficient to reject the hypothesis that the population is evenly split?
Student Response	[They simulated 500 samples $n = 60$ from an urn with $p = 0.5$ and generated the SSD.] The hypothesis does not hold because we took the surveys from 31 responses onward that preferred Coke, which gives a total of 233 surveys. If this is multiplied by the 60 people in each survey, we obtain a total of 13,980. Meanwhile, from 30 (which represents 50%) and below, there are 267 surveys, and multiplying this by the 60 people gives a total of 16,020, which represents a quantity greater than 50%. That is, more than 50% of the population does not prefer Coca-Cola. We are confident in this conclusion because we relied on the results shown in the graph, considering that the total number of people in this case is 30,000.

Results

In this section we present the categories that constitute the evaluation framework proposed in this study. The categories and their properties emerged through an open coding process in which common response patterns were identified in at least two responses from different teams, whether from one activity or another. The categories are not mutually exclusive, since some responses display characteristics that allow them to be classified into two or more categories.

Below, we provide a brief definition of each category and include a characteristic example of a teacher response classified within it. The number in parentheses indicates the frequency of responses classified in the corresponding category or subcategory.

As an example of how the coding happened, the category “brief” was initially called “conclusive” and later “immediate.” The term “conclusive” was assigned because teachers appeared to be oriented toward reaching conclusions; however, it did not reflect the brevity of their responses. The term “immediate” was then considered to describe the absence of mediation. However, since the synthetic subcategory could be both mediated and brief, “brief” was ultimately chosen as a more general characteristic that encompasses both the reductionist and synthetic subcategories.

Brief Response

Responses classified in this category (24) are short, concise, and sometimes blunt statements in which the teacher highlights an isolated feature of the student’s response to comment on the

underlying reasoning. This category is divided into two subcategories, *reductionist* and *synthetic*.

The reductionist responses (19) are general and superficial, since although they mention a feature that is relevant to the student's response, they fail to capture a key aspect of the argument behind it. For example, in response to the question *How did the student reason?* one pair answered: *Through the distributions generated in Fathom.* This response does not reflect any key element of the students' argument.

In contrast, the synthetic responses (5), despite being brief, point out or interpret specific aspects of students' reasoning. For instance, other team's response is as follows: *By taking the 500 samples of size 60 as a population.* This team identifies an important aspect of the students' reasoning—namely, that they confuse the SSD with the population—although no further elaboration is provided.

Evaluative

Responses in this category (30) judge whether the student's response, or part of it, is correct or not according to the teacher's expectations. Their explanations are likely variants of the argument that a teacher would give to a student when assigning a grade on an assessment. This category is also divided into two subcategories: *absent* and *present*.

Responses classified as absent (20) judge students' productions based on aspects that do not appear in the response but are considered by the teacher as elements that should be present. This subcategory is usually identified through negations, for example: *does not consider variability*.

Responses classified as present (10) judge what is explicit in the response, mentioning at least one aspect to support the judgment. For example: *They did correctly interpret and analyze what was obtained in the simulation.*

Descriptive

Responses in this category (43) merely describe part or all of the procedure represented in the student's response without adding anything beyond what is written there. Typically, they paraphrase some or all of the expressions present in the student's response and occasionally replace vague terms with more precise ones. In general, these responses consist of two or more sentences. This category is divided into *procedural* and *explanatory* subcategories.

The procedural subcategory includes responses that focus on general aspects of the procedure by paraphrasing students' expressions and using their same vocabulary. For example, one pair of teachers wrote: *They consider the sample size and multiply it by the number of samples.* This operation is indeed performed by the students, but the response adds nothing further.

On the other hand, responses classified as explanatory translate words from the students' vocabulary into the corresponding statistical or mathematical terms or add formal terminology to specify what the students are doing. In addition, explanations or hypotheses about how students reasoned are sometimes formulated. For example, one pair of teachers stated:

> Their reasoning in the significance test was correct; the procedure is correct, since they calculated the percentage of defective pieces of at least 16 and contrasted it with the p-value, finding that the result is typical in the sample, and concluding that the machine should not be repaired.

In this response there is some confusion on the part of the teachers, since the "percentage of defective pieces of at least 16" corresponds to the p-value, and the comparison should be made with the 5% significance level. What is important in this response is that the teacher introduces the term *p-value* to indicate aspects of what the student is doing.

Interpretative

Responses in this category (47) make interpretations; that is, they go beyond what is

observed by offering an interpretation of the students' responses, or parts of them, either from a statistical or a didactical perspective. Two subcategories are identified: *statistical* and *didactical*.

Teacher responses classified in the statistical subcategory (25) include statistical concepts and terms that are not explicitly mentioned in the analyzed response. For example, one team's answer is: *They use the method correctly, although we observe a slight confusion with the 5%, since they consider defective pieces when the value actually refers to samples.* In this response, the teachers suggest that the students should have used the term *sample* when referring to the 5% significance level.

In the didactical subcategory (22), interpretations are made about students' knowledge, reasoning, difficulties, or curricular factors. For example, another pair of teachers stated: *They reasoned using it as a statistical rule for decision making. The rule is: the result must be less than or equal to 5% of the pieces in order to consider a result.* The teachers recognize and highlight the central point of the reasoning in the student's response and judges it as appropriate.

Table 2 presents the categories, subcategories, and their frequencies. The total frequency is 144 because some responses were assigned to two or more categories.

Table 2: Frequencies of Categories and Subcategories

Brief (24)		Evaluative (30)		Descriptive (43)		Interpretative (47)	
Red.	Syn.	Abs.	Pres.	Proc.	Expl.	Stat.	Did.
19	5	20	10	25	18	25	22

Note. Red. = Reductionist; Syn. = Synthetic; Abs. = Absent; Pres. = Present; Proc. = Procedural; Expl. = Explanatory; Stat. = Statistical; Did. = Didactical.

Discussion and Conclusion

The categories of the framework are similar to some categories proposed in other studies that share purposes and methodological features with ours. For example, Crespo (2000) initially asked preservice teachers to differentiate between descriptive and interpretative responses. She also noted that students held evaluative attitudes that, because of the course she implemented during a long period, shifted toward interpretative attitudes. We also found several points of convergence with Nagle et al. (2018), who worked with in-service secondary school teachers on the line of best fit. In their results, they also proposed four codes to describe teachers' interpretations of student work. However, their framework contrasts with ours in that, although we agree on several observations, the way of organizing them is different. The codes they propose are: (1) evaluative affordances (2) evaluative deficits, (3) interpret students process, and (4) interpret students understanding. Making a general comparison and setting aside caveats related to differences in the statistical content of the tasks, the first two codes proposed by Nagle et al. can be grouped within our Evaluative category. However, whereas their framework emphasizes identifying affordances or deficits in students' answers, ours focuses on whether teachers evaluate what is present in students' answers or what is missing, so that what is present may still reveal deficit. Also, their last two codes are similar to our descriptive procedural and our interpretative category, in the case of describing process we identify that teachers can do it by paraphrasing student's work or by adding teachers' knowledge, we believe this distinction is important for purposes of developing teachers' knowledge of content and students. In their framework, Nagle et al. introduce levels of sophistication based on whether teachers' claims are supported by evidence from student work, an aspect that could strengthen our framework. By contrast, our interpretative category does not require teachers to infer students' understanding

which we regard as more aligned with research-oriented analysis; instead, it centers on interpreting the reasoning expressed in the student's response, either from a statistical or pedagogical perspective. This distinction becomes particularly relevant when aiming to support formative assessment practices.

It has been reported that most teachers tend to describe or evaluate students' productions and, in general, find it difficult to interpret their reasoning (Crespo, 2000; Nagle et al., 2018; Talanquer et al., 2015; Yilmaz, 2022). Our results show that interpreting students' reasoning in the context of significance testing with simulation is also challenging for teachers. This difficulty may be associated with teachers' limited content knowledge (Nagle et al., 2018; Talanquer et al., 2015), the complexity of students' reasoning (Talanquer et al., 2015), or teachers' lack of experience both teaching the content and analyzing students' responses (Jacobs et al., 2010). It is plausible that the cause involves a combination of these three factors.

In the argument that follows, some plausible hypotheses are proposed. It is worth noting that in-service teachers may respond to the tasks somewhat differently from preservice teachers (Crespo, 2000). For example, in-service teachers have experience repeatedly facing the problem of grading large numbers of students' exams. This implies that they develop evaluative habits consisting of deciding whether a response is correct or not and providing a brief, and probably stereotyped, justification to convince the student of the error. Thus, when confronted in this study with the task of analyzing students' responses, their evaluative habits naturally emerged; that is, they associated the task with the practical problem of assessment, and their responses fell into the "brief" or "evaluative" categories.

By contrast, some of the experienced in-service teachers who participated in our study had completed graduate studies, some in Statistics and others in Mathematics Education. The latter were highly likely to have been introduced to tasks involving the interpretation of data derived from students' work and were therefore more inclined to offer responses situated at the interpretative level. Although these teachers may initially have applied their evaluative habits, within the dynamics of the course they shifted toward an interpretative stance (Crespo, 2000).

Under these considerations, we formulated the hypothesis that the categories of the framework are of three types. The first comprises the brief and evaluative categories, which reflect dispositions oriented toward solving practical problems of assessment. The second consists of the descriptive category, which is transitional in nature, as its purpose is merely to clarify the student's response without an interpretation intent. The third type is formed by the interpretative category, which reflects a theoretical disposition in which the problem to be addressed is that of increasing teacher's understanding. This is because the identification of a pattern in students' thinking or of a misconception does not, in itself and immediately, suggest consequences for the teacher's instructional practice (Groth, 2013; Jacobs et al., 2010).

In line with grounded theory, the central concept that emerged from our analysis was evaluation. Our initial research question asked *How do teachers develop knowledge of content and students by evaluating students' written responses to problems with statistical content?* Yet, constructs such as models of students' thinking, commonly used by researchers to conceptualize knowledge of content and students (Groth, 2013; Liang, 2023), may be experienced by teachers as foreign or abstract, whereas assessment is deeply embedded in their professional practice. This tension itself constitutes one of the findings of our study.

One limitation of this study is that teachers' work was examined outside the context of classroom practice. We were also unable to observe how the course influenced teachers' instructional practices. Nonetheless, the findings may inform future professional development

initiatives. A further limitation concerns the limited time available to support the development of teachers' understanding of the informal approach. Increased teacher content knowledge would likely lead to different research outcomes.

ChatGPT was used to refine the grammar and enhance the clarity and fluency of the English in this manuscript, following ethical guidelines.

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ASSESSING PRIMARY SCHOOL TEACHERS' PERCEPTIONS OF PROFESSIONAL DEVELOPMENT USING A NOVEL RESEARCH INSTRUMENT

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Professional development (PD) is widely regarded as a key component of teacher growth. Existing measures assess teachers' satisfaction with PD, but few do so while also capturing the PD opportunities available to teachers, teachers' participation in PD, or the challenges teachers encounter in PD. The current study describes a researcher-designed instrument used to survey 228 public and private primary school teachers in Nigeria. Descriptive revealed that a range of PD formats were available to teachers, but overall teacher participation was moderate. Major challenges were inadequate financial incentives, insufficient institutional support, excessive workload, and a lack of interest. The results show that the designed instrument produced actionable evidence on teachers' access to PD, particularly in less-studied settings.

Keywords: Professional Development, Teachers' Engagement, Primary School

Today, the demands of the changing educational ecosystem have redefined the competencies and roles expected from teachers (Ekinci & Akar, 2019). In addition to traditional pedagogical competencies, educators are expected to demonstrate mastery of digital pedagogy and STEAM-based teaching, inclusive pedagogy, and the ability to understand the changing curriculum and assessment systems. Consequently, continuous professional development (PD) is needed to support teachers' learning.

The quality and availability of PD are critical to teachers' effectiveness and growth, especially when classroom demands are evolving. High-quality PD is characterized by being practice-based, collaborative, and effective in the long-term and must be facilitated by coaching or follow-up to enhance the chances of significant changes in teaching (Darling-Hammond et al., 2017; Desimone and Garet, 2015). The study of PD access is particularly instructive in the context of Ilorin South Local Government Area (LGA) of Kwara State in north-central Nigeria. The primary schools in this LGA comprise government-funded public schools and privately owned schools. Continuous professional learning is a requirement of the Teachers Registration Council of Nigeria (TRCN) for all registered teachers, but schools' capacity to implement this requirement varies across the region (Ajadi, 2020; Ojo & Emeka-Nwosu, 2023). This context is important to consider when interpreting the patterns of PD access reported in this study.

Notably, educators can make rational professional choices not to participate in PD, such as when they believe a training is not relevant to classroom realities, offered as isolated experiences with no follow-up, scheduled in a manner that is incompatible with their workload or necessitates personal expenses that are not institutionally subsidized. Studies show that the interaction of PD design, school conditions, and professional judgement of teachers determine their level of participation (Darling-Hammond et al., 2017; Opfer & Pedder, 2011). This implies that enhancing learning outcomes is not just about pushing attendance but also providing equitable access to well-designed PD with sufficient resourcing and long-term support, particularly at the primary level where the quality of teaching is foundational (Ssh & Jacob, 2020).

The availability of PD opportunities varies significantly among teachers. Despite its importance, access to PD is uneven, resulting in unequal opportunities for teachers to engage in

meaningful learning experiences. Although nations invest significant financial resources in PD programs to improve teachers' competencies and students' academic performance (Popova et al., 2022), access to high-quality PD remains unevenly guaranteed. While studies found that professionally designed PD models, including workshops, in-service training, mentoring, professional learning communities, and technology-based learning environments, can positively affect the quality of the instruction provided to professionals (Darling-Hammond et al., 2024; Yoon et al., 2020; Lauer et al., 2014), the involvement of teachers however, is usually determined by the resources, leadership support and organizational structures that are present within their educational institutions.

Beyond unequal access, structural and situational barriers limit teachers' meaningful participation in PD. Empirical research also suggests that insufficient funding, unequal budgetary distribution, ineffective professional support structures, and lack of follow-up mechanisms are also major limiting factors to teacher participation in PD programmes (Bowman et al., 2022; Saleem et al., 2021). Even when equitable access and active participation are present, situational constraints may hinder sustained engagement. Structural barriers that teachers might face include a lack of funding, excessive workload, poor leadership support or poorly planned programmes. In other instances, where PD opportunities might exist, they are scheduled at inappropriate times or are not relevant to teachers' needs. Such obstacles may decrease motivation, limit long-term involvement, and eventually limit the influence of PD on instructional practice.

In spite of the acknowledged importance of PD, existing research instruments employed to study teachers' experiences of PD are limited (Ritcher & Ritcher, 2024; McChesney & Aldridge, 2018). It can be estimated that the most prevalent forms of large-scale surveys and institutional assessments assess the level of teachers' satisfaction or their impressions about the quality of PD (Popova et al., 2022). Scholars have developed a variety of instruments to measure different aspects of teachers' PD. Indicatively, scales developed within the Teaching and Learning International Survey (TALIS) instrument assessed the professional practices, background, development activities, and teaching conditions of teachers on an international scale, which provided insight into how contexts influence teachers' level of engagement with PD opportunities (Ainley & Carstens, 2018). Likewise, national survey instruments like the TALIS 2018 teacher questionnaire have included subscales of PD barriers, with teachers having to rate the importance of factors like the absence of institutional support, time-related conflicts, and relevance of subject content to their participation in professional learning (Yuner, 2022). Additionally, large-scale surveys have also come up with instruments that focus on the features of PD programmes.

Although these tools provide significant information regarding teachers' PD, they possess significant limitations in comparison to the objectives of the present study. Few instruments are available for the systematic evaluation of whether teachers are able to access PD equitably and participate in it, or capture the barriers that constrain their involvement in PD (Ndlovu et al., 2024). Large-scale measures like TALIS are cross-nationally comparable and tend to capture general measures of PD participation and satisfaction rather than the systematic measures of accessibility, barriers to engagement, and participation patterns. Other instruments, including those that focus on the characteristics of PD or workplace learning, are more likely to concentrate on the quality or characteristics of PD programmes, as opposed to the circumstances in which teachers can fairly access and participate in PD opportunities or the circumstances that hinder participation. Such a gap in measurement tools highlights the necessity of a specific tool, such as the Assessment of Professional Development Programmes among Primary School

Teachers Questionnaire (APDPPSTQ). In contrast to generic PD surveys, the APDPPSTQ is designed to systematically measure the availability and type of PD, as well as the level of participation and the barriers teachers face.

Theoretically, the APDPPSTQ is based on three complementary frameworks. The availability subscale was informed by Desimone's (2009) four core features of effective PD: content focus, active learning, coherence, and duration, as well as collective participation. Second, the participation subscale was guided by the TALIS conceptual framework (Ainley & Carstens, 2018), which was used to guide the items, which were parallel to internationally validated indicators of frequency and type of engagement. Third, the barriers subscale was based on the complexity model of teacher professional learning developed by Opfer and Pedder (2011), which posits that barriers (such as workload, funding, and institutional culture) lie between the availability of PD and teacher participation.

Thus, the three dimensions of access, willingness and barriers are interrelated and influence teachers' participation in PD. Access is the base, without access, teachers can't be involved in opportunities that are not available to them. Availability does not, however, ensure participation, but rather teachers' willingness to participate is dependent on the relevance, quality and supportiveness of the opportunities offered. Meanwhile, structural and situational barriers can be both access barriers and can also dampen motivation over time. So, measuring just one or two of these dimensions might lead to a partial assessment of teachers' PD involvement. All three dimensions need to be taken into account for a comprehensive assessment.

Methodology

For this research study, the researcher adopted a quantitative descriptive survey research design to assess PD programmes among primary school teachers in Ilorin South Local Government Area of Kwara State, Nigeria. This context was chosen because it is a microcosm of the overall primary education system in Nigeria as it has both publicly funded schools, managed by the State Universal Basic Education Board (SUBEB) and privately owned schools, with teachers ranging from the Nigeria Certificate in Education (NCE) to postgraduate education. I used a structured questionnaire to sample teachers' perceptions about PD programmes. The population of the study comprised all public and private primary school teachers within Ilorin South Local Government Area. Stratified and simple random sampling techniques were used to select 228 primary school teachers.

I constructed an instrument titled "Assessment of Professional Development Programmes among Primary School Teachers Questionnaire" (APDPPSTQ). APDPPSTQ was subdivided into four sections: Section A: demographic distribution of the respondents (Gender, teachers' qualification and school type). Section B: with 10 items measuring available PD programs on four Likert scales of Strongly Agree, Agree, Disagree and Strongly Disagree. Section C with 10 items measuring the level of participation in PD programs on a 3-point Likert scale of Always, Sometimes, and Never. Section D challenges hindering primary school teachers from participating in PD programs, on four Likert scales of Strongly Agree, Agree, Disagree and Strongly Disagree.

The items of the APDDPSTQ were developed based on the existing conceptual and empirical frameworks of teacher PD. The availability and participation sections were informed by the framework of Desimone (2009) on the key characteristics of effective PD and survey designs of major international surveys, including the OECD Teaching and Learning International Survey (TALIS) (Ainley & Carstens, 2018). The participation items are conceptually related to Garet et al. (2021), who also note that frequency and collective engagement are important aspects of PD

involvement. Empirical literature that informed the challenges section was the identification of structural and contextual barriers to PD participation that include time constraints, funding limitations, administrative support, and motivational factors (Avalos, 2011; Opfer & Pedder, 2011). All these frameworks were used to inform the item generation and provide conceptual consistency with the existing research.

The research was carried out in Ilorin South Local Government Area of Kwara State in Nigeria, an urban district, comprising of both publicly and privately owned primary schools. These schools differ in terms of administrative structures, funding schemes, and institutional support, and these differences can affect teachers' access to PD opportunities. The sample consisted of 228 primary school teachers in both the public and the private schools in the district. The demographic profiles of the respondents differ in their academic qualifications, years of teaching experience, and type of school, which represents the diversity of the teaching workforce in the region. All respondents were actively involved in classroom teaching at the primary level during data collection and had different levels of exposure to PD programmes. Their responses revealed how PD opportunities are offered, whether and to what extent teachers participate, and what difficulties they encounter during this process.

The face and content validity of the instrument were assessed by three experts in education; their comments and observations were used to draft the final instrument. The data obtained from the study were analyzed using descriptive statistics (frequency, percentage, mean and standard deviation) and inferential statistics, including an independent sample t-test and analysis of variance at a 0.05 level of significance. The internal consistency of the instrument was assessed using Cronbach's alpha. The questionnaire was pilot-tested with 20 participants who were not included in the main study. The analysis indicated good reliability for the overall instrument ($\alpha = .89$).

The Instrument

The demographic distribution of teachers comprised gender, school type and qualifications. Out of 228 primary school teachers who participated in the study, 90 (39.5%) were male, while 138 (60.5%) were female; however, more female teachers participated than their male counterparts owing to the high concentration of women in the teaching sector in this region. In addition, out of 228 primary school teachers who participated in the study, 82 (36.0%) were from public schools, while 146 (64.0%) were from private schools; however, more private school teachers participated than their public school counterparts. Finally, of the 228 primary school teachers who participated in the study, 182 (79.8%) were Nigeria Certificate in Education (NCE) holders, 26 (11.4%) were BSc/B.Ed/Higher National Diploma (HND) holders, while 20 (8.8%) were M.Ed/ PhD holders.

Professional Development Availability and Access

The overall data from the availability of the PD opportunities in the primary schools in Ilorin South LGA in Table 1 show that the opportunities for PD are reasonably wide in the primary schools in the LGA (item means ranging from 2.96 to 3.28 on a 4-point scale). In particular, the most highly rated items, Kwara Learn Initiatives ($M = 3.28$), academic conferences ($M = 3.28$), and further training opportunities ($M = 3.28$), indicate that both government-coordinated and institutionally supported formats are considered relatively accessible. Exposure to global best practices was rated the lowest ($M = 2.96$), suggesting that international alignment in PD is a gap area. The patterns indicate that the instrument is sensitive to variation in PD provision within contexts, rather than yielding uniform patterns of response, which is an indicator of discriminative validity.

Table 1: Mean and Standard Deviation of the Available Professional Development Programmes among Primary School Teachers in Ilorin South Local Government Area

S/N	Items	Mean	SD
1	Government-sponsored STEAM capacity-building seminars are available for teachers.	3.18	0.65
2	Continuous Professional Development training programmes abound for teachers.	3.09	0.75
3	Teachers can easily partake in development courses for in service.	3.17	0.69
4	School administrators sponsor professional development programs for teachers from time to time.	3.05	0.76
5	Teachers are often exposed to current global best practices in the teaching profession from time to time.	2.96	0.81
6	Teachers have a chance to attend Kwara Learn Initiatives to improve their teaching practices.	3.28	0.63
7	Webinars are available for teachers to improve their teaching methodologies.	3.13	0.68
8	School managers sponsor teachers to attend academic conferences related to their field of specialization.	3.28	0.69
9	There are several opportunities for teachers to embark on further training to improve their academic credentials.	3.28	0.67
10	Teachers are exposed to digital skills training for teachers.	3.21	0.64

Professional Development Engagement

According to Table 2, results indicate that primary school teachers engage in PD programmes that are available occasionally. In this context, average participation is a weighted average of 2.14 on a 3-point scale (1 = Never, 2 = Sometimes, 3 = Always) meaning that most teachers engage in PD sometimes, but not consistently or reliably. This is not reflective of teachers' general lack of interest in professional learning, but rather reflects the episodic nature of professional learning in the context of structural conditions, such as financial pressures, workload stress, and insufficient institutional support, as evidenced by the challenges data. For instance, for in-service courses (M = 2.11) and Kwara Learn Initiatives (M = 1.95), the participation rates are among the lowest but the perceived availability rate is among the highest (Table 1), indicating a meaningful gap between availability and participation that should be addressed through institutional and policy interventions.

Table 2: Mean and Standard Deviation Showing the Level of Participation in Professional Development Programmes among Primary School Teachers in Ilorin South

S/N	Items	Mean	SD
1	I am a beneficiary of government-sponsored STEAM capacity-building seminars for teachers.	2.04	0.81

2	I have had the opportunity to attend Continuous Professional Development training programmes.	1.99	0.76
3	I have embarked on development courses for in-service teachers.	2.11	0.76
4	The administrators of my school sponsor professional development programs for teachers from time to time.	2.25	0.72
5	I have been exposed to current global best practices in the teaching profession in recent times.	2.16	0.79
6	I have had a chance to attend Kwara Learn Initiatives to improve my teaching practices.	1.95	0.89
7	I have upgraded my teaching methodologies through webinars.	2.11	0.74
8	I have been sponsored by my school to attend academic conferences related to my field of specialization.	1.92	0.84
9	I have embarked on further training to improve my academic credentials.	2.38	0.73
10	My digital skills have improved significantly as a result of my exposure to digital skills training for teachers.	2.45	0.69
	Weighted Mean	**2.14**	

Challenges Associated with Access to Professional Development

Results from Table 3 show that primary school teachers are experiencing a number of challenges that restrict their involvement in the PD programmes. Significant limitations include insufficient funding by the government and the administration of the private schools, the absence of regular promotion of teachers who are professionally developed, and the absence of institutional support. Other identified challenges include limited digital skills to develop professionally online, the absence of fringe benefits, and the unconcerned attitude of the school administrators towards PD programs. Teachers are further limited by high costs of in-service programmes, negative attitude towards change, workloads and administrative duties. The average scores of all the items were higher than the benchmark value, which proved that these factors are a major barrier to the participation of teachers in PD programmes.

Table 3: Mean and Standard Deviation Showing the Challenges Hindering Primary School Teachers from Participating in Professional Development Programmes

S/N	Items	Mean	SD
1	The Government and private school administrators do not allocate enough funds to engage teachers in professional development programmes.	3.29	0.69
2	Teachers do not have substantial time to participate in professional development programmes due to excessive workload.	3.05	0.76
3	Lack of institutional support discourages teachers from harnessing the diverse potentials of various professional development programmes.	3.24	0.64

4	Lack of the required digital skills hinders teachers from engaging in online professional development programmes	3.18	0.73
5	Negative attitude towards change impairs the rate of teachers' professional development.	3.06	0.74
6	Lack of fringe benefits reduces teachers' motivation for engaging in professional development programmes.	3.18	0.68
7	High costs of in-service professional development programmes limit teachers' professional development.	3.11	0.70
8	Lack of regular promotions for qualified and professionally developed teachers discourages teachers from enrolling in in-service professional development programmes.	3.25	0.71
9	The administrative work of teachers leaves no time for further professional development.	2.89	0.82
10	The lackadaisical attitude of school administrators towards professional development programmes reduces the morale of teachers to partake in professional development programmes.	3.15	0.69

Discussion

The survey instrument captured a wide range of PD opportunities, including government-sponsored STEAM programmes, capacity-building initiatives, continuous PD training, in-service programmes, webinars, academic conferences, digital skills training, and the Kwara Learn Initiative. Rather than a direct description of teachers' experiences, these results indicate the survey's capacity to record the types and perceived accessibility of PD formats in the local school environment. The response pattern aligns with previous literature showing that teachers are typically provided with various PD formats across different systems (Venter et al., 2024; Abd Ghani et al., 2023; Saleem et al., 2021). In this regard, the survey was able to determine the scope of PD programmes available within the region.

The instrument, however, fails to distinguish the programmes based on depth, duration and instructional coherence. Although the survey differentiates between event-based formats (e.g., seminars, webinars, conferences) and more structured initiatives (e.g., continuous PD training, in-service programmes, and the Kwara Learn Initiative), it does not capture actual measures of programme quality or long-term instructional effects. The current literature highlights the idea that long-term, consistent PD with follow-up elements is more likely to affect the instructional practice compared to the short-term, isolated training sessions (Darling-Hammond et al., 2017; Desimone & Garet, 2015). Thus, although the survey is a good way of recording the scope and comparative eminence of PD opportunities, it cannot be used to measure the substantive quality or long-term efficacy of those programmes.

The survey shows that teachers' level of involvement in PD programmes is average. Importantly, "average" participation does not mean that there were equal numbers of engaged and disengaged teachers, but that the weighted average score of 2.14 (on a 3-point scale) should be interpreted as episodic, irregular, rather than sustained, participation in professional learning. Qualitatively meaningful: Teachers seem to be engaged in PD when institutional conditions are conducive (e.g., administratively sponsored programmes, $M = 2.25$) but not when structural barriers are present (e.g., attendance at Kwara Learn Initiatives, $M = 1.95$, or academic conferences, $M = 1.92$). The differences between high availability scores and low participation

scores on the same items highlight a structural disconnect that the APDPPSTQ is well suited to uncover, which would not be as apparent in the aggregate TALIS-style data that would be likely to be seen in more diverse institutional settings. Although this trend aligns with previous research that has found moderate levels of engagement in other settings (Awodiji et al., 2022; Opper & Greer, 2022), the survey data raises more questions than just a mere comparison.

Findings also revealed that there are a number of challenges that impede the involvement of teachers in the PD programmes. These are poor funding by government and private school administrators, infrequent promotion of and incentives for professionally developed teachers, institutional support, low digital skills, and administrative dedication to PD programs. Other limitations like the high programme costs, negative attitude towards change, work overload and administrative burden also minimise the participation of teachers. These results are in line with Geldenhuys and Oosthuizen (2015), who found weak school management support and teacher-related constraints to be significant obstacles to PD. Similar difficulties were also noted by Pham et al. (2024), who identified financial limitations, insufficient access to digital technologies, and unfavorable conditions, and by Tayag (2020), who also pointed out the pressure of work and the absence of funds as the main barriers to the professional growth of teachers.

In spite of its contributions, the survey should be viewed against a number of methodological limitations that could influence validity. The sampling was done among both public and private primary school teachers, but only those who were available and willing to fill in the questionnaire were sampled. This brings about the risk of self-selection bias, where teachers with more workloads, less institutional support, or more professional restrictions, which are some of the factors found in the survey as obstacles to PD, might have been less inclined to take part. It should be noted that the APDDPSTQ was administered in only one local government area (Ilorin South), a practical decision that also constitutes a methodological constraint. When an instrument is administered in a relatively homogeneous setting, it may not demonstrate sensitivity to change across different PD settings.

Pedagogical Implications

The results of this study have implications for educational practice, school leadership and policy formulation in Nigeria and other low- and middle-income countries. The discrepancy between the number of PD opportunities available and teachers' participation in PD indicates that more PD is not enough. Financial, workload and release time are all structural barriers that need to be overcome to promote meaningful engagement. Supportive policies and incentives, such as transportation allowances or PD time allocations, can enhance participation at school.

The results also underscore the importance of providing more digital literacy support in online PD programs. Digital skills proved to be a key barrier, so it is important to include basic technology training in programme design to make it accessible for teachers in under-resourced settings. In addition, the high score for limited recognition and promotion opportunities suggests that career advancement systems may not be sufficiently designed to recognize professional learning. Connecting the PD to promotion requirements may help to reinforce teachers' motivation to engage in ongoing development.

Future Directions

Future studies should administer the survey in more diverse settings, including rural and urban schools, or schools in different states with varying PD policy conditions, to further test whether the tool is capable of measuring the full spectrum of variation in teacher access, participation, and barriers to PD. These would enhance the construct validity of the instrument

and increase its applicability beyond the Kwara State. Qualitative study could also be applied to future research to capture the lived experiences and personal narratives of participants that closed-survey items are not able to fully represent.

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INTERTEXTUALITY AS LEARNING IN TEACHER STORIES AND REPRESENTATIONS OF PRACTICE

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This paper examines teachers' naturalistic storytelling as a representation of practice in a professional development program for early career secondary mathematics teachers. It argues that storytelling reveals the intertextual nature of learning from representations of practice. The paper draws on an analysis of 48 episodes of teacher talk and presents a microanalysis of one episode. The analysis reveals how teachers narrated accounts of practice based on different evidence sources to support learning that the participants marked as valuable. The paper concludes that intertextuality is a valuable concept for future analysis of representations of practice more generally and storytelling in particular.

Keywords: Professional Development

Introduction and Purpose

There remain significant questions about the value of teachers' conversational stories as representations of practice to support teacher learning (Little, 2003; Pulvermacher & Lefstein, 2016; Segal, 2019). Teachers often rely on stories and narrated accounts to communicate about practice when other representations, such as video or student work, are inadequate or unavailable (Hall & Horn, 2012; Horn, 2010; Topham, 2024). However, narrative is understudied in mathematics education in general (Cannon & Dubbs, 2024) and recent studies of teachers' conversational stories have found mixed value for storytelling. For example, while Mendler and Vedder-Wiess (2023) found that video recordings and teacher stories could both lead to productive engagement, Segal's (2019) study of story exchanges found that teachers' sharing of stories could let teachers avoid the work of argumentation and reasoning about teaching practice, meaning storytelling could undermine learning in some cases.

To explore how teachers' storytelling can lead to learning, this study treats teachers' stories as "small stories" (Bamberg & Georgakopoulou, 2008), following Segal's (2019, p. 4) definition of a story as "a recounting of a past or hypothetical event as unfolding action, with an element of specificity (regarding time, place, and/or actors, or including a replay of enacted or imagined speech)." When applied to teacher talk, these stories include similar narrated accounts, such as what Horn (2010) called "replays" and "rehearsals."

Moving beyond Segal's framing of "story exchange," I treat teacher learning from stories as a form of intertextuality (Bloome et al., 2022; Hodges, 2015; Lemke, 1992). Intertextuality is the recognition that "we can make meanings through the relations between two texts; meanings that cannot be made within any single text" (Lemke, 1992, p. 257). Thus, teachers learn through story when they move beyond exchanging individual stories and toward connecting and synthesizing various perspectives on teaching practice represented in different stories. I ask: how do teachers use small stories, including re-narrations of other representations of practice, to achieve an intertextual understanding of teaching practice?

Data Sources and Methods

I draw on video recordings from a two-year professional development program that enrolled early-career secondary mathematics teachers from across the United States, analyzing data from

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

one cohort's (22 teachers) time in the program from 2017-2019. The analysis focuses on the program's small group coaching sessions carried out over a videoconferencing platform during each school year. A mathematics coach facilitated the small group sessions. Each group met three times per year for about 90 minutes and consisted of the coach plus three or four teachers. Most of the meeting time focused on discussing video clips from each teacher's classroom which everyone had previously viewed and annotated on a shared platform.

My analysis covers a sample of three small group meetings from the teachers' first year in the program. The sampled meetings are from three different time points during the year, and each contains different participants. There are ten teachers in the sample, plus the coach. I ultimately analyzed 5 hours and 16 minutes of video data.

To carry out this study, I first identified all the stories in the meetings (n = 116), then identified the larger conversational episodes (n = 48) surrounding the stories, bounding the episodes based on natural conversational shifts (Horn, 2005), with most episodes including more than one story. To analyze this data, I used methods from conversation analysis (Jordan & Henderson, 1995; Pomerantz & Fehr, 2011) and a recursive coding process (Charmaz, 2014; Corbin & Strauss, 1990; Saldaña, 2016) to identify how stories achieved social action and led to intertextual understandings. In this paper, I present a brief overview of some findings and a microanalysis of one episode of talk.

Findings

Similar to Segal (2019), I found that stories were often clustered in episodes, with several stories often occurring close together and spaces of time with no stories. The mean episode was two minutes and 45 seconds long and contained about 2.5 stories. 19/48 episodes contained three or more stories. My qualitative analysis revealed that the various stories in these episodes often synthesized preceding arguments, added new evidence, and identified connections with other stories or representations. This building of intertextuality often continued until one or more members of the group signaled that some form of valuable learning had been reached.

To answer my research question of *how* this played out across a conversation, I present and analyze an extended excerpt from a conversation where one small group was discussing a video shared by a teacher named Wesley (all names are pseudonyms). The group included Garima, the coach and facilitator, and Wesley, Jesse and Amit, who were all teachers. The episode started with Garima, the facilitator, prompting Wesley to share a question for discussion. After Garima's prompt, Wesley told the first story in the episode by recounting his experience planning the video-recorded lesson while hoping to see specific responses from students to show understanding. Wesley asked if he should keep hoping to see specific ideas and words from students or be more open and accepting of however students phrased their ideas. Jesse then initiated the group's discussion:

(1) Jesse: Yes, I'd love to comment on that or I think it's related to what you're getting at, where I felt like students [in the video] kept saying remainder theorem and factor theorem but I wasn't sure. […] They're not emphasizing it because they're like, "Oh, we learned this thing called remainder theorem so it's related to remainder theorem." They know that it's related to the problem. […] I would've loved to hear them take that extra step somehow and go from like, "Oh, this is related to factor theorem because if it's equal to zero, it's a factor." That's what you're looking for in terms of the explanation but then I'm

wondering if students are just thinking their reasoning is good enough, if they say, “Well, remainder theorem.” Right?

(2) Wesley: Right. You're saying, I should probably be more intentional in making sure that their explanations include their more reasoning than citing the said the said theorem, about why it works or why I would use this particular theorem on this particular problem rather than this is what we’ve been working on. Is that what you’re saying or?

(3) Jesse: Well, I guess I guess I’m not sure because you said you were looking for something really specific, right? I’m curious about what would be surfaced if you asked them, “Well, what does the remainder theorem help us do here, or why is it that this is appropriate in this?” because they’re citing it. They clearly know it. I have the exact opposite problem in my classroom, where if I taught them something like remainder theorem and then was like, “What are we using?” they’d just be like, “I don’t know. It’s equal to zero. We just know it’s supposed to equal zero.” […]

(4) Wesley: All right. That makes sense. Advice?

In his response (1), Jesse told the second story in this episode, creating an intertextual connection by re-narrating some of the details of Wesley’s story and drawing on other details from the video. Wesley then used this connection to draw implications for his practice (2).

Then, Jesse used another story to both downplay specific assertions about Wesley’s teaching and provide more evidence (3). This new intertextual connection gave insights from a story about Jesse’s own class (3). Wesley’s next turn (4) acknowledged the value of this new story but also indicated a need for more “advice.” The interaction continued:

(5) Garima: Amit, what are you thinking about?

(6) Amit: I’m thinking about how [in the video] you asked the closing question at the end because you were like, “Exit ticket.” [chuckles]

(7) Wesley: I had a real problem I wanted to solve. I saw that clock and it was over with.

(8) All: [Laughter]

(9) Amit: Again, all of our classes.

(10) Jesse: Every one.

(11) Amit: Every single one. So, I’m wondering if the exit ticket question could be something like what you're saying, Jesse, something along the lines of, let’s say, somebody were absent today. How would you explain to them that this problem was solved by remainder theorem? How would you draw that connection for a person who wasn’t here for our discussion? Clearly, we all know what we’re talking about. Can you make the connection for a third-party kind of thing, I guess. That’s the first thought that came to my mind because it's something you already used was this verbal exit ticket. I don't know how quick of an answer that would be, though, is the only thing. Under time, I feel like kids would probably take some time to think about that question. I don’t know how quick that could go, but maybe it could be an entrance ticket for another day, I don’t know.

(12) Garima: I think that would give you some interesting formative assessment data around when you say, “Why did we use it?” do kids end up giving a response that tells you how they used it, or that makes that conceptual connection that you’re really looking for. I think kids often get explanation or narration of process

rather than justification, and they're not sure where the line is. I think that's something we have to help define for them of what's a sufficient explanation.

(13) Amit: That's a good point. Can you write that down, Garima?

(14) Wesley: That was pretty dope.

After Garima's invitation (5), Amit re-narrated further details from the video (6), thus introducing another story to the conversation. At this point, Wesley jumped in (7) to provide details not available in the video by narrating his thought process in the moment. The group laughed (8), and Amit and Jesse's comments (9, 10) signaled that the laughter arose because running out of time had previously been a topic of conversation. As the laughter subsided, Amit presented an idea for how to use the exit ticket approach to generate student comments and provide Wesley with a more detailed sense of his students' understanding, explicitly connecting his own idea with Jesse's previous thoughts (11). However, after presenting this idea, Amit also hedged it by bringing back up the idea of running out of time (11). After this hedge, Garima re-entered the conversation to support Amit's suggestion and then connected it back to Jesse's original thoughts (1) about students not explaining enough, reinforcing the intertextual connections present in the conversation. After Garima's turn, in contrast to the original response to Jesse's ideas (4), both Amit and Wesley marked Garima's final summation as a "good point" (13) and "pretty dope" (14), suggesting that the group had gained some new insights.

Significance

The analysis of this excerpt reveals a fascinating trajectory of teachers' learning through storytelling. Stories told by Wesley, Jesse, and Amit drew on different sources to create intertextual connections that supported the group in grappling with practice. The lens of intertextuality reveals how teachers learn as they move beyond exchanging individual stories to building connections between them. Specifically, this analysis reveals how contrasting stories, such as the story Jesse told about his own classroom (3), are not necessarily in conflict with each other but can come together to produce a nuanced understanding of teaching practice based on multiple perspectives. This does not mean that stories will always produce valuable intertextual perspectives or that they cannot devolve into conflict. However, recognizing these intertextual connections for what they are can provide valuable tools for analyzing teachers' use of stories to support their learning.

This study is limited in scope by focusing on one small conversation and not attending explicitly to questions of power, such as the position of Garima as the coach and how it affects the reactions to her contributions to the conversation (especially turn 12). However, this paper still has implications for the study of teachers' use of representations of practice more generally and suggests important lines of future research. Recognizing teachers' naturalistic stories as representations of practice means recognizing that teachers are constantly representing practice in different ways and for different ends in their talk. To do this, teachers draw on their own experiences and other available representations, such as video in this case. Future research could examine the different ways teachers' stories draw on, enhance, or contradict other representations to identify specific types of storytelling that are valuable for learning, recognizing that drawing material from another representation of practice is a form of intertextuality. Indeed, it is possible that narrating actual and possible classroom events is necessary for teachers to connect representations of practice to their own experiences (see Oliver et al., 2018). If this is the case, designers, facilitators, and teacher educators should take teachers' storytelling seriously as a form of learning and find ways to support teachers in making

intertextual connections as they tell stories. This could mean inviting teachers to share experiences as stories but also finding ways to invite accounts that synthesize disparate perspectives to narrate new practices.

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FROM TEXTBOOKS TO THE REAL WORLD: TEACHING GRAPH LITERACY WITH AUTHENTIC DATA

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This study examines how professional development can support teachers in advancing students' graph literacy through real-world graphs. We conducted an online workshop with 20 middle and high school mathematics and science teachers focused on using news and media graphs in instruction. Teachers selected graphs from a New York Times resource and generated questions for classroom discussion. We found that most questions emphasized mathematical features (e.g., trends and comparisons), with limited attention to contextual and critical dimensions. In response, we propose a research-informed questioning framework and discuss implications for professional development and teacher education.

Keywords: Professional Development, Graph Literacy, In-Service Teacher Education

We live in a data-driven world where graphs appear daily in news, social media, and public reports. As consumers of information presented in graphical form, individuals must develop graph comprehension—the ability to derive meaning from graphs created by others—and engage in making sense of quantitative information represented visually (Friel et al., 2001). This involves not only reading values and trends but also interpreting how information is structured and communicated through the graph. To navigate these responsibly, one must be able to read, evaluate, and make informed decisions based on the data presented (Weiland, 2017). Students, therefore, need experience interpreting real-world media graphs. In contrast, graphs encountered in school mathematics classrooms often appear in decontextualized forms and emphasize standard structural features, such as clearly labeled axes, axes intersecting at (0, 0), and uniform scales (Paoletti et al., 2022; Weiland, 2017). This difference may leave students underprepared to interpret the graphs they encounter outside of school-one needs to be graph literate.

Theoretical Framework

So, what does it mean to be graph literate? Curcio (1987) proposed a framework of reading the data, reading between the data and reading beyond the data that measures students' ability to extract information, identify pattern, and making predictions from a given graph. This framework guides educators to design tasks that deepen students' graph comprehension (Friel et al., 2001). Shaughnessy (2007) later introduced a fourth category: reading behind the data, which emphasizes examining the causes of variation and making connections between the data's context and its graphical representation. However, students encounter graphs not only in textbooks but also in media and public discourse. Gal (2002) and Watson (1997, 2003) argue that statistical literacy involves understanding statistical information in social and media contexts and questioning claims made in contexts that lack proper justification.

Drawing on scholarship in critical mathematics education, Weiland (2017) defines critical statistical literacy as the ability to interpret, interrogate, and evaluate statistical arguments, particularly in relation to issues of bias, power, equity, and how data shape social structures. Bailey and McCulloch (2023) extend this perspective by articulating what such engagement looks like in practice through critical statistical literacy habits of mind, drawing on Lee and Tran's (2015) work on statistical habits of mind. Together, this body of research suggests that

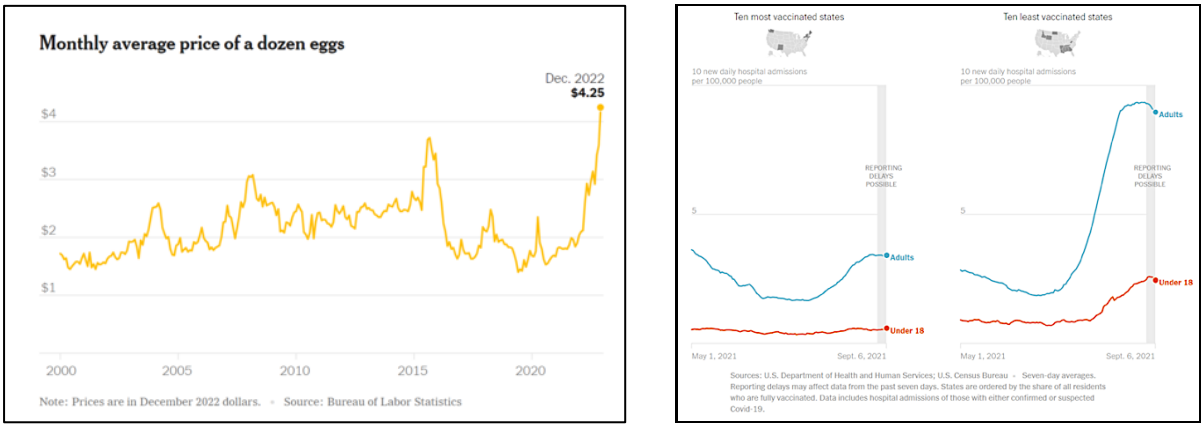
graph literacy requires both mathematical interpretation and critical contextual evaluation. In this paper, we conceptualize graph literacy as comprising two interconnected dimensions: (1) mathematical and statistical analysis of patterns, relationships, and inferences within a graph; and (2) critical contextual analysis of data sources, representation, and social implications. This conceptualization informs the questioning framework introduced in our workshop.

Methods and Session Overview

We conducted a two-hour online workshop for 20 in-service middle and high school mathematics and science teachers from schools in a large Northeastern U.S. city. Of the 20 participants, 15 taught high school and five taught middle school; there were 13 mathematics teachers and seven science teachers. Their teaching experience varied, with 70% having 8–15 years of teaching experience, 20% with more than 16 years, and 10% with 4–7 years of experience. We designed the workshop to support teachers in using real-world graphs to strengthen students' graphical reasoning and develop their skills to critically evaluate the contexts represented in the graphs.

Preliminary Results

We administered a pre-session survey (n =13) in which teachers reported how and why they use graphs in their classrooms and wrote questions for a sample graph from the NYT (Figure 1a). We opened the workshop by comparing and contrasting textbook graphs with real-world graphs from news and media sources. We then introduced the New York Times "What's Going on in This Graph?" resource (The Learning Network, n.d.). As a first Activity, teachers explored the site in break out rooms, selected a graph of their interest, and discussed how they might engage students in analyzing the graph if they were to use it in their classrooms. In Activity 2, we presented a graph (Figure 1b) and asked teachers to generate questions to guide classroom discussion. Here we describe the survey results and observations from the two activities.



(a) (b)

Figure 1: Graphs from NYT (The Learning Network, n.d.) – (a) Monthly Average Price of a Dozen Eggs, (b) Rate of Covid Hospitalizations in the U.S.

Pre-Session Survey Results

The pre-session survey examined teachers' instructional goals, classroom practices, and use of real-world graphs. Responses revealed a tension between teachers' stated goals and their teaching practices. Many participants described connecting mathematics to real-world issues and

encouraging discussion as important goals when teaching graphs. Yet several reported rarely using real-world or media-based graphs in their classrooms. Teachers reported several challenges such the "messier" nature of real-world data, noting that "many graphs students deal with *have* to be exact—perfectly linear, perfectly quadratic—and the real world is not like that." One teacher wrote when using graphs from real world contexts "I sometimes also struggle to come up with interesting questions for students to ponder as they examine these graphs." Several teachers described difficulties related to context and questioning, including "lacking background knowledge to make graphs relevant."

When asked to generate questions for a graph on monthly average price of a dozen eggs in the US in 2022 (Figure 1a), most participants focused on mathematical features (e.g., "During which month was the price highest?", "What is the average rate of change between January and March?", "Is the trend increasing or decreasing?"). Only a few questions were on contextual factors (e.g., "Why do you think this graph was created?", "What argument or claim can you make with this graph?"). Together, these responses suggest that while teachers value real-world engagement, they may need support in framing questions that move beyond mathematical interpretation toward contextual analysis.

Observations from Activity 1

In Activity 1, teachers explored the NYT graph website (The Learning Network, n.d.). The purpose was to introduce them to graphs based from media on real and current issues and to think about whether and how they might use such graphs in their classrooms. Teachers selected graphs on topics such as food and health, U.S. imports from China, air pollution, time use during the pandemic, and electric vehicle adoption. These graphs looked different from typical textbook graphs by using colored images, complex layouts, multiple variables, and artistic non-linear layouts to highlight trends in real-world data. Teachers noted that the graphs were different from the traditional textbook graphs but were timely and could lead to discussion. Nevertheless, most groups focused on why the graph was interesting rather than on the kinds of questions they would ask students to prompt thinking about the context in the graph.

Observations from Activity 2

To move from exploration to structured questioning, in Activity 2, we asked teachers to generate questions for a graph on Covid hospital admissions in the most and least vaccinated states (Figure 1b). Specifically, we asked teachers to distinguish between questions that support mathematical reasoning and questions that invite real-world discussion. Across all groups, teachers produced numerous mathematical questions. These focused on identifying variables, interpreting rates per 100,000, determining intervals of increase and decrease, considering function type or best fit, explaining slopes and trends, and determining the rate of change.

In contrast, contextual questions were more limited. Teachers from two groups generated a few reflective questions. For example: "What message is this trying to send?", "Who's recording/sharing this data?", "What feelings might this evoke in people?", and "How might interpretation change if only one graph were shown?" Other groups wrote as few as one or two broad prompts (e.g., "COVID-19 era"), and one group did not generate any contextual questions. During whole group discussion, several participants noted that they had not previously separated mathematical and contextual questioning in this way and that generating such questions required more deliberate thinking. This pattern suggests that while teachers are comfortable framing mathematical questions, they may need clearer structure or sample prompts to help them write questions that guide students to think about the context.

A Framework for Structuring Classroom Discussions

After participants completed both the activities, we introduced our graph literacy framework for structuring a classroom discussion (Figure 2). The framework operationalizes the two dimensions described in the theory section—mathematical/statistical analysis and critical contextual analysis—and translates them into concrete question types that teachers can use in the classroom. It was presented as a practical tool to help teachers move more intentionally between analyzing patterns in a graph and examining its context, source, and implications.

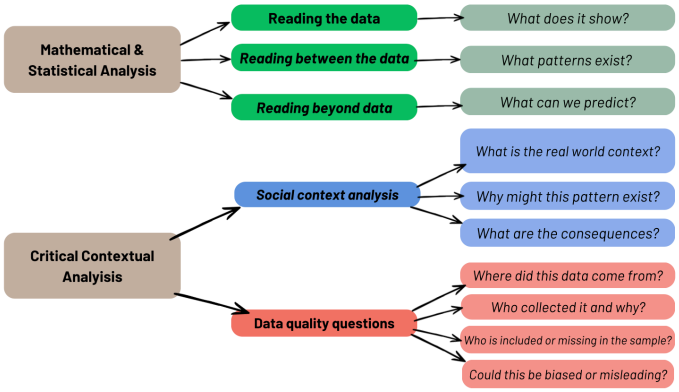


Figure 2: A Framework for Structuring Classroom Discussions

After introducing the framework, we modeled how it could be used in practice by providing sample questions for the Covid admissions graph. For example, teachers might ask: What was the approximate hospitalization rate for adults in the least vaccinated states in early September 2021? (reading the data); How do hospitalization rates compare between the most and least vaccinated states? (reading between the data); What might happen to hospitalizations if vaccination rates increased? (reading beyond the data); Where did this data come from, and how are "most" and "least" vaccinated states defined? (data quality); or why might rates differ between these states, and who is most affected? (context and impact). These examples show how teachers can move from trends to context and consequences.

Preliminary Discussion and Implications

This study examined how in-service teachers generated questions around socially situated graphs drawn from NYT and introduced a questioning framework to support graph literacy. Participants described the framework as useful and practical, and several indicated that they would try using it in their classrooms. While teachers readily generated mathematical questions about trends, slopes, and comparisons in the given graph, few framed questions about its context, construction, or implications. Teachers noted that this type of questioning required deliberate effort and was not part of their usual practice. Our observations echo prior research highlighting the need for professional development and curricular support (Watson & Callingham, 2003; Weiland, 2017) for teachers to assist their students in developing graph literacy. Preservice preparation, professional development, and curriculum materials that include socially situated graphs with modeled questioning strategies may better support teachers in engaging students in both mathematical analysis and critical contextual evaluation. In an era where students regularly encounter graphs in news outlets and social media, the ability to question how data are selected, categorized, and framed is central to graph literacy. This study contributes a practical, research-informed questioning framework that translates theoretical dimensions of graph literacy into

classroom-ready prompts. By modeling how teachers can move from identifying trends to examining context and implications, the framework offers a structured approach to integrating critical graph analysis into everyday instruction.

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ALGEBRA KNOWLEDGE OF IN-SERVICE MATHEMATICS TEACHERS FOR SENIOR HIGH SCHOOL TEACHING

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This study investigates the algebra knowledge of 153 in-service mathematics teachers at the senior high school level in Ghana using the Knowledge of Algebra for Teaching (KAT) framework. Utilizing a cross-sectional survey design across 40 schools, the study assessed teachers through a 74-item achievement test. Results reveal a significant discrepancy between procedural fluency and conceptual depth: while 83% of teachers correctly ordered the derivation of the quadratic formula, 79.1% failed to solve a basic algebraic rate problem, and 56.9% could not identify correct pedagogical explanations for student misconceptions. The findings indicate that while "School Knowledge" is relatively high, specialized "Teaching Knowledge" and "Advanced Knowledge" remain underdeveloped. The study advocates for a shift in teacher training toward the multidimensional knowledge types identified in the KAT framework.

Keywords: Algebra and Algebraic Thinking, High School Education, Teacher Knowledge, Advanced Mathematical Thinking

Background

Algebra is positioned as a foundational strand of school mathematics, with standards discourse emphasizing reasoning, representation, and conceptual understanding rather than procedural reproduction alone (National Council of Teachers of Mathematics [NCTM], 2000). Research on teacher knowledge has long argued that teaching quality depends on more than subject matter competence; it also depends on specialized knowledge that supports explaining ideas, anticipating learner thinking, and selecting representations (Shulman, 1986). Subsequent work has elaborated how effective instruction involves interweaving content and pedagogy in ways that sustain mathematical integrity while making ideas learnable (Ball & Bass, 2000), which is especially consequential in algebra where teachers must manage both symbolic procedures and conceptual structure (Ferrini-Mundy et al., 2005). Internationally, questions of alignment between classroom practice and standards expectations remain persistent (Jacobs et al., 2006), reinforcing the relevance of professional development that targets teachers' knowledge and its instructional enactment (Mewborn, 2003). In the Ghanaian context, earlier doctoral work provides local evidence that teachers' algebra-for-teaching knowledge relates to student performance, motivating renewed attention to the profile of knowledge domains required for instruction (Wilmot, 2008).

Objectives and Purposes of the Study

Algebra is positioned as a foundational strand of school mathematics, with global standards emphasizing reasoning, representation, and conceptual understanding over procedural

reproduction (NCTM, 2000). In the Ghanaian context, earlier evidence suggests that teachers' algebra-for-teaching knowledge is a significant predictor of student performance, motivating renewed attention to the profile of knowledge domains required for effective instruction (Wilmot, 2008).

Despite its importance, Ghanaian students have historically underperformed on international assessments such as the TIMSS in 2003, 2007, and 2011 (Wilmot, 2008). Chief Examiners for the West African Examination Council (WAEC) frequently cite students' inability to perform algebraic tasks, yet it remains unclear if these difficulties stem from student-level deficiencies or a lack of conceptual depth in the teachers' own knowledge (Wilmot, 2008). This study investigates the algebra knowledge possessed by in-service mathematics teachers at the Senior High School (SHS) level within the Knowledge of Algebra for Teaching (KAT) framework. Specifically, it addresses:

1. The overall proficiency level of SHS mathematics teachers in algebra based on the KAT framework.
2. The extent to which the boundaries between the three knowledge types (School, Advanced, and Teaching) are sustained or overlapping.
3. The alignment between teacher knowledge and the conceptual demands of the Ghanaian curriculum.

This research addresses the PME-NA 2026 theme, "Changing Mathematics Education by Educating for Change," by identifying where teacher training must evolve to move beyond procedural stagnation and provide equitable learning experiences (PME-NA, 2026).

Perspective and Theoretical Framework

The theoretical foundation of this study is the KAT framework, which re-conceptualizes Shulman's (1986) notions of content knowledge and pedagogical content knowledge specifically for school algebra. Subsequent elaborations have emphasized that effective instruction involves interweaving content and pedagogy to maintain mathematical integrity while making ideas learnable (Ball & Bass, 2000). The framework posits three intertwined domains:

1. *School Knowledge:* Mastery of the intended K-12 mathematics curriculum (NCTM, 2000; Wilmot, 2008).
2. *Advanced Knowledge:* College-level mathematical perspective (e.g., Abstract Algebra, Calculus) that provides a trajectory for mathematical ideas (Ferrini-Mundy et al., 2005).
3. *Teaching Knowledge:* Specialized mathematical insights used to identify student misconceptions and select appropriate algebraic tasks (Ferrini-Mundy et al., 2005).

Recent literature has expanded this by introducing the Technological Knowledge of Algebra for Teaching (T-KAT) framework, arguing that modern instruction requires teachers to integrate digital tools to unpack complex concepts (Osei, 2024; Osei & Agyei, 2024). Furthermore, contemporary work on Pedagogical Content Knowledge (PCK) highlights the necessity of understanding how students conceptualize specific topics to apply strategies that enhance deep learning (Li & Copur-Gencturk, 2024).

Methods and Modes of Inquiry

This study utilized a cross-sectional survey design to capture a snapshot of teacher knowledge (Wilmot, 2008).

Participants and Sampling

The target population included SHS mathematics teachers in the Ashanti, Western, and Central Regions of Ghana. A multi-stage sampling technique categorized schools (A, B, C, D) per Ghana Education Service (GES) classification. From 40 randomly selected schools (10 single-sex, 30 co-educational), a final sample of 153 in-service teachers currently teaching core or elective mathematics was achieved.

Data Collection and Analysis

The instrument was a 74-item multiple-choice achievement test adapted from the KAT project and contextualized for the Ghanaian academic environment (Wilmot, 2008). Items were revised to reflect local currency (Cedis) and market commodities to ensure context validity. Reliability was established with a KR-20 coefficient of 0.855. Data were analyzed using descriptive statistics and Analysis of Variance (ANOVA) to compare performance across the three KAT domains.

Results

The analysis reveals a profile characterized by high procedural fluency but notable conceptual and pedagogical gaps.

Domain-Specific Performance

Teachers demonstrated appreciable competence in School Knowledge (M = 1.0688, SD = 0.1900), significantly outperforming their levels of Advanced Knowledge (M = 0.3159, SD = 0.1780) and Teaching Knowledge (M = 0.2369, SD = 0.1623), which were comparatively low.

Procedural Fluency against Conceptual Application

Standard procedural tasks showed high proficiency; 83% of teachers correctly identified the steps to derive the quadratic formula. This is attributed to the fact that 72% of the participants have a formal education background where such derivations are heavily emphasized (Wilmot, 2008). However, conceptual application was weak. On Item 27, which required solving an algebraic rate problem involving two pipes filling a tank, 79.1% (121 out of 153) of teachers answered incorrectly, often failing to realize that two pipes working together must take less time than the fastest individual pipe (Wilmot, 2008).

Pedagogical Content Knowledge and Misconceptions

Regarding student misconceptions (Item 39), 56.9% of teachers were unable to correctly explain why dividing both sides of an equation by a variable (e.g., $2x^2 = 6x$ by x) is problematic. Many selected incorrect justifications or could not identify the risk of losing the solution $x = 0$, indicating a lack of the specialized "Teaching Knowledge" required to navigate student errors while maintaining mathematical integrity (Ferrini-Mundy et al., 2005; Wilmot, 2008).

Discussion and Conclusions

The findings suggest that the knowledge base of SHS teachers in Ghana is "thin" and predominantly curriculum-bound, supporting earlier claims that subject matter competence alone is insufficient for reform-oriented teaching (Shulman, 1986; Wilmot, 2008). The high procedural success vs. conceptual failure on rate problems mirrors international concerns regarding the alignment between classroom practice and standard-based expectations (Jacobs et al., 2006).

Educating for Change

In alignment with the "Educating for Change" theme, this study highlights that current systems produce teachers who are procedurally competent but conceptually limited, which

perpetuates mathematics as a gatekeeper (PME-NA, 2026). If teachers cannot conceptually "unzip" content, they cannot address the systemic injustices that exclude learners from advanced STEM trajectories. Current research emphasizes that teachers need knowledge of mathematical problem-solving not just for themselves, but as a broader pedagogical skill to help students become resilient problem solvers (Lit et al., 2026; Schreiber, 2025).

Synthesis and Recommendations

The study concludes that the boundaries between knowledge types are indeed fuzzy and overlapping, with weaknesses in one domain (Advanced Knowledge) directly limiting effectiveness in another (Teaching Knowledge).

1. *Curricular Reform:* Training institutions should develop courses helping teachers explore all seven knowledge types required for algebra, including functional thinking (Twohill et al., 2025).
2. *PCK Focus:* Professional development must shift from "what" to teach to "how" students conceptualize topics, using representations that enhance learning (Li & Copur-Gencturk, 2024).
3. *National Restructuring:* Findings should inform the restructuring of university mathematics programs and the integrated mathematics curriculum in Ghana to prioritize conceptual application over memorization (Wilmot, 2008).

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DO RACE AND GENDER MATTER?: TEACHER LEADER RESPONSES TO MICROAGGRESSIONS

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Keywords: Teacher Knowledge; Equity, Inclusion, and Diversity

Microaggressions convey negative attitudes toward people from nondominant groups (Nadal et al., 2014; Sue et al., 2007) and are subtle, common occurrences that can be intentional or unintentional. We investigated, "How do developing STEM teacher leaders respond to a hypothetically observed teacher-to-student microaggression?" The focus of this study is a cohort of nine teacher leaders (TLs) in a five-year professional development (PD) fellowship aimed at further developing their leadership. As part of the PD, the TLs participated in five experiences: coaching on the topic of Professional Noticing (Jacobs et al., 2010), equitable teaching practices for urban schools (Johnson et al., 2019), engaging in a university laboratory setting, apprenticing as instructor in a methods class, and a STEM Education Leadership EdD.

Methodology

The participants were seven mathematics and two science teachers who teach grades ranging from elementary to high school; all taught at urban schools in high needs districts. Six identify as women and three identify as men and the TLs come from a mix of cultural backgrounds. We conducted semi-structured interviews in which the teacher leaders were presented with two scenarios. In one scenario we asked TLs if and about how they would respond to a colleague's remark about a student of a particular background. We followed this task by asking about how their response might change based on the race and/or gender of the teacher and student, which allowed us to capture variation in their perceptions (Leyva et al., 2021). The interviews were coded and organized into emergent themes through axial coding (Creswell, 2013).

Results and Discussion

Many TLs considered their own backgrounds (e.g., queer, Jewish) in giving the teacher in the scenarios the benefit of the doubt or in choosing when to respond. Coding categories included: Responses, Considerations for Response, Motivation to Respond, and Strategies for Response. Under Responses, for example, in the theme of *Clarifying Question*, were codes that distinguished a presumed intent: C1-Curiosity about Meaning, C2-Reflecting a mirror to the speaker, and C3-Making sure the TL heard correctly. Although most of the TLs described similar responses, their reasoning differed when asked about race and gender. One TL centered their differing backgrounds while acknowledging she may not possess the lived experiences that situate the exchange, thus rooting her response in developing understanding. In contrast, many TLs expressed that racial/gender background doesn't matter because the microaggression was harmful regardless of the backgrounds of either party. The research helps us understand how TLs center their own and others' identities when thinking about how to respond to microaggressions.

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INTEGRATING SPATIAL REASONING INTO MIDDLE AND SECONDARY MATHEMATICS CLASSROOMS

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Spatial thinking—the ability to generate, manipulate, and reason about spatial relations between and within objects—is both highly malleable and strongly associated with mathematics performance (Hawes et al., 2022). Yet, spatial reasoning activities remain largely absent from typical mathematics instruction. To address this, we engaged in a research-practice partnership with grades 5-12 teachers at a New Brunswick school, partnering with 16 teachers to co-develop a curated collection of learning activities to support the implementation of spatialized approaches in their classrooms. Professional learning began with a two-day in-person intensive and continues through monthly sessions delivered both in person and virtually. Teachers expressed strong enthusiasm for the project during the initial intensive but also reported frustration with the lack of ready-to-implement spatialized lessons. In response, we adjusted our approach shifting from broader research dissemination to developing a collection of classroom-ready materials to support implementation and modeling concrete strategies for spatialization within existing curricula. Here, we share preliminary reflections derived from both qualitative analysis of pre and post teacher survey data as well as qualitative narrative and thematic analysis of teacher-researcher discussions and shared classroom artefacts as a proxy for implementation. These analyses point to increased teacher understanding of spatial thinking concepts as well as iterative refinement to classroom implementation of spatial approaches over the course of the year. Further, additional analyses of teacher reflections point to the promise of collaborative design as an effective tool for professional learning (Voogt et al., 2015).

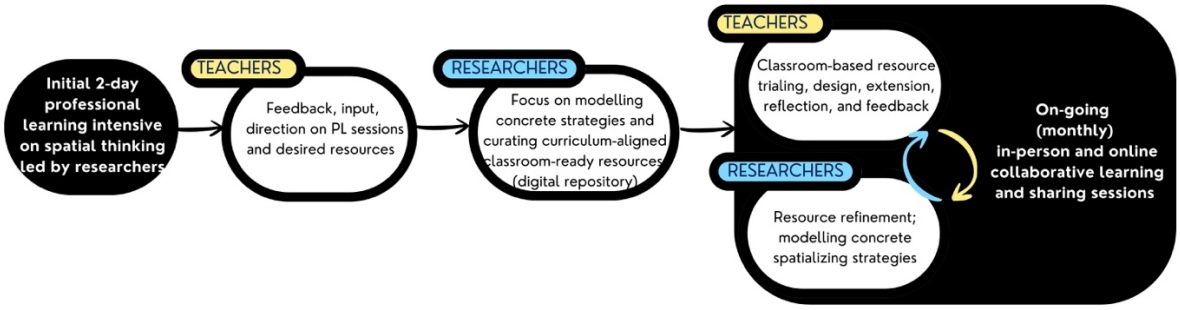


Figure 1: Design-research Approach for Professional Learning, Showcasing the Use of Reflections, Surveys, and Artifacts in Assessing Implementation Cycles.

This design-research (Figure 1) is part of a larger study investigating how integrating spatialized activities into middle- and high-school mathematics and science classrooms can

support student learning and performance, and how teachers adopt and adapt such practices. As analyses of teacher and student learning data continue, this work aims to contribute empirical insight into how spatial reasoning can be meaningfully integrated into middle school and secondary mathematics instruction and how such integration may influence student outcomes.

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DESIGNING PROFESSIONAL LEARNING FOR MATHEMATICS TEACHER LEADERS: EVIDENCE-BASED FACILITATION OF JOB-EMBEDDED PROFESSIONAL LEARNING

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Teacher learning is central to instructional improvement in mathematics, and job-embedded professional learning is increasingly recognized as a sustainable context for supporting that learning (Darling-Hammond et al., 2017). However, research indicates that discourse within these settings is often anecdotal and insufficiently grounded in evidence, limiting opportunities for meaningful professional learning (Horn & Kane, 2015; Lefstein & Perret-Clermont, 2014).

The Learning to Lead in Math (LTL-M) Project is designed to address this limitation by developing mathematics teacher leaders' (MTLs') capacity to facilitate evidence-based, student-centered discussions in job-embedded professional learning contexts. Central to the design of the project are active teacher learning routines in which MTLs engage with evidence from practice and develop the capacity to facilitate teacher learning that is grounded in evidence about instruction, student thinking, and mathematics learning (Darling-Hammond et al., 2017; Ball & Cohen, 1999).

As part of LTL-M, MTLs participate in professional learning focused on facilitating active teacher learning routines: analyzing instructional practice, analyzing student work, and lesson planning. These routines utilize classroom video, student work, and curricular materials. While these forms of evidence are frequently used in professional learning, they represent practice in different ways and therefore create different opportunities and challenges for facilitation (Grossman et al., 2009; Borko et al., 2014). Little is known about how these different forms of evidence shape opportunities for teacher learning or the specific demands they place on MTLs as facilitators. Understanding the affordances and constraints of different forms of evidence is critical for designing professional learning that prepares MTLs to facilitate evidence-based discussions.

In this poster, we present a design-oriented analysis of how different forms of evidence function within these three active teacher learning routines. The different forms of evidence make particular aspects of practice visible, creating distinct opportunities for teacher learning while also requiring specialized mathematical knowledge for teaching (Ball et al., 2008) and facilitation knowledge (Borko, 2016; Grossman et al., 2009; Jacobs et al., 2010). Across these routines, we analyze evidence in terms of representation type (static or dynamic), affordances and constraints, and the associated knowledge demands for facilitation. The goal of this work is to inform the design of professional learning experiences that build MTLs' capacity to facilitate productive, evidence-based discussions in job-embedded professional learning contexts.

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LEARNING TO NOTICE POSITIONING IN AN ONLINE ASYNCHRONOUS PROFESSIONAL DEVELOPMENT

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Mathematics teachers need professional development (PD) throughout their careers to support their continued learning (Feiman-Nemser, 2001). High-quality PD can support teachers' mathematical knowledge for teaching (MKT; McDuffie et al., 2014) alongside the development of equitable teaching practices, such as positioning of students (Bartell et al., 2017). Unfortunately, high-quality PD is not easily accessible to many teachers (Nite & Bicer, 2020). Online asynchronous PD has the potential to remove cost and time barriers. In this poster, we present research related to an online asynchronous PD designed to support teachers to learn the equitable practice of positioning students as competent within the mathematical context of a transformations-based approach to linearity and similarity (Seago et al., 2013).

The Video in the Middle course engaged teachers in six modules (approximately 20 hours total) with each module focused on analyzing the positioning practices in one classroom. The central activities in each module consisted of teachers watching and analyzing classroom video with preloaded facilitator comments that highlighted (Goodwin, 1994; Grossman et al., 2009) and discussed important moments related to interpreting students' positions and storylines in classrooms. These activities took place within a purposeful sequence of 21 activities; although all activities were influential in supporting teachers' noticing skills, we do not fully describe them here due to space limitations. Overall, teachers watched and asynchronously discussed 10 different classroom videos. We designed these activities to cultivate teachers *professional vision* (Goodwin, 1994) related to the practice of positioning. The data we report here is from an initial noticing activity given to teachers prior to beginning the PD and a final noticing activity that utilized the same initial video as a part of the final module. In these activities we asked teachers to respond to the question, *What do you find noteworthy and why?*

We drew on van Es (2011) coding scheme to analyze teachers' depth of noticing. Level one included responses that primarily attended to the whole class environment and level two attended primarily to teacher pedagogy. Responses coded to level three attended to students' and/or teacher's actions; Level four responses attended to the relationship between students' and teachers' (interactions). All responses were double-coded by the authors and discrepancies were resolved through discussion until there was mutual agreement.

Preliminary results suggest that the PD did support teachers to notice at deeper levels. On the pre-PD noticing activity, six teachers' responses were coded as level one, seven as level two, and one as level four. In the final noticing activity, two teachers' responses were coded as level one, five as level two, five as level three, and two as level four. A majority (eight) of teachers' responses were deeper at the post-noticing activity, while four remained steady and two decreased. Although many teachers' depth of noticing increased throughout the PD, we also noticed that their reasoning was not supported by interpreting students' possible positions and classroom storylines. In this poster we discuss implications for the design of subsequent iterations of this online asynchronous PD.

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AI-SUPPORTED PROFESSIONAL LEARNING OF PRESERVICE MATHEMATICS TEACHERS: IMPLICATIONS FOR SUPPORTING STUDENTS' ZONE OF PROXIMAL DEVELOPMENT

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The rapid expansion of generative artificial intelligence (GenAI) has entered mathematics teacher preparation programs, where preservice teachers use tools that draft lessons, suggest solution paths, simulate student responses, and provide instructional feedback. Although AI's role in mathematics teacher professional development has received limited attention (Joshi & Heinold, 2025; Joshi, 2026), it is important to examine how AI-supported professional learning shapes preservice mathematics teachers' ability to support students' learning within the Zone of Proximal Development (ZPD). This review asks: How does AI-supported professional learning shape preservice mathematics teachers' capacity to design and enact instruction aligned with principles of the ZPD? Vygotsky's (1978) ZPD guided the analysis of this review.

A systematic literature review was conducted following PRISMA protocols (Page et al., 2021). Six databases were searched for peer-reviewed empirical studies on AI in preservice mathematics teacher education published from 2023 to January 2026. 178 records were identified, and only 13 studies met the inclusion criteria after duplicate removal and screening. Data were analyzed using reflexive thematic analysis (Braun & Clarke, 2006). This poster conference proceeding included the highlights of four representative studies (Bernardi et al., 2025; Flavin et al., 2025; Gurl et al., 2025; Son et al., 2024) that illustrate the three central themes identified across the review.

The first theme was that AI functioned as a mediational resource rather than an instructional authority. Preservice teachers used professional judgment to interpret AI outputs rather than treating them as authoritative. AI-supported simulations enabled responsive teaching practice through interactions with virtual students (Son et al., 2024), while GenAI broadened lesson-planning possibilities and promoted reflection on instructional alternatives (Bernardi et al., 2025; Gurl et al., 2025). The second theme highlighted how teacher knowledge shaped the quality of this mediation. Participants differed in how they questioned, revised, or accepted AI-generated outputs (Flavin et al., 2025). Those with stronger mathematical knowledge engaged more critically, whereas others were more likely to accept outputs uncritically, sometimes leading to errors in lesson plans. Studies also found that pedagogical outputs were evaluated more accurately than mathematical outputs (Gurl et al., 2025; Bernardi et al., 2025).

The third theme integrated these findings within the ZPD framework. Because the quality of AI mediation depended on teachers' mathematical and pedagogical knowledge, AI did not function as scaffolding independently. Instead, preservice teachers transformed AI outputs into meaningful scaffolds by evaluating, adapting, and orchestrating them in response to students' needs. In this sense, AI expanded opportunities for instructional support, but its contribution to student learning depended on informed teacher mediation.

AI rarely served as an instructional authority; rather, its contribution to instruction aligned with the ZPD depended on preservice teachers' ability to critically evaluate and adapt AI-

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generated outputs (Flavin et al., 2025; Son et al., 2024). From a sociocultural perspective, AI functions as a mediational artifact within the ZPD (Vygotsky, 1978), whose value is realized through knowledgeable teacher mediation. This review is based on 13 empirical studies. Future research should examine how AI-supported teacher mediation contributes to students' mathematical learning within the ZPD.

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INVESTIGATING INSTRUCTIONAL CIRCLES AS A PROFESSIONAL DEVELOPMENT MODEL FOR MATHEMATICS TEACHER EDUCATORS

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Our poster aims to present the *Instructional Circle* (Melville, 2022) model and discuss its affordances and obstacles for mathematics teacher educator (MTE) professional learning. MTEs need an additional layer of expertise for working with prospective teachers, beyond the knowledge needed for K-12 teachers to work with students (Castro Superfine & Li, 2014a). Lave and Wenger (1991)'s community of practice offers a promising approach for supporting MTEs' professional development (Castro Superfine & Li, 2014b; Dick et al., 2022; Castro Superfine & Pitvorec, 2021; Olanoff et al., 2021). We propose instructional circles (Melville, 2022) as an innovative structure that leverages our community of practice to build MTEs' knowledge and skills for preparing mathematics teachers.

Instructional circles are a common professional learning opportunity in Japan, which helps teachers deprivitize their practice, address specific problems of practice, and meet their professional goals. Instructional circles are often made up of teachers with the support of math instructional coaches and/or university professors serving as a *knowledgeable other*. During an instructional circle, teachers present their research lesson that address a desired problem of practice (Takahashi, 2015). Other group members and the knowledgeable other then engage in feedback-giving practices towards improving the identified problem. Not only does the presenting teacher receive support, but the discussion could have implications for the other members' practice.

The instructional circle we started as university-based MTEs is a novel use of the model. Initially, our group needed support in teaching teacher education courses. Similar to K-12 instructional circles, each month we alternate presenting research lessons and problems of practice, giving feedback, and reflecting on our experiences. The novelty of the model for MTEs is that group members draw on each other's expertise because there is no knowledgeable other to guide the conversation or provide insight into theory.

After each instructional circle model, each member wrote a reflection on their takeaway from that meeting. We were curious to see the benefits from the MTEs not presenting their specific problem of practice as well as how instructional circles afforded the development of a community of practice among university-based MTEs. We then analyzed our reflections to look for common themes among our experiences, and did member checks to verify our findings.

Initial analysis of our reflections surfaced two levels of support experienced by the MTEs. The first is concrete support for answering the problem of practice that was proposed. For example, Author 1 received advice from the group about using manipulatives in a fractions lesson, and a member of the group shared a useful article to assign to students related to the class activities. Application of the group's advice and resources resulted in a more interactive and cohesive lesson in Author 1's methods class. Secondly, MTEs found a supportive space for

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collaborative thinking, innovative problem solving, and a collective effort to improve math teacher education across universities.

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(RE)DEFINING TEACHER LEADERS' UNDERSTANDING AND IMPLEMENTATIONS OF EQUITY IN THE CLASSROOM

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Understanding equity is an essential component in the improvement of mathematics and science teacher education (Gutiérrez, 2012; Jacobs, 2012). Although teachers have reported strong links between equity theory and practice in their credential programs (Athanases & Martin, 2006), the context and aims of equity-oriented education vary (Levinson et al., 2022). Additionally, there is little established research concerning equity and teacher leadership (Milner et al., 2015). We adopt Philip's (2011) *ideology in pieces*, which proposes that individual conceptualizations of equity are informed by lived experiences as well as shared social understandings, to understand teacher leaders' (TLs) shifting understandings of equity.

We studied a cohort of seven mathematics and two science TLs, working in urban schools, participating in a five-year teacher leader professional development. Participants included six who identify as women and three who identify as men from a mix of cultural backgrounds: four White/non-Latine, three Latine, one Vietnamese American, and one reporting mixed ethnicities. The TLs participated in four primary experiences, including meetings exploring equitable teaching practices for urban schools (Johnson et al., 2019).

Data included an open-response survey administered in the first year of the fellowship and an interview at the end of the fourth year. The survey asked the TLs to select amongst common historical definitions of equity (Pelaez, 2022), while interview allowed the TLs to elaborate on, and describe any changes to, their definition. The interview also asked the TLs to provide examples of how their understanding of equity is expressed in their teaching and to explain how the fellowship experiences influenced their thinking and practices related to equity. Preliminary analysis categorized responses into whether the TLs kept, changed, or modified their definitions. Additionally, we open coded the TLs' interview responses, looking for distinct ideas about how they enacted their definition in the classroom and what aspects of the fellowship they attributed their definition to. Open codes were thematically organized across participants, identifying themes related to changes in their definitions, enactment in the classroom, and their attribution to aspects of the fellowship (Creswell & Poth, 2018).

Eight of the nine TLs suggested changes to their chosen definition with many modifications focused on identifying student needs. When asked about how their definition manifests in their classroom, seven of the nine discussed taking time to get to know their students, arguing for definitions that ensure "every kid gets what they need." When reporting on components of the fellowship, they often attributed their definition of equity and teaching practices, to learning about equitable teaching practices in urban schools (Johnson et al., 2019). One limitation, particularly through the lens of ideology in pieces, is an omission of how each TLs' social positioning via their intersecting identities might impact their shifting definitions of equity.

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DEPRIVATIZING INSTRUCTIONAL PRACTICE USING INSTRUCTIONAL CIRCLES

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Background

The effectiveness of professional learning opportunities depends on their content and how the content is facilitated, *what* teachers' study, and *how* they study it (van Es et al., 2014; Kazemi et al., 2008; Kennedy, 2016). In my study, I focus on the way in which professional learning opportunities are facilitated, on how teachers are engaged in studying teaching.

Instructional circles are somewhat informal gatherings of small groups of teachers to discuss instructional practices (Melville, 2022). These circles seem oriented to analyzing specific practices and sharing advice rather than collectively designing a lesson or learning about a specific instructional method. Some local teachers and educators voluntarily attend instructional circle sessions that usually meet once every few months. Japanese teachers have described them as important professional learning activities that serve an essential function quite different from lesson study. This session will look at how instructional circles support the development of teachers' ability to critically discuss improvements to instructional practices enabled through the deprivatization of their practice.

U.S. teachers often have a difficult time deprivatizing their practice among other professionals (de Jong et al., 2019). Often when teachers do deprivatize their practice, three main types of interactions occur: giving praise, giving advice, and (uncommonly) engaging as challenging colleagues (Males, 2009). Engaging as challenging colleagues is less about critiquing the professionalism of the teacher but working together to improve the instructional practices found within a lesson. Thus, taking the focus from the teacher and put onto *teaching* (Hiebert & Grouws, 2007). This session will present how instructional circles are able to help teachers deprivatize their practice and engage as challenging colleagues to improve their instructional approaches to teaching.

Methods and Results

An ethnographic approach was taken to learn about how instructional circles engage U.S. teachers in deprivatizing their practice in a community of learners. We focused specifically on what teachers brought to the instructional circle, what they taught, and then how they felt the process beneficial for their learning. We did participant observations, interviews, artifact collection, and journaling.

So far, we have found that instructional circles addressed the need for a safe place for teachers to share their practice with others. In other words, teachers felt comfortable deprivatizing their practice to allow for high-quality discussions about the improvement of instructional practices as identified by the teachers themselves. Teachers were able to develop a community of learners through their focus on improving a specific aspect of the lesson plan. For example, one teacher was trying to increase student engagement and asked the other teachers for

their input. Teachers offered advice, but the focus was more detailed through the lens of trying to accomplish the learning goals.

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LEARNING TO LEAD IN ELEMENTARY MATHEMATICS: DEVELOPING TEACHER LEADER CAPACITY TO SUPPORT TEACHER LEARNING

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Keywords: Professional Development

Teacher learning is instrumental to math instructional improvement, and research sheds light on the kinds of teacher learning experiences that generate such practice change. We have known for decades that more ambitious instructional improvements, like those needed in math, require a sustained emphasis on teacher learning (Spillane & Thompson, 1997). While research has established that schools are critical contexts for teacher learning, many lack the infrastructure or culture to provide robust professional learning. As a result, fostering and systematizing the kinds of within-school learning experiences that are vital to teacher development has proven to be challenging. "Job-alike teams" (Gallimore et al., 2009), such as formalized grade-level or subject-oriented teacher teams, provide a natural structure for teacher learning (Weddle, 2022). Because they provide the kinds of ongoing time and space of value for sustained teacher learning throughout the year (Kyndt et al., 2016), teacher team meetings hold strong potential for providing learning experiences that are vital to teacher learning. Although research highlights the key roles that teacher leaders and principals play in supporting teacher learning, few interventions have been designed and tested to build the collaborative capacity of these leaders to develop teacher teams as effective teacher learning environments.

In this poster, we illustrate the design and implementation of a professional learning intervention, Learning to Lead in Math (LTL-Math) Project, focused on developing both math teacher leaders (MTL) and principals as co-leaders in and contributors to the support of teacher learning and instructional improvement in schools. The design of LTL-Math includes three components: (1) math teacher leader professional learning, (2) principal professional learning, and (3) school-based group coaching. In this poster, we focus exclusively on the first component, the design of the MTL professional learning. Drawing on prior teacher learning research (Horn, 2010), MTL learning focuses on the facilitation of a set of active teacher learning routines during teacher team meetings (i.e., lesson planning, analyzing instructional practice, examining student learning artifacts). Related to these routines, MTL learning experiences focus on promoting an understanding of how to support teachers in using instructional practice and artifacts as objects for collective instructional inquiry (Lesseig et al., 2017). This occurs as teacher practice becomes public within a teacher team (Kruse et al., 1996), all teachers on the team adopt an inquiry stance (Lieberman & Miller, 2001), and the team uses instructional practice as a learning resource.

During professional learning sessions, MTLs engage in active learning experiences using a modified version of the Learning to Lead cycle (Borko et al., 2021) and effective facilitation practices (van Es et al., 2014). These experiences include five phases in which MTLs *experience*, *analyze*, *practice*, *enact*, and *reflect* on teacher team meetings. MTLs begin by *experiencing* a simulated teacher team meeting modeled by a project coach, then *analyze* the facilitation of meeting activities, then plan and *practice* the facilitation of a teacher team meeting with targeted feedback on their rehearsals (Kazemi et al., 2016), then *enact* facilitation of a teacher team meeting in their school context, and finally *reflect* on the enactment. Throughout, MTLs learn to design, adapt, and use tools and practice artifacts to support teacher learning.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

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FUNDS OF KNOWLEDGE: MATHEMATICS TEACHERS PRACTICES

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Keywords: Culture and Asset-Based Pedagogies; Equity, Inclusion, and Diversity

Asset based practices in mathematics instruction emphasize the importance of teachers identifying and using students lived experiences and experiential knowledge as foundational resources for teaching and learning. One of such frameworks is Funds of Knowledge (FoK) which that leverages students' family, community and cultural experiences for teaching (Moll et al., 1992). Research has reported higher student engagement, motivation and potential improvement in achievement when teachers connect the school mathematics to students' cultural, community, and family knowledge (Civil, 2007; Turner & Drake, 2016). This study investigates mathematics teachers views on FoK, specifically how they conceptualize it, elicit and integrate students FoK in mathematics instruction.

Methods

This study forms part of ongoing broader research involving K-12 mathematics teachers participating in a master teaching fellowship the United States. We interviewed seven mathematic teachers and observed three lessons from them. The interviews examined teachers understanding of FoK and the observations examined how Fok was integrated in classroom instruction. Data from the interviews and observations were coded and analyzed thematically.

Discussion and Implications

Three interrelated themes emerged around mathematics teachers' understanding, enactment and challenges with FoK.

Understanding FoK

Mathematics teachers conceptualized FoK as (1) students' prior academic knowledge (Denton & Borrego, 2021), (2) community-based cultural knowledge and (3) a process of formalizing everyday mathematics into school mathematics. These are family (farming, ranching etc.) and cultural (music, clothing, beadwork) based knowledge which were elicited through relationship building and involvement in school and community activities.

Enacting FoK

The analysis revealed an emerging instructional cycle: eliciting and recognizing → connecting and integrating → reflecting and refining. Teachers identified and connected students FoK to lesson plans before instruction, translated these FoK into mathematical representations, examples and tasks during instruction and the reflected on these connections and identified newly surfaced FoK to inform subsequent instruction.

Integration challenges

Teachers identified institutional constraints such as curriculum demands and limitation of instructional time as barriers to planning for FoK integration. Teachers' capacity (Hunter et al., 2024), family perception and parent expectation of what should be taught further complicated efforts to adopt FoK in instruction. These results imply that integrating FoK goes beyond the use of real-life examples in mathematics lessons, but it is an ongoing process of connecting students' experiences which is shaped by teachers choices and institutional constraints.

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STATISTICS, PROBABILITY, AND DATA SCIENCE

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EXAMINING THE COGNITIVE DEMANDS OF LLM-MEDIATED EXPECTED VALUE TASKS

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Given students' growing use of large language models (LLMs) for academic work, research is needed to examine how the cognitive demands of statistics tasks are mediated by LLMs. We examine the cognitive demands at which ten undergraduate students, in think-aloud interviews, solved two expected value problems with access to ChatGPT. We found that choosing not to use ChatGPT did not guarantee that students resolved tasks at higher levels of cognitive demands than those who did use it. We also found evidence that LLMs can support students in connecting task language to relevant constructs and making progress when they lack an initial strategy. The interviews also highlight the role students play in verifying LLM-outputs and in self-monitoring their problem-solving processes, both of which are critical for high-demand problem-solving.

Keywords: Statistics, Large Language Models, Cognitive Demand

A long-standing discussion in mathematics education research concerns the cognitive demands of mathematics tasks (Doyle & Carter, 1984). This discussion has distinguished the demands as intended by task designers from the demands that emerge as instructors enact tasks with their students (Stein et al., 1996; Stein & Smith, 1998). Existing research has noted the impacts that students' uses of technology have on cognitive demands of tasks as they solve them (Sherman, 2014). Recognizing that statistical activities, while distinct from mathematical activities, blend mathematical and non-mathematical knowledge and thinking (Groth, 2007), we extend this discussion to the cognitive demands of students' problem-solving on statistics tasks.

The growing prevalence of AI tools, particularly large language models (LLMs), among undergraduate students (Morell-Mengual et al., 2025) and their increasing adoption by universities (Ford & Knox, 2025) have raised questions about their impact on students' problem-solving practices. LLMs occupy a unique position compared to other technologies. LLMs can serve functions like tools commonly used in mathematics and statistics classrooms, such as calculators. However, they also introduce features and limitations not captured by existing descriptions of other technologies. For instance, LLM-generated outputs can mimic student thinking and reasoning and, in some cases, correctly explain concepts or solve problems. These outputs, however, are generated stochastically and require user verification. These differences invite the question of how LLMs may impact the level of cognitive demands at which students solve tasks, which may differ from what their teacher or the curriculum designer intended.

This study investigates the cognitive demands that students take on as they solve problems with LLMs. We also build on discussions that consider how features of LLMs, such as their ability to generate explanations and illustrative examples of statistical concepts, may support students' statistical reasoning (Ellis & Slade, 2023). Here, we examine if these kinds of benefits (or limitations) emerge as students solve these problems. To that end, we ask: *How does an LLM mediate the level of cognitive demands at which undergraduate students solve probability tasks?*

Theoretical Framing

Cognitive demands refer to the "cognitive processes in which students actually engage in as they go about working on the task," (Stein et al., 1996, pg. 461). There has been a historical

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focus on sustaining high-demand tasks because the types of thinking and reasoning that students do ultimately influence what mathematics they will learn (Doyle & Carter, 1984). Stein and Smith (1998) characterized cognitive demands in their Mathematics Tasks Rubric (1998), an artifact of the QUASAR project, organized in a four-level hierarchy that describes low-to-high thinking demands that students may employ, depending on how the task is written and how they interpret what they are asked to do. A task may require students to rely on *memorization (low demands)*, involving the reproduction of previously learned facts, rules, formulas, or definitions with no connection to underlying concepts. Tasks may emphasize *procedures without connections (low demands)*, in which students apply algorithmic procedures in routine ways that focus on obtaining correct answers. *Procedures with connections (high demands)* requires students to use procedures to make sense of mathematical ideas in context, often requiring attention to underlying concepts and representations. Finally, tasks categorized as *doing mathematics (high demands)* require non-algorithmic thinking, including exploring relationships, navigating multiple solution pathways, and engaging in sustained reasoning and self-monitoring. These levels can be used to characterize the cognitive demands of tasks at various stages of enactment. This suggests (and prior work supports that) cognitive demands of tasks can be mediated temporally from their written, set up, and solved forms (Smith et al., 2007).

The level of cognitive demands at which a student solves a task may differ from that intended by designers of the task. One reason could be the tools that students use while solving. Sherman (2014) extended Stein et al.'s work to show that students' uses of technologies can further mediate the cognitive demands at which students solved tasks: technologies only help students sustain high cognitive demands when they reorganize (Pea, 1985) student thinking in ways that shift their focus to high-demand aspects. For instance, when students use graphing calculators, their thinking demands may shift to interpreting multiple graph-based models in context (*high demands*). Whether students can sustain high cognitive demand while working on tasks with LLMs remains an open question, since many of today's LLMs can solve many undergraduate mathematics tasks correctly most of the time. Recent work examining students' problem-solving with LLMs has suggested that cognitive demands largely depend on what students prompt them to do (e.g., Wu et al., 2025; Yoon et al., 2024). In Wu et al.'s study, students who checked their chatbot's work with resources from their statistics class retained more thinking demands than students who solved the task solely through LLM-mediated interactions. However, these studies have not measured cognitive demand directly or examined how students' thinking processes may alter the cognitive demands of tasks. We address this gap through the following analysis.

Methods

Interview Structure and Task

The work reported here is drawn from a larger study investigating how undergraduate students solve mathematics problems with access to GenAI tools (Delaney et al., 2025). Twenty participants were recruited via flyers posted in public spaces at a public university in the western United States. Students were eligible if they had taken at least one mathematics, probability, or math-intensive STEM course within the past 12 months; the final sample included both STEM and non-STEM majors. We conducted individual think-aloud cognitive interviews (Goldin, 2000) in which each participant solved three multi-part tasks spanning college algebra, probability, and calculus topics. One researcher served as the lead interviewer while the other managed the recording and took notes. The interview setup included a laptop open to ChatGPT-4o, which was selected because the institution provided students with ChatGPT subscriptions. However, students were allowed to use any generative AI tool they preferred, such as Claude or

Gemini, or not use any generative AI tools at all. Two researchers conducted each interview. The interviewer asked follow-up questions to clarify participants' ideas and to understand how GenAI, when used, supported their problem-solving.

This paper focuses on the second and fifth items of the probability task, named the *ATM Task*, adapted from a question in a previous AP Statistics examination (College Board, 2015; see Figure 1). The second item, which we label as the *expected value item*, was still intended to motivate the calculation of the expected value. However, we replaced "expected value" with "average" to raise the item's cognitive demand, requiring participants to either recognize that the expected value formula or infer what "average" means in the context of a discrete probability table. The fifth item, labeled the *conditional expected value item*, was given as written.

A shopping mall has three automated teller machines (ATMs). Because the machines receive heavy use, they sometimes stop working and need to be repaired. Let the random variable X represent the number of ATMs that are working when the mall opens on a randomly selected day. The table shows the probability distribution of X.

X = Number of ATMs working when the mall opens	0	1	2	3
Probability	0.143	0.286	0.429	0.143

Item 2. On average, how many ATMs will be working on any given day?
Item 5. Given that at least one ATM is working when the mall opens, would the average number of ATMs working on any given day be less than, the same, or greater than the average you found in question 2? Explain your answer.

Figure 1: Excerpt of the ATM Task

Data Analysis

Our findings come from an analysis of 10 participants who completed both the expected value and conditional expected value items. This group consisted of eight female students, one male student, and one non-binary student, representing both STEM and non-STEM majors. All participants had taken a mathematics course, or a course involving mathematics, during the academic year in which the data were collected. The first and third authors served as raters and reviewed video recordings, written work, and field notes from data collected from these ten participants. They assigned a cognitive demand rating to each participant's solution. Ratings were based on criteria from the Mathematical Tasks Framework (Stein & Smith, 1998), adapted to account for the forms of thinking and reasoning relevant to solving the expected value and conditional expected value items of the ATM Task (see Table 1).

Table 1: Stein & Smith (1998) Framework Applied to Expected Value Items

Level	Expected Value Item	Conditional Expected Value Item
0 - N/A *Guessing, non-mathematical reasoning.*	Guessing without procedures or mentioning previously learned facts, rules, or definitions about expected value.	Guessing without procedures or mentioning previously learned facts, rules, or definitions about conditional probability or expected value.
1 - Memorization *Involves either reproducing facts, rules, formulas, or definitions.*	Reproduces a fact or definition about averages or expected value. *Example:* A student describes the expected value as the long-run average of a random variable.	Reproduces memorized ideas about conditional probability (e.g., a formula) and/or expected value. *Example:* A student only recalled the conditional probability formula.

2 - Procedures without Connections *Focused on correct answers, does not connect to underlying meaning.*	Produces an answer using a procedure without making sense of the computed value or the underlying reasoning. *Example:* A student claims that $E(X) = 0(0.143) + 1(0.429) + 2(0.429) + 3(0.143) \approx 1.57$ working ATMs, on average.	Attempts to solve the problem via a procedure with little to no conceptual explanation of how conditioning changes the probability distribution. *Example:* A student computes conditional probabilities and obtains $E(X) = 1(0.334) + 2(0.500) + 3(0.167) \approx 1.83$ working ATMs if $X \geq 1$.
3 - Procedures with Connections *Procedures for the purpose of developing deeper levels of understanding of mathematical concepts.*	Produces an answer using the standard formula and reasons or makes sense of the computed value or the process behind obtaining the value. *Example:* A student completes the calculation above and checks the result by identifying the median of the distribution.	Solves the problem via a procedure but connects the procedure with an explanation showing how conditioning changes the distribution (e.g., describing the impact of removing $X = 0$ on the other probability values). *Example:* A student completes the calculation above and connects this value with the assumption $X \geq 1$.
4 - Doing Mathematics *Complex, nonalgorithmic thinking, make use of relevant knowledge and experiences*	Determines the average using conceptual reasoning rather than formulaic computation; justifies the result using the structure noticed in the item. *Example:* Without a formula, a student explains that the heavier probabilities of $X = 1$ and 2 pull the average to be between those values.	Uses structural reasoning (not a procedural method) to justify the comparison, for instance, by explaining how eliminating $X = 0$ produces a conditional probability distribution. *Example:* Without performing the standard calculations, a student reasons that, given $X \geq 1$, the probabilities for $X = 1$, 2, and 3 increase proportionally, leading to a greater average.

We recognized that students can engage with a single item at multiple levels of cognitive demand while solving (Boston, 2006). Thus, we assigned ratings solely according to the participants' *resolution* of the item (Boston & Candela, 2018). For instance, a student might initially reason about the conditional expected value item in ways that reflect *doing mathematics*, but then abandon that reasoning in favor of applying statistical formulas, resulting in a code of *procedures without connections*. We also assigned ratings based on the nature of participants' problem-solving (as detailed by Stein and Smith's (1998) descriptions, not correctness. The raters initially coded the solutions independently, reaching agreement on 70% of the ratings. They then clarified the assigned ratings and the criteria themselves until reaching full agreement. For example, these discussions clarified when a resolution should be coded as not applicable and what kinds of non-algorithmic work and explanation could constitute doing mathematics.

Findings

We begin with the intended cognitive demands of the two items as a baseline. We determined these demands by reviewing solutions published by the Advanced Placement program. The published solution to the expected value item employs the formula for the expected value of a discrete random variable. We rated this solution as *procedures without connections*, since the written solution did not require students to make sense of the expected value in relation to the

data distribution. The written solution to the conditional expected value item, on the other hand, required students to observe that $X \geq 1$ would result in a new distribution where the probability values would be rescaled proportionally to sum up to 1, thus increasing the expected value. We rated this solution as *doing mathematics* as it entailed a non-algorithmic approach that notices the relationship between a conditional event and expected value.

Figure 2 summarizes the cognitive demand ratings the authors assigned to each participant's solution to the expected value and conditional expected value items, along with the time spent solving each item and whether ChatGPT was used on either item. The findings are organized below by task item. As emphasized in the research question, we pay special attention to how participants' resolutions were mediated by their interactions with ChatGPT.

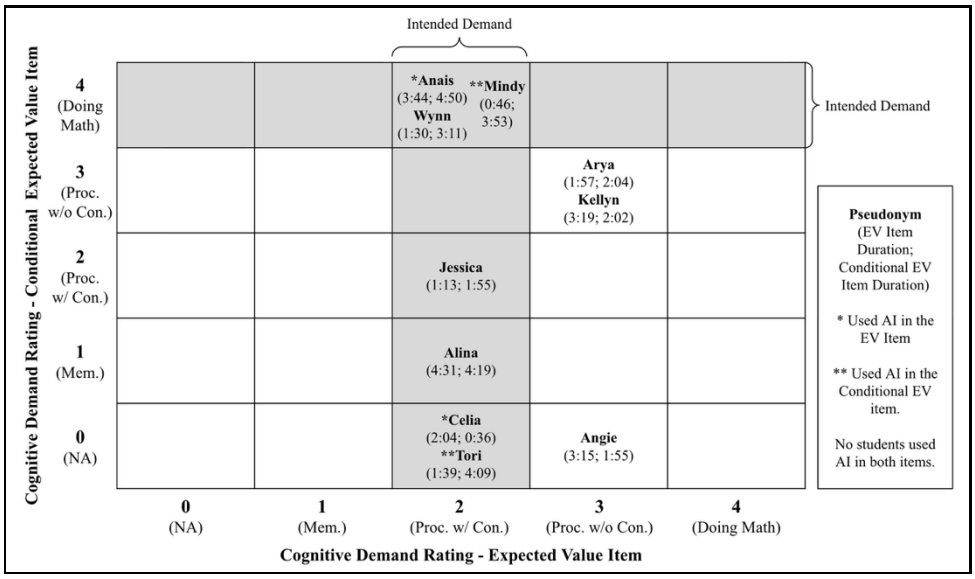


Figure 2: Summary of Cognitive Demand Ratings

Expected Value Item

This item asked for the average number of ATMs working on a given day. All ten participants had resolutions coded at the intended level of cognitive demands, *procedures without connections*, or higher. Because of this, we divide our findings related to this item based on whether the participants' resolution was rated as solved with high cognitive demands.

High cognitive demands. Three participants resolved the item at the level of *procedures with connections*, with none using ChatGPT. These participants computed the expected value using the standard algorithm and reasoned aloud about their final answer by triangulating it with another measure of center for the probability distribution. For example, Kellyn completed an identical procedure, obtaining 1.57, and argued that the value was reasonable since "looking at the probability values, I would expect [the average] would probably fall between 1 and 2." Here, she was using her notion of center in the probability distribution to triangulate her work.

Low cognitive demands. Seven participants resolved the expected value item using solutions we rated as *procedures without connections,* aligning with the task item's intended level. Five of these participants did not use ChatGPT and employed a range of strategies. Notably, only one participant, Wynn, applied the formula for the expected value of a discrete random variable, doing so correctly but without articulating any reasoning to justify the

procedure or the reasonableness of the resulting value. The remaining four participants relied on non-normative procedures to determine an average, such as selecting the X value associated with the greatest probability or computing the average of the probability values.

The other two participants in this group, Celia and Anais, used ChatGPT while solving the item. Celia, who was in the AI-Only condition, used ChatGPT solely as a calculator to implement her own approach of averaging the given probability values. Anais, in contrast, prompted ChatGPT in ways that supported her solution to the problem beyond computation.

Upon reading the task item, Anais expressed uncertainty about how to proceed, stating, "It's something that's catching me. [The problem] doesn't look familiar. I don't have a sense of direction on where to go." Anais then prompted ChatGPT with a general query about how to find an average given X values and their probabilities (Figure 3). ChatGPT generated an output that connected the notion of "average" with expected value and provided a worked example.

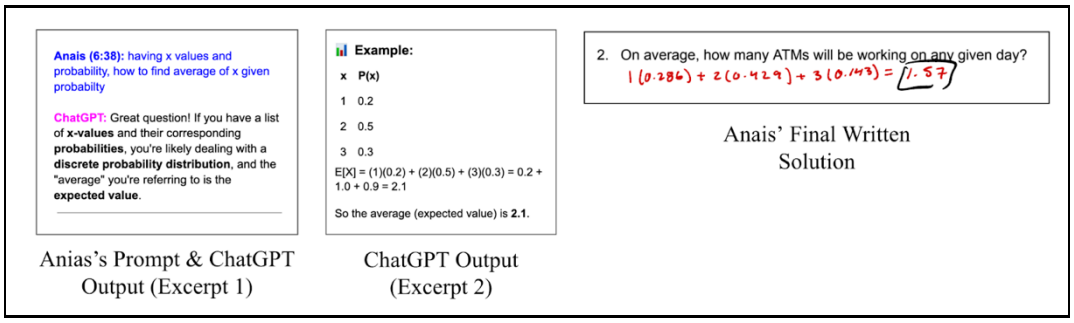


Figure 3: Excerpts of Anais' Interactions with ChatGPT and Solution

Anais reflected on how ChatGPT's example aligned with the task components: "[ChatGPT] gave me an example of essentially the information I have, which is X values and probability, and finding the expected value, which is what the question is asking." She adopted ChatGPT's connection between "average" and "expected value" and correctly applied the procedure to obtain 1.57. Discussing her solution, she emphasized the familiarity of the formula rather than interpreting the result in relation to the context or the given probability distribution, stating, "I [have seen this formula] in high school … These symbols right here, everything looks familiar."

Conditional Expected Value Item

This item asked if and how the average number of working ATMs would change under the condition $X \geq 1$. We organize findings related to this item based on whether participants resolved the task item at or below the intended level of cognitive demand, *doing mathematics*.

At intended cognitive demand. Three of the participants (Anais, Mindy, and Wynn) solved the item at the level of *doing mathematics*. In general, these participants hypothesized, without assistance from ChatGPT, about how conditioning would affect both the probability distribution and average, with varying levels of accuracy in their implementations and explanations. Wynn, for example, produced diagrams illustrating how the probability values would change proportionally under the condition to reason that the average would increase.

Mindy was the only participant in this group who used ChatGPT while solving the item. Her use of ChatGPT was limited to using it as a calculator for an initial attempt at the problem. However, she abandoned this attempt and instead solved this item using similar reasoning as the other two participants, which led to her resolution being rated as *doing mathematics*.

Below intended cognitive demand. The remaining seven participants resolved the conditional value item below the intended written level of cognitive demands. Two of these participants, Arya and Kellyn, had resolutions rated as *procedures with connections*, thus still representing high-demand solutions. These solutions contained similar ideas to those described above, but rooted in the completion of a procedure as opposed to non-algorithmic reasoning about the condition. Arya, for example, argued that the same calculation applied to the expected value and conditional expected value items. He reasoned that outcomes with a value of 0 "don't contribute to the average" because "anything you multiply by zero is just going to be zero," and therefore only including the values of X where $X \geq 1$ and their probabilities would not change the result. On this basis, he concluded that the average, under the condition, would not change.

The resolutions from the remaining five participants (Tori, Celia, Angie, Alina, and Jessica) were rated at lower levels of cognitive demand. While the thinking from these participants varied widely, these participants generally struggled to interpret and account for the condition $X \geq 1$. For example, two treated the item as a comparison of previously computed values rather than reasoning about how the condition affected the probability distribution and the expected value. Tori was the only participant in this group to prompt ChatGPT in response to the condition.

Tori immediately expressed uncertainty about how to solve the problem, indicating difficulty interpreting the word "given" in the item statement after struggling with a prior conditional probability item. She then prompted ChatGPT (Figure 4) with the text of the conditional expected value item, but copied the prompt incorrectly (e.g., writing "greater than the average of 2 ATMs" instead of "greater than the average you found in Question 2"). ChatGPT's response, organized into two steps (headings not pictured). ChatGPT first calculated an expected value without considering the condition (an unnecessary step given the prompt). It then computed the conditional expected value by determining conditional probabilities and recomputing the expected value and concluded that the conditional expected value was less than two.

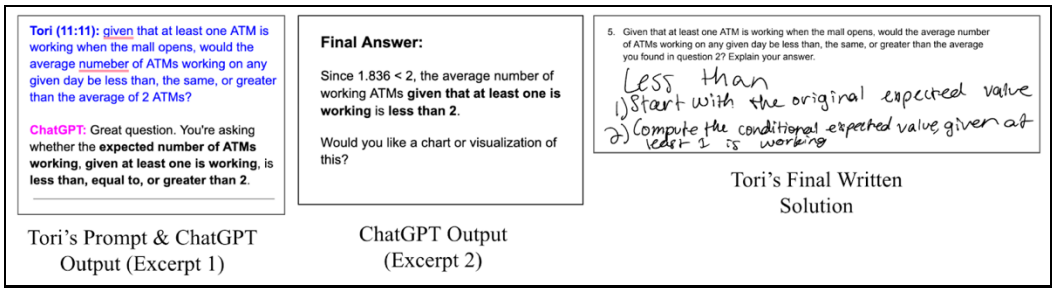


Figure 4: Excerpts of Tori's Interactions with ChatGPT and Solution

In resolving the task item, Tori accepted ChatGPT's answer of "less than" and copied the headings on her written work. In doing so, she wanted "to explain [her] answer" and that she might "just use the steps that it gave [her]." Tori clarified, however, that without understanding what "given" implies for solving the item, she had no way of verifying whether or not ChatGPT's output was correct. Thus, she herself noted the potential fallibility of ChatGPT's output, while also acknowledging that her limited knowledge left her unable to verify the solution. However, we note that she did not use ChatGPT to attempt resolve this missing understanding. Tori's resolution of the task item was rated as *not applicable,* as she accepted ChatGPT's output with little evidence of sense-making about its contents.

Discussion

Our findings extend the field's understanding of how cognitive demands are mediated in statistics and how some students' LLM uses contribute to changing probability tasks' demands. Notably, six of the ten participants did not use ChatGPT on either task item. However, abstaining from LLM-use did not seem to influence the demands at which they solved the tasks with respect to the designers' intentions. This introduces questions of what LLMs could have afforded to students who were less accurate or worked at lower levels of demand on either item. For instance, Celia, Angie, and Alina resolved the conditional expected value item at the levels of *memorization* and *NA*. They did not prompt ChatGPT, despite struggling with accounting for the condition. Had they used ChatGPT as a reference to account for the condition, could they have been better positioned to engage with the item at a higher level of cognitive demands?

Anais and Tori used LLMs to mediate the task's demands, and were similar in that they prompted ChatGPT to progress when they did not know how to proceed. For Anais, ChatGPT served as a reference to obtain a general procedure for the expected value item that she could apply and eventually resolve the task at the level of *procedures without connections*. For Tori, ChatGPT provided a full solution to an incorrect version of the conditional expected value item that she was unable to make much sense of, leaving her resolution rated as *NA*. We also saw that for Celia and Mindy, LLM-use did not appear to alter the cognitive demands of their solutions, as they had their own approaches and solely used ChatGPT as a calculator.

Our descriptions of students' problem-solving also suggest that students engaged in self-monitoring (Flavell, 1979) while reviewing outputs from ChatGPT, albeit to a limited extent. Specifically, Anais and Tori expressed gaps in their knowledge, turned to ChatGPT accordingly, and appeared to reflect on what they did and did not understand (Fan et al., 2025) about the statistical content of the task. However, we also observed that none of the participants who used ChatGPT checked the results or generated follow-up prompts to triangulate the information they obtained from their initial queries. For instance, Tori acknowledged her limited subject-matter knowledge for the conditional expected value item but did not pursue follow-up prompts that might have deepened her understanding of the relevant probability concepts. This pattern raises concerns regarding cognitive demands in LLM- mediated problem-solving, as high-demand problem-solving requires students to "access relevant knowledge" (p. 348) necessary for solving and understanding problems and monitor their knowledge and thinking (Stein & Smith, 1998). These findings suggest that students may need explicit motivation to monitor their understanding of and follow-up with LLM-generated outputs to better emulate high-demand problem solving.

Of course, our findings and discussion should be interpreted in light of the study's limitations. The low-stakes interview setting may restrict the applicability of the findings to other contexts, such as problem-solving in the context of coursework where participants obtain a grade. The think-aloud structure of our interviews may have prompted students to think more metacognitively than in settings outside of the interview-context. Additionally, the conditional expected value item was placed at the end of the *ATM Task*, and some students may have felt hurried to complete it, which may have impacted their approach to solving the problem.

Nevertheless, our analysis suggests that LLM-use does not inherently shift the cognitive demands of students' problem-solving. Rather, cognitive demands may be more contingent on the ways students employ LLMs as well as their own metacognitive thinking and actions. Future work should examine how students can learn to recognize gaps in their knowledge, strategically employ LLMs to address these gaps and obtain information needed to progress in a problem, and engage in verification and reflection about LLM outputs when such tools are permitted.

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UNDERSTANDING STUDENTS' PROBABILITY VOCABULARY: OVERCOMING LINGUISTIC AND CONCEPTUAL BARRIERS IN A MULTILINGUAL NIGERIAN CONTEXT

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This study examined how Nigerian secondary school students understand key probability vocabularies—fair, independent, likely, and random—within a multilingual and culturally embedded context. Guided by an explanatory sequential mixed-methods design, data were collected from 120 students using a Probability Vocabulary Meaning Test and from 24 students through semi-structured interviews. Quantitative findings showed a moderate overall understanding of probability terms, with marked difficulties in technical concepts such as mutually exclusive (30%) and independent (33%). Qualitative results revealed pervasive semantic interference, as students frequently relied on every day and culturally shaped interpretations of fairness, chance, and fate. Analysis informed by Vosniadou's Framework Theory of Conceptual Change indicated that most students assimilated probabilistic vocabulary into naïve cognitive frameworks, with few achieving full conceptual accommodation.

Keywords: Probability

Today, the world is driven by data, and the ability to think probabilistically is an important aspect of critical reasoning and mathematical literacy (Garfield & Ben-Zvi, 2008; Gal, 2005). Comprehending probability provides people with the opportunity to make informed decisions in various daily life contexts, ranging from assessing risk in health and financial domains to interpreting weather forecasts. Accordingly, many nations have incorporated probability concepts into the primary school curriculum to develop foundational skills for innovative statistical thinking (Batanero et al., 2016). Notwithstanding its rising significance, probability appears to be one of the most difficult aspects of school mathematics. Unlike algebra or arithmetic, where methods and solutions are exact and often obvious, probability deals with prediction, chance, and uncertainty. The intellectual swing from deterministic reasoning—imagining patterns and certainty—to probabilistic reasoning, which embodies degrees of beliefs and randomness, is significant (Pratt, 2000; Kahneman & Tversky, 1972). This theoretical intricacy is echoed in dogged challenges students come across when learning the concept of probability (Shaughnessy, 1992). An important source of challenge lies in the linguistic context embodying probability concepts. Textbooks and classroom discourse in probability present many specialized technical terminologies—such as mutually exclusive, independent, fair, likely, and random—that have twin or several connotations. In plain linguistic context, these terminologies convey informal or traditionally rooted elucidations that often deviate from prescribed scientific definitions (Watson & Moritz, 2003; Batanero et al., 2014). For example, while random in statistics depicts results created by a distinct probabilistic procedure, in everyday plain conversation, it might mean disorganized, erratic, or influenced by chance or fate events.

Equally, fair can imply moral equality or social justice rather than the equiprobable nature of outcomes in probability education.

This language vagueness is a source of semantic interference, where students hold on to their everyday connotations even after formal education, obstructing complete conceptual understanding (Gal, 2005). Students may have excellent response to probability problems procedurally, yet their elucidations or casual reasoning show mixed contexts uniting subjective beliefs, moralistic, and deterministic views (Batanero et al., 2016). Such twin connotations and everyday extrapolations produce blockades to deep understanding of probability concepts. This challenge of probability vocabulary is further compounded in polyglot didactic settings. In some countries, together with Nigeria, students are often taught mathematics in a language different from their mother tongue, thereby heightening the intricacy of language mediation (Watson & Moritz, 2003). Nigeria is culturally and linguistically diverse, with English as the lingua franca alongside numerous dominant mother-tongues such as Hausa, Igbo, and Yoruba, which are default languages in students' communities and homes (Federal Ministry of Education, Nigeria, 2013). The multilingual nature of mathematics classrooms in Nigeria may present translation challenges or variable elucidations formed by homegrown cultural tales and beliefs about fairness, luck, and chance (Fischbein & Schnarch, 1997). These cultural beliefs may intensely impact probabilistic reasoning among Nigerian students. In many societies in Nigeria, events may be professed as ruled by divine will, destiny, or fate—a deterministic worldview that contradicts the probabilistic reasoning needed in mathematics. In addition, moral elucidations of justice and fairness often interweave with the statistical construct of "fairness" in probability, obscuring students' capability to extricate informal connotations from prescribed characterizations (Gal, 2005).

Research on probability misconceptions has largely focused on routine mistakes, intellectual preconceptions, or conceptual misinterpretations gleaned from cultural and linguistic dimensions (Fischbein, 1975; Konold & Pollatsek, 2002). Although a rising body of literature recognizes the importance of language in teaching and learning of probability concepts (Batanero et al., 2014; Watson & Moritz, 2003), very few studies have meticulously investigated how probability terms or vocabularies are construed in polyglot, customarily varied African settings. This dearth of empirical evidence is particularly more pronounced in Nigeria, where the intersection of cognition, culture, and language needs cautious pragmatic examination to inform effective teaching and learning. Comprehending students' elucidations of probability terminology in this setting is important for fashioning educational pedagogies that promote real conceptual change rather than superficially shallow learning. In the absence of clear attention to the cultural narratives and language nuances students bring to the teaching space, mathematics instruction risks strengthening misconstructions concealed in superficial accuracy (Jones et al., 2007). Teachers often assume common denotations of probability vocabularies, which may be acceptable in similar language sceneries but not in polyglot teaching space where word connotations are exchanged and disputed (Gal, 2005). This study is theoretically grounded in the Framework Theory of Conceptual Change (Vosniadou, 1994; 2007), which postulates that students unify knowledge into lucid but often inexperienced cognitive models that contradict simple replacement by official constructs. Conceptual change encompasses the addition of new facts and the reorganization of existing frameworks through accommodation, synthesis, and assimilation. Applying this theory to probability vocabulary, predicts that students' linguistic preference shows their causal conceptual frameworks—whether they genuinely restructure their understanding to accommodate mathematical concepts, create hybrid models blending old and

new ideas, or retain everyday meanings (Vosniadou et al., 2001). Supplementing this mental viewpoint is the theory of sociocultural learning (Vygotsky, 1986), which postulates language as a tool of mediation. Teaching and learning of probability terminology in a polyglot context includes circumnavigating not only mental mathematical characterizations but also translation procedures and social connotation-making. Traditional beliefs about causality, fairness, and chance shape how students understand constructs and engage with instruction in formal setting (Batanero et al., 2016). Incorporating these frameworks shows how linguistic and conceptual factors are intertwined in the teaching and learning process of mathematics. Therefore, this study investigated how Nigerian secondary school students construe key probability vocabularies through mixed-methods explanatory sequential research design.

Research Questions

The following research questions were answered in this study:

1. What meanings do students attribute to probability vocabularies like fair, independent, likely, and random, and how do these vary by gender and grade?
2. How do students' explanations reflect conceptual frameworks of chance and uncertainty according to the Framework Theory of Conceptual Change?
3. What cultural and linguistic factors influence students' ability to differentiate mathematical and everyday connotations of probability terminologies?

Methodology

Participants

The study population comprised 22,832 senior secondary school (SS1–SS3) students enrolled in 28 public senior secondary schools within the Ikorodu Local Government Area of Lagos State, Nigeria. This locale was selected due to its representative linguistic and cultural diversity, encompassing urban, suburban, and rural communities, thus providing a varied sociolinguistic context for studying probability vocabulary learning. A purposive stratified sampling strategy was used to recruit participants to ensure representation by school type (urban public, rural public), grade level (SS1, SS2, SS3), gender (male, female), and language background (Yoruba, Igbo, and Hausa) from one senior secondary school. The total sample consisted of 120 students (40 per grade level), for the quantitative phase balanced for gender and linguistic diversity, including students who spoke English as a second language alongside Yoruba, Igbo, or other indigenous languages at home. For the qualitative phase, a subsample of 24 students was selected using maximum variation sampling based on PVMT performance (high, medium, low scorers), gender, and language background to provide a rich cross-section of students for semi-structured interviews. This nested sampling enhanced methodological rigor by aligning qualitative inquiry closely with quantitative patterns (Creswell & Plano Clark, 2018).

Research Design

This study adopted an explanatory sequential mixed methods design to investigate how Nigerian secondary school students interpret and use probability vocabulary terms such as random, likely, independent, and fair. The rationale for employing a mixed-methods approach stems from the need to obtain both breadth and depth of understanding about students' conceptual frameworks as well as the linguistic and cultural factors shaping their interpretations. Mixed methods research combines quantitative and qualitative techniques to allow for triangulation and complementarity (Creswell & Plano Clark, 2018; Dawadi & Giri, 2021). This approach enhances interpretive validity by cross-verifying findings from different data sources

and methods. The exploratory design is particularly suited to uncovering nuanced interpretations embedded in students' everyday language and cultural contexts, which purely quantitative measures alone cannot reveal (Batanero et al., 2016). Similarly, the qualitative strand gains strength and generalizability by being embedded within a larger quantitative assessment framework. This design aligns well with the study's guiding framework, the Framework Theory of Conceptual Change, which implies that student ideas are complex and organized within coherent but naïve frameworks requiring deep investigation (Vosniadou, 1994).

Instruments

Two instruments were used for data collection in this study. They were Probability Vocabulary Meaning Test (PVMT) and Semi-Structured Interview Protocol. The PVMT is a researcher-developed diagnostic tool designed to quantitatively assess students' understanding of key probability vocabulary terms within contextually relevant scenarios. The instrument consists of 15 items combining multiple-choice and open-ended questions, constructed to elicit definitions, explanations, and applied reasoning for terms such as random, likely, fair, independent, and mutually exclusive. The PVMT was developed based on a literature review, curriculum analysis, and consultation with mathematics education experts. Care was taken to embed vocabulary items within realistic probabilistic contexts familiar to Nigerian students (e.g., coin toss, dice rolls, spinners). The instrument was piloted with 30 students from a non-participating school to evaluate clarity, cultural appropriateness, and psychometric properties. Item analysis led to refinements that improved internal consistency, with the final version attaining a Kuder-Richardson 20 reliability coefficient of 0.86, indicating good reliability for research purposes. The multiple-choice items provided objective measures of vocabulary comprehension, scored dichotomously (1 = correct, 0 = incorrect). Open-ended questions complemented these by allowing expression of reasoning and identification of the semantic models used.

Semi-structured Interview Protocol

To deepen insight into the quantitative findings, semi-structured interviews were conducted with 24 students selected for diverse performance profiles and backgrounds. The interview protocol was developed after preliminary analysis of PVMT responses and was aligned conceptually with the Framework Theory of Conceptual Change. Open-ended questions probed students' interpretations of probability terms in their own words, experiences with probability concepts in and outside the classroom, perceived differences between every day and mathematical meanings, and cultural beliefs influencing their understanding. Interview questions were piloted and revised to ensure clarity and cultural sensitivity. Interviews were conducted individually during school hours in English, with students permitted to use local languages for explanations where necessary. This bilingual flexibility helped capture authentic linguistic reasoning. Interviews averaged 30-45 minutes, were audio-recorded, transcribed verbatim, and translations incorporated during transcription where needed.

Data Collection Procedure

Data collection was strategically staged to align with mixed-methods integration principles (Fetters et al., 2013). Ethical approval was obtained from the relevant Institutional Review Board and the State Ministry of Education. Permission was secured from school authorities, and parental consent and student assent were obtained prior to data collection. The PVMT was administered in classrooms under standardized conditions during scheduled mathematics lessons. Instructions emphasized honest and thoughtful responses, particularly for open-ended items. Students were given 60 minutes to complete the test to allow adequate time for reflection.

Immediately after PVMT administration, item scoring and descriptive statistics informed the selection of interview participants to represent performance and demographic diversity. Interviews were conducted within two weeks following PVMT upon participant availability. Researchers established rapport, ensured confidentiality, and clarified that there were no "right" or "wrong" answers to encourage candid discussion. All data were anonymized using pseudonyms. Digital files were password-protected, and transcripts were stored securely. Only authorized researchers had access to identifiable data.

Data Analysis

PVMT data were entered and analyzed using SPSS (version 27). Initial descriptive statistics (frequencies, percentages, means, and standard deviations) assessed overall vocabulary comprehension levels. Inferential statistics of one-way ANOVA and independent samples t-test tested for differences in vocabulary understanding across grades and genders respectively. Interview transcripts were analyzed thematically. The coding framework was developed deductively from the Framework Theory of Conceptual Change constructs (assimilation, synthesis, accommodation) (Vosniadou, 1994) and inductively from emergent linguistic and cultural themes. Two researchers independently coded a subset of transcripts to establish intercoder reliability, obtaining Cohen's kappa of 0.87, considered high agreement. Discrepancies were discussed and resolved through consensus. Coding continued iteratively with emerging themes refined accordingly. Major themes included semantic interference patterns, cultural beliefs influencing vocabulary interpretations, negotiation of everyday versus school meanings, and evidence of conceptual reshaping.

Results

Research Question 1

What meanings do students attribute to probability vocabularies like random, likely, independent, and fair, and how do these vary by grade and gender?

Table 1: Percentage of Students Selecting Correct Definitions by Term and Grade Level

Probability Term	% Correct (SS1)	% Correct (SS2)	% Correct (SS3)	Overall % Correct
Random	30	42	55	42
Fair	46	61	70	59
Likely	40	55	65	53
Unlikely	38	49	58	48
Certain	62	74	81	72
Independent events	20	33	45	33
Mutually exclusive events	18	31	42	30

Table 1 displays the percentage of students choosing the correct mathematical meanings for each target vocabulary term across grades SS1, SS2, SS3, and overall. Statistical analyses (one-way ANOVA) confirmed significant grade-level progression for all terms ($p < 0.01$), indicating that older students demonstrated better mastery of technical vocabulary. However, even at SS3, less than half correctly identified independent and mutually exclusive terms. Gender differences were not statistically significant ($p > 0.05$).

In the qualitative phase, interview responses revealed that students frequently used every day and culturally informed meanings for vocabulary, particularly at lower grade levels. On random, Ada (SS1) said, "Random means something that just happens anyhow without reason, like when someone suddenly shouts in class," reflecting everyday notions of spontaneity and chaos rather than mathematical randomness. For fair, Chika (SS2) explained, "A fair game is when everybody agrees on the rules, and nobody cheats." This social justice interpretation was common, overshadowing the equiprobable meaning. Regarding this, Mary (SS2) stated, "It is likely that my team will win because they are the best." This subjective, belief-based interpretation contrasts with probabilistic likelihood. On independent, Musa (SS2) noted, "Two events are independent when they don't know each other, like two friends living far apart," revealing a misinterpretation equating independence with causal or social unrelatedness rather than probabilistic independence. This qualitative data illustrates the semantic interference and everyday conceptual frameworks that underlie students' vocabulary use.

Research Question 2

How do students' explanations reveal conceptual frameworks of chance and uncertainty according to the Framework Theory of Conceptual Change?

Table 2: Quantitative Summary Coding

Conceptual Category	% of Interviewees
Assimilation	58
Synthetic models	30
Accommodation	12

Using Vosniadou's conceptual categories, students' responses were coded (Table 2) into assimilation (retaining everyday meanings), synthetic models (hybrid interpretations), and accommodation (scientifically accurate understanding). Most students exhibited assimilation, using probability vocabulary with everyday meanings without restructuring underlying concepts. Synthetic models involved partial accommodation but unstable blends of formal and informal ideas. Only a minority achieved full accommodation. This is supported by qualitative analysis. For Assimilation: Sola (SS1) said, "Random is when something happens without planning, like my brother surprising me." The term retained everyday unpredictability without probabilistic depth. For Synthetic Model: Tunde (SS3) on random stated, "Random means every outcome has the same chance, but sometimes it feels like luck controls it too." This hybrid view blends formal equiprobability with intuition about fate. For Accommodation: Funmi (SS3) explained, "Random means all outcomes are equally likely; for example, a fair die roll where each face has a 1/6 chance independently." This reflects a restructured probabilistic framework. These patterns confirm the difficulty in achieving true conceptual change, as predicted by framework theory, and underscore the linguistic and cognitive complexity of probability learning.

Research Question 3

What linguistic and cultural factors influence students' ability to distinguish every day and mathematical meanings of probability terms?

Qualitative thematic analysis identified several recurrent themes explaining semantic interference. First is the linguistic translation and polysemy. Many students struggled with translating the English probability terms into their home languages, which often lacked precise equivalents or introduced additional meanings. Kehinde (SS2) remarked, "In math class, random

means like dice and numbers, but outside it just means something weird or unexpected." This highlights the challenge of parallel semantic systems where students fail to connect formal definitions with everyday language. Second is the cultural beliefs and deterministic frameworks. Cultural narratives about fate, luck, and divine control infused students' interpretations of chance. Nnamdi (SS3) said, "Some things are random because God decides them; that's why people have luck." This spiritual attribution inhibits the accommodation of mathematical randomness, anchoring students in deterministic schemas. Third is the social and moral reasoning on fairness. Students often infused social fairness into the concept of equiprobability. Chika (SS2) noted, "A game is fair if everyone agrees and no one is cheated," blending moral and mathematical notions.

Discussion

This section interprets the key findings of the study in relation to the three research questions, framed within the Framework Theory of Conceptual Change (Vosniadou, 1994; 2013) and informed by sociocultural perspectives on language and cognition (Vygotsky, 1986). The discussion highlights how linguistic ambiguity, cultural beliefs, and cognitive frameworks interact to shape Nigerian secondary school students' understanding of probability vocabulary.

For research question one, the quantitative results revealed moderate overall comprehension of basic probability terms, with higher mastery for certain (72% correct) and much lower accuracy for technical terms like independent (33%) and mutually exclusive (30%). Grade-level progression was significant, indicating that exposure and maturation positively influence vocabulary acquisition, though challenges persist even at the upper secondary level. Gender differences were statistically negligible, suggesting similar struggles across male and female learners. The qualitative data offers critical insights into the nature of these misunderstandings. Students predominantly employed everyday language and culturally grounded interpretations rather than formal probabilistic definitions. For example, the term random was commonly associated with unpredictability, chaos, or spontaneity ("something that just happens anyhow"—Ada, SS1), consistent with findings by Fischbein and Schnarch (1997) and Batanero et al. (2014). Similarly, fairness was more often interpreted through a moral or social lens ("everyone agrees and nobody cheats"—Chika, SS2) than as equiprobable outcomes, corroborating previous research highlighting social justice connotations as obstacles to mathematical fairness comprehension (Gal, 2005). Interpretations of likely reflected subjective belief and judgment rather than quantified probability ("likely means my team will win because they are the best"—Mary, SS2) align with Childers and Viswanathan's (2000) observations of verbal quantifier vagueness. The conceptually complex vocabulary, such as independent and mutually exclusive, was the least understood, with students conflating stochastic independence with causal or social unrelatedness. Musa's (SS2) analogy of "events being independent when they don't know each other" typifies this misinterpretation, similar to Konold's (1989) findings. From a theoretical standpoint, these findings exemplify the dominance of naïve deterministic and social frameworks over probabilistic ones in student reasoning. The progression with grade level indicates increasing, but incomplete, accommodation of formal vocabulary meanings, underscoring the challenge of deeply restructuring conceptual frameworks embedded in everyday experience and language.

For research question two, applying Vosniadou's framework theory, students' responses were categorized to reveal the dominant cognitive positioning towards the target vocabulary. Over half of the participants demonstrated assimilation, fitting new probabilistic vocabulary into pre-existing everyday conceptual systems without significant restructuring. For example, Sola's

explanation of random as “something happens without planning” retains the colloquial notion of spontaneity without internalizing the formal equiprobability concept. This reflects the resistance of established naïve frameworks to replacement (Vosniadou, 1994). A substantial portion exhibited synthetic models, blending formal and informal meanings inconsistently. Tunde’s simultaneous description of random as “every outcome having the same chance" and being “controlled by luck” illustrates the unstable hybrid conceptualization typical of partial accommodation. Only a minority, predominantly among older students, achieved accommodation, restructuring their conceptual frameworks to adopt normative mathematical meanings. Funmi’s articulate explanation of random as “equal chance in a fair die roll” exemplifies this big conceptual change. This cognitive distribution aligns with the complexities of achieving accommodation described by Vosniadou et al. (2001) and the findings of Batanero et al. (2014), who emphasized the challenges posed by entrenched intuitive beliefs. The prevalence of assimilation suggests that vocabulary teaching in probability often leads to surface-level learning unless explicitly linked to conceptual restructuring. This also highlights the importance of recognizing that conceptual change is gradual and nonlinear. Students negotiate meanings across contexts and languages, consistent with Vygotsky’s model of language mediating thought development (1986). The hybrid conceptions indicate the dynamic interplay of linguistic, cultural, and cognitive factors in learning probabilistic language.

For research question three, linguistic challenges emerged as a primary factor contributing to semantic interference. Students often failed to reconcile the dual meanings of probability vocabulary in English and their mother tongues. Kehinde’s comment that random “means dice and numbers in math class, but outside class it just means something weird” encapsulates this dichotomy. This linguistic split mirrors findings by Watson and Moritz (2003) on translation dilemmas in multilingual classrooms. Moreover, the lack of precise equivalents in indigenous languages and the explanatory burden placed on English as a second language intensified cognitive load, impairing concept internalization (Cummins, 2001). Instructional language itself—often informal and unstandardized—added another complexity layer, as vocabulary was inconsistently linked to formal definitions during teaching. Cultural beliefs infused students’ understandings with deterministic and moralistic interpretations. Many participants invoked notions of divine intervention, fate, or luck to explain randomness (“God decides; that is why people have luck”—Nnamdi, SS3), reinforcing non-probabilistic frameworks. These culturally anchored beliefs presented conceptual obstacles to adopting probabilistic thinking. Similarly, fairness was largely conceptualized in social and moral terms, blending equity and justice with the mathematical idea of equiprobability. This conflation reflects Fischbein and Schnarch’s (1997) insights into cultural influences on probability vocabulary and underscores the necessity for culturally sensitive pedagogies. Together, linguistic ambiguity and cultural determinism inhibit clear differentiation between everyday and mathematical meanings and consequently hinder accommodation. Addressing these requires explicit, scaffolded instruction which clarifies linguistic nuances and respectfully integrates cultural narratives, as advocated by Batanero et al. (2016) and Gal (2005).

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"SOME SORT OF AVERAGE": A MICROANALYTIC ANALYSIS OF REPRESENTATION, SCHEMA, AND AI-MEDIATED THINKING PROCESSES

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This paper presents a moment-by-moment analysis of a series of interactions between Kamden, an undergraduate engineering student, and ChatGPT as he solved an unfamiliar statistics problem. Kamden spoke aloud during this episode, which gave us insights about how he interpreted what the task was asking, how he drew upon his prior knowledge to make sense of the task, and his knowledge of ChatGPT. We analyzed each interaction turn between Kamden, the task, and ChatGPT to explain how he connected the expected value of a discrete probability distribution (a concept he did not know) to his knowledge of Riemann sum approximations (a familiar concept). We offer fine-grained insights about how students think and reason mathematically with AI and the role that their own expertise plays in the process.

Keywords: Statistics, Probability, Cognition, Technology

Today's students are expected, even encouraged, to learn with AI tools (OECD, 2026). Yet there is little consensus on what thinking with AI means or looks like in practice, including how we might identify episodes of joint student-AI cognition from instances where students use AI without thinking deeply about the subject-matter they are reasoning about. It seems imperative that the field distinguish between *AI use* to accomplish a task versus *thinking and reasoning with AI* in service of gaining new knowledge, as clarifying this distinction could delineate what human-AI arrangements are conducive for learning. Bastani and colleagues' randomized control trial (2025) illustrated the consequences of AI *using versus thinking*, as students who did math with chatbots experienced a 17% decline in assessment scores after the chatbots were taken away. If students are expected to *use AI* while in school, but do not show robust evidence of *thinking with AI,* this invites larger questions about the purpose of school (Labaree, 1997).

This work attempts to draw such a distinction with empirical data on a statistics task. While video-recording 36 undergraduates who were asked to solve a probability task and use AI if desired, we encountered Kamden, who used ChatGPT to reason about an unknown statistics concept. During roughly 6 minutes of AI-mediated interactions, Kamden linked ideas about the expected value of a discrete probability distribution (previously unknown) to Riemann sum approximations for the area under a function curve. His case is atypical because most other students in the study did not use AI to build new conceptual connections. We present his case to illustrate processes of Kamden's thinking, interactions with ChatGPT, and knowledge that may have helped these conceptual connections to form. We ask: *What knowledge of (a) mathematics and statistics, and (b) AI tools appear to support student thinking in AI-mediated interactions?*

Conceptual Framework

We frame our work within the broader notion of thinking and reasoning as a function of students' own highly interconnected networks of knowledge. Hiebert and Carpenter, who unpacked the notion of what "mathematical understanding" means from a cognitive perspective, suggest that students demonstrate mathematical understanding when facts, concepts, ideas, and procedures are more strongly linked with one another through networks of mental representations (1992, pg. 67). Practically speaking, this suggests that students' mathematical

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

thinking and reasoning should encompass (a) what they understand to be the goal state of the problem, as related to their prior knowledge, and (b) how their prior knowledge is structured and organized in memory to support their mathematical thinking. Chi and Glaser refer to these ideas as *representations* and *schema*, respectively (1985).

An open question is how and to what extent AI-mediation impacts students' representations and schema when they are still building skill and expertise. Since AI tools are designed to procure instantaneous, probabilistically-produced information in response to a student query, we hypothesized that AI-mediated thinking may depend less on students' highly developed schema and more on their ability to monitor their conception of tasks' representations while triangulating them with features of AI-generated reasoning. In short, this means that students should ask, "What facts, procedures, ideas, or concepts is AI giving me, how do I know they are correct, and to what extent do they help me think and reason about what the task is asking?"

To ask such a question, students must develop additional knowledge about AI tools: how they work, what their affordances are, and in what ways they can support or undermine thinking processes. Vérillon and Rabardel refer to such knowledge of tools' affordances as *instrumental genesis* (1995), and Drijvers et al. expanded their conception to include math-specific uses (2010). For instance, a student with high instrumental genesis would know that their TI-84 calculator can accept inputs for a two-sided t-test and calculate the respective *p*-value, but would realize that the TI-84 calculator cannot interpret what the *p*-value means in the context of the statistical test itself. We hypothesize that students must develop instrumental genesis of AI tools, but suggest that developing this expertise may depend more on their awareness of what they prompt and how their prior knowledge is embodied within it.

To date, research on AI-mediated mathematical thinking is emerging and leaves the field with open questions (Pepin et al., 2025; Walkington, 2025). Early empirical work has shown that AI tools can aid students' mathematical thinking in modeling and proof tasks, depending on how students consider and adopt AI solutions (Yoon et al., 2024). In addition to the Bastani et al.'s randomized control trial mentioned earlier (2025), Wu et al. demonstrated that when students reasoned about statistics only with AI tools (and disregarded non-stochastic resources, like class notes), they forfeited epistemic agency because they did not appear to fully understand how information from AI related to their prior knowledge (2025). Relatedly, Schorcht et al. (2024) demonstrated that the contents of students' prompts affected the mathematical quality of AI output, implying that students' ability to articulate their thinking to an AI may impact their subsequent reasoning. We contribute these findings by illustrating how AI-mediated thinking processes unfold with attention to students' knowledge and instrumental genesis.

Methods

Our goals were to examine with immaculate, temporal detail students' thinking and reasoning processes with AI tools. At no point did we require students to use AI. We conducted this study throughout spring 2025, after students at our (large, U.S., mostly-undergraduate) university were given ChatGPT-Edu subscriptions. Our university was an early adopter of AI tools and we wished to learn how students might use them (Delaney et al., 2025).

Study Design

The first author and three graduate research assistants constructed a 90-minute task-based interview (Goldin, 2000). We selected tasks with features that we believed would give students opportunities to decide whether and how to use ChatGPT. They contained algebra, probability, and calculus topics and were designed by curriculum professionals. Here we focus solely on the probability task, named "The ATM Task" by its authors from The College Board (Figure 1).

ATM

A shopping mall has three automated teller machines (ATMs). Because the machines receive heavy use, they sometimes stop working and need to be repaired. Let the random variable X represent the number of ATMs that are working when the mall opens on a randomly selected day. The table shows the probability distribution of X.

X = Number of ATMs working when the mall opens	0	1	2	3
Probability	0.143	0.286	0.429	0.143

Column names and first row of the second table

Day Index	Num. ATM repairs 2023	Num. ATM repairs 2022	Num Mechanics working per day	Daily ATM Usage	Num Working ATMs (beginning of day)
Day 1	2	1	5	43	2

Subtasks

1. What is the probability that at least one ATM is working when the mall opens?
2. On average, how many ATMs will be working on any given day?
3. The data table on the next page contains information about ATMs at the mall. What information from the table do you think was used to create the probability table, and how?
4. What is the probability that all three ATMs are working when the mall opens, given that at least one ATM is working?
5. Given that at least one ATM is working when the mall opens, would the average number of ATMs working on any given day be less than, the same, or greater than the average you found in question 2? Explain.

Figure 1: The ATM Task. Kamden's Analysis Focuses Solely on the Second Question

All researchers held at least three years of collegiate math and statistics teaching experience at the time of this study and were familiar with undergraduates' problem-solving skills. The first author holds a graduate degree in machine learning, experience in clinical interviewing methodologies, and over a decade of experience teaching statistics and mathematics. Our discipline-specific expertise enabled us to conduct this work with high fidelity (Bourke, 2014).

Each interview was conducted as follows: the student was invited into a quiet room with the first author and one graduate research assistant, who led the interview. The interviewer followed a semi-structured protocol designed to inquire about students' uses of AI, and during the tasks, encouraged them to "think aloud" without providing any additional help or hints. For the first 10 minutes, the interviewer asked students about their AI uses and beliefs. Then the student was invited to solve each task (algebra, probability, and calculus, in that order) for up to 20 minutes, and use ChatGPT however they wished. If and when students used ChatGPT, the interview asked why and what they were hoping to learn. After all three tasks were complete, the interviewer debriefed with the student for 10 minutes and paid them $50 for their time. The first author served in a supporting role by video-recording each problem-solving episode, keeping time, and asking clarification questions as needed.

Recruitment, Sample, and Case Selection

We invited undergraduate students who had taken at least one collegiate (or high school equivalent of collegiate) statistics or mathematics course in the past 12 months to participate. It was not a requirement that students needed to have used an AI tool for school before partaking in our study. We recruited students across majors by posting recruitment flyers around the physical campus. Students filled out a brief screening survey, detailing their prior math courses, before we invited them to the study. 36 total students participated in our study from March to May 2025.

Of these 36 participants, we selected a student whom we will refer to as "Kamden" for a fine-grained analysis because his interactions with ChatGPT helped him to begin to connect novel statistics topics to his prior knowledge. This was not the case for the vast majority of other participants. Kamden was a sophomore mechanical engineering major at the time of his interview. He had recently completed Calculus 3 and was currently enrolled in an aerospace course whose description includes "ordinary differential equations, the Laplace Transform, Fourier Series, and linear algebra, with engineering applications" (San Diego State University, n.d.). He also reported that he used AI tools "for just about everything," including mathematics coursework, and described in detail the limitations of AI tools in advanced mathematics due to a lack of training data on those topics. During his interview, Kamden reported that he never took a statistics course (including high school). Thus, Kamden represents a unique case among our sample of relatively high mathematics domain knowledge, relatively high knowledge about AI tools as related to solving mathematics problems in school curricula, and a knowledge gap in probability and statistics, which we appraised from his articulation of prior statistics experience.

Data Collection and Analysis

Kamden was audio, video, and screen-recorded while solving the task, and his ChatGPT log was saved. The video camera was placed behind him to maximize privacy and capture details of his ChatGPT interactions; for instance, where within its output he attended to at any given time. The first author transcribed each task segment first with otter.ai, then by hand, including interactional details about when Kamden interacted with AI between the transcript and his ChatGPT log (Jordan & Henderson, 1995). These types of details helped us to identify which segments of Kamden's episode were enacted with AI-mediation.

The first author and a graduate student who was not one of the three study designers analyzed Kamden's case, which began around 57 seconds into the ATM task and lasted approximately six minutes. We use microanalytic knowledge analysis to examine moment-by-moment what Kamden appeared to know (diSessa et al., 2016). Knowledge analysis stems from the Piagetian tradition of knowledge construction (Piaget, 1972), is context-specific, treats knowledge as a medium that can take on diverse (and sometimes unstable) forms, and requires nuanced interpretation. Our microanalytic episode was selected from the larger corpus for a short time scale where Kamden's prior knowledge about calculus may have been cued by features of the ATM task and the ChatGPT environment/interactions (Sherin, 2001). In analyzing this episode, our goal was to generate ideas about how Kamden's interactions with ChatGPT impacted his knowledge of the ATM problem representation and how this knowledge was generated.

The first author segmented the episode into phases by ChatGPT prompts or interviewer questions that elicited additional clarification from Kamden. Then each researcher analyzed Kamden's interactions individually and iteratively. While watching, re-watching, and memoing about the episode, researchers observed how Kamden's problem representation changed over time, how his schema appeared to support his problem representation, and how his interactions with ChatGPT's information appeared to influence his representation, including his evolving conceptions about what the "average" means in a discrete probability distribution. We also hypothesized connections between elements of Kamden's knowledge at various time points via sketched graphs. Then both researchers met and discussed our interpretations of Kamden's evolving knowledge. We referred back to transcripts and collected artifacts until we reached agreement on what Kamden appeared to know, grounded in evidence of what he stated aloud. For instance, we needed to affirm in several cases what Kamden knew about the meaning of probability distributions, as opposed to instances where he was simply re-reading portions of the task aloud. We discussed until we reached intersubjective agreement about Kamden's representations and schema (Krippendorff, 2018). We reduced the data to findings that highlight interactions with and without ChatGPT that appeared to influence changes in his knowledge.

Findings

We present the findings by phases to present a descriptive narrative of how Kamden's new statistics knowledge emerged and the interactions with ChatGPT that may have cued it.

Phase 1: Initial Representation

Kamden read the second ATM task question and almost immediately stated, "I don't know what question to ask, so I'm just going to ask ChatGPT." Asking ChatGPT, in this context, meant typing the question into the prompt and including all information from the probability table, which Kamden explained aloud. Figure 2 shows his prompt and ChatGPT's response. At this point, it was not yet clear to us what Kamden wanted to learn from ChatGPT, as his prompt contained only the task and its corresponding information, and omitted language that would have

signaled a request for a particular form of response. His spoken utterance suggests that he has not yet formed an initial representation ("I don't know what question to ask").

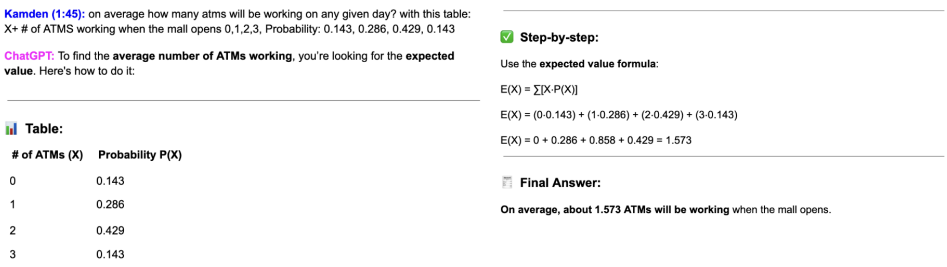


Figure 2: Kamden's First ChatGPT Interaction

Phase 2: Representation After First ChatGPT Prompt

We observed that ChatGPT's output immediately inferred a representation: "to find the average number of ATMs working, you're looking for the expected value." It further provided a tabular and symbolic schema for calculating the expected value. Kamden, interestingly, did not attend to its representation, nor did he mention "expected value" at any point. He did notice the table and appeared to connect it to prior knowledge about *x*-*y* table representations: "My probability's the *y*-axis, the number of ATMs is the *x*-axis, so from the table I got that. And then I figured it was an average of them, or some sort of average … so the answer is 1.573."

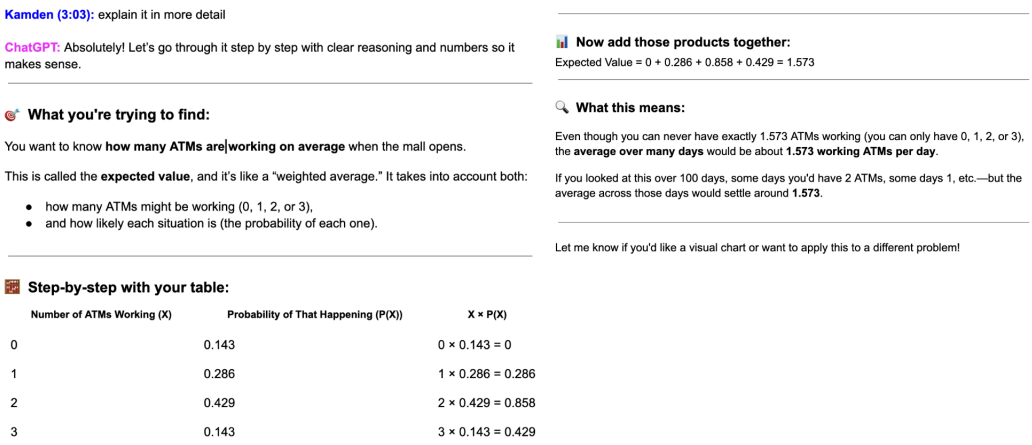
Some sort of average suggested that Kamden may have held a prior belief about what "average" is, such as the arithmetic mean as it is often presented to middle school students as a measure of centrality. Furthermore, it suggests that ChatGPT's table did not present "average" as Kamden may have known it, because he does not associate it with a centering feature within the table, or more broadly, any mathematical characteristic of averages. At this point, Kamden has processed ChatGPT's representation, read information that ChatGPT has (stochastically) determined to be associated with his query, and is connecting it to ideas that he remembers.

Phase 3: Representation After Second ChatGPT Prompt

Kamden then realizes that he still does not understand the question and states: "I'll just ask [ChatGPT] to explain it … honestly just what the question is asking." "What the question is asking" (in other words, a more specific request for a representation) is what Kamden means when he says "explain it in more detail" in his second prompt (Figure 3).

Again, Kamden appeared to ignore the portions of ChatGPT's output that explained the expected value concept ("the expected value [is] like a "weighted average") and instead focused on the table and computations associated with the table. He reacted to the information aloud:

Kamden:	There's a specific probability for each one, and there's a specific number of ATMs for each *x*-axis. It just did the computations … that's (gestures to screen) when there's zero, that's the probability that they're working when it's one. That's the probability that they're working when it's two. That's the probability that it's working when it's three. So, it just did some sort of function, and it solved it, and that's how many ATMs there would be.
Lead Int.:	Have you seen this function before?
Kamden:	I mean, it's just a sum of the *x* times the *y*. So, it's the number of ATMs times the probability of there being an ATM working, and then just added them all together to get the total number of ATMs.



Kamden (3:03): explain it in more detail

ChatGPT: Absolutely! Let's go through it step by step with clear reasoning and numbers so it makes sense.

What you're trying to find:

You want to know **how many ATMs are working on average** when the mall opens.

This is called the **expected value**, and it's like a "weighted average." It takes into account both:

- how many ATMs might be working (0, 1, 2, or 3),
- and how likely each situation is (the probability of each one).

Step-by-step with your table:

Number of ATMs Working (X)	Probability of That Happening (P(X))	X × P(X)
0	0.143	0 × 0.143 = 0
1	0.286	1 × 0.286 = 0.286
2	0.429	2 × 0.429 = 0.858
3	0.143	3 × 0.143 = 0.429

Now add those products together:

Expected Value = 0 + 0.286 + 0.858 + 0.429 = 1.573

What this means:

Even though you can never have exactly 1.573 ATMs working (you can only have 0, 1, 2, or 3), the **average over many days** would be about **1.573 working ATMs per day**.

If you looked at this over 100 days, some days you'd have 2 ATMs, some days 1, etc.—but the average across those days would settle around **1.573**.

Let me know if you'd like a visual chart or want to apply this to a different problem!

Figure 3: Kamden's Second ChatGPT Interaction

In making sense of this information, Kamden connected how the value of 1.573 was obtained by viewing the $X * P(X)$ calculations per row in the output. He attributed these calculations to ChatGPT "doing some sort of function." We note that there was no function given in this AI interaction and that Kamden associated the series of calculations with function behavior either in the prior ChatGPT interaction or his prior knowledge. He interpreted the function in the context of the problem, however it was still not apparent that Kamden knew how the "total number of ATMs" was related to the concept of average or centrality. By this point, Kamden had associated a particular algorithm for computing "some sort of average" with his prior knowledge of bivariate relationships and function relationships. It was not yet clear if he knew what the task was asking, why the computation made sense, or how it was related to the notion of "average."

Phase 4: Clarifying the Meaning of "Average"

Realizing this discrepancy, the supporting interviewer (Supporting Int.) engaged in the following dialogue with Kamden where he appeared to make connections between "average" and his prior knowledge. This time, he referred to the probability table in the task while making the connections:

Supporting Int.: Can you say…why that explanation makes sense?
Kamden: The answer makes sense that there would be 1.573 [ATMs]. Because, if we're only looking at these four specific [columns of data] … half of the data is over two, half of the data is under two. The average was 1.573 because the probability of it not working goes up with two [to 0.429].

This clarified the value of 1.573 as the approximate "center" of the data table for Kamden (relating to the centering aspect of "average") and may have explained why he trusted the value 1.573 even though he did not appear to know why the expected value formula yielded "some sort of average." The supporting interviewer pressed once more to attempt to understand how Kamden made sense of this:

Supporting Int.: How did you come to trust ChatGPT's expected value formula … because I think you'd said you have never taken a statistics course?
Kamden: No yeah, it was new. But I've seen stuff like that, the sum part … so calculating the average looked similar to a right Riemann sum or left

> Riemann sum … so the number of ATMs times the probability and adding those together to get a total average probability. So that made sense.

At last, we observed that Kamden's mental model of average in this context combines his knowledge of bivariate relationships, functions, calculations of and with Riemann sums, and what Riemann sums represent (a "total" approximation). Kamden explained that he linked ChatGPT's calculations for "some sort of average" to calculations for how right or left Riemann sums are calculated. Both contain computations of the form *quantity 1 multiplied by quantity 2* repeated over a fixed interval. This in turn led Kamden to construct a notion of "total average probability" (his phrasing) akin to approximations through Riemann sums as an accumulation of a function. More specifically, Kamden's task representation is a conception of "total average probability" as related to algebraic and geometric ideas of accumulation over an interval, and measures of center as the "midpoint" between the interval of x-values in the given probability table. ChatGPT appeared to support his conception by illustrating how "average" calculations over this interval could be constructed and how those calculations were related to mathematical ideas (e.g., summation functions and bivariate relations) that were, by chance, part of his schema.

Discussion

Our analysis presents an unfolding of Kamden's mathematical thinking with ChatGPT about a statistical concept that he did not appear to have prior knowledge about before the ATM task. By the end, Kamden connected the expected value of a discrete probability distribution ("some sort of average") to his prior knowledge of summations, functions, bivariate relationships, and calculus approximations. We attempt to model changes in his knowledge network in Figure 4, noting that our representation is naturally limited by its inferential and interpretive properties.

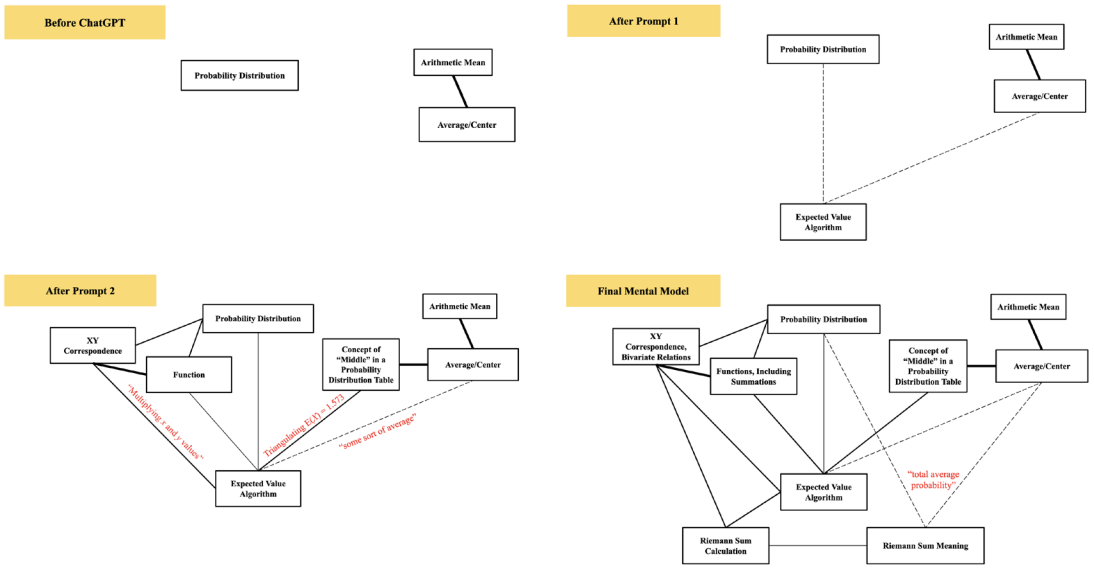


Figure 4: Hypothesis of Kamden's Evolving Knowledge During This Episode

We discuss features of Kamden's AI interactions that may interest education researchers and teachers. First, Kamden spent much of the episode iteratively reasoning about the task representation. Upon reading ChatGPT's first output, he made a few conceptual connections to

bivariate distributions and functions, but these connections alone were not enough for him to make an assessment about the task's goal. Rather, he knew that he did not have enough prior knowledge about averages derived from a probability table, and prompted AI again. This interaction combined his knowledge of mathematics and statistics, showed evidence of progress monitoring, and revealed his instrumental genesis about AI tools, because he appeared to know that there was no discernable disadvantage to re-prompting ChatGPT for a different explanation.

Second, we observe that Kamden's progress monitoring of the task representation had limitations from the perspective of the interviewers, who were mathematics teachers. While Kamden stated, "it makes sense" after ChatGPT's second output (Figure 3), the interviewer was unclear *what* made sense because they had not yet heard evidence of how Kamden connected "average" to his prior knowledge. It was only after Phase 4 that it became apparent how Kamden organized "average" with his schema of Riemann sums: he associated the center of an accumulated probability with the idea of accumulated area of a function, as approximated by rectangle areas over a fixed interval, which he appeared to infer from the $X * P(X)$ calculations given by ChatGPT. We took this to be a mathematically legitimate construction, although it likely shares instability properties described by diSessa et al. (2016).

It is unclear if Kamden would have reached his conceptual connection without the interviewer. He appeared ready to move on before the interviewer requested him to re-articulate what made sense. The substance of the interviewer's questioning suggests that Kamden's knowledge in Phase 4 was co-constructed between himself and the interviewer (Koichu & Harel, 2007). This points toward a broader finding that students' own progress monitoring and metacognition (Flavell, 1979) may only extend so far in tasks containing unfamiliar mathematics or statistics. Because Kamden may have had less of a sense about how to organize the expected value "average" in his schema of algebraically-cued ideas, he did not appear to know where or how to organize this average within his mathematical knowledge network (Hiebert & Carpenter, 1992), and likely still does not understand this "average" in a statistically-rigorous sense.

We view this finding as important for teachers. It reveals that even a student who has taken advanced mathematics courses and maintains (relatively) high instrumental genesis of AI can struggle to independently configure new statistics and/or mathematics ideas into their schema. Part of this struggle may be a function of human attention, as ChatGPT (Phase 3) gave output that explained a notion of "weighted average" that Kamden did not attend to, which could have aided him. Research on high-quality math teaching shows that teachers direct students' attention toward mathematically salient concepts, procedures, and connections, and orchestrate instruction that makes mathematical connections stronger (Stein et al., 2008). Our study shows that while AI tools can jumpstart the process of building mathematics connections, students may not exhibit follow-up behaviors to strengthen connections on their own (or at least, not in the short-term).

There are limitations that should impact how the results are interpreted. This study took place in a clinical setting, and Kamden's interactions with AI tools may differ in classrooms, where his learning goals and orientations toward mathematics tasks are likely different. Yet this setting was necessary in order for Kamden to explain his uses of ChatGPT, and as such, we view our study ecology as a necessary tradeoff for what we wanted to learn. Additionally, the remaining 35 students in our sample did not tend to think and reason as deeply as Kamden did with ChatGPT. We encourage the reader to view his interactions with AI as one instantiation of possible AI-mediated reasoning, not as what most students typically do when they use AI for statistics or mathematics. This also suggests that much more research is needed to illustrate the range of possibilities that may occur during thinking processes between students and intelligent agents.

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USING PEIRCE'S CASE-RULE-RESULT FRAMEWORK TO CHARACTERIZE STUDENTS' REASONING ABOUT SAMPLING DISTRIBUTIONS

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I used Charles S. Peirce's three classic forms of inferential reasoning—deduction, induction, and abduction—defined by case, rule, and result to examine 11 undergraduate students' understanding of sampling distributions. In this paper, I report on a subset of these findings. During one clinical interview, I engaged each student in a task designed to investigate their reasoning when estimating an unknown population parameter. Although all three forms of reasoning were present, abductive reasoning was particularly useful in drawing conclusions from sample data to the population with an unknown parameter.

Keywords: Data Analysis and Statistics, Cognition, Undergraduate Education

Data are everywhere. Developing statistical reasoning is essential for students to evaluate data-based claims and become well-informed citizens (Bargagliotti et al., 2020). A critical component of statistical reasoning is inferential reasoning, or drawing conclusions from data. Described as a "fundamental skill for participation in a global, technological, and data-driven society" (Noll & Hancock, 2015, p. 362), inferential reasoning involves multiple ideas, such as repeated sampling, variability, and randomness. Understanding these ideas is important for improving students' statistical literacy and productive citizenship (Franklin et al., 2007). Despite its importance, research indicates that reasoning inferentially is complex for students (e.g., Sotos et al., 2007). Although many students can perform calculations for formal statistical inference procedures, they have difficulty understanding the underlying logic (Chance et al., 2004). Part of this difficulty stems from the complex and abstract nature of sampling distributions, which requires coordinating multiple ideas such as sample, population, distribution, and variability (Noll & Shaughnessy, 2012; Saldanha & Thompson, 2002; 2007). Furthermore, statistical reasoning requires reasoning with uncertainty. This type of reasoning differs from students' many and varied experiences in mathematics classes, which include an emphasis on deterministic reasoning.

Theoretical Framework

Definitions of reasoning include some process for arriving at a conclusion or making an inference (National Council of Teachers of Mathematics, 2009). Deduction, induction, and abduction are three classic forms of inferential reasoning. I adopt definitions from Peirce (1878) and others (e.g., Reid & Knipping, 2010). *Deductive reasoning* is the process of drawing logically valid conclusions based on stated assumptions or rules and is thought to be the only form of reasoning that establishes a conclusion with certainty. *Inductive reasoning* involves generalizing a rule from several observations. *Abductive reasoning* starts with an observation and involves hypothesizing what rule might lead to that particular observation. Peirce (1878) defined deduction, induction, and abduction in terms of a triadic structure involving a *case*, *rule*, and *result* that are linked in a particular order. Interpreting Peirce's definitions, Reid & Knipping (2010) defined a *case* as a specific observation that a condition holds. A condition describes an attribute of something, or a relation between things. A *rule* is a general proposition that states that if one condition occurs then another one will also occur. A *result* is a specific observation,

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similar to a case, but referring to a condition that depends on another condition linked to it by a rule. The order in which one links a case, rule, and result determines the kind of inference needed to gain more knowledge about the situation. In deductive reasoning, a rule and a case conclude a result. In inductive reasoning, a case and a result (or often, many cases and many results) lead to a rule. In abductive reasoning, a result and a rule infer a case.

Reasoning about data involves reasoning under uncertainty. This kind of reasoning is different from what is expected and highlighted in mathematics, such as deductive reasoning and proof. Although deduction, induction, and abduction are seen in both mathematics and statistics, deduction is privileged in mathematics whereas induction and abduction should be emphasized in statistics because these forms of reasoning include uncertain conclusions. More specifically, reasoning abductively is a productive way to reason when drawing conclusions from sample data because abductive reasoning shares aspects of inferential reasoning. Abductive reasoning is the inference of a case from a rule and result: We start with an initial observation, or result. This result could be explained by the supposition that it is a case of some general rule. Therefore, we adopt that supposition (Peirce, 1878). Similarly, in statistical inference, we start with the outcome of a sample (result). By considering the probability of the sample outcome under multiple hypothetical parameters, we conclude the sample outcome could be explained by (is a case of) the supposition that it was drawn from a population whose parameter is equal to p (general rule). Therefore, we adopt that supposition. This paper reports on a subset of findings from a larger project in which I investigated novice statistics students' understanding of sampling distributions. For this paper, I studied the research question: When given a population with an unknown parameter, what forms of reasoning (i.e., deduction, induction, and abduction) do students employ when making inferences from one sample outcome to the population from which it was drawn, and what do these forms of reasoning reveal about their understanding of sampling distributions?

Methods

Participants for the larger study were 11 undergraduate students who had recently completed an introductory statistics course. I engaged each student in two clinical interviews, each 60-75 minutes in length, in which students worked through a series of statistical tasks related to sampling distributions. The context of the tasks across both interviews was about the proportion of intended business majors at two large universities: North University's website states that 20% of their undergraduate students are intended business majors while South University's website does not have this information posted. In the first interview, students reasoned about sample outcomes from North University, a population with a known parameter. They made predictions for sample outcomes, physically drew samples from a large box of beads with proportions corresponding to the population of interest, simulated drawing many samples using a web-based applet (Rossman & Chance, 2021), and continually updated their estimates of sampling variability. In the second interview, students reasoned about the unknown parameter at South University. They drew one sample from a large box of beads that modeled the South University context, made an initial prediction for the unknown parameter, tested and refined one or more predictions using the web-based applet, and provided a range of values they believed captured the unknown parameter. This paper reports on data collected from the second interview.

After checking and timestamping interview transcripts against audio and video recordings, I adapted Simon's (2019) three-level approach to analyzing qualitative data. In the first level, I analyzed the transcripts line-by-line and identified segments of data in which students provided an explanation or a justification. Taking a convergent approach (Clement, 2000) in the second

level of analysis, I applied Peirce's (1878) case-rule-result framework to each segment to infer the form of reasoning students used. This involved identifying what the student used as a rule and how they applied the rule to a case (deductive reasoning), developed the rule from multiple cases (inductive reasoning), or hypothesized a rule to explain a case (abductive reasoning). In the third level of analysis, I identified emergent themes that provided additional insight into students' understanding of sampling distributions beyond the three broader forms of reasoning.

Results

All three forms of reasoning were evident across the participant pool: three students reasoned abductively, one student reasoned inductively, and five students reasoned deductively. Two students did not show evidence of these three broader forms of reasoning. Due to limited space, the idiosyncratic nature of these two students' reasoning, and the focus of the research question, I describe and compare the reasoning of a subset of nine participants who engaged in abductive, inductive, and deductive reasoning. For complete results, see Miller (2024).

Nine students engaged in repeated patterns of reasoning as they engaged with the box of beads and the web-based applet. Although students were tasked with estimating the unknown population parameter based on one sample they drew from the large box of beads, not all students used this information. Seven students repeatedly referred to the outcome of their single sample as they engaged with the task. Of these seven students, three reasoned abductively and four reasoned deductively. Other than making an initial prediction for the unknown population parameter, two of the nine students did not use the information from their single sample to estimate the unknown parameter. Of these two students, one reasoned deductively and the other reasoned inductively. Beyond these three broader forms, I identified four components that were important in understanding differences in students' reasoning:

1. Students related one or more sampling distributions to a hypothetical population;
2. Students assessed the plausibility of one or more values for the unknown parameter by considering the likelihood of one sample outcome;
3. Students assessed the plausibility of multiple values for the unknown parameter by considering visual characteristics of the graph (e.g., shape, highest peak) of one or more sampling distributions;
4. Students provided a range of values for the unknown population parameter based on the outcome of multiple random samples.

I first describe and compare the abductive and deductive reasoning of the seven students who repeatedly used their sample outcome in their reasoning. Then I describe the deductive and inductive reasoning of the two students who did not use the result of their sample to determine their range of values for the unknown parameter. Throughout both sections, I support the major findings by identifying what the broader forms of reasoning and the more nuanced differences in reasoning revealed across the student pool.

Drawing Inferences About a Population Using the Result of One Random Sample

Seven students used the likelihood of their sample outcome to assess the plausibility of multiple values for the population parameter (Component 2). However, the form of reasoning the students employed revealed a subtle yet critical component of their understanding of sampling distribution. Those who reasoned abductively showed evidence of coordinating at least two levels of distribution—population distribution and sampling distribution—indicating they understood that each sampling distribution they constructed represented a collection of sample

outcomes that could have been produced by some hypothetical population (Component 1). Those who reasoned deductively showed evidence of understanding that each sampling distribution represented a collection of sample outcomes, but they did not relate the collection to the population that could have produced the collection—a critical understanding in drawing inferences from sample data to the population from which it was drawn.

Abducing plausible values for the unknown population parameter. Three students related one or more sampling distributions that they constructed using the applet to a hypothetical population that could have produced the collection of samples outcomes (Component 1). These students abduced a range of values based on the likelihood of the outcome of their one random sample that they drew from the box of beads that modeled the South University context. They repeatedly predicted and tested multiple values for the unknown parameter and assessed the plausibility of each value by considering how often their sample occurred in the many samples they drew using the web-based applet (Component 2). Each time they determined that a hypothesized value was plausible, they abduced a case from a result and a rule. Starting with their one sample outcome (result), they hypothesized a value for the unknown population parameter, then constructed a sampling distribution with the probability of success equal to the hypothesized value and drew many samples. Because their sample outcome was likely to have occurred under the hypothesized value (rule), they abduced that their sample could have been drawn from a population whose parameter was equal to the hypothesized value (case).

For example, when Eric (pseudonym) tested his prediction of 40% for the unknown population parameter, he used the applet to simulate 500 random samples of size 100 using a probability of success of 0.4. Figure 1 shows the results of his simulation. He explained that the 40% represented "the assumption that if South University had their proportion online…that proportion is 40%," thus relating the sampling distribution he constructed to a hypothetical population whose true parameter was equal to 40% (Component 1). To assess the plausibility of his prediction, he considered the likelihood of his sample outcome of 50% (Component 2) under this assumption. He said, "If you say it's 40, then 50 could be a possibility," pointed to a sample outcome of 50 on the dot plot, and said, "So we could say, oh, that one [sample] that I drew was that one." Given his sample outcome of 50% (result), Eric assessed the likelihood of his sample outcome, determining his sample outcome "could be a possibility" if the true parameter were 40% (rule). He abduced that his sample outcome could have come from a population whose true parameter was 40% (case) and included 40% in his range of values for the unknown parameter. Eric, Lorraine, and Lyla repeatedly used the applet in this way to abduce multiple values that were plausible for the true percentage of undergraduates at South University who were intended business majors, ultimately providing a range of plausible values for the unknown parameter.

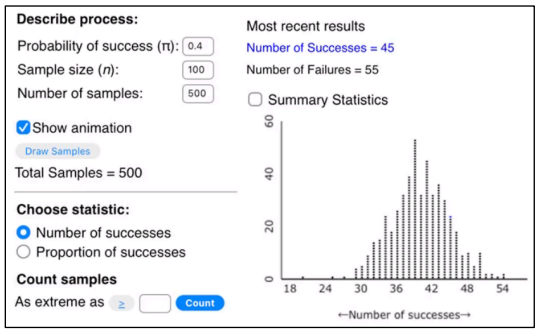


Figure 1: The Results of Eric's Simulation When Testing 40%

The conclusion of this abductive inference is that the random sample these students drew could have been drawn from a population with some parameter, *p*. Without connecting a collection of sample outcomes to the population from which they were drawn, this conclusion is not possible. Without this connection, considering the likelihood of the outcome of the one sample drawn from the box of beads can be used as a procedure to deduce a range of values for the unknown population parameter. This was the case for Tami, Tyra, Mindy, and Corrina.

Deducing plausible values for the unknown population parameter. Four students used the likelihood of their sample outcome to assess the plausibility of multiple values for the unknown parameter (Component 2) and applied a general rule (rule) to each of their predictions for the unknown parameter (case) to deduce whether they should include the prediction in their final range of plausible values (result). These students reasoned that if their sample outcome occurred often when the probability of success was equal to *p*, then *p* should be included in their range of values, without connecting the collection of sample outcomes to a population whose parameter is equal to *p*. This connection is the subtle yet critical distinction between an abductive inference and deductive inference in this context.

Corrina's reasoning is representative of the group. When testing her prediction of 40%, Corrina considered the likelihood of her sample outcome of 51. She said, "It's kind of on the higher end. It's not as likely, but it totally could happen" and included 40% in her range at the end of Task D (see
Figure 2a). When she tested 60%, she said, "Yeah, so that 51, again, is kind of on the lower end, but still likely, so I think it could be 60" (see
Figure 2b) and, again, included 60% in her final range of 38% to 62% for the unknown parameter.

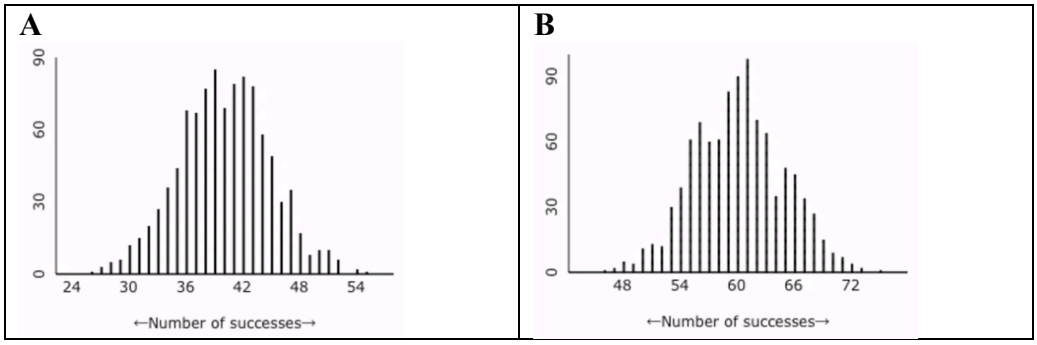


Figure 2: The Results of Corrina's Simulations When Testing 40% and 60%

Corrina applied a general rule to each of her predictions to deduce whether she should include the prediction in her range of values for the unknown parameter at South University: If my sample outcome occurred in a large collection of outcomes when the probability of success is equal to *p*, then *p* should be included in my range of values. She applied her rule to her prediction of 40% and reasoned that because her sample outcome of 51 "could totally happen" when the probability of success was equal to 40% (case), then 40% was a plausible value for the unknown parameter (result). Similarly, because her sample outcome of 51 was "on the lower end, but still likely" when the probability of success was 60% (case), then 60% was a plausible value for the unknown parameter (result).

Drawing Inferences About a Population Without Using the Result of One Random Sample

Other than in their initial prediction for the unknown parameter, two students did not use the result of their one sample to determine a range of values for the unknown parameter. Instead, Julie used the shape and highest peak of each sampling distribution she constructed using the web-based applet (Component 3) and Jess used the results of five random samples she drew from the box of beads (Component 4). Although both students understood that random samples will produce outcomes that vary from the population parameter, they did not apply this understanding to a new context—one in which the parameter was unknown—to determine the likelihood of their sample outcome under multiple hypothesized values for the unknown parameter.

Using visual characteristics of a graph: Julie's deductive reasoning. Julie estimated the unknown parameter by considering visual characteristics of the dot plots she constructed using the applet (Component 3). She constructed multiple sampling distributions, identified the "highest peak," and included her prediction in her final range if the "highest peak" was close to her predicted value. Julie's initial prediction for the unknown parameter was 55% because it was close to her sample outcome of 60%. To test her initial prediction, Julie input 0.55 as the probability of success and drew 525 samples (see Figure 3). After asking Julie if any of the samples she drew resulted in samples like the one she drew from the box of beads, she said, "I definitely had at least five were at 0.6" and explained this meant her initial prediction of 55% was "more accurate" than her sample outcome because there were "more near 0.55 than 0.6."

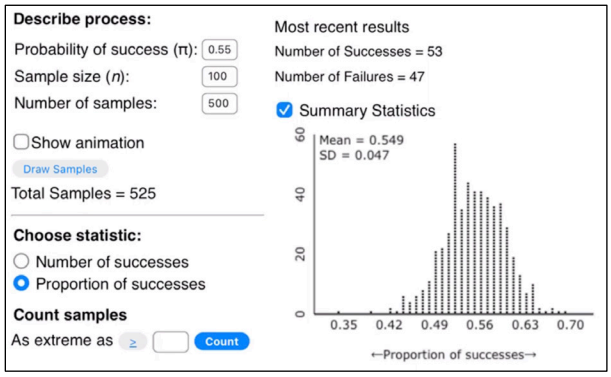


Figure 3: The Results of Julie's Simulation When Testing 55%

Because there were more sample outcomes that resulted in 0.55 than 0.6, Julie reasoned that her initial prediction of 0.55 was a better estimate for the unknown parameter than her sample result of 0.6. This is evidence she may have viewed the sampling distribution as a collection of sample outcomes from the actual South University. As I tried to understand Julie's thinking, I asked her to interpret the 0.55 she input in the applet. She explained, "So, essentially if the university had reported the statistic and how many they believed were intended business majors, that's what the 0.55 is. If they had reported it, but right now it's what my prediction was." During the interview, I interpreted Julie's explanation to mean she understood she used the applet to draw 525 samples from South University, *if* South University's true percentage of intended business majors was 55%. However, only through retrospective analysis did I gain a different understanding of Julie's thinking; I hypothesize she viewed each sampling distribution she constructed as a collection of sample outcomes from the actual South University. This would explain why Julie based her subsequent predictions on the mode, or what she called the "highest peak," of one or more

distributions she constructed. For example, after Julie tested her initial prediction of 55%, she chose to test 52%, which was the mode of the distribution shown in Figure 3. Julie tested multiple predictions, such as 52%, 60%, and 47%, among others. Each time she input a new prediction in the applet and drew 500 or more samples, she identified the mode(s) of the distribution and used this information to make future predictions and to determine whether she should include her predicted value in her interval estimate for the unknown parameter. Julie ultimately provided a range of 47% to 63% and summarized her process and justified her confidence by explaining, "Every time we do in the range of 47 to 63, it starts to centralize around that number. So that's why I am pretty confident it would be in that range."

Julie's repeated reasoning indicates she deduced her range of values from a rule involving the mode, or "highest peak," of each distribution: If the mode is approximately equal to the probability of success, p, then p is a good estimate for the true parameter (rule). When testing her prediction of 47%, she identified 0.47 as being "tied for the highest" (case), so she deduced that 47% should be included in her range (result). Based on her understanding of how samples vary from the population parameter, if Julie viewed each sampling distribution as a collection of sample outcomes from South University, she would identify the center of the distribution as the parameter. Throughout her predicting and testing, Julie identified the outcome with the "highest peak" as a plausible value for the true parameter, supporting my hypothesis that she viewed each sampling distribution she constructed as a collection of sample outcomes from the actual South University, rather than from a hypothetical population whose parameter was equal to the probability of success she input.

Using the result of multiple samples: Jess's inductive reasoning. Like Julie, Jess initially identified the mode of each distribution and used this information to make future predictions. However, Jess was perturbed when she noticed that each time she changed the probability of success in the applet, the center of the distribution shifted. She said,

> I think in my head, the probability of success, it's like whatever number I'm going to change it to, it's going to kind of situate it around that. So, if I put it to 0.7, it's going to shift it to 0.7…Because I'm telling it there's a 70% chance of success, therefore it's more likely to be situated around 70…So, I guess thinking about that now, I feel like maybe there is a way using this graph to determine the proper prediction, but I wouldn't know how based on just one thing (points to her sample) because I'm telling it this is the prediction or this is how many business majors there are at the school.

Jess explained multiple times that she was not confident in any value she tested using the applet and said, "I think for me to give a range and feel more confident, I would want to scoop out a couple more and see truly how many I get each time." Although this was not the intent of the task, I allowed Jess to draw more samples. Jess drew five random samples from the box of beads (cases), observed that her sample outcomes were in the "high fifties and low sixties" (results), so she induced a range of 57 to 62 for the unknown parameter at South University (rule). Jess was the only student to draw multiple samples from the box of beads and use these sample outcomes to induce a range of values she thought was plausible for the unknown population parameter. Reasoning inductively in this way is productive, but unrealistic. In other words, Jess could easily draw multiple samples and estimate where they clustered because the samples she drew came from a box of beads representing South University undergraduates. However, it is not feasible or cost-effective to draw multiple samples from a population of interest to estimate the unknown parameter.

Discussion

When given a population with an unknown parameter, students employed all three forms of reasoning when making inferences from one sample outcome to the population from which it was drawn. However, students who reasoned abductively showed evidence of having a more sophisticated and flexible understanding of sampling distributions than those who reasoned deductively or inductively.

I identified a subtle yet critical distinction in students' reasoning across the two groups who used the likelihood of a sample outcome to determine a range of plausible values. Although the reasoning in both groups initially appeared similar, additional analysis and comparisons of students' explanations and justifications revealed that students who reasoned abductively explicitly hypothesized multiple populations that could have produced their sample outcome. These three students reasoned about the multi-tiered repeated sampling process (Saldanha & Thompson, 2002) and coordinated at least two levels of distribution—population distribution and sampling distribution. In contrast, students who reasoned deductively included the likelihood of their sample outcome in their rule for determining whether to include a particular prediction in their range of values for the unknown parameter but did not connect a hypothetical population with the resulting sampling distribution.

Following Peirce's (1878) definitions, I argue that statistical inference is an abductive process: concluding the sample (result) could have been drawn from (is a case of) a population with some parameter *p* (rule). This requires the student to connect a collection of sample outcomes back to a hypothetical population. Only those who reasoned abductively made this connection explicit, treating each applet-generated sampling distribution as a collection of outcomes from a population with a specific hypothetical parameter. This result brings up an important question about the causal mechanism between coordinating multiple levels of distribution and reasoning abductively: Were students able to reason abductively *because* they understood the relationship between the hypothetical population and the sampling distribution, or did they understand the relationship between the hypothetical population and the sampling distribution *because* they reasoned abductively? My initial hypothesis was that these students were able to reason in this way because they understood the relationship between the hypothetical population and the sampling distribution. However, Lyla's final remarks suggest the latter. She said,

> That was just really cool. I don't know. It was cool to go backwards from, you just know when you take stat, it's like, oh, we have this confidence that this number is going to fall in this population because of blah, blah, blah, blah. But working backwards shows the logic behind it even more. I thought it made sense to me before, it made perfect sense, our rationale for drawing these conclusions. But going backwards, it's another level of understanding.

Lyla described the process of "working backwards" from a sample to a population as "another level of understanding" compared to the traditional calculation-based procedures of confidence intervals taught in introductory statistics courses like the one she had recently taken. Her reflection on her own understanding suggests that tasks requiring students to repeatedly predict and test parameters support the complex coordination of different levels of distribution.

Surprisingly, not all students used their sample outcome to inform their inferences. These students appeared to view the applet-generated sampling distributions as collections of outcomes from the *actual* population rather than a *hypothetical* one. Although they understood that samples vary and cluster around the population parameter, they did not apply this understanding to a

context where the parameter was unknown. They likely interpreted the probability entered into the applet as a "true" value rather than a testable hypothesis. It is possible that these students' existing meanings did not enable them to productively engage with the applet to further their understanding. This suggests that while simulation tools are intended to support students in visualizing abstract concepts (e.g., Garfield et al., 2012; Saldanha & Thompson, 2014), they do not automatically produce the intended meanings. More research is needed to understand how students' existing meanings influence their engagement with simulation tools.

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MAPPING EQUITY: THE IMPACT OF PLACE-BASED DATA USE IN SYSTEMS ENGINEERING CLASSROOMS

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Many entrepreneurship and systems-engineering programs prioritize profit over social responsibility when teaching location-based decision-making. This study explored means of fostering data-informed, other-oriented purpose and motivation among systems-engineering undergraduates using place-based data visualizations. Industrial and Systems Engineering students in Mexico (N = 63) participated in a quasi-experimental study with a pre-post-followup control group design. Participants in the treatment classroom engaged with geospatial data visualization software to select a location for a hypothetical restaurant in ways that balanced profit with social responsibility for providing accessible services to the community. Compared to a business-as-usual comparison classroom, students demonstrated lower levels of self-oriented motives and cost perceptions, but shared that the technology was difficult to use.

Keywords: Data Literacy, Data Visualizations, Motivation, Purpose, Systems Engineering

In engineering economics education, students learn to make fiscally sound decisions. However, courses often emphasize profit maximization without sufficiently addressing social and environmental responsibilities, such as ensuring community accessibility when choosing a business location (Neal, 2017). For instance, when proposing a new food and beverage venture, entrepreneurs may consult geospatial visualization data (represented in maps) to gain an understanding of quantities such as population density, income levels, and traffic patterns. Engaging with such representations requires learners to interpret distributions, compare data layers, and weigh tradeoffs across multiple quantitative indicators—core practices in statistics and data science. While these metrics can guide profit-driven decisions, they can also inform more socially responsible choices. To help students navigate the tensions between profit and accessibility, this study introduces a place-based learning experience using Google Earth Pro—an open-source geospatial data visualization platform—to engage future entrepreneurs with the ethical dimensions of data- and location-based decision-making. The purpose of this project is to investigate how place-based learning with open-source geospatial visualization software (Google Earth Pro) can help systems engineering students consider more socially responsible goals and adopt "other-oriented" self-transcendent motives as well as their value for learning.

Theoretical Framework

To investigate how data visualization tools influence students' motivation and socially responsible decision-making, we integrate four theoretical perspectives: Data Literacy, Place-Based Education, Self-Transcendent Purpose, and Expectancy Value Theory.

Data Literacy

Data literacy can be defined as the statistical competencies, methods, and techniques that facilitate decision-making (Gould, 2017). These include: acquiring, managing, visualizing, interpreting, evaluating, reading, and using data (Börner et al., 2019; Carlson & Johnston, 2014; Prado & Marzal, 2013; Ridsdale et al., 2015); dimensions that are critical to advancements in science and engineering (Qiao et al., 2024). Competencies that are central to data interpretation

and decision-making is *translating* real world problems into data problems and using results to *advocate* and argue for social benefits (Thacker & Herrick, 2025; Börner et al., 2019; Weiland, 2017; Rubel et al., 2021). In this study, we aimed to offer students opportunities to translate pressing social issues into place-based geospatial data and develop and apply their data literacy skills to select a location of a hypothetical restaurant.

Place-Based Education

A useful framework for contextualizing map data and creating personally relevant data experiences for students is to connect to their "sense of place" (Semken et al., 2017; Semken & Freeman, 2008; Sobel, 2004). A sense of place refers to the emotional, cultural, ecological, historical connections that students develop associated with a geographical location. Map data can tap into one's sense of place because maps can represent one's lived environment, both cartographically and culturally (Duxbury et al., 2018). Further, connecting content to learners' sense of place has the potential to expand learners' attention beyond their own self-interest, and connect to something larger than oneself.

Self-Transcendent Purpose

Self-transcendent purpose refers to the motivation to pursue goals that contribute to the well-being of others or society at large, beyond one's own self-interest (Yeager et al., 2014). In educational contexts, interventions that emphasize self-transcendent goals have been shown to enhance persistence, engagement, and deeper learning—particularly in STEM domains where students may not immediately see the societal relevance of the material (Yeager et al., 2014). Self-transcendent purpose is particularly relevant for future entrepreneurs and systems engineers, where students in these domains are often encouraged to prioritize personal gain or market efficiency.

Expectancy Value Theory

We also draw from *Expectancy Value Theory* (EVT), which proposes that student motivation is driven by learners' expectancy for success and task value (Eccles et al., 1983; Wigfield et al., 2017). According to EVT, there are four different types of task value: *intrinsic value* is when a learner values a task because they find the activity enjoyable for its own sake, *attainment value* is perceived personal importance of a task as it relates to one's identity, *utility value* refers to perceptions that a task may be useful to a learner to achieve their present or future goals, and *cost* is the extent of time and effort that is perceived to complete a task. Evidence suggests that higher value and lower cost perceptions are associated with student persistence, engagement, and achievement STEM (Thacker, 2024, Flake et al., 2015; Wigfield et al., 2017). In this study, we investigate students' the effect of implementing place-based learning visualizations to explore equity issues on students' expectancies for success, value of learning, and cost perceptions.

Gaps in the Literature

Though there are several studies of place-based approaches to science education at the undergraduate level, particularly in biology, geoscience, chemistry, and physics (Thacker et al., 2025, Deringer, 2017; Gosselin et al., 2016; Semken & Freeman, 2008), to our knowledge there are no studies that investigate its effectiveness in systems engineering classrooms nor for promoting self-transcendent motives. Further, the implementation of place-based approaches in diverse cultural contexts remains understudied (Semken et al., 2017). In this project, we address these research gaps by addressing the following research questions: *(1) To what extent does using geospatial data visualizations in a systems-engineering course influence undergraduate students' perceptions of: self-transcendent purpose, expectancy, value, and cost compared to a comparison group? (2) To what extent are these outcomes retained after 5 months?*

Methods

Data & Analysis

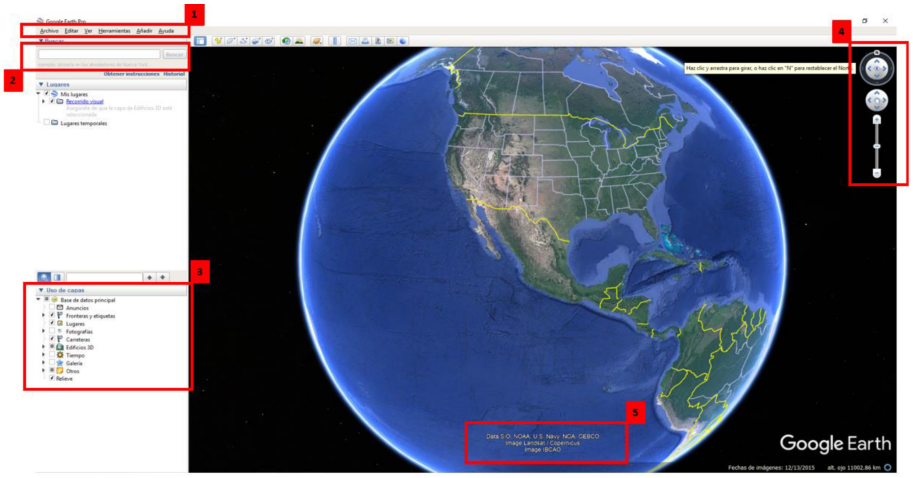
To address our research question, we conducted a quasi-experimental study at Tecnológico de Monterrey, a large university in Northwestern Mexico. A total of $N = 63$ undergraduate students in Industrial and Systems Engineering were recruited from three sections of a five-week course, *Economic Evaluation of Projects* (see Table 1 for participant characteristics and descriptive statistics for all analytic variables). A sensitivity analysis using G*Power revealed that this sample size is capable of detecting an effect size of $f^2 = 0.13$ using an F test with two predictors, power of .80, and alpha level of .05 (Faul et al., 2009). The course culminated in a final project where students applied engineering economic principles to propose a new food and beverage venture. This study was embedded within a one-week module focused on market feasibility analysis, where students selected a strategic location for their proposed restaurant. During this week, one of the three courses continued with instruction as usual and selected a restaurant location without using map software (control group; $n = 22$), while the other two courses made use of geospatial visualization tools in selecting a restaurant location (treatment groups; $n = 39$). The study received approval by the *Comité Institucional de Ética en la Investigación*, Tecnológico de Monterrey (P-EIC-202505-001).

Table 1: Descriptive Statistics by Condition

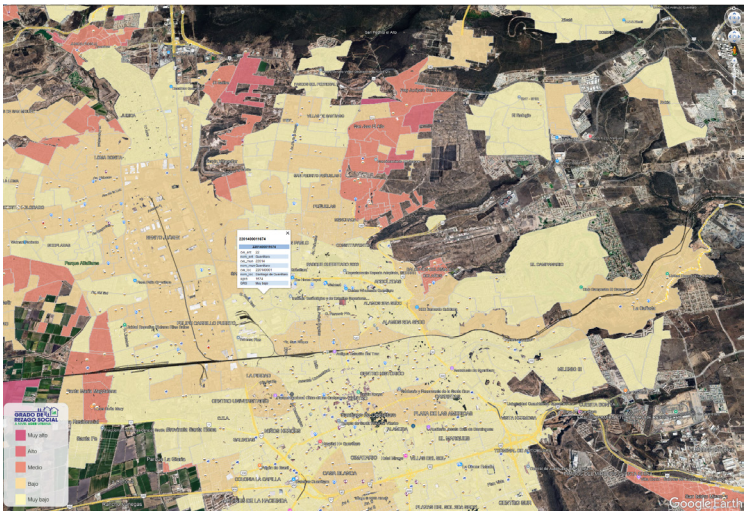
	Total									Treatment			Control		
	n	M / %	SD	Min	Max	Skew	Kurt	items	α	n	M / %	SD	n	M / %	SD
Comfort Speaking English	61	4.2	0.94	2.0	5	-0.8	-0.6	1	-	39	4.3	0.9	22	4.1	0.9
Age	61	20.3	0.98	19.0	24	1.4	2.8	1	-	39	20.3	0.8	22	20.2	1.2
Female	61	38%	-	-	-	-	-	-	-	39	23%	-	22	15%	-
Male	61	62%	-	-	-	-	-	-	-	39	41%	-	22	62%	-
Fourth year	61	7%	-	-	-	-	-	-	-	39	0%	-	22	7%	-
Other	61	2%	-	-	-	-	-	-	-	39	2%	-	22	0%	-
Second year	61	89%	-	-	-	-	-	-	-	39	61%	-	22	28%	-
Third year	61	3%	-	-	-	-	-	-	-	39	2%	-	22	2%	-
Usability	61	4.17	0.72	2.5	5	-0.35	-1.08	4	0.81	22	4.39	0.6	39	4.04	0.76
Self-Transcendent (Pre)	43	4.22	0.7	2.33	5	-0.8	0.01	3	0.83	17	4.12	0.6	26	4.29	0.76
Self-Transcendent (Post)	61	4.49	0.69	2.33	5	-1.38	1.31	3	0.92	22	4.52	0.84	39	4.48	0.61
Self-Transcendent (Followup	52	4.62	0.55	3	5	-1.3	0.74	3	0.86	19	4.63	0.61	33	4.61	0.53
Self-Oriented (Pre)	43	4.47	0.55	3.33	5	-0.54	-1.12	3	0.51	17	4.25	0.55	26	4.62	0.51
Self-Oriented (Post)	61	4.72	0.44	3.67	5	-1.32	0.3	3	0.73	22	4.76	0.41	39	4.69	0.46
Self-Oriented (Followup)	52	4.84	0.33	4	5	-1.77	1.58	3	0.8	19	4.88	0.3	33	4.82	0.34
Expectancy (Pre)	43	4.42	0.66	2.67	5	-0.84	-0.38	3	0.88	17	4.27	0.68	26	4.51	0.65
Expectancy (Post)	61	4.41	0.63	3	5	-0.46	-1.28	3	0.85	22	4.52	0.57	39	4.35	0.66
Value (Pre)	43	4.54	0.52	3.33	5	-0.56	-1.18	3	0.74	17	4.47	0.59	26	4.59	0.47
Value (Post)	61	4.53	0.56	3	5	-0.66	-0.84	3	0.82	22	4.65	0.5	39	4.46	0.58
Value (Followup)	52	4.42	0.64	3	5	-0.59	-0.93	3	0.87	19	4.3	0.75	33	4.49	0.56
Cost (Pre)	43	2.63	1.13	1	5	0.74	-0.56	4	0.86	17	2.44	0.95	26	2.75	1.24
Cost (Post)	61	3.23	1.22	1	5	0.06	-1.19	4	0.93	22	3.5	1.39	39	3.07	1.1
Cost (Followup)	52	3.27	1.15	1.5	5	0.2	-1.33	4	0.89	19	3.04	1.18	33	3.4	1.13

Figure 1 provides a summary of the activities in the treatment condition with screenshots of the visualizations. Students in the treatment groups received a tutorial on using Google Earth Pro to conduct their analysis. They learned to import and overlay shapefile data onto the map, specifically focusing on the Social Backwardness Index (*Grado de Rezago Social*, GRS) at the census-tract level. This enabled them to visualize socioeconomic disparities across the city. They were prompted with guiding questions to analyze this data, challenging them to identify a location that not only promised profitability but also served a community need. To supplement their analysis, students also used the 15-Minute City Web App (Bruno et al., 2024) to explore additional data layers, such as population density and walkability. The assessment was the justification for their final restaurant location, which was a key deliverable in their final project.

Figure 1: Examples of the Place-Based Learning Activity Guiding Students Through Socially Responsible Restaurant Site Selection, In Chronological Order



(A) An In-Class Tutorial for Google Earth Pro

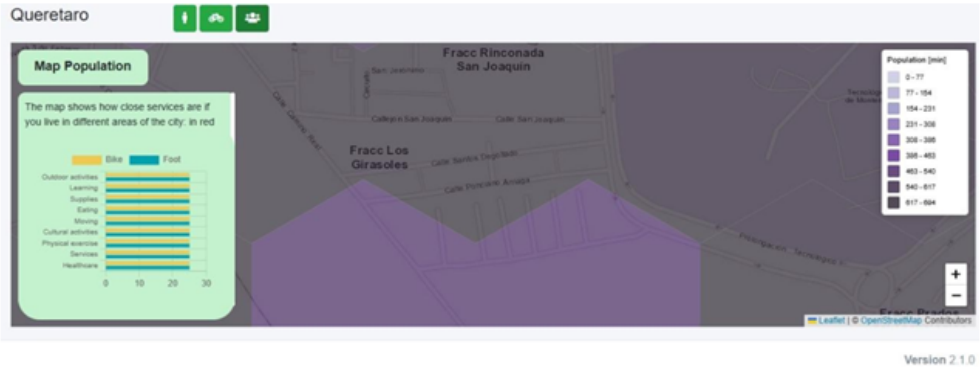


(B) Social Backwardness Index (GRS), Where Darker Colors Indicate Greater Social Disadvantage

Strategic location: Querétaro, Tec de Monterrey Zone

After some research and using some tools like google earth and the 15 minutes city web app, we were able to ascertain that the area of Prolongación Tecnologico and Plaza San Pablo has the following benefits:

- High foot traffic of students and office workers.
- Existing successful taquerias (providing customer demands)
- A constant mix of daytime and nighttime consumers.
- Good population and economic growth.



(Screenshot from the 15 minutes city web app of the maps and its population near Tec de Monterrey)



(Screenshot from the google earth app of the economic level found in the area near Tec de Monterrey)

(C) Example of a Map Produced by a Student Team for Their Restaurant Location Proposal

In contrast, students in the control group did not receive training in geospatial tools or access to the 15-Minute City Web App. Instead, they spent the time brainstorming possible restaurant locations based on general knowledge and intuition. During these sessions, the instructor provided comments to steer their thinking toward customer demand and market potential, but without the structured data-driven approach available to the treatment group.

Materials

Prior to engaging in the restaurant project, students completed a survey of their self-transcendent motives (Yeager et al., 2014), expectancy, value, and cost (Kosovich et al., 2015), and a demographics questionnaire. Immediately after the project, as well as five months later, students completed all items other than the expectancy measure. The *Self-Transcendent Motives Scale* was adapted from Yeager et al., (2014) and consisted of six items across two subscales: Self-transcendant motives (motivation derived from improving the well-being of others; three items) and self-oriented motives (that prioritize personal needs and interests; three items). The *Expectancy Value Cost Scale* was adapted from Kosovich et al., (2015) and included ten items across three subscales (expectancy for success, task value, and cost perceptions). Both scales required participants to respond to statements using a five-point scale ranging from 1 = *not at all true* to 5 = *completely true.* Further, we included five items from the *Systems Usability Scale* (Lewis et al.; 2018) to assess the extent that students' liked using the technology. Internal reliability for all scales at pretest and posttest were judged using Cronbach's alpha, and are reported in Table 1; all items are reported in the Supplemental Materials, Appendices A–C.

Prior to analyses, we assessed if student characteristics differed by condition. Chi-Squared and t-test analyses showed that student gender, comfort speaking English, and STEM major status were independent of condition (all $p > .564$). However, students' year of study was significantly higher in the control group ($p < .039$; namely there were fourth-year students in the control group than in the treatment). Thus, we included "year of study" as a covariate in all analyses. See Table 1 for descriptive statistics for all analytic variables by condition.

Analytic Strategy

We used two-level multilevel models with time nested within individuals to examine treatment effects on each outcome. Models included fixed effects of time, condition, and their interaction, with random intercepts for participants, and year of study as a covariate. Model assumptions were evaluated by inspecting residual distributions and residual-versus-fitted plots to assess normality and homoscedasticity, and no substantial violations were observed. Multilevel modeling was used to accommodate the dependence of repeated measures and to retain cases with incomplete data under a missing-at-random assumption. The reduced-form model can be described by:

$$\begin{aligned} Y_{ti} = {} & \gamma_{00} + \gamma_{10}(Posttest_{ti}) + \gamma_{20}(Followup_{ti}) + \gamma_{01}(Treatment_i) + \\ & \gamma_{11}(Posttest_{ti} \times Treatment_i) + \gamma_{21}(Followup_{ti} \times Treatment_i) + \\ & \gamma_{02}(Year) + r_{0i} + e_{ti}, \end{aligned}$$

where subscripts t and i denote time points nested within individuals. The coefficients γ_{10} and γ_{20} represent the main effects of time, γ_{01} the baseline group difference, and γ_{11} and γ_{21} the primary interaction effects of interest. r_{0i} accounts for participant-specific random intercepts, while e_{ti} represents the residual error.

Expectancies and usability outcomes were only measured at posttest, and were thus analyzed using linear regression models. In these models, the treatment condition was the main predictor, and covariates were year of study, and for expectancies, pretest scores.

Results

Intervention effects across three time points are presented in Table 2. Compared to pretest scores, we found significant main effects of time in self transcendent purpose, self-oriented purpose, and cost at posttest and follow-up. Self-oriented purpose was higher in the treatment

group compared to the control at pretest, but at posttest and follow-up, significantly lower—suggesting that the intervention inhibited self-oriented purpose over time. We also found lower cost perceptions in the treatment group at posttest, though this effect was not retained at the follow-up measurement. There was no significant effect of time or condition on value.

Students also reported significantly lower levels of technology usability in the treatment compared to the control condition (b = –0.448, SE = 0.192, p = 0.024). We found no significant treatment effects on expectancy.

Table 2: The Effects of the Mapping Intervention at Three Time Points After Adjusting for Year of Study

	Self-Oriented Purpose	Self-Transcendent Purpose	Cost	Value
Posttest	**0.476*** **	**0.368** **	**1.145*** **	0.179
	(0.114)	**(0.136)**	**(0.325)**	(0.152)
Followup	**0.591*** **	**0.486*** **	**0.710***	–0.166
	(0.118)	**(0.142)**	**(0.338)**	(0.158)
Treatment	**0.380** **	0.213	0.532	0.103
	(0.138)	(0.198)	(0.374)	(0.183)
Posttest × Treatment	**–0.453** **	–0.258	**–0.855***	–0.319
	(0.145)	(0.174)	**(0.414)**	(0.194)
Followup × Treatment	**–0.457** **	–0.316~	–0.036	0.033
	(0.151)	(0.182)	(0.431)	(0.202)
Marginal R^2	0.127	0.083	0.104	0.038
Conditional R^2	0.455	0.624	0.360	0.388
ICC	0.4	0.6	0.3	0.4

Note. Pretest is the reference variable regarding time, control regarding experimental treatment. All models include year of study as covariate to adjust for baseline differences between classrooms. Rows in bold represent factors that are significant at the .05 level. ~ p < 0.1, * p < 0.05, ** p < 0.01, *** p < 0.001

Significance

Findings from this research contribute to both theory and practice in data-oriented STEM and entrepreneurship education, and place-based pedagogy. Theoretically, our study provides novel empirical support for the idea that place-based, geospatial data visualization activities centered on social issues can shape self-oriented motives and cost perceptions, two motivational constructs that have been shown to predict persistence, meaningful engagement, and achievement in STEM fields (Flake et al., 2015; Yeager et al., 2014). Prior research on place-based education and self-transcendent motives has largely focused on U.S.-based student populations and science disciplines (e.g., biology, chemistry, physics; Deringer, 2017; Gosselin et al., 2016; Semken et al., 2017); our findings expand this work by demonstrating similar benefits in systems engineering education and within a Latin American cultural context.

Practically, this study provides a model and lesson structure for integrating free and open-source data-visualization software (Google Earth Pro) into systems engineering coursework to support socially responsible entrepreneurship and equity-mindedness. Rather than positioning business decision-making as solely profit-driven, this approach encourages students to consider trade-offs between economic gain and social impact, such as the placement of businesses in areas

with fewer resources or higher need. Our results suggest that this kind of spatial, ethically informed, data-focused decision-making task may reduce students' self-focused motives and perceived task costs—two barriers to sustained engagement in STEM and entrepreneurship coursework (Flake et al., 2015; Yeager et al., 2014).

Finally, by highlighting the usability challenges students experienced with the technology, our study underscores the need for scaffolded instruction when introducing geospatial tools in undergraduate settings. As interest in place-based and data-informed education grows, especially in applied disciplines like engineering and social sciences, educators must strike a balance between cognitive demand and accessibility to ensure that such tools enhance rather than inhibit learning and motivation.

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INVESTIGATING HIGH SCHOOL STUDENTS' EMERGENT PROBABILISTIC REASONING USING A DIE SIMULATION ACTIVITY

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Traditional probability instruction often emphasizes rote, formulaic procedures over conceptual reasoning, leaving students with persistent misconceptions such as equiprobability bias and confusion between empirical and theoretical probability. This multiple case study examined high school students' probabilistic thinking during a simulation-based dice activity. We used a priori theoretical coding to assess how students' reasoning evolved across the activity. Our analytical framework distinguished between local and global resources, with each category further characterized as either compatible or incompatible. Findings revealed that all students demonstrated some understanding of the law of large numbers by the end of the activity, though some misconceptions persisted. Results suggest that probabilistic reasoning is nonlinear since students shifted between compatible and incompatible thinking throughout.

Keywords: Probability, High School Education, Simulation-Based Learning

In our data-driven world, understanding probability is essential for equipping students with the tools to make informed decisions, interpret information critically, and recognize patterns. Despite its importance, probability is often taught in isolation, reduced to rote calculations and formulae memorization. When instruction focuses primarily on symbolic manipulation and procedural tasks, research suggests students struggle to reason probabilistically beyond textbook scenarios (Fischbein & Schnarch, 1997; Konold et al., 2011). As a result, they are left with an incomplete understanding of probability and find it difficult to connect intuitive ideas with formal representations, limiting their grasp of foundational concepts such as randomness, variability, and uncertainty.

Attending to students' misconceptions about probability is crucial for designing instructional materials that address and mitigate these misunderstandings. In this study, we drew from work of Groth et al. (2021) and Pratt (2000) to develop a framework for assessing students' thinking about probability as they engaged in a simulation-based probability activity. The framework captures how students' reasoning evolves, from short term probability events, to understanding of the law of large numbers. We aimed to understand how students build connections between empirical evidence and theoretical probability.

Background Literature

Decades of research highlight persistent challenges in how students reason about probability. Despite formal instruction, students rely on flawed, intuitive ideas, leading to errors in reasoning (Fischbein & Schnarch, 1997; Groth et al., 2021; Konold et al., 2011). One common misconception is equiprobability bias, which is the tendency to assume all outcomes in a probability situation are equally likely, even when evidence suggests otherwise. This bias becomes more pronounced with age (Fischbein & Schnarch, 1997), suggesting that without targeted instruction, incorrect probabilistic reasoning becomes more deeply entrenched over time. Some students struggle to differentiate between empirical and theoretical probability. They often generalize based on very small datasets, not noticing that experimental results get closer to theoretical ones as the number of trials increases (Konold et al., 2011). Simulations, while

promising as instructional tools to represent theoretical probabilities, present their own challenges. Some students view simulations as perfect representations of theoretical probability, discounting the role of chance and randomness. They may struggle to articulate what the simulation represents or how it relates to theoretical models (Lee et al., 2016). Without opportunities to explore simulations, interpret distributions, and practice application in real-world scenarios, research suggests that students: (a) fail to develop an intuitive understanding of randomness and variability, (b) over trust small samples, (c) struggle to reconcile empirical probability with theoretical models, (d) remain stuck at procedural fluency, and (e) do not internalize the Law of Large Numbers (Fischbein & Schnarch; 1997; Konold et al., 2011).

Framework

The aim of this study is to understand and characterize students' probabilistic thinking through simulation-based learning (SBL). SBL is a pedagogical approach that replicates key features of real-world scenarios, enabling learners to engage in authentic practices while bypassing real-world constraints such as time, safety risk, cost, or replicability (Chernikova et al., 2020). The SBL activity used in this study recreates the context of rolling a die multiple times to facilitate the exploration of probability concepts and is grounded in constructivist views of learning, in which students build upon and strengthen their existing knowledge of probability through active engagement with meaningful tasks (Piaget, 1964).

Pratt (2000) argued that learners often hold multiple, and sometimes contradictory, intuitions simultaneously when making sense of probability concepts. For instance, a student may understand that each number on a fair six-sided die is equally probable (1/6 chance), but simultaneously expect even numbers to be more probable. Drawing on situated cognition (Lave, 1988; Nunes et al., 1993), Pratt (2000) suggested that these conflicting, yet coexisting intuitions can be understood as products of participation in different learning environments, where the development of knowledge is constructed by the tools, people, and affordances specific to that environment. Groth et al. (2021), suggested that early probabilistic thinking unfolds in cycles where children move back and forth between different conceptions and forms of reasoning as contexts change. Thus, students' intuitions may change when the problems they encounter feature slight variations in wording, question structure, and the type of reasoning required. These subtle contextual changes can trigger inconsistent, or even contradictory, ideas as students figure out how to apply new concepts across tasks.

Additionally, students' thinking develops over time in ways that can be described by developmental theories. Piaget's theory of development (1964) conceptualizes learning as a gradual progression through different stages as a child ages, with each stage characterized by distinct forms of reasoning. From this perspective, formal probabilistic thinking emerges gradually. A Neo-Piagetian elaboration of this perspective is described by Biggs and Collis' (1991) Structure of the Observed Learning Outcome (SOLO) taxonomy, which models five cognitive modes: sensorimotor, ikonic, concrete-symbolic, formal, and post-formal. Groth et al. (2021) drew on the SOLO taxonomy to frame early probabilistic thinking within the ikonic mode. Within this mode, children often rely on mental images, stories, their imagination, and prior experiences to make sense of probability concepts. Groth et al. (2021) argued for an important distinction between two types of ikonic reasoning in probability, that is, normative-compatible and normative-incompatible; the former describes children's informal, early conceptions that are productive when further developed, and the latter describes conceptions that are unproductive and unaligned with probabilistic principles. In the next section, we describe

how we adapted these frameworks and stages of thinking to develop a novel codebook that characterizes students' thinking as they progressed through our SBL activity.

To frame our work, we drew from Pratt's (2000) local and global resources and Groth et al.'s (2021) types of reasoning, and Fishbein and Schnarch's (1997) research on misconceptions. Pratt (2000) introduced local and global resources. Local resources (i.e., unpredictability, irregularity, unsteerability, and fairness) capture intuitions developed from experience with short term probability, while global resources (i.e., probability, large numbers, and distribution) reflect more formal concepts grounded in long-term probability.. *Unpredictability* captures the recognition that the next outcome is unknown. *Irregularity* refers to the absence of an obvious pattern in the sequence of outcomes. *Unsteerability* reflects the understanding that outcomes cannot be controlled. *Fairness* describes a perception that outcomes are balanced or equal, without talking about probability (e.g., when rolling a die six times, assuming each number must appear once, as this is fair). *Probability* refers to students' recognition that the proportion of outcomes for each possibility is predictable. *Large numbers* captures the understanding that while short-term results are unpredictable, frequencies stabilize and converge as the number of trials increases. *Distribution* refers to the idea that outcomes can be controlled by how you represent the possibilities.

Groth and colleague's (2021) study focused on proposing a theoretical framework called the Structure of Early Probabilistic Thinking (SEPT) to guide curricular decisions and characterize young children's learning. The appeal of Groth et al.'s work is the distinction between compatible and incompatible reasoning. Since we are focused on secondary students' probabilistic reasoning, we drew from Fischbein and Schnarch (1997) who described the evolution of misconceptions that arise when students engage with probability tasks from grades 5–11. Their hypothesis was that misconceptions would dissipate as students learned more formally about probability. They found that to be partially true, while some misconceptions grew weaker, others grew stronger. To operationalize compatible and incompatible reasoning for the context of this study, we drew from Fishbein and Schnarch (1997) and decided to categorize incompatible reasoning by the persistent use of misconceptions and personal beliefs (e.g., consistently ignoring sample size when estimating probability, believing one can influence random events). While Fishbein and Schnarch (1997) did not explicitly address compatible reasoning, we operationalized compatible reasoning as the capacity to engage with a probability task without the interference of intuitive-based misconceptions.

Methods

In this exploratory multiple case study (Yin, 2018), we aimed to answer the research questions: (1) How do high school students utilize probabilistic resources and reasoning when engaging in an SBL probability activity? (2) How do their resource use and reasoning change throughout the course of the activity? In the subsequent sections we will describe the cases (e.g., participants), the SBL activity, data collection, and analysis.

Cases

Each of the four cases features a rising high school senior. Three of the four cases were female students: Juliana, Annie, and Hannah. One case was male, Michael (all names are pseudonyms). The participants all attended a large urban high school in the Northeast U.S. at the time of the study. The cases were bound by time and context. All participants were enrolled in a program in which they were dually enrolled in high school and a large university. To be part of this program, the students had to apply and meet G.P.A. requirements. They took an introductory

statistics course at the university with a college professor during the Summer 2025 semester. The SBL activity took place after the students completed a unit on descriptive statistics.

The SBL Activity

The activity began with a prompt that asks students to predict and sketch the outcome of 10 die rolls, and were asked to respond to the prompt: What do you think will happen if you roll a die 10 times? Then, they ran a 10-roll simulation and were asked: What did you observe in your 10 rolls?. In the second phase of the activity, students were asked: If we rolled this die 60 times, what would you expect to see? Then they ran 50 more rolls using the simulation for a total of 60 rolls (see Figure 1 for sample output), and were asked to compare their expectations with the simulated outcomes by responding to the prompt: What actually happened when you rolled the die 60 times? Then they were directed to compare their results from 60 simulated rolls with a neighbor. In the final phase of the activity, students simulated 240 rolls (see Figure 2) observing how the results converge toward the theoretical probability and a more uniform distribution, despite some residual variation. Students were given the following prompt: Now compare your charts with your group. How different are your results from others? Why do you think these differences occurred? Although students completed the activity independently in Amplify, certain prompts required brief comparisons with nearby peers to discuss similarities and differences. Mentions of peers in the findings reflect how each student individually described or interpreted discussion with a peer.

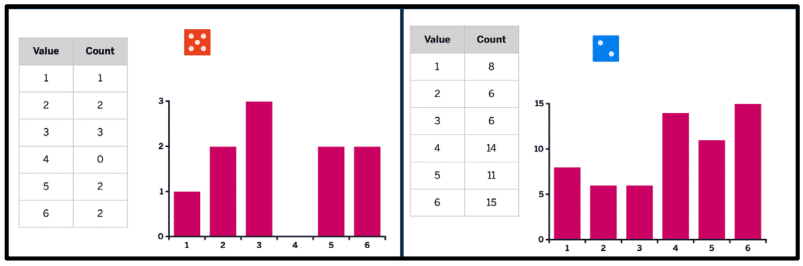


Figure 1: Sample Output after 10 and 60 Simulated Rolls

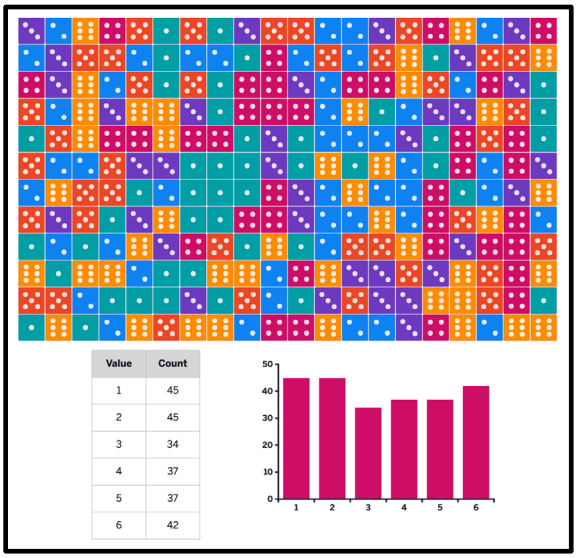


Figure 2: Sample Output after 240 Simulated Rolls

Data Collection

Each student individually completed the *Die Hard* task in Amplify during class (movie and misconception puns intended). Amplify saves student responses and simulation results. The data was extracted from Amplify after final grades posted to adhere with IRB. Data included the students' simulated and written work from the *Die Hard* Amplify task. Screenshots of each page of the activity were blinded for analysis.

Analysis

Blinded screenshots of the students' written responses to each prompt were coded using a priori theoretical coding using an analytic framework based on Pratt's (2000) global and local resources layered with Groth et al.'s (2021) compatible and incompatible reasoning. In total, the analytical framework included codes for six resources (e.g., large numbers), each paired with reasoning classifications (e.g., compatible). For most resources, responses were categorized as either compatible or incompatible, resulting in 14 initial codes. For the *large numbers* global resource, we introduced a third code, *pre*, thus resulting in three possible classifications (pre-compatible, compatible, incompatible) for that resource, and 15 codes overall. Coding proceeded in two stages. First we coded for the presence of local and global resources (e.g., *irregularity*). Second, we coded for compatible or incompatible reasoning (e.g., *irregularity-compatible*). The codebook was refined using the protocol developed by DeCuir-Gunby et al. (2011). After independently coding a random subset of the data, we reviewed all responses we initially coded as *other* to identify any potential new themes. This process led to the addition of a *language* code to capture moments when the students were using words like *chance* or *random* in ways where it was not clear if they meant the colloquial use or the probabilistic use of the word. While Groth et al.'s (2021) compatible and incompatible reasoning was helpful to tease out levels of Pratt's (2000) resource codes, we found that we needed a third level for the global resource, *large numbers.* There were several instances where students did not demonstrate enough evidence to support a compatible code, but did demonstrate some evidence of moving toward such reasoning. These instances led to the development of a *pre-compatible* code that captured instances of partial or emergent reasoning about *large numbers* that did not meet the criteria for *compatible* reasoning, but indicated movement in that direction. For example, a student who noted that as the sample size increases the counts of each outcome begin to level would be coded as *large numbers-compatible*. A student who notices that each outcome is more likely with larger samples, and that each value is not necessarily going to occur in a small sample would be coded *large numbers-pre* to indicate that they are noting ideas that move them toward an understanding of the law of large numbers. After consistency was achieved on a random subset of data, all three authors independently coded all data and discussed discrepancies until consensus was achieved.

Findings

The *Die Hard* Amplify activity was structured around three phases: (1) short term probability; (2) stabilizing, where empirical results begin to approach the theoretical probabilities; and (3) convergence to theoretical probability or the Law of Large Numbers (LLN). We will summarize the findings for each case by activity phase followed by the cross-case analysis.

Case 1: Julianna

In phase one, Julianna demonstrated *unpredictability compatible* and *probability compatible* reasoning when she noted: "You never know what number you get on a die since there is a 1 in 6 chance on getting any number," acknowledging both the equal likelihood of each face and not having a way to know what the next outcome will be. She also predicted "some numbers will be

coming up more than others" and was coded for *unpredictability incompatible*, since short term outcomes can show uneven frequencies. Julianna was later asked to compare her prediction to the 10-roll simulation outcome, she wondered, "why there wasn't only one number that had the most rolls," we coded her response as *irregular incompatible* because she assumed that there was a way to predict an outcome and that it would favor a specific value.

In phase two, her reasoning shifted toward *large numbers compatible* when she predicted that with 60 rolls, "many different number[s] would have the same amount of rolls," and that one or more values "would have the same amount or the most amount of rolls," recognizing that the counts for each outcome/face would get closer to equal as the rolls increased. She also showed *unpredictability compatible* reasoning when comparing the results of her simulation with her peer, "since all of the sides of the die have the same chance it is randomized and there will be a different output for everyone."

In phase three, Julianna showed *irregular incompatible* reasoning when predicting that "three of the same numbers were going to have the same amount of rolls" when rolling the die 240 times, because she tried to predict a pattern and arbitrarily chose three. Finally, when simulating thousands of rolls, she noted, "when I increased the number of rolls the amount of rolls for each side kind of became equal," reflecting *large numbers compatible* reasoning as she recognized that frequencies stabilize with more trials.

Case 2: Annie

In the first phase of the activity, Annie stated that she believed that certain numbers, like 5 or 6, would appear more often, due to the "randomizing formula" which we coded as *irregular incompatible*. She assumed a pattern (i.e., expecting to see 5's or 6's) where no obvious pattern should exist. Annie wondered "[if] there is a way to find out what number will be next for sure" which was indicative of *unpredictability incompatible* reasoning since there is no way to know or predict the next outcome.

In the second phase, Annie assumed higher or lower numbers would show more frequently because "extremes are more common," thus assuming a pattern in a sequence of outcomes where one should not exist. She was also coded for *language* because we could not discern the meaning of "extremes." Later, Annie shifted to *probability compatible* reasoning when she simulated 60 rolls, recognizing that "every number has an equal chance." Her prediction that increasing rolls "would show an even greater difference between the most common rolls and the least common rolls" was coded as *large numbers* and *distribution incompatible* because she assumed that difference would be persistent and noticeable even after we increase the number of rolls significantly.

By the final phase with 240 rolls, Annie noticed that "the graph begins to even out and have no extremes," reflecting *distribution* and *large numbers compatible* reasoning as she recognized that frequencies stabilize with more trials and that the graph is uniformly distributed.

Case 3: Hannah

In phase one, Hannah showed *probability incompatible* reasoning when she assumed numbers 3 and 4 "have a higher probability [of] being rolled" because they appeared more frequently in the simulation. When asked to predict 10 rolls, she demonstrated *unpredictability compatible* reasoning, noting that "I think I will get the same numbers more than once and some won't be rolled at all," demonstrating awareness that the next outcome cannot be known.

In phase two, when comparing her 60-roll simulation with peers, Hannah said, "I think these differences occurred because the probabilities of a number being rolled vary a lot." Her response was coded for *language* as use of "probabilities" and "vary a lot" was unclear. When predicting

the effect of increasing rolls, she showed *large numbers compatible* reasoning, expecting "all numbers showing up more than once as there are more trials and therefore more chances of them being rolled."

In the final phase, after simulating 240 rolls of the die Hannah noted "all of us [are] noticing a more even graph … these differences occurred because the probability of the numbers showing up increases as we add more trials," this was coded as *distribution compatible* since she noticed that graphs are getting more even, and *language* for unclear use of the word "probability." When simulating as many rolls as possible she described the graph as "almost a straight line with only small bumps or peaks" which we coded as *distribution compatible*. Her final reflection that "increasing the number of rolls also increases the probability of the numbers showing up, which makes the graph more even" was coded as *large numbers compatible* since she recognized that frequencies stabilize as the trials increase, and *distribution compatible*.

Case 4: Michael

In phase one, Michael showed *unsteerability incompatible* reasoning when he assumed that number four appeared the most due to "the strength that you roll the die at," because he assumed other unrelated elements (i.e., strength or force) would influence the outcome. When predicting 10 rolls, he wrote, "it'll give me a random assortment of numbers that may or not repeat," and this was coded as *unpredictability compatible*. In comparing his prediction to the simulation, he wrote, "my prediction was completely different from my outcome since the odds of the die are made to be random." We coded his response as *language* since the use of the word "odds" was unclear and we could not determine if he meant the odds of faces or the odds of the theoretical probability.

In phase two, after simulating 60 rolls, Michael wrote, "every outcome is equally possible and no one outcome will be the same," and we gave his response three codes: (a) *probability compatible*–he recognized that each face is equally probably, (b) *language*–it is unclear what "outcome" he is referring to, and (c) unpredictable compatible–he sees there is no way to predict the outcomes of his group members. When predicting 240 rolls; he expected "more variety in the graph." We coded his response as (a) *large numbers compatible* because he describes the count of each face increasing as the number of rolls increases and (b) *language* because we could not discern what kind of "variety" he was anticipating.

In phase three when simulating thousands of rolls, Michael wrote, "increasing the rolls increases the odds that all the numbers will roll an equal amount of times," we coded this as (a) *large numbers compatible* since he acknowledged that the counts for each face stabilize as the number of rolls increases, and (b) *language* since we are not clear what "odds" he is referring to.

Cross-Case Analysis

As the activity progressed, students started showing evidence of the *language code* as they began to grapple with more complex concepts and terminology. It was unclear if the students were confusing colloquial and probabilistic definitions for their word choices (e.g., chance, odd, vary) or just using more informal colloquial language when they felt challenged. Simultaneously, they started to show more nuanced understanding of evening distributions, and convergence to theoretical probability concurrently with some incompatible reasoning. By phase three, each student evidenced *large numbers compatible*, which is indicative of some movement toward understanding of LLN. Every student alternated between various compatible and incompatible probabilistic reasoning, indicating at least for these students probabilistic reasoning develops non-linearly.

Discussion

In the first phase we mostly saw instances of *unpredictable compatible*, indicative of the students' awareness that we can't know what happens in the short term, and *irregular incompatible* thinking, where some students assumed that the patterns that emerge in the short term are a basis for making assumptions about the theoretical probability. Together this is indicative of what previous research pointed to, that probability is challenging, or in Pratt's words "probability is hard" (2000, p. 602).

In the second and third phases, most participants were coded for *language* which potentially indicates that students lean on informal or colloquial language when they are challenged. It is possible that some students were aware of the differing definitions between the colloquial and probabilistic uses of terms, such as odds, vary, and chance. In those instances, it is possible that students move back and forth between colloquial and probabilistic definitions. Just as students move in a non-linear, sometimes contradictory fashion when engaging in probabilistic reasoning (Pratt, 2000), the language they use to communicate such understandings is also likely to be unstable. Future research is needed to understand the role of how students navigate using probabilistic terms that are used in different manners colloquially.

By the end of the activity, every student exhibited some form of large numbers compatible thinking. Students moved from inconsistent or incompatible reasoning, to reasoning that aligned with the idea of the law of large numbers. Regardless of how some students began thinking (compatible or incompatible) they all shifted back and forth throughout the activity in their own ways. All students concluded the activity with varying levels of compatible LLN thinking. This highlights how the students can hold both compatible and incompatible ideas as they work through new concepts. This finding corroborates Pratt's (2000) finding that students navigate probabilistic reasoning by moving between local and global resources such that compatible and incompatible reasoning coexist throughout the learning process rather than following a linear progression. Our findings are also consistent with findings by Fischbein and Schnarch (1997) who demonstrated how some misconceptions remain persistent or even grow stronger as students gain formal probability instruction (e.g., ignoring sample size).

This study adds to a growing body of research affirming that probabilistic thinking develops in a nonlinear way. For K–12 education, this means that probability cannot be confined to the upper-level grade bands only. All K–12 students need the opportunity to refine their probabilistic reasoning, especially as it grows nonlinearly. Prior research has persistently shown that the intuitions students form early shape how they engage with probabilistic thinking for years to come (e.g., Pratt, 2000). We added nuance to existing frameworks for probabilistic reasoning by showing how compatible and incompatible reasoning can coexist. Future research should examine how patterns of probabilistic thinking manifest across more diverse student populations and educational contexts.

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IMAGE-BASED AND SOUND-BASED MACHINE LEARNING INTERVENTIONS: DIFFERENCES IN STUDENTS' EXPERIENCES AND UNDERSTANDING

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Machine learning (ML) remains absent from high school math classrooms, due to its perceived technical complexity and reliance on coding. Google's Teachable Machine makes core ML processes accessible through data-driven, sensor-based approaches without programming. We examined high school students' (n = 51) perceptions of website usability, perceived ML understanding, and performance on a multiple-choice assessment following two interventions: image-based and sound-based model training. Students in both conditions reported similarly positive experiences, with no statistically significant differences in usability or perceived understanding. Although image-based students scored higher on the assessment, the difference was not statistically significant. Item analyses showed stronger data and training performance in the image condition, with similar understanding of behavior and generalization across groups.

Keywords: Machine Learning, High School, Data Science

Machine learning (ML), a subset of data science, involves training computational models on prior data to generate predictions or classifications (Alpaydin, 2020). ML concepts such as feature selection, training and testing data, generalization, and model accuracy are becoming important components of K–12 data science education (Touretzky et al., 2025), yet ML remains absent from secondary math classrooms. This gap persists for several reasons: ML is inherently technical and code-intensive, obscuring students' access to core conceptual ideas (Sanusi et al., 2022); classroom-appropriate tools and visualization supports are limited; and teacher preparation in ML content and pedagogy remains inadequate (Amaniampong & Oyelere, 2024).

Web-based ML tools that require no coding foreground data practices (e.g., collecting, labeling, training, and evaluating) while making model behavior visible. Less is known, however, about how different data representations shape students' understanding of these processes. Image-based training foregrounds visual features (patterns in space), whereas sound-based training highlights temporal patterns and signal variation (patterns over time)—differences that may influence how students conceptualize features, interpret performance, and reason about generalization. In this study, we examine students' experiences with a web-based ML environment (Google, n.d.) implemented through a data-driven instructional approach (Morales-Navarro & Kafai, 2024) across two conditions: image-based and sound-based model training. This study addresses two questions: (RQ1) How do students' perceptions of website usability and ML understanding differ between image- and sound-based interventions? (RQ2) To what extent do students demonstrate understanding of core ML processes (e.g., data preparation, training, classification), and does this differ by condition?

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Literature Review

Hands-on image recognition tasks can support students' understanding of core ML concepts while surfacing important conceptual tensions (Morales-Navarro & Kafai, 2024). Through structured image-classification activities, middle schoolers demonstrated mastery of image representations, feature maps, kernels, and matrix-based computations (Rajarajan & Martin, 2026), and students in Živković et al. (2025) deliberately added samples for underrepresented categories, adjusted lighting, and retrained models to improve performance. Such engagements can also reshape how students conceptualize ML itself: when high schoolers worked with simplified tools foregrounding features, labels, and training–test distinctions, they began framing ML as a human-assisted, data-dependent process rather than an autonomous system (Kaspersen et al., 2022). For example, students using a one-feature image-recognition model saw that a long fish and a tall giraffe could be misclassified as similar, and that adding a second feature improved accuracy (Toivonen et al., 2022).

Students' conceptions of ML also remain varied and sometimes unstable. Surveying 155 students, Mertala et al. (2024) identified three dominant conceptions—ML as "teaching machines" (34.0%), as coding (7.6%), and as machines learning from data (37.1%)—and found that misconceptions about automation, agency, and human decision-making persisted even among partially accurate definitions. Kaspersen et al. (2022) similarly reported that some students distanced themselves from ML as "very complex," while others recognized human responsibility in feature selection and bias, noting that machines only examine what they are directed to. Toivonen et al. (2022) likewise found that students initially attributed classification errors to their own deficiencies (e.g., drawing ability) before recognizing that performance depends on model structure and training data quality.

The recent work has explored sound-based ML environments that foreground temporal data and signal variation, positioning students not merely as users but as designers who curate, label, and iteratively refine training data (Tedre et al., 2021). Speech recognition and sensor-based tools offer "low-floor, high-ceiling" entry points that make core ML processes visible. Hsu and Chen (2022), for example, had undergraduates train spectrogram-based voice-recognition models to control robotic cars in a computational thinking board game, yielding gains in learning effectiveness and engagement. Such findings suggest sound-based contexts may surface different aspects of data reasoning than image-based environments, warranting closer examination of how representational modality shapes students' understanding of ML.

Method

We introduced ML concepts in three high school classrooms across two southern U.S. districts in Fall 2025; approximately 95% of students identified as African American. One classroom implemented the sound-based condition and two the image-based condition. Each 90–120-minute lesson occurred during regular instruction and followed the same learning objectives, instructional sequence, and design challenges using a web-based ML platform (Google, n.d.). Of 59 survey completers, eight responses were excluded for missing data or uniform response patterns (six image-based, two sound-based), yielding final samples of 34 image-based and 17 sound-based students. A post-intervention survey and assessment compared students' perceptions and conceptual understanding of core ML processes across conditions. The survey included 39 six-point Likert-scale items (strongly disagree to strongly agree); we focus here on website usability, perceived ML understanding, and multiple-choice outcomes. Survey constructs showed strong reliability ($\alpha = .80–.95$), and the 19-item multiple-choice assessment demonstrated good internal consistency (KR-20 $\alpha = .83$), with item difficulty ranging from .28 to .80.

Results

RQ1: Website Usability and Perceived ML Understanding

We compared the image-based and sound-based instructional conditions on students' perceptions of website usability and perceived ML understanding using Welch independent-samples t-tests (Table 1). Mean scores were numerically higher in the image-based condition for both website usability and perceived ML understanding. However, these differences were not statistically significant ($p > .05$).

Table 1: Comparison of Image-Based and Sound-Based Conditions

Construct	Image M (SD)	Sound M (SD)	Mean Difference [95% CI]	Effect Size *d* [95% CI]	*t*	*p*
Website Usability	4.63 (0.66)	4.29 (0.72)	0.34 [–0.09, 0.76]	0.49 [–0.10, 1.08]	1.62	.117
Perceived ML Understanding	5.04 (0.84)	4.71 (0.99)	0.33 [–0.24, 0.91]	0.37 [–0.22, 0.95]	1.18	.249

Perceived ML understanding was moderately correlated with website usability ($r = .59$, $p < .001$). These findings indicate no statistically reliable differences in students' perceptions across conditions, though effect sizes suggest a possible advantage for the image-based condition.

RQ2: Understanding of Machine Learning Process

We compared total scores on the 19-item multiple-choice assessment using Welch's t-test (Table 2). Students in the image-based condition scored higher, though the difference did not reach conventional significance, $t(41.4) = 1.80$, $p = .079$, Mean Difference = 2.26 items, 95% CI [–0.27, 4.80]. The small-to-moderate effect size ($d = 0.49$) suggests a possible but inconclusive advantage for the image-based condition.

Table 2. Comparison of Multiple-Choice Assessment Scores by Condition

Image	Sound-Based	Mean Difference	95% CI	*t* (df)	*p*	Effect Size
11.85 (5.04)	9.59 (3.76)	2.26	[–0.27, 4.80]	1.80 (41.4)	.079	0.49

Subscale analyses showed image-based students outperforming sound-based peers overall (62.4% vs. 50.5%), with the largest gaps in Data and Representation and Training and Model Improvement—particularly data variation, training/test splits, and iterative improvement. Differences were smaller in Model Behavior and Evaluation and minimal in Generalization and Conceptual Understanding, where both groups scored lower. Notably, sound-based students slightly outperformed image-based peers in pattern-based classification and responding to poor results, suggesting domain-specific strengths.

Table 3: Subscale Performance on the Multiple-Choice Assessment by Condition

Scale	Image-Based *M* (SD)	Sound-Based *M* (SD)	Mean Difference	95% CI	Effect Size (d)	*t*	df	*p*	Holm-Adjusted *p*
Overall (19 items)	11.85 (5.04)	9.59 (3.76)	2.26	[–0.27, 4.80]	0.49	1.80	41.4	.079	—
Data	0.58 (0.27)	0.40 (0.17)	0.18	[0.05, 0.30]	0.74	2.83	45.8	.0068	.027

Scale	Image-Based *M* (SD)	Sound-Based *M* (SD)	Mean Difference	95% CI	Effect Size (d)	*t*	df	*p*	Holm-Adjusted *p*
Training	0.75 (0.31)	0.53 (0.35)	0.22	[0.01, 0.42]	0.66	2.13	29.2	.042	.126
Model Behavior	0.70 (0.29)	0.61 (0.26)	0.09	[–0.07, 0.26]	0.34	1.17	34.8	.252	.504
Generalization	0.47 (0.32)	0.43 (0.23)	0.03	[–0.13, 0.19]	0.11	0.42	43.0	.675	.675

Discussions

This study examined how image-based versus sound-based model training shapes students' perceptions and demonstrates understanding of core machine learning processes. Although overall differences between conditions were not statistically significant, the pattern shows that representational modality may influence which aspects of ML reasoning become salient during instruction.

Perceptions of Website Usability and Perceived Understanding of Machine Learning

We found no statistically reliable differences emerged between image- and sound-based conditions in reported website usability or perceived ML understanding. However, mean scores were consistently higher in the image-based condition, with small to approaching moderate effect sizes. We also found that perceived ML understanding was moderately correlated with website usability, suggesting that students who felt they understood the underlying ML processes found the interface more accessible. This relationship reinforces prior work indicating that interpretability and transparency of tools play a central role in shaping students' engagement with ML systems (Kaspersen et al., 2022; Morales-Navarro & Kafai, 2024). These findings suggest that perceptions of usability may be intertwined with conceptual understanding.

Core Understanding of Machine Learning Processes

Students in the image-based condition outperformed those in the sound-based condition on the overall assessment, though this difference was not statistically significant. The moderate effect size and positive mean difference suggest a possible advantage for image-based training.

Subscale analyses revealed a more nuanced pattern. Differences were most pronounced in the Data and Training domains, where image-based students demonstrated a stronger understanding of data variation, training/test splits, and iterative model improvement. After adjustment for multiple comparisons, only the Data subscale remained statistically reliable. In contrast, differences in Model Behavior and Generalization were small, and both groups demonstrated comparatively lower performance in Generalization and Conceptual Understanding.

This pattern suggests that data modality may shape students' reasoning about data practices more than their understanding of abstract model behavior or generalization principles. Image-based intervention, which foregrounds visible spatial features and class distinctions, may make issues of data diversity, feature salience, and misclassification more immediately observable. In contrast, sound-based environments rely on temporal and spectral variation, which may be less perceptually transparent without additional scaffolding. As a result, differences in feature interpretability could influence how readily students reason about the quality of training data and the refinement of models.

These results suggest that the representational format may influence which aspects of the ML pipeline become cognitively salient. Image-based tasks may support reasoning about data composition and feature differentiation, whereas both modalities appear to leave persistent challenges in reasoning about generalization. This aligns with prior findings that students often struggle to move from procedural engagement with ML tools to deeper conceptual understanding of how models abstract patterns from data (Mertala et al., 2024; Toivonen et al., 2022).

This study has several limitations: the sample was modest and uneven across conditions, limiting statistical power; the subscale analyses were exploratory, warranting caution in interpreting domain-specific differences; and the multiple-choice assessment may not fully capture students' ML reasoning processes. Future research should examine modality effects in larger, balanced samples and use qualitative analyses to explore how students articulate feature selection, generalization, and evaluation across representational contexts.

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PROSPECTIVE TEACHERS' MODEL-BUILDING OF STUDENT THINKING ON SAMPLING DISTRIBUTIONS

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Prospective secondary teachers (PSTs) should have opportunities to analyze student work and construct "second-order models" of student thinking—that is, the PSTs' conceptions of students' thinking as informed by their observations of student work and verbalizations. We detail a qualitative analysis of group discussions from a teaching experiment in a curriculum course for PSTs, where PSTs constructed second-order models of student thinking in response to two hypothetical students' statements about the sampling distribution. Results indicate that PSTs can build diverse types of second-order models of students' statistical thinking but need support to identify a range of evidence of students' knowledge as embedded within their thinking.

Keywords: Sampling Distributions, Prospective Teachers, Second-Order Modeling

Research in teacher education shows that analyzing student work can help pre-service teachers (PSTs) refine their views of what counts as developing mathematical knowledge (Crockett, 2002), recognize students' assets (Krebs, 2005), and use students' thinking in instruction (Crespo, 2000). Given their future role in supporting students' learning of statistics concepts, this raises questions and extends these research directions to how PSTs develop skills in interpreting and responding to students' *statistical* thinking.

We investigate how statistics PSTs interpret and respond to students' thinking, components of teaching noticing (Jacobs et al., 2010), about sampling distributions, a statistical topic known to be challenging for learners (Sharma, 2005; Noll & Shaughnessy, 2012; Ajao et al., 2023). Sampling distributions are central to developing students' understanding of statistical inference (Bargagliotti et al., 2020). Consequently, programs serving PSTs have been encouraged to incorporate sampling distribution pedagogy into their coursework (Lovett & Lee, 2017; Franklin et al., 2015). Research on PSTs and sampling distributions has shown that they attend to some of their statistical properties, like sample representativeness, and not others, like sampling variability (Begué et al., 2023). We combine insights from PSTs' learning of sampling distributions with their ability to notice and respond to student work containing ideas about sampling distributions. We frame our inquiry with Steffe & Thompson's (2000) formulation of first- and second-order models and ask: *In what ways do PSTs model students' statistical thinking in a task designed to elicit teacher noticing about student thinking?*

Theoretical and Analytical Frameworks

According to Steffe and Thompson (2000), *first-order models* are interconnected bodies of knowledge students construct as they make sense of their mathematical and statistical activity. As teachers do not have direct access to this knowledge, they must observe students' thinking through their work to build *second-order models* that aim to approximate students' first-order models. We view PSTs' processes of reviewing student work, hypothesizing about student thinking, and planning responses to that thinking as enactments of second-order model building.

As an analytical framework, we used Wilson et al. (2011) framing of processes that PSTs use to develop second-order models while watching students work on statistics tasks. In *describing*, PSTs state what students did or said without conjecturing about their thinking. In *comparing*,

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PSTs identify similarities and differences between students' work or verbalizations and their own work or thinking on the task (Didis et al., 2016). In *inferring*, PSTs generate hypotheses about students' thinking and understanding of the relevant concept. Through *restructuring*, they use these inferences to extend their own subject matter and pedagogical knowledge. Following Didis et al., we also include *questioning* about any aspect of students' work or thinking.

Methods

We conducted a teaching experiment (Brown, 1992) as part of a statistics unit of a PST curriculum course at a large, Western U.S. public university, with the featured lesson focusing on sampling distributions. We introduced the sampling distribution of sample means as the distribution of means from repeated samples of a fixed size drawn from a population, engaging in group and whole-class discussions about its center at the population mean and the reduced variability that results from averaging. They then analyzed a dataset of 777 institutions reporting estimated annual personal expenses per student, denoted *EAP spending values* (James et al., 2021), and examined histograms of the population distribution and an approximate sampling distribution generated by simulating 1,000 sample means from samples of 15 colleges (Figure 1).

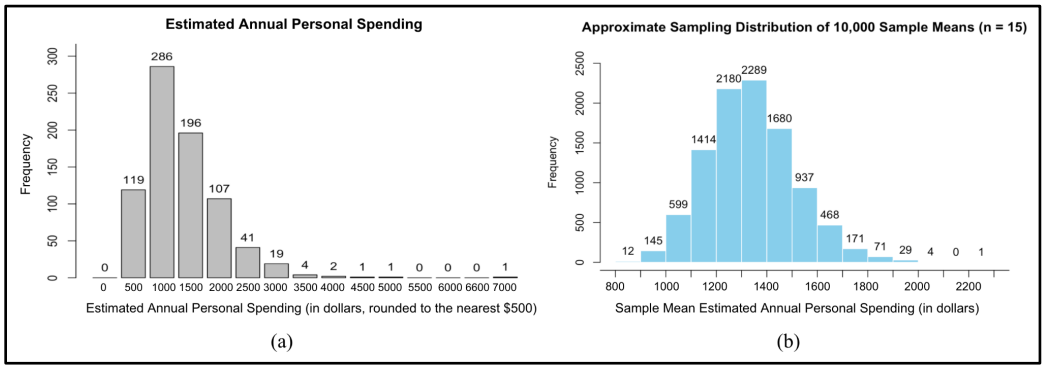


Figure 1: Graphs Given to PSTs

The PSTs were then shown statements from "Lynette" and "Alberto," designed by the first author to reflect common misconceptions about sampling distributions identified by Chance et al. (2004). Alberto's statement read: "*The approximate sampling distribution tells us that (12 + 145 + 599 + 1414)/1000 · 100 ≈ 21.7 percent of colleges in the population have EAPS values greater than $1,200*." Lynette's statement read: "*I don't think these graphs make sense together. I don't understand how there are so many values in between $0 and $1000 in the first histogram, but so few values in between the amounts $0 and $1000 in the second histogram*." We asked PSTs to identify what each student understood or misunderstood and to craft questions they could ask the student to learn more about their thinking. We intended for the PSTs to notice that Alberto calculates a valid percentage from the sampling distribution but references the wrong tail and frames it as a proportion of individual colleges rather than sample means, conflating the population and sampling distributions. For Lynette, we aimed for PSTs to notice that she correctly identifies the difference in proportions within [0, 1000] across the two distributions, yet overlooks that this difference reflects what each distribution measures. PSTs could then build on her observation by acknowledging how averaging reduces the variability of sample means.

Automated transcripts (verified and cleaned by the authors) from four groups of two to four PSTs each served as the primary source of analysis for this study. The data was condensed to

include only on-topic discussion and analyzed at the level of the talk turn. The first and second authors coded two group transcripts (46 of 81 talk turns) using a codebook that defined criteria for the five types of second-order model building described above or *NA* if the PST did not appear to be interpreting the statements. When multiple codes could apply to a single talk turn, we assigned one code according to the hierarchy *describing* < *comparing* < *inferring* < *restructuring*. The first two authors obtained agreement on 85% of the 46 turns. After the first two authors resolved discrepancies through discussion, the first author coded the remaining transcripts. We inductively coded each PST talk turn for its focal statistical idea to identify which aspects of the sampling distribution were emphasized. Examples included "*reading a histogram*" and "*identifying an interval a calculated proportion represents*." Through pattern coding (Miles et al., 2014), we noted connections we observed between the deductive (i.e., the codes describing second-order model-building) and inductive codes (i.e., the codes describing statistical ideas). This allowed us to report the relationship between the content of Alberto's and Lynette's statements, the PSTs' focal statistical ideas, and their second-order model building.

Findings

Figure 2 provides details on PSTs' second-order model-building in talk turns aggregated by groups and statements. For Alberto's statement, PSTs' second-order model-building varied, but most often centered around *describing* and *comparing*. Lynette's statement, in contrast, largely centered on *inferring*. We elaborate more on these patterns with PSTs' responses below.

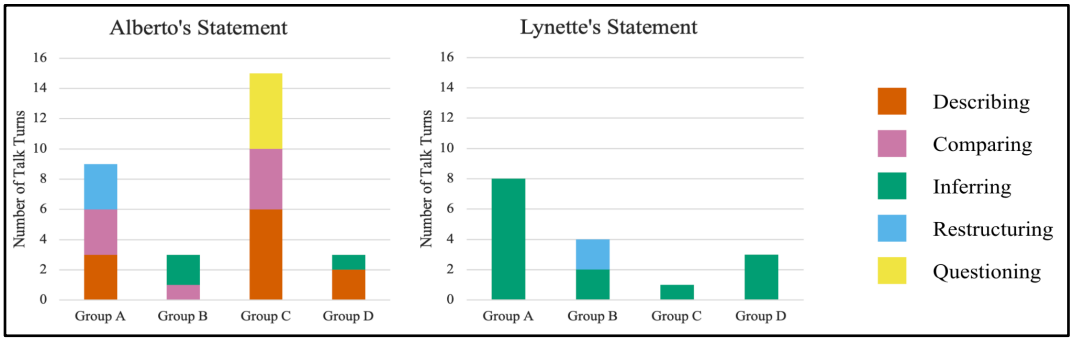


Figure 2: Distribution of Second-Order Model-Building Types Across PSTs' Groups

Responding to Alberto's Statement

The PSTs were attentive to the calculation embedded in Alberto's statement and the tail of the histogram that his proportion, 21.7%, represents (Figure 1(b)). More than half of the talk turns were coded as *describing* and *comparing* in Groups A, C, and D, with many PSTs parsing out and evaluating the steps Alberto completed to obtain 21.7%. For example, Kaylee (Group C) stated, "It's not greater than 1200… It's the ones that fall between 800 and 1200," *comparing* implicitly between Alberto's statement and her own statistical conclusion. Interestingly, none of the PSTs recognized that Alberto did not distinguish that the unit represents a sample mean from a group of colleges rather than a single college's observation.

Some of the talk turns for Alberto's statement did emulate *inferring* and *restructuring*. For instance, Rodney (Group A) engaged in *restructuring* by suggesting the question "Does the right side of 1200 look like 20% of the graph?" and following up with, "There definitely is a nicer way to ask that, but that's what I would ask." Here, Ryder considered how Alberto could benefit from making sense of his calculation and how percentage values might appear on a histogram.

Responding to Lynette's Statement

Unlike their discussion of Alberto's statement, the PSTs explicitly discussed the difference in the type of data the population and sampling distributions illustrate. This knowledge, combined with the ambiguity of Lynette's statement compared to Alberto's, seemed to motivate PSTs toward *inferring* as they constructed second-order models of Lynette's statistical thinking. For instance, Ilyas (Group D) *inferred* from Lynette's thinking and argued that "she understands that this range shows where the center, or majority, of student spending is, but she doesn't get that this isn't what we're measuring. It's a different picture." However, we noticed that while we intentionally designed Lynette's statement to be more ambiguous, we were surprised to see that the PSTs did not engage in *questioning* to inquire about and address these ambiguities.

Additionally, we observed that PSTs primarily focused on how Lynette did not distinguish between the population and sampling distributions, without elaborating on what this difference implies about Lynette's observations—namely, that the averaging process is precisely what produces the reduced variability in the sampling distribution she is observing. Although this point was explicitly discussed prior to the activity, students did not take up this point during their analysis of Lynette's thinking. We hypothesize that PSTs may have still been developing their understanding of sampling distributions while also choosing to examine one facet of Lynette's thinking in depth, rather than extending their analysis to other aspects of her reasoning.

Discussion

Exposure to multiple statements appeared to be productive for facilitating different kinds of model-building activity. Alberto's statement appeared to orient students toward all of Wilson et al.'s (2011) second-order model-building types, whereas Lynette's statement was dominantly followed by PSTs *inferring* about her thinking. However, the PSTs' discussions of Lynette's work highlighted how PSTs may not immediately *question* aspects of work they may need to inquire about. Additionally, we found that the PSTs often did not *restructure* their statistical knowledge for teaching, despite the activity inviting the PSTs to pose questions in response to the statements. These results suggest that PSTs may need further support to inquire about and reflect on the breadth of evidence of student thinking they have access to, so that they may in turn learn how to apply responsive pedagogies characteristic of *restructuring* more consistently.

Moreover, analyzing student statements with distinct conceptions of sampling distributions appeared to cue different statistical ideas in PSTs' discussions. However, we noticed that although PSTs sometimes correctly identify students' struggles with interpreting sampling distributions, they often focused on only one component of those misconceptions. For example, most groups *described* or *compared* Alberto's computational mistake but overlooked his incorrect meaning of a sampling distribution. These observations highlight how PSTs may need support to consider multiple ways that students can (mis)interpret statistical information, visualizations, etc., as statistics is a discipline that relies heavily on interpretation.

These findings should be interpreted considering the study context. First, PSTs in this course regularly practiced interpreting and responding to student thinking (Jacobs et al., 2010), which likely impacted PSTs' discussions. The lesson focused on sampling distributions, a topic PSTs were likely less familiar with, given their limited statistics coursework. Because subject matter knowledge shapes teachers' analyses of student work (Heritage, 2022), results may have differed with more familiar topics, such as algebraic reasoning. This raises an interesting concern, as the PSTs, even with multiple opportunities to build second- order models of the course, experienced challenges in accounting for the multi-dimensional nature of students' statistical thinking.

Consequently, teacher educators may need to draw more attention to the distinctions between teaching mathematics and statistics (Groth, 2007).

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UNDERGRADUATES' EVOLVING ATTITUDES TOWARD MATHEMATICS IN A DATA SCIENCE-FOCUSED QUANTITATIVE LITERACY COURSE

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In this study, we investigated how undergraduate students' attitudes and perceptions towards mathematics changed over a semester in a data science-focused Quantitative Literacy course. We combined quantitative and qualitative approaches to understand changes in students' attitudes from the beginning to the end of the course. Data sources for the study consisted of undergraduate students' ($n = 39$) responses to the Mathematics Attitudes and Perceptions Survey (MAPS), administered at the start and end of the semester, and pre-/post-interviews with selected students ($n = 14$) to explain the observed patterns. We conducted statistical analyses to examine changes in attitudes, together with a case study analysis to examine how changes in attitudes occurred. The results indicated a statistically significant increase in students' overall attitudes and perceptions toward mathematics.

Keywords: Mathematics Attitudes, Student Perceptions, Quantitative Literacy, Data Science Education

Introduction

Quantitative literacy (QL) refers to developing habits, dispositions, and skills needed to interpret quantitative data in ways that support one's daily responsibilities (e.g., Gillman, 2006; Steen, 2004). Clearly, QL serves an important purpose in today's society, where interpreting quantitative data affects one's voting choices, personal finance, and health management, among other daily decisions. Because of its importance, the broader community has long recognized the significance of QL for postsecondary education. About 30 years ago, organizations such as the Mathematical Association of America (MAA) and the National Numeracy Network (NNN) created focused efforts to promote quantitative literacy in college students (e.g., Sons, 2006). These efforts eventually led to the creation of quantitative literacy requirements at most colleges and universities, where all students were required to fulfill a certain number of quantitative credits to complete their degree.

Since the united effort to require QL courses for college completion was initiated nearly three decades ago, colleges and universities have used various QL curricula. One of the more mainstream approaches was to require traditional mathematics courses, such as college algebra and introductory statistics, to fulfill the QL requirement. Yet these approaches have been met with criticism; for instance, Agustin et al. (2012) provided empirical evidence suggesting that traditional mathematics courses alone do not foster meaningful gains in quantitative reasoning because students failed to demonstrate adequate quantitative reasoning skills. Consistent with this concern, Tunstall (2018) argued that college algebra, even in modified formats, is not inherently designed to develop quantitative literacy. These courses tend to heavily emphasize analytical techniques that are at odds with the very purpose of QL in this modernized era—to

support learners to develop quantitative skills, including computational skills, needed for practical decision making. Further, courses like college algebra and introductory statistics often require intense remediation for learners who do not have the abstract prerequisite knowledge for success. Such remediation has been shown to increase costs for many students and delay graduation (Jiminez et al., 2016; Logue et al., 2019).

More recently, some postsecondary institutions have modernized the QL curriculum to focus on real-world phenomena. These courses have included data science, mathematical modeling, problem-solving, and other curricula that centralize application (e.g., Anderson, 2006; Hartman et al., 2025; Small, 2006; Tunstall et al., 2016). While such courses are clearly consistent with the overall aim to support students in developing quantitative skills for the real world, the modern QL curriculum has been critiqued for its potential to become a remedial enterprise (Steen, 2004). In other words, there is concern that modernized QL courses could become non-rigorous and unsuitable for collegiate education.

Furthermore, notwithstanding the importance of QL to postsecondary education, there has been relatively little empirical research examining students' QL course experiences. Student experiences are vital for understanding the curricula that support learners' skills, attitudes, and dispositions toward quantitative information. Despite growing efforts to modernize quantitative literacy curricula, empirical research has not adequately examined the attitudinal consequences of data science–focused QL courses. Existing studies either emphasize traditional mathematics pathways or rely primarily on quantitative measures without investigating the underlying processes of attitudinal change. Students' attitudes towards quantitative literacy, particularly mathematical attitudes, play an important role in both their mathematics performance and their persistence in STEM (Cohen & Kelly, 2020). Therefore, studying how a modernized QL course shapes students' mathematics related attitudes can offer important insights into its impact on the STEM pipeline—a concern raised by the community.

In light of this, in this study, we share results from students' experiences in *one* data science QL course that focused on developing data skills relevant to students' personal, civic, and professional lives while simultaneously having no pre-requisite requirements. We specifically focus on learners' attitudinal shifts toward mathematics within the course because attitudes are a dispositional dimension of QL (Steen, 2001) and hence a critical component of QL development. Our purpose is not to argue for the supremacy of modern approaches to QL—rather, we aim to show that modern approaches to QL *can* support students' QL development. The following research question guides the overall goal of the research: *How do undergraduates' attitudes and perceptions about mathematics change over one semester in a data science-focused QL course?*

Background

Data Science as a Mathematical Practice

Data science is an interdisciplinary field that connects mathematics, statistics, computer science, and realistic contexts (Conway, 2010; Hazzan & Mike, 2023). Engaging with data science implies conducting analysis on data that emerged as a consequence of a real-world occurrence, to describe, interpret, and predict the behavior of that phenomenon. In doing so, data scientists typically formulate investigative questions about the real-world domain, collect, consider, and retrieve data, perform computational analysis on the data, and make inferences connecting the results to the real-world domain (Bargagliotti et al., 2020; Baumer, 2015). These investigative processes reflect the mathematical and statistical practice standards emphasized in policy documents and by professional organizations (e.g., Bargagliotti et al., 2020; Common

Core State Standards Initiative, 2010), positioning data science as a modern mathematical practice.

Mathematics Attitudes and Perceptions

A growing body of research identifies students' attitudes toward mathematics as an important determinant of academic performance and broader educational outcomes (e.g., Gonzales et al., 2008; Gjicali & Lipnevich, 2021; Mazana et al., 2019; Wakhata et al., 2024). In general, attitudes (understood broadly) have a reciprocal relationship with students' achievement—attitudes support achievement and achievement supports attitudes (e.g., Bayaga & Wadesango, 2014; Ma, 1997). Given the significant role of mathematics attitudes in learning and achievement, researchers have extensively examined the factors that influence their development. For example, students' attitudes toward mathematics have been linked to factors such as perceived value of mathematics (Peixoto et al., 2016), confidence (Barkatsas et al., 2009), self-concept (Stankov et al., 2012), enjoyment (Ahmed et al., 2013), and anxiety (Ashcraft & Faust, 1994), all of which shape how students engage with and respond to mathematical learning experiences. Consequently, across the literature, many researchers have defined attitudes towards mathematics in different dimensions, incorporating affective, cognitive, and behavioral components (Di Martino & Zan, 2011; Haladanya et al., 1983; Hemmings et al., 2010; Koyuncu & Dönmez, 2018; Moyer et al., 2018; Stampoltzis & Giannoulas, 2021; Wen & Dubé, 2022).

In this study, we adopt a multi-dimensional view of mathematical attitudes and beliefs as proposed by Code et al. (2016). In Code et al.'s (2016) framework, mathematical attitudes and perceptions consist of *Confidence* (perceived ability to successfully engage in mathematical task), *Persistence* (Keep going when its challenging), *Growth Mindset* (belief about whether mathematical ability is innate or can be developed), *Interest* (active willingness to engage in mathematical situation), *Real World* (transference of domain-specific knowledge to novel situations), *Sense Making* (students' perspectives on the nature of their personal mathematical knowledge), and *Answers*, hereafter Nature of Mathematics, (views on the ontological nature of mathematics and nature of solutions to mathematics problems), each of which reflects the extent to which students hold *expert-like views* about learning and doing mathematics. For this study, to measure students' mathematical attitudes, we adopted Code et al.'s (2016) Mathematics Attitudes and Perceptions Survey (MAPS) because the instrument was designed and validated with undergraduate students who resembled our target population, and the MAPS framework explicitly treats attitudes as a multidimensional attribute, aligning with our theoretical stance.

Methods

We employed a multiple-method research design (Creswell, 2003) to examine the changes in students' attitudes and perceptions towards mathematics over the course of one semester.

Study Setting and Participants

Our study's context is situated within a data science-focused QL course (STAT 1080) at a midsize research university in the Western United States. The STAT 1080 course was inspired by the success of the University of California Berkeley Data Science curriculum (Adhikari et al., 2022) and teaches select topics from Irizary (2019). The course provided students with a contemporary introduction to foundational statistical ideas through computational practice rather than manual calculation. The curriculum emphasized relevance, real-world inquiry, and the use of R programming to analyze, visualize, and interpret data. Major concepts included data exploration and visualization, correlation vs causation, randomness and simulation, and hypothesis testing and prediction. A total of 77 students were enrolled and completed the course, which met twice per week. The fourth author served as the instructor. A typical class session

began with an introduction to a real-world example (e.g., election polls), after which students learned how to use R programming to explore similar types of data and worked collaboratively to explore data and interpret their results.

The participants were students enrolled in STAT 1080 during Fall 2025. A total of 63 students completed the pre-survey at the beginning of the semester, and 39 completed the pre- and post-survey, resulting in a retention rate of 62%. For the qualitative data collection, 14 students participated in interviews from the pool of 39 students at the beginning and end of the semester.

Data Collection and Analysis

Quantitative data were collected using MAPS, developed and validated by Code et al. (2016). In MAPS, students respond to 32 questions using a 5-point Likert format: 'Strongly Disagree' (1), 'Disagree' (2), 'Neutral' (3), 'Agree' (4), and 'Strongly Agree' (5). The survey was administered online twice during the semester through the web-based Qualtrics system, once at the start of the course and other at the end, to evaluate changes in undergraduate students' attitudes and perceptions. Qualitative data were collected through semi-structured interviews conducted via Zoom. An interview protocol was developed to guide the interview process. Each interview lasted for 30–35 minutes and was recorded with the participants' consent. All recordings were transcribed through the Zoom transcription service.

Data analysis was conducted in two phases. We first conducted a quantitative analysis of the survey data using descriptive and inferential statistics. Descriptive statistics provided the magnitude and direction of attitudinal change across participants. The paired sample t-test was used to determine whether the change in students' pre- and post-survey scores was statistically significant. These quantitative results guided the selection of a potential case for deeper qualitative analysis by highlighting a student who demonstrated meaningful shifts in attitudes across multiple MAPS dimensions. For qualitative analysis, for this preliminary stage, we analyzed the pre- and post-interviews of three participants using the deductive coding approach, applying the seven MAPS categories as *a priori* codes. Three researchers collaboratively coded all interviews through iterative cycles in which each independently coded the transcripts and then met to discuss and resolve any disagreements. Next, we selected a case (Sophia) that enabled detailed descriptions of changes in attitudes, allowing for a nuanced examination of how attitudes evolved across all seven MAPS dimensions. We selected Sophia's case because she exhibited alignment between quantitative change and qualitative evidence, enabling us to illustrate not only *that* attitudes changed but also *how* and *why* they shifted.

Findings

We first provide a summary of the quantitative results showing how our participants' attitudes changed across the seven MAPS dimensions. Building on these patterns, next we present Sophia's in-depth qualitative changes in attitudes across the same seven categories.

Quantifying Changes in Students' Mathematical Attitudes Using MAPS

As suggested by Code et al. (2016), the overall attitudes and perceptions was computed by averaging the scores for all items, excluding Items 19, 22, and 31, while the subscales were calculated based on the set of question items that belonged to each category. The instrument was assessed for internal consistency for the seven subscales and overall instrument using Cronbach's alpha (α). For the subscales, Cronbach's α values for the pre- and post-surveys ranged from 0.35 to 0.81, indicating acceptable internal consistency due to the small number of items in most categories (Code et al., 2016). For the overall instrument, Cronbach's α was 0.89 for the pre-

survey and 0.87 for the post-survey, consistent with the reliability of Code et al.'s (2016) validation study.

As shown in Table 1, Pre-survey means ranged from 2.54 (Interest) to 3.69 (Growth Mindset), indicating a neutral to slightly positive disposition towards mathematics. Post-survey scores showed shifts, with notable positive gains across all subscales, as well as the overall score, and a slight decline in the Sense Making subscale (see Figure 1). These descriptive trends were followed by an inferential analysis to examine whether the observed differences from pre- to post-survey were statistically significant.

Table 1: Descriptive and Inferential (t-test) Statistics for MAPS ($N = 39$)

	Pre-Survey			Post-Survey				
MAPS Categories	Mean	SD	Cronbach α	Mean	SD	Cronbach α	t	p-value
Growth Mindset	3.69	0.69	0.81	3.82	0.53	0.48	–1.15	0.255
Real World	3.48	0.63	0.56	3.81	0.51	0.35	–3.23	0.003*
Confidence	2.66	0.84	0.80	2.97	0.61	0.62	–3.34	0.002*
Interest	2.54	0.94	0.80	2.68	0.90	0.77	–1.14	0.260
Persistence	2.55	0.84	0.78	2.93	0.73	0.71	–3.34	0.002*
Sense Making	3.17	0.61	0.69	3.04	0.51	0.61	1.15	0.255
Nature of Mathematics	3.02	0.60	0.69	3.21	0.56	0.60	–2.07	0.045*
Overall Attitudes and Perceptions	2.98	0.47	0.89	3.17	0.42	0.87	–2.84	0.007*

Note. *p-value < 0.05

A paired sample t-test was used to examine pre-post differences in students' mathematical attitudes and perceptions. Results indicated significant positive differences in the MAPS categories of Real World, Confidence, Persistence, Nature of Mathematics, and Overall Attitudes and Perceptions ($p < 0.05$). No significant differences were found for Growth Mindset, Interest, and Sense Making as shown in Table 1. Thus, the results showed positive changes in students' attitudes and perceptions toward mathematics from the beginning to the end of the course.

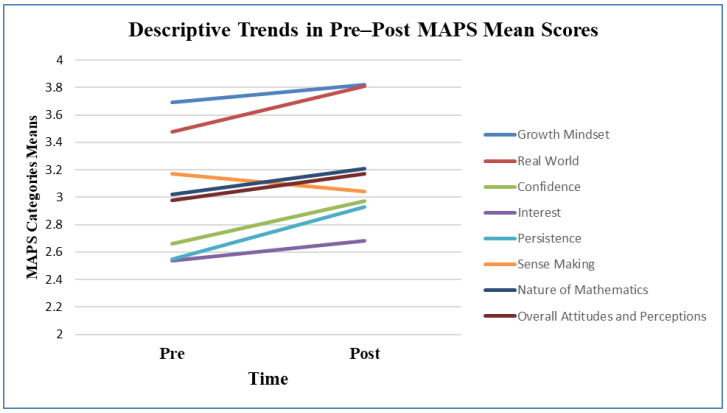


Figure 1: Pre and Post MAPS Categories Mean Scores

Qualitative Changes in Sophia's Mathematical Attitudes and Perceptions

Confidence. Sophia entered the course with low confidence in mathematics, shaped by years of feeling unsuccessful. She reflected that "Growing up, I never wanted to do math, I was never good at math, I've always found it hard to start a problem, or fully grasp the concept of it," a statement that captures both her early avoidance and the deficit-based views she internalized. Despite dedicating "numerous hours" to studying, she still felt that her work "didn't really get to the point that [she] wanted it to," interpreting her struggles as evidence of limited innate ability. Mathematics, for her, "takes a lot of practice, trial and error," and it "is not something that comes easy to her," signifying deeply rooted doubt about her mathematical capacity. After completing the course, however, Sophia's framing of her competence began to shift. Although she continued to acknowledge challenges, she described a growing belief in her ability to make progress, stating, "I feel more confident; I still feel like I'm not the best in it, but I do have that confidence of working through it still." This reflects an important development toward increased confidence and trust in her ability to learn.

Growth mindset. Before the course, she expressed a fixed mindset rooted in comparison to peers who she believed grasped concepts easily. She explained, "I'm not one of those people that look at a problem and know exactly what to do. I always need reteaching," viewing her need for additional time as a lack of mathematical ability. After the course, she articulated a more growth-oriented stance. Although she acknowledged that her challenges had not disappeared, she described a deeper motivation, noting, "I think it's a little bit the same, but I think I'm more motivated to, no matter what math I'm in, to continue to work hard, and not be like, 'oh, I just suck at it'… continue to have the mindset that it is gonna help me, it is gonna be useful, it'll just help me in the future." She now sees mathematics as something worth investing in because of its long-term utility, signaling a meaningful transformation in her mindset.

Persistence. Sophia's persistence has long been apparent. Even before the course, she described pushing herself academically despite negative experiences with mathematics, saying, "I always took the honors classes, because I wanted to push myself, and for that GPA boost." Her persistence, however, was often driven by external motivators such as grades rather than by confidence or internal factors. After the course, her persistence became more intrinsic. She emphasized her motivation to continue working hard regardless of perceived difficulty, because "it is gonna be useful." Her persistence began to stem from a sense of relevance, indicating a shift toward more resilience.

Real-world relevance. In the pre-interview, Sophia made no direct references to mathematics being relevant to real life, and her comments provided little evidence that she viewed math as meaningful beyond the classroom. This changed after the course. She repeatedly emphasized how statistics and mathematics connected to her daily life, her major, and her future career, stating, "Never once have I felt like 'why am I even taking this? I can apply it to my daily life." She identified the concepts learned in class as both familiar and academically important and described course projects as making mathematics personally meaningful. This realization represented a major shift from seeing no real-world relevance to understanding mathematics as applicable, useful, and strongly connected to her future.

Interest. Sophia's interest in mathematics also evolved. Growing up, she "never wanted to do math" and often "hated" it when she did not understand a concept. Although she experienced a slight increase in interest over the year prior to taking the course, her interest remained fragile and heavily tied to her perceptions of success. After the course, however, she expressed more consistent and positive interest, stating, "I've appreciated math more through this course." She

attributed this shift to the instructor, course content, and especially the real-world applications embedded in the curriculum. Her interest moved from conditional and negative toward s appreciation.

Sense making. Before the course, Sophia did not describe how mathematics made sense to her personally; instead, she focused on procedural fluency and the ability to produce solutions quickly as indicators of mathematical competence. After the course, she described mathematics as more meaningful, often emphasizing the role of visuals and context. She explained that although the course incorporated familiar ideas, it used them "in a way that kind of made more sense to [her]; it uses it in a way that has meaning." Visual representations helped her shift away from seeing math as "just numbers," and she contrasted her prior understanding as "surface level" with more technical comprehension. She described analyzing problems, breaking them down, and recognizing multiple solution pathways, as well as improved fluency in interpreting graphs as all part of understanding mathematics.

Nature of mathematics. Finally, Sophia's conception of the nature of mathematics broadened. Before the course, she described mathematics primarily as numbers and patterns, noting that "math is everywhere" and can help people "understand things in a different way." While true, this perspective was relatively limited. After the course, she described mathematics as more visual, interpretive, multifaceted, and connected to computational analysis. She emphasized that the course presented a kind of mathematics she had never encountered before: "It's not like any math that I've taken … it looks at math at a different view." She recognized the discipline as flexible, offering multiple approaches and strategies, and generalized that the course could help others "realize the importance of math."

Taken together, Sophia's journey reflects a strong qualitative shift in mathematical attitudes and perceptions characterized by struggle, self-doubt, and negative emotions to a position defined by motivation, relevance, appreciation, and broadened understanding. Sophia developed confidence, sense-making, and interest that was supported by the course instructor, real-world applications of course content, visual and computational tools, and an expanded view of the nature of mathematics. These illustrate a shift towards positive mathematical attitudes and perceptions. Although she still experiences mathematics as challenging, she increasingly views those challenges as workable, worthwhile, and meaningfully connected to her life.

Discussion and Conclusions

Our study has three major contributions. *First*, our findings confirm that a data science-focused QL course, which differs from conventional mathematics coursework, has the capacity to shift undergraduates' mathematical attitudes and perceptions. Although undergraduates' overall mathematical attitudes and perceptions increased significantly, one striking finding was the decline observed in the *Sense-Making* subscale, which captures "students' perspectives on the nature of their personal mathematical knowledge." However, this decrease was not statistically significant. A plausible explanation for the decrease in sensemaking is that students may not have readily associated the core concepts they learned in STAT 1080 with what they traditionally recognize as "mathematics." As a result, they may have underreported their own sense-making in relation to mathematics, even though they engaged in data-centered reasoning, a core mathematical practice. That is, this behavior could be attributed to students assessing their abilities more conservatively than their actual level of understanding (León et al., 2021). However, despite this decline, the *Nature of Mathematics* subscale showed a statistically significant improvement, suggesting that while students' self-perceptions about sense-making shifted negatively, their conceptualization of what mathematics *is* expanded in productive ways.

The case of Sophia illustrates the ways in which a student's mathematical attitudes can change as a result of engaging in a data science-focused QL course. Notably, Sophia's mathematical confidence grew from initial self-doubt to a stronger sense of self-efficacy, a critical factor for influencing STEM persistence (Lin et al., 2018). Literature notes persistence as also influential for performance in mathematics and overall persistence in STEM (Cohen & Kelly, 2020). For Sophia, although she described herself as an individual with persistence before the course, it was driven by external factors such as grades. After engaging in the course, Sophia developed a sense of persistence shaped by intrinsic factors, such as genuine engagement with the course activities. Although externally driven persistence can motivate students to perform in the short term, it is generally not sustainable. Long-term persistence requires intrinsic motivation (Deci & Ryan, 2012). For Sophia, this course enabled the development of persistence shaped by intrinsic factors. Sophia's reflections on real-world relevance further reinforced this point. Sophia noted that "never once have I felt like 'why am I even taking this?'" highlighting how the authenticity of the course context supported her engagement.

Second, our study promotes a potential novel pathway to STEM persistence for undergraduate learners. Research has shown that positive mathematical attitudes can shape an individual's mathematical well-being (an individual's desire) and STEM persistence (a societal desire). While there exist various research documenting the factors that support learners' mathematical attitudes (e.g., Barkatsas et al., 2009; Stankov et al., 2012; Ahmed et al., 2013), those studies have not focused on quantitative literacy requirements at college, which is a promising avenue for recruiting and retaining students in STEM. Through this study, we have demonstrated that a data-science focused course, that emphasizes computational skills, data literacy, and quantitative competencies, all rooted in real-world applications, can shape students' mathematical attitudes positively. Together with prior research showing the connection between attitudes and STEM persistence, our study implies a potential connection between modernized QL and STEM persistence.

Third, our study contributes to educational research in undergraduate data science education. Although the movement to revolutionize undergraduate mathematics curriculum through the integration of data science was initiated in 2015 (Cassel & Topi, 2016), the educational research to improve undergraduate students' experiences in data science is scarce. That is, within the body of data science education scholarship at the undergraduate level, "there is lack of empirical data and identifiable research questions" and therefore empirical studies in data science education must be prioritized (Dogucu et al., 2025, p. 467). By documenting shifts in undergraduates' mathematical attitudes within a data science learning environment, our study offers an initial response to this pressing research need.

In conclusion, our study shows that a data science-focused QL course, which focuses on engagement with real-world phenomena and computational skills, can support undergraduates' development in mathematics, at least along the attitudinal dimension. It is imperative that the field continues to conduct concurrent research alongside novel curriculum efforts in QL to discern how to best support college learners to develop QL competencies. QL is becoming increasingly more important in today's data-driven society, and the present study contributes to the need for empirically grounded QL curriculum.

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HIGH SCHOOL STUDENTS' CONCEPTUALIZATIONS OF MACHINE LEARNING ACROSS IMAGE- AND SOUND-BASED MODALITIES

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This study investigates how image-based versus sound-based data modalities shape high school students' core understanding of machine learning (ML). Using a teaching intervention, students participated in either an image- or a sound-based ML intervention. Data sources included post-survey assessments (n = 51) and follow-up interviews (n = 2). We report findings from a thematic qualitative analysis of an open-response survey item and interviews. Findings revealed modality-based differences in conceptualization: students in the image-based condition more frequently referenced variation, accuracy, and classification, while those in the sound-based condition emphasized variation and the learning process. The image-based focal student recognized classification limits due to overlapping features, whereas the sound-based student viewed ML as a human-driven "feeding" process and believed that enough data would result in perfect accuracy.

Keywords: Machine Learning, Image-Based, Sound-Based, High School

Machine learning (ML) is positioned as an important component of the K–12 curriculum (Touretzky et al., 2019) and is closely linked to pattern recognition and problem-solving practices (Lee et al., 2024; Susanti et al., 2019; Wu & Yang, 2022). Defined as computational techniques that build and refine models to generate data-driven predictions or classifications from prior examples (Alpaydin, 2020), ML holds promise for cultivating students' interest and preparing them for data-intensive futures (Oh et al., 2024). Yet in practice, ML integration in mathematics classrooms remains limited and often superficial. Teachers frequently lack accessible tools and pedagogical support that make core processes conceptually visible to students (Sanusi et al., 2023a; Yue et al., 2022). One overlooked potential factor is the data modality used in teaching ML. Because images and sounds represent information in fundamentally different ways and require distinct forms of interpretation (Adhegaonkar et al., 2026; Kwak et al., 2023), the modality itself may shape how students conceptualize model learning and accuracy. However, little empirical research has examined how different classroom data modalities influence students' developing understandings of ML processes. This study examined students' definition of ML, the role of data in model performance, and expectations of machine effectiveness.

Literature Review

Research on K-12 ML education has grown considerably, with studies showing that students can engage with ML concepts through structured and hands-on experiences even without prior knowledge or strong STEM interest (Sanusi et al., 2023b). Participation in activities such as data labeling, model training, and evaluation supports students' understanding of foundational ML

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

principles (Hitron et al., 2019). In image-based contexts, students recognized how dataset size and balance affect model accuracy and demonstrated early awareness of responsible AI practices (Živković et al., 2025), while also developing understanding of image representation and feature maps (Rajarajan & Martin, 2026). Sound-based experiences similarly produced gains in learning effectiveness, creativity, and engagement when students trained audio classifiers to recognize speech (Hsu & Chen, 2022). Beyond hands-on skills, research suggests that ML learning experiences shape how students conceptualize intelligent systems, with youths recognizing that ML learns from training data and identifies patterns, though their explanations of how learning occurs often remained underdeveloped (Morales-Navarro & Kafai, 2024). As students begin to understand how machines learn, they become more motivated to build mental models for interpreting technology in everyday life (Lin et al., 2020). However, attitudinal and logistical barriers, including disengagement when tasks felt demanding and limited instructional resources, can hinder participation (Karalekas et al., 2023; Alvarez et al., 2022). Collectively, students' understanding tends to remain focused on observable system behavior rather than on the underlying learning processes, and limited empirical work has examined how the type of data used in ML environments shapes students' conceptualizations of how models learn and make classifications.

Method

This study employed a teaching experiment design (Steffe & Thompson, 2000) in which students participated in either an image-based or sound-based instructional condition using Google's Teachable Machine (Google, n.d.), with both conditions sharing identical learning objectives and instructional sequences. Approximately 95% of participants identified as African American, recruited from two high school classrooms open to Grades 9–12. Of 59 survey completions, eight were excluded due to missing or uniform response patterns, yielding 34 image-based and 17 sound-based participants. Eight students were invited for semi-structured post-intervention interviews; two consented, resulting in STUDENT_A (9th grade, image-based) and STUDENT_B (11th grade, sound-based) completing 26 questions covering ML concepts, model learning, accuracy, and ML understanding. Interviews were video recorded, transcribed, and analyzed alongside survey data using thematic analysis (Ahmed et al., 2025), with codes capturing students' ideas about ML processes, data use, and model behavior, which were then grouped into broader themes.



(a) (b) (c)

Note. (a) Instructor teaching the students how to use the model, (b) Students learning through image recognition. One student uses white paper to improve the background (c) Learning through sound-based classification. Student shaking the cars as the model records the sound.

Figure 1: Student Data Collection Procedures for Image- and Sound-Based Machine Learning Model Training

Results

Student Responses on "Why Should We Give the Model Many Pictures/Recordings Instead of Just One?"

The students exposed to the image-based intervention most frequently referenced variation/multiple perspectives (n = 7; 20.6%), followed by accuracy improvement and classification/decision-making (n = 6, 17.6%) (Figure 2). In contrast, most of the students in sound-based intervention either had unclear or no response (n = 5, 29.4%), with the learning process and variation (n = 3, 17.6% each) the next referenced. Overall, image-based intervention elicited more responses related to variation, accuracy, and classification, whereas sound-based intervention elicited more responses reflecting the learning process and variation, while providing limited elaboration.

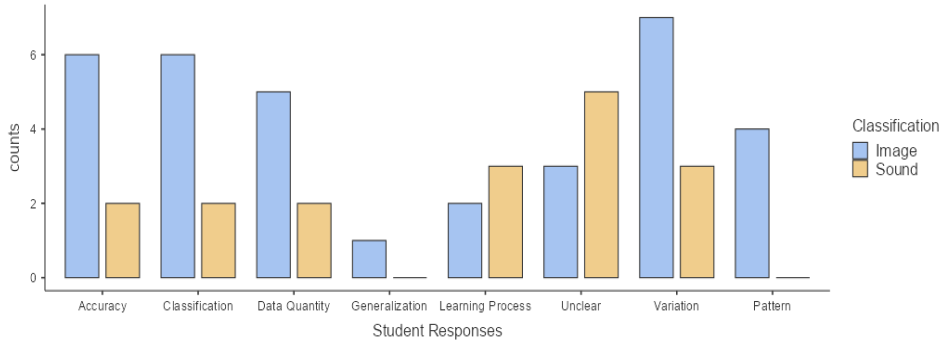


Figure 2: Distribution of Coded Student Responses by Teaching Intervention

Comparative Case Analysis

To further examine how students conceptualized ML across modalities, we analyzed interviews from two focal students: STUDENT_A (image-based) and STUDENT_B (sound-based)

Students' Definition of Machine Learning

STUDENT_A, who was trained on image-based classification, had no prior experience with ML before the intervention. He stated, "*I've never learned about teachable machines before.*" STUDENT_A defined ML as a system for interpreting and "classifying data into various categories. In explaining what ML is, STUDENT_A stated "*ML is basically, uh, how machines, uh, interpret the data we have.*" He continued talking about features of the data that "*a category could be their appearance, their color, based on that, and then we can just, like, sort them, and then we can classify them.*"

On the other hand, STUDENT_B has little experience with ML, having played with a tool while growing up. She said, "*I had a little robot when I was little that we used, and I could teach it how to do things.*" In describing what ML is, the student emphasized repeated "feeding" of the model until it starts to learn by saying, "*you feed it information and it basically gives you information back, and you basically keep feeding it and feeding it until it starts learning.*" The description of the student is that ML is a data-driven, human-fed process that gives an output after constant training.

Role of Data in Model Performance

STUDENT_A viewed model performance as dependent on both the quantity and variety of training data. He emphasized adding more samples and using different types of examples to

improve recognition, validity, reliability, and accuracy. Similarly, STUDENT_B believed that model performance improves when more information is provided, stating that a model trained on 100 samples would outperform one trained on 10. She also emphasized the importance of human involvement in training the model and suggested that factors such as sound waves and speed could influence accuracy in sound-based classification. Both students associated improved model performance with increased training data, although STUDENT_B placed greater emphasis on the role of human training.

Expectations of Machine Effectiveness

STUDENT_A described a machine that can generalize effectively as a model that can easily recognize objects. He stated, "*I would describe it as an effective model which can easily recognize objects… so that, uh, the results can be accurate.*" However, the student thinks that models cannot always be relied on to classify, as some objects can be very similar to each other. He stated, "*I don't think that the model can, like, 100% be accurate because, uh, there can be, like, some objects which are, like, similar.*" In contrast, STUDENT_B believes a model can be perfect if trained over and over. She said, "*I feel like if you keep giving it data of the same thing over and over, it will be*." The student described a model that can generalize effectively as "*you put in the information, and it automatically gives you an answer back.*"

Discussion

Both students described ML as data-driven and referred to variation in training data, which may indicate some awareness that models can benefit from exposure to diverse examples during training (Hitron et al., 2019; Morales-Navarro & Kafai, 2024). Their explanations also differed. STUDENT_A focused more on classification, visual features, and model accuracy, whereas STUDENT_B emphasized the learning process. These findings suggest that data modality influences how students reason about ML and model performance (Morales-Navarro & Kafai, 2024; Wan et al., 2020). The findings also showed that STUDENT_A was able to engage effectively with ML tools despite having no prior exposure, consistent with research suggesting that students can understand ML concepts without prior knowledge or strong STEM interest (Sanusi et al., 2023b). Continued engagement with ML may further support understanding of classification, feature selection, and iterative training (Živković et al., 2025; Hsu & Chen, 2022). Therefore, introducing ML concepts early in K-12 curricula may help strengthen students' understanding and participation in ML. The students expressed contrasting views about model effectiveness. STUDENT_A believed that perfect accuracy is unattainable because similar objects can be difficult to distinguish, reflecting an awareness of the limitations of ML models (Živković et al., 2025; Rajarajan & Martin, 2026). However, STUDENT_B believed that a model could become perfect with enough data, emphasizing the importance of data quantity in improving performance. Providing opportunities for students to compare inputs and observe prediction outcomes may help deepen their understanding of how ML models make decisions. A limitation of this study is that only two students were interviewed, and participants were not randomly selected, which may limit the transferability of the findings.

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DESIGNING FOR LEARNER AGENCY IN AN AI-INTEGRATED DATA SCIENCE COURSE

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This study examines how an undergraduate mathematics instructor redesigned her data science course to cultivate learner agency with AI tools. Using our conceptual framing of AI-mediated cognition, we analyzed a course syllabus, instructor reflection memos, and a post-course interview transcript to describe how instructional decisions encouraged students to position AI as a cognitive capacity builder and critical companion. Findings revealed three categories of instructor decision making intended to promote learner agency: making students' thinking visible through AI interactions, revising assessment to promote learner agency, and framing AI-mediated integration within the structure of course assignments. Implications for preserving authority, accountability, and sensemaking in AI-integrated data science courses are offered.

Keywords: Data Science, Artificial Intelligence, Agency

Data science offers a distinctive space for AI engagement in undergraduate STEM education because the requisite statistical thinking is grounded in variability, uncertainty, modeling, and contextual interpretations (Lee & Tran, 2015). Unlike procedural computing tasks or mathematical problems where AI tools execute predetermined steps to produce a known answer, data science projects can grant students opportunities to engage in statistical sensemaking as they use AI tools to assess multiple plausible models and iterate data science experiments on a greater scale. The interpretive character of data science creates a constructive context for examining how undergraduate course instructors can cultivate learner agency when students use AI. Learner agency has been defined as an individual's will and capacity to act (Gao, 2010) in ways that reflect autonomous, self-regulated engagement with learning (Bown, 2009). The level of learner agency that a student exhibits when using AI in data science varies as AI can reshape how models are generated and how decision making is distributed.

Opportunities to learn data science deeply do not develop automatically simply because a student has access to AI tools. Students who show low learner agency might use AI as a tool for creating a model that does not include in-depth reasoning about variables and constraints within statistical models. They might also use AI tools as a shortcut to analyze the data rather than using it as a resource to reason about the data. Students who show high learner agency might substantially engage in decision making about contextual relevance, constraints, coherence with the data, data quality and their analysis aim.

An instructor's decision making about the use of AI in a data science course (e.g., how students engage with AI, what counts as evidence for students' thinking and learning, and how students use and assess AI-generated outputs) can position students with high learner agency

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who critically evaluate and reflect on AI use, rather than depend on AI for task completion. This study explores the decisions one instructor made to foster students' agency in engaging with AI within a "Statistical Programming in Data Science" course. By describing varying levels of learner agency within a rich data science course context, the mathematics education community can gain insight into how students might delegate reasoning and sensemaking responsibilities to AI (Mouta et al., 2025, Wu et al., 2025). Exploring such delegations can generate insights into how student authority and accountability might be strengthened or diminished with the use of AI.

Methods

To explore instructional decision making and the cultivation of high learner agency within AI-integrated data science learning, we capitalize on our conceptual framing of AI-mediated cognition (Rhodes et al., under review) as our analytical lens. Within this framing (adapted from Cukurova's (2025) work on the interplay of human intelligence and artificial information processing), students can interact with AI in ways that foreground or background different forms of cognition, depending on the activity and their learning goals. The student might use an AI tool as a c*ognitive load reducer, cognitive capacity builder*, or *critical companion*. With *cognitive load reduction*, the student might rely on AI tools in ways that undermine conceptual understanding and limit the potential for an instructor's learning goals to be met. With *cognitive capacity building*, the student prompts AI tools in ways that increase opportunities for metacognition, creative thinking, or self-regulation. When the student positions AI as a *critical companion*, they maintain control and direction while iteratively exploring ideas, testing hypotheses, and co-constructing models with AI. High learner agency is characterized by a keen sense of what the task is asking, what contributions AI can potentially offer in solving the task, how to evaluate the AI's output, what thinking the learner is responsible for, and ultimately, how to merge their own thinking with the AI tool output (Wu et al., 2025).

Context and Instructor Positionality

Statistical Programming in Data Science was the second course in a five-course undergraduate data science sequence. Twenty students enrolled in the course in Fall 2025. All students had completed one prior data science course and several statistics courses, and about one-third had computer science experience or internships. The instructor participant (third author) was an experienced data science educator with a graduate degree in machine learning and a background in learning theory. She designed an AI usage policy that provided flexibility while maintaining alignment with course objectives. She aimed for students to learn how to build data science pipelines (Hardin et al., 2015) through real-world projects and iterative work with messy datasets. Recognizing the varied levels of Python experience of her students, from extensive practice to none, she structured the course to support multiple pathways of engagement including use of AI for coding. Three or four students worked collaboratively on the projects, with AI permitted as a tool to support analysis and exploration.

Data Collection and Analysis

The data science course syllabus provided the first of three data sources for this study, documenting AI usage expectations, opt-out language, assessment structures, and learning goals. Instructor reflection memos documented how teaching a mathematics content course (MATH 414) informed her design decisions for the data science course. Her reflections captured policies regarding AI use, student engagement with AI, and assessment challenges in AI-integrated learning environments. After the first two authors reviewed the reflection memo and syllabus, they conducted a semi-structured instructor interview with the third author. The interview protocol was designed to clarify and deepen understanding of the instructional decisions

articulated in those materials. Questions focused on how prior teaching experiences informed data science course design, how AI-integrated learning was conceptualized, and how assessment structures reflected her goals for fostering learner agency in AI-integrated learning environments. Collaborative qualitative data analysis (Richards & Hemphill, 2018) guided iterative coding, pattern identification, and reflexive interpretation across data sources, with analytic discussions informed by the instructor participant's perspective.

Findings

Analysis of the three data sources suggested three categories of instructor decision making with the aim of fostering learner agency in AI-integrated data science course: (1) Making students' thinking visible through AI interactions rather than restricting AI use, (2) Revising assessment to promote agency within an AI-integrated learning environment, and (3) Framing AI-mediated cognition within course assignments.

Making Students' Thinking Visible Rather Than Restricting AI Use

A central instructional decision was to design tasks where students were required to provide evidence of their interactions with AI. This decision was shaped by instructors' prior experience in teaching MATH 414. She recalled that their reflections often suggested one-shot prompting of AI rather than iterative reasoning and substantive interactions. Detection of AI alone could not resolve her deeper question of "where did their [students] thinking end and the AI's thinking begin?" The MATH 414 students primarily used AI as *cognitive load reducer* who could generate responses for students. Such experiences led the instructor to reflect that simply requiring disclosure of AI use did not necessarily make students' thinking visible in AI-integrated learning environments. This realization prompted a fundamental shift on what counts as evidence of students' understanding and thinking. Rather than attempting to regulate AI use directly, she concluded: "For AI in and out of class, I need to design my assignments in such a way that it's not going to matter what they do with AI outside of class." In designing the data science course, the instructor moved away from student reflection use and instead redesigned course activities so that the AI output was one part of a larger body of evidence of student learning.

Promoting Learner Agency within an AI-Integrated Learning Environment

In the data science course, the instructor promoted learner agency by framing student decision-making through the lens of professional competence. Students were encouraged to use AI in their projects but were also reminded that professional contexts require independent reasoning, justified decisions, and the ability to respond to scrutiny. Project assessments required interpretation and evaluation of the data-model relationship. Students were free to prepare their projects using AI. However, students were required to provide evidence interpreting data, critiquing models, and recommending improvements.

The instructor's new assessment approach shifted AI's role from a *cognitive load reducer* who could support the student in generating a correct response to a *cognitive capacity builder* and/or *critical companion* who interacted with a user to work toward a reasonable response. By requiring students to provide evidence reasoning, the instructor fostered learner agency and gained a window on conceptual sense making. Biweekly project updates required detailed descriptions of data sources, modeling decisions, and emerging challenges. When updates lacked specificity, the instructor reshared her expectation that general "AI-like" responses did not count as evidence of demonstrated understanding. For instance, if an update contained statements such as "This analysis will give us a comprehensive understanding…" the instructor would reply with, "A comprehensive understanding of what data science phenomenon?" implicitly suggesting that

their response (whether it was AI generated or not) lacked sufficient detail. Emphasizing what students would need to do in data scientist job interviews, the instructor set rigorous expectations. "AI can help you, but at the end of the day, it's on you to explain what all of this means… I should be able to ask you anything, and you should be able to answer anything."

Student groups who relied heavily on AI-generated approaches without understanding the data set–model relationships struggled to engage in deep discourse during the oral assessments. In contrast, other groups demonstrated iterative refinement and sustained engagement with their data and models, sometimes with the support of AI. The instructor recalled how students "constantly identified aspects of the data that were causing their models to fail," and considered "what part of the model do we need to change, or is the data set bad?" Their patterns of testing, diagnosing, revising, and justifying suggest that when students positioned AI as a *critical companion*, they retained authority to direct the course of engagement with the AI.

Framing AI-Mediated Cognition within Course Assignments

Examination of the instructor course design shows intentional framing of AI use consistent with the three roles in our conceptual framing: *cognitive load reducer, cognitive capacity builder, and critical companion*. When revising her data science course syllabus for Fall 2025, the instructor evaluated each course learning objective. If AI use rendered an objective obsolete, it was removed. If cognitive demand was preserved with the use of AI, the objective was kept. This evaluation positioned AI not as a tool that lowers cognitive demand, but as one that required deliberate design to ensure cognitive engagement.

AI functioned as a *cognitive load reducer* for students who were stuck with coding because it was not "productive for students to sit in class and not be able to test a model because they didn't have the correct code." The instructor put emphasis toward project pipelining and higher-order reasoning. As a result, she rarely assessed coding syntax, highlighting that students could offload the coding to AI to create more cognitive space for reasoning. The instructor also recalled that several students used AI as a *cognitive capacity builder* to generate practice questions. While the instructor permitted AI use as a study tool to support understanding, she also positioned human feedback as necessary to ensure that students' reasoning aligned with course objectives.

The instructor's decision making also created the potential for AI to serve as a *critical companion* in their projects. She organized students in groups so that AI would not become their sole thinking partner; instead, students could reconcile ideas generated with AI through peer discourse. She emphasized that understanding, not code production, was the goal: "I don't think that coding the algorithm is the only way to understand how the algorithm works or why it's useful for a particular type of problem." AI could assist with coding and analysis, but students remained responsible for explaining modeling choices and interpreting results. The instructor positioned AI not as a substitute for student reasoning but as a tool to support collaboration.

Discussion and Conclusion

Data science, with its emphasis on uncertainty and multiple plausible solutions, offers a productive setting for developing assessments suited to AI-integrated learning environments. These assessments prepare students to think, decide, and justify their work in professional contexts where AI is already part of the expected skill set. The findings of this study suggest that the central challenge for data science instructors is not whether to permit AI use, but how to design learning environments where students' thinking remains visible and students are held accountable for AI use. When instructors require students to justify their analytical choices and engage with AI as a *capacity builder* or *critical companion*, they can position their students as a doers and learners of data science. These findings also challenge the mathematics education

community to redefine what counts as evidence of student learning in AI-integrated learning environments.

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CRITICAL STATISTICAL LITERACY HABITS OF MIND ENACTMENT: WHEN STATISTICAL KNOWLEDGE AND CRITICAL CONSCIOUSNESS VARY

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Over multiple decades a consistent call for statistical literacy persists (e.g., Engel, 2017; Gal, 2002; Watson & Smith, 2022), and a call for Critical Statistical Literacy (CSL) emerged (e.g., Weiland, 2017). CSL encompasses making sense of and evaluating statistical messaging from both the production (i.e., generating the statistics) and consumer (i.e., digesting statistical messages produced by others) orientations to better understand the world and issues of power (Weiland, 2017; note that critical implies a nod to critical theories and issues of power). Beyond Gal (2002, 2019), few have investigated how adults engage in statistical literacy or critical statistical literacy. Since we live in a data rich world, with a sea of information at our fingertips, where "data driven" claims permeate our social media streams, CSL research is needed. This study aims to explore if and how adults enact Critical Statistical Literacy Habits of Mind (CSLHM) when interpreting statistical messages from the social media posts.

The CSLHM framework is a set of six habits of mind necessary for critically interpreting statistical information from the media (Bailey & McCulloch, 2023). Habits of mind are thinking behaviors that are employed with ease and frequency (e.g., Goldenberg, 1996). This instrumental multiple case study (Yin, 2018) aimed to better understand adult CSLHM enactment when interpreting statistical messages from the media. The cases were defined using two dimensions: (1) critical consciousness and (2) statistical self-efficacy/knowledge. We administered the Short Critical Consciousness Scale (S-CCS; Diemer et al., 2020; Rapa et al., 2020) and the Current Statistical Self-Efficacy Scale (CSSS; Finney & Schraw, 2003). Since self-efficacy is not necessarily correlated with knowledge, we also included four items from the Intermediate/ Advanced Statistical Literacy form of the Levels of Conceptual Understanding in Statistics (LOCUS) assessment online (Jacobbe et al., 2014). Strong statistical self-efficacy/knowledge was defined as a CSSS sum of 56 or higher with 75% or higher on the LOCUS items. Of the 89 eligible participants (snowball sampling), each was classified into one of four cases (emergent /emergent, emergent/strong, strong/emergent, strong/strong). We randomly selected five participants from each case for the study. The interviews were coded using a priori theoretical coding based on the CSLHM descriptions following DeCuir-Gunby et al.'s (2011) protocol.

Preliminary analysis reveals some distinct trends. Participants with emergent critical consciousness and emergent statistical self-efficacy/knowledge (Case 1) and participants with strong critical consciousness and emergent statistical self-efficacy/knowledge (Case 2) almost never enacted all six CSLHM on a social media post and emergent enactment was more common. In contrast, participants with emergent critical consciousness and strong statistical self-efficacy/knowledge (Case 3) enacted 4 of the 6 CSLHM frequently. Finally, participants with

strong critical consciousness and strong statistical self-efficacy/knowledge evidenced more consistent enactment across posts and more robust enactment.

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EXTENDING THE SEPT FRAMEWORK TO THE CONCRETE-SYMBOLIC MODE

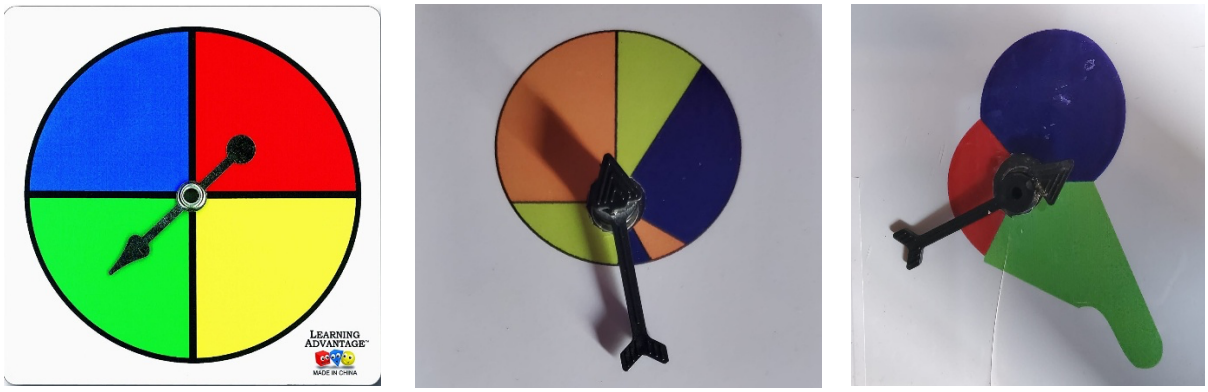
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Keywords: Conceptual Development, Probability

Earlier models of probabilistic thinking often treat responses within the Ikonic (IK) mode as incorrect and unaligned with curricula agenda (Tarr & Jones, 1997). Students that operate in this mode ascertain chance through preferences, beliefs, or personal experiences (Amir & Williams, 1999). In contrast, Concrete-symbolic (CS) thinking utilizes symbols typical of formal schooling (Biggs & Collis, 1982). In the Structure of Early Probabilistic Thinking (SEPT) framework, Groth et al. (2021) argued that IK thinking is not entirely homogenous in quality. They classified such responses into either normative compatible or incompatible cycles based on their resource potential for latter modes, specifically sampling. This study investigates whether compatibility extends beyond IK. We ask: (1) How do responses within and across modes differ by normativity? (2) How might normative-compatible reasoning serve as resources for learning?

Data was obtained from seven Grade 10 students on their judgments on three fair spinners: one common (*S1*) and two deliberately perturbed (*S2*, *S3*).



Spinner 1 **Spinner 2** **Spinner 3**

Figure 1: Common, Skewed, and Unusual Spinners

We found that (1) Normative-compatible and incompatible cycles permeate both IK and CS modes, (2) Students frequently judged fairness using area, shape, or center position. Although incorrect for probability, these ideas were legitimate within geometry, and (3) Some students simultaneously drew on IK imagery and CS reasoning, supporting the view that representational modes coexist rather than replace one another. Illustrative excerpts include that *S2* is unfair because "the circle is not in the center and some part are bigger than the other" (CS-incompatible), and *S1* is fair, "90 degree angles; equal chances" (CS-compatible). These excerpts show that incorrect reasoning may contain legitimate disciplinary resources, perturbed tasks reveal conceptual resources hidden by conventional symmetric spinners, and normative compatibility may extend across mathematical domains.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

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STUDENT LEARNING AND RELATED FACTORS

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USING STRUCTURAL EQUATION MODELING TO EXAMINE RELATIONSHIPS BETWEEN MATHEMATICS CAPITAL AND OTHER VARIABLES

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We investigate mathematics capital and its relationship with mathematics teaching and learning experiences, mathematics mindset, mathematics self-efficacy, cultural capital, and subjective social status. Survey data were collected from 927 undergraduate students in Fall 2024. Mathematics capital was modeled as a latent factor in our model with acceptable fit indices via structural equation modeling. The results of our model show that mathematics teaching and learning experiences and mathematics self-efficacy account for most of the variability in mathematics capital, and that subjective social status accounts for variability in mathematics self-efficacy and mathematics teaching and learning experiences. Moreover, cultural capital and mathematics mindset are positively related to mathematics teaching and learning experiences. Overall, mathematics capital has the most associations in our final model.

Keywords: Mathematics Capital, Structural Equation Modeling, Subjective Social Status

Much research has already considered prior achievement in mathematics, mathematical classroom experiences, mathematical dispositions, and how these constructs influence student success in mathematics education (Anthony, 2000; Ashcraft, 2002; Boaler, 2016; Cribbs et al., 2021; Darragh, 2016; Graven & Heyd-Metzuyanim, 2019; Montague et al., 1991; National Council of Teachers of Mathematics [NCTM], 2014). We take a Bourdieusian perspective and aim to consider these constructs jointly (Bourdieu, 1973, 1977, 1984, 1986, 1989, 1990, 1993, 1994, 1997, 2005; Bourdieu & Wacquant, 1992). This approach parallels Archer and colleagues' research in science education (Archer et al., 2012, 2014, 2015). We use a Bourdieusian-style lens to frame and define *mathematics capital*, similar to how *science capital* was defined by Archer and colleagues (2012, 2015), and subsequently widely researched (e.g., Archer et al., 2024; DeWitt et al., 2016; Godec et al., 2017; Godec et al., 2024; King et al., 2015; Kontkanen et al., 2025; Moote et al., 2020; Moote et al., 2021; Nomikou et al., 2017). In prior work, we modeled mathematics capital as a latent factor, constructed a structural equation model (Anderson et al., 2025), and performed a path analysis and confirmatory factor analysis to provide validity evidence for our instrument (Anderson et al., 2024). Our goal in this paper is to extend previous analyses with new data and refine our conceptual framework by adding subjective social status and mathematics identity. We address the following research questions:

1. What variables explain differences in the mathematics capital of undergraduate students?
2. What interrelations, if any, exist between the other measured variables in our model to explain such differences in mathematics capital?

Background and Conceptual Framework

Our framework of mathematics capital extends Bourdieu's (Bourdieu, 1973, 1977, 1984, 1986, 1989, 1990, 1993, 1994, 1997, 2005; Bourdieu & Wacquant, 1992) theorization of capital and Archer and colleagues' (2012, 2014, 2015) application of Bourdieusian capital to the field of science education. Bourdieu (1986) theorized that multiple forms of capital, such as economic, cultural, and social capital, operate within and are valued differently across fields. Fields are understood as structured social spaces in which participants seek to accumulate, exchange, and deploy capital (Bourdieu, 1977). Prior researchers have applied a Bourdieusian lens to the field of science education, empirically examining students' science capital through a theoretical model that integrates scientific forms of cultural capital, science-related dispositions and practices, and science-related social capital (Archer et al., 2015; DeWitt et al., 2016).

We conceptualize mathematics capital as an integral construct within the field of mathematics education (Anderson, 2024; Anderson et al., 2024; Anderson et al., 2025; Anderson & Uhing, 2025). Accordingly, we frame mathematics teaching and learning experiences (MTEAC), mathematics mindset (MMIND), mathematics self-efficacy (MSE), and mathematics identity (MIDT) as established variables within the field of mathematics education that interact with mathematics outcomes, cultural capital (CC), subjective social status (SSS), and mathematics capital (see Figure 1). We include MIDT in our conceptual framework to acknowledge recent research connecting MIDT and mathematics capital (Archer & Mendick, 2024; Valero et al., 2025).

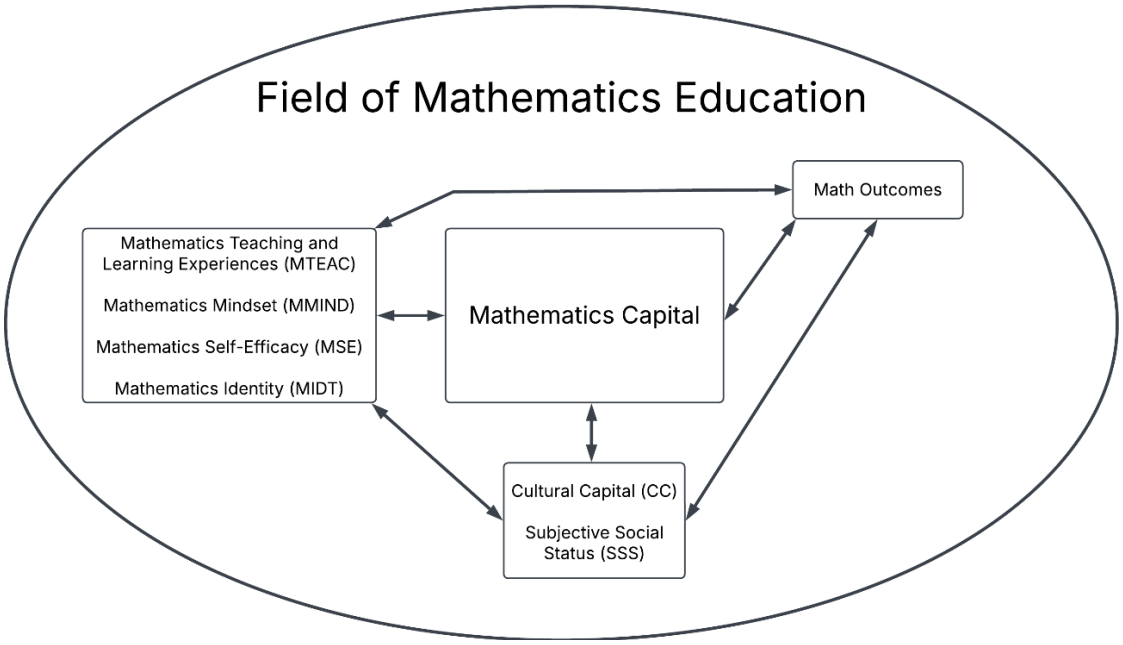


Figure 1: Conceptual Framework for the Field of Mathematics Education

Mathematics Capital

Bourdieu (1989) discussed mathematics capital in relation to access to advanced academic programs and the sciences, noting that within *doxa*, that is, the taken-for-granted, socially accepted cultural norms that often go unquestioned (Bourdieu, 1977), mathematics is positioned as valuable and exclusive, which creates a form of prestige (cf. Bourdieu, 1984). Archer and

Mendick (2024) argued that examining mathematics capital through a Bourdieusian lens is important, contending that such an approach productively challenges dominant orthodoxies within the field (Gutiérrez, 2012). Our work aligns with this perspective by situating mathematics capital within a structural model that examines its relationships with CC, thereby empirically supporting Archer and Mendick's (2024) theoretical claims.

Researchers have also linked mathematics capital to MIDT (Archer & Mendick, 2024; Black & Hernandez-Martinez, 2016; Valero et al., 2025). Black and Hernandez-Martinez (2016) argue that students' access to and activation of mathematics-related capital influence their identification with mathematics and their decisions to pursue mathematically demanding pathways, highlighting how these resources shape both participation and persistence. Archer and Mendick (2024) conceptualize identity and capital as mutually constitutive constructs that mediate participation and engagement in STEM fields. They argue that multiple forms of capital (social, cultural, economic, and symbolic) operate in conjunction with MIDT to shape students' mathematics outcomes, including postsecondary attainment (Archer & Mendick, 2024). Valero et al. (2025) found that mathematics capital shapes students' mathematical engagement (e.g., enrollment in mathematics courses) and their self-perceptions as mathematicians, or MIDT. Jorgensen (2018) highlighted the compounded challenges faced by Indigenous learners in accessing and engaging with mathematics and discussed mathematics capital as a construct that can increase or decrease, although mathematics capital was not explicitly defined. Still, Jorgensen (2018) conceptualized mathematics and school mathematics as a field, aligning with our framing of mathematics education as the field in which mathematics capital operates (Anderson, 2024; Anderson et al., 2024; Anderson et al., 2025; Anderson & Uhing, 2025).

Our research on mathematics capital focuses on defining and describing mathematics capital to disrupt deficit-oriented narratives that label students in mathematics (Anderson, 2022; Anderson, 2024; Anderson et al., 2024; Anderson et al., 2025; Anderson & Uhing, 2025). *We define mathematics capital as the mathematical assets that students draw upon in their everyday lives, which in turn influence their positioning within the field of mathematics education.* We conceptualize these mathematical assets as a constellation of students' collective experiences, dispositions, and intrinsic or extrinsic expectations related to mathematics (del Rosario Zavala & Aguirre, 2024). We contextualize and operationalize mathematics capital within the field of mathematics education by examining its relationships with established constructs in the literature (Anderson, 2024; Anderson et al., 2024; Anderson et al., 2025).

Mathematics Teaching and Learning Experiences

We conceptualize MTEAC as the formal and informal opportunities through which students engage with mathematics across their lifespan (Anderson & Uhing, 2025; Anderson et al., 2024; Anderson et al., 2025). This construct is grounded in the National Council of Teachers of Mathematics' (NCTM) Process Standards (2000), which provides a framework for understanding how students experience and engage with mathematics. We argue that students' engagement, or lack thereof, with these Process Standards reflects their exposure to, and participation in, methods for solving problems, reasoning, communicating, and working with mathematical ideas. The five NCTM (2000) Process Standards (Problem Solving, Reasoning and Proof, Communication, Connections, and Representation) articulate multiple ways students can engage with mathematics, which in turn may enhance their capacity to learn, retain, and develop productive mathematical knowledge and dispositions.

NCTM (2014) reaffirmed and extended the original Process Standards by articulating eight mathematics teaching practices for teachers of mathematics. Whereas the Process Standards

describe how students engage with mathematics, the mathematics teaching practices focus on how teachers enact instruction. Together, these eight teaching practices offer guidance on critical instructional approaches that support high-quality mathematics teaching (NCTM, 2014). Collectively, the NCTM (2000, 2014) publications serve as the theoretical lens through which MTEAC are conceptualized in this study (Anderson, 2024).

To examine students' MTEAC, we initially drew on NCTM's student- and teacher-centered recommendations as well as Ottmar et al.'s (2014) study, which similarly used the NCTM (2000) Process Standards to inform the development of survey items. A key finding of Ottmar et al.'s (2014) work was the identification of a latent factor labeled Instructional Practices for Teaching Mathematics, which measures "best practice in methods for delivering high-quality mathematics instruction to students" (p. 135). Using these theories, we established the reliability of the seven items associated with this construct, yielding Cronbach's alpha values of .81 (Anderson et al., 2024) and .84 (Anderson et al., 2025). For the current study, these items were reused directly from these prior analyses (Anderson et al., 2024; Anderson et al., 2025).

Mathematics Mindset, Mathematics Self-Efficacy, and Mathematics Identity

Drawing on the work of Dweck (2006, 2008) and Boaler (2016), we conceptualize MMIND as individuals' beliefs about their intellectual abilities in relation to mathematics. Individuals with a growth mindset view mathematical intelligence and ability as malleable and believe these capacities can be developed through effort, persistence, and learning, whereas individuals with a fixed mindset perceive mathematical intelligence as an innate and unchangeable trait (Dweck, 2006). Prior research demonstrates that students' MMIND are predictive of mathematics achievement, with students holding a growth mindset consistently outperforming those with a fixed mindset (Dweck, 2008). In addition, MMIND is closely connected to students' beliefs about themselves as learners and their perceptions of the nature of mathematics as a discipline (Boaler, 2016).

MSE is another construct strongly associated with students' mathematics achievement (Hall & Ponton, 2005) and has frequently been examined in relation to MMIND (Cribbs et al., 2021; Anderson et al., 2024). Using structural equation modeling, Cribbs et al. (2021) examined the relationships among MMIND, MSE, and related constructs, including mathematics anxiety and MIDT. In the present study, we employed their instrument that demonstrated validity evidence to measure MMIND (eight items) and MSE (12 items).

Recent research has described the relationship between MIDT and mathematics capital (e.g., Valero et al., 2025), specifically referring to the constructs as "self-reinforcing virtuous loops" (Archer & Mendick, 2024, p. 134). Therefore, we include MIDT in our conceptual model and acknowledge it as a relevant variable within the field of mathematics education. MIDT is defined by Cribbs et al. (2022) as "how students see themselves in relation to mathematics, based upon their perceptions and navigation of everyday experiences with mathematics" (p. 70). In prior work, Cribbs et al. (2015) modified an identity framework that included factors of recognition, competence, performance, and identity. Cribbs et al. (2015) found that students' competence and performance had an indirect effect on students' MIDT, primarily by association with students' interest and external recognition in mathematics. In addition, Cribbs et al. (2021) developed an instrument that demonstrated validity evidence to measure MIDT, which we intend to use in future work.

We differentiate MIDT and mathematics capital by *clarifying a key characteristic of mathematics capital: the daily use and leveraging of mathematical abilities.* While we hypothesize that a person's MIDT and their mathematics capital are related (cf., Archer &

Mendick, 2024; Valero et al., 2025), MIDT is described as a student's self-perception in relation to mathematics, built on their prior experiences with mathematics (Cribbs et al., 2022). Mathematics capital, however, includes an external component: *the person must be utilizing their mathematical abilities.* Furthermore, these *abilities must be valued in the field of mathematics education to provide the person with potential for upward mobility within the field of mathematics education.* In other words, the mathematical ability that a person possesses because of mathematics capital is the type of ability that is valued by others in mathematics education.

Cultural Capital

CC is widely recognized as a construct with varied and sometimes inconsistent definitions across the literature (Dumais & Ward, 2010). In response to these conceptual differences, Dumais and Ward (2010) drew on foundational work by Swartz (1997) and Bourdieu (1973, 1997) to develop survey items designed to operationalize CC. Consistent with their approach, we drew on survey items from Dumais and Ward (2010), Noble and Davies (2009), and Zabihi and Pordel (2011) to inform the development of a set of measures for CC in our own work (Anderson et al., 2024; Anderson et al., 2025).

In prior studies, we included items related to participation in cultural activities, consistent with quantitative research on CC (Dumais & Ward, 2010), as well as items capturing familial support, reading and being read to, and patterns of screen use (Noble & Davies, 2009; Zabihi & Pordel, 2011). Together, these items yielded an acceptable level of internal consistency, with a Cronbach's alpha of .75 (Anderson et al., 2025). A complete list of the 11 items used to measure CC is reported in our earlier works (Anderson et al., 2024; Anderson et al., 2025).

Subjective Social Status

In psychological and educational research, social class is frequently treated as a control variable rather than examined as a central construct or tested as a mediator or moderator of key relationships (Evans, 2004; Liu et al., 2004). Two common approaches to operationalizing social class are objective socioeconomic status (SES) and SSS. SES typically includes indicators such as educational attainment, occupational status, and income, and has been consistently linked to mathematics achievement and educational outcomes (e.g., Sirin, 2005). However, individuals with similar SES profiles often report markedly different perceptions of their social standing, suggesting that objective indicators alone do not fully capture lived experiences of social positioning (Adler et al., 2000; Diemer et al., 2013; Singh-Manoux et al., 2003).

Diemer et al. (2013) discussed SSS as capturing how individuals internalize structural inequalities and make sense of their opportunities and constraints within social systems, distinguishing it conceptually from SES. Because SSS reflects perceived access to valued resources and recognition, it has been shown to predict a range of educational, career, and health-related outcomes beyond objective socioeconomic measures (Adler et al., 2000; Diemer et al., 2013; Singh-Manoux et al., 2003). From a Bourdieusian perspective, SSS can be understood as an indicator of how individuals perceive their position within the field and their capacity to mobilize capital. In mathematics education, students' perceived social standing may shape their sense of legitimacy in mathematical spaces, their interpretation of instructional experiences, and their use of mathematical assets as capital (Archer et al., 2012; Bourdieu, 1986; Nasir & Hand, 2006). Accordingly, we include SSS in our model to examine its associations with mathematics capital, MSE, MTEAC, and CC, providing insight into how perceived social positioning relates to variation in students' mathematical trajectories (Adler et al., 2000; Diemer et al., 2013).

Methods

Participants and Data Collection

Participants in this study were college students enrolled in a psychology, teacher education, or mathematics course at a public, metropolitan university in the Midwestern United States in Fall 2024. A Qualtrics survey was distributed via email to all students enrolled in a mathematics course. We received 1,168 responses from 4,555 students, for a response rate of 26.2%. Only participants who completed the entire survey were included in the analysis, for a total of 927 participants and a response rate of 20.8%.

Variables

Mathematics capital, MTEAC, MMIND, MSE, CC, and SSS were the six variables that we included in this study. Participants responded to questions using a 5-point Likert scale for mathematics capital, MTEAC, MMIND, and MSE. For SSS, participants responded using an adapted Cantril Ladder (Cantril, 1965; Levin & Currie, 2014) that allowed them to select whole numbers from 1 through 10 to rate themselves, their family, and families in their neighborhood growing up in terms of social status (cf. Graham & Ciciurkaite, 2025; Stanford SPARQtools, n.d.). Existing instruments for MMIND and MSE (Anderson et al., 2024; Anderson et al., 2025; Cribbs et al., 2021) were used with established validity evidence. Further, existing instruments for MTEAC (Anderson et al., 2024; cf. Ottmar et al., 2014) and CC (Anderson et al., 2024; Anderson et al., 2025; cf. Dumais & Ward, 2010; Noble & Davies, 2009; Zabihi & Pordel, 2011) were used, which also had established reliability. For the SSS variable, our measurement instrument was based on prior research on SSS and established methodologies (Cantril, 1965; Diemer et al., 2013; Graham & Ciciurkaite, 2025; Krieger et al., 1997; Levin & Currie, 2014; Stanford SPARQtools, n.d.). Our Cronbach's alpha values were as follows: MTEAC $\alpha = .84$, MMIND $\alpha = .92$, MSE $\alpha = .95$, CC $\alpha = .79$, SSS $\alpha = .75$. Our prior instrument to measure mathematics capital showed validity evidence, which we have discussed in our prior work (Anderson et al., 2024; Anderson et al., 2025). Specifically, we reported a Cronbach's alpha value of 0.94 (Anderson et al., 2024) and we reported an omega value for mathematics capital modeled as a latent factor of 0.90 (Anderson et al., 2025). See prior works for the list of survey items (Anderson et al., 2024; Anderson et al., 2025).

Analysis

We collected data using Qualtrics, then exported these data to SPSS (Ver. 29). Initial descriptive analyses were performed as an initial analysis and assessment of the survey measures. We then created a latent factor for mathematics capital using M-Plus Ver. 7 (Muthén & Muthén, 2012). Next, using a path analysis, we examined the associations between the study variables to the latent factor of mathematics capital (Kline, 2015). Indirect effects of the associations with mathematics capital were delineated. The aim of these analyses was to explain differences in participants' mathematics capital.

Results

First, the latent factor of mathematics capital was tested. An acceptable model fit was obtained for the six indicators loaded onto a single factor by adding one association between the errors of two of the items (cf., Anderson et al., 2025). All of the factor loadings were acceptable (i.e.: positive and significant; loadings $\geq .76$). The resulting model was a good fit to the data ($\chi^2_{(8)} = 51.30$, $p < .05$, CFI = .99; RMSEA = .08, SRMR = .01). In general, the study variables were related to each other significantly in the expected directions. Notably, the two strongest associations with mathematics capital were MSE ($r = .82$, $p < .01$) and MTEAC ($r = .41$, $p < .01$). See Table 1 for a complete review of the zero order correlations.

Table 1: Zero-Order Correlations Between the Study Variables.

Variables	1.	2.	3.	4.	5.	6.
1. Mathematics Capital (latent)	-	.12**	.03	.31**	.82**	.41**
2. SSS	.12**	-	.20**	.01	.15**	.14**
3. CC	.03	.20**	-	.09**	.05	.34**
4. MMIND	.31**	.01	.09**	-	.48**	.17**
5. MSE	.82**	.15**	.05	.48**	-	.35**
6. MTEAC	.41**	.14**	.34**	.17**	.35**	-

Note. * p < .05, ** p < .01.

Given the strong associations between the latent factor of mathematics capital with both MSE and MTEAC, a structural equation model using both as principal predictors of mathematics capital was created. As such, SSS, CC, and MMIND were included in the model, with their effects on mathematics capital being indirect, where the direct relationship was seen through MSE or MTEAC. See Figure 2 for the full structural equation model.

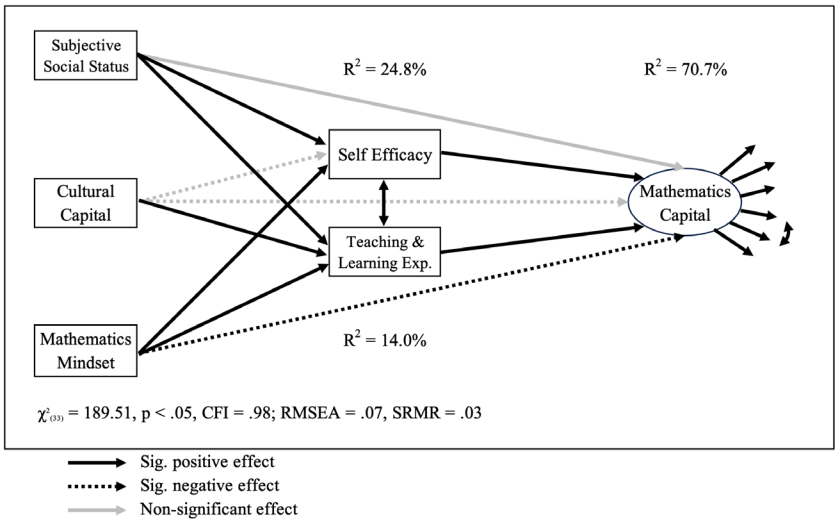


Figure 2: Full Structural Equation Model with Mathematics Capital as a Latent Factor

There were several relations between the other variables with MSE and MTEAC (replicating the correlation matrix). More interesting is that part of the associations between these variables were significantly indirectly through MSE and MTEAC. Specifically, the association between SSS with mathematics capital was indirectly through MSE ($\Delta\beta = .12$, $p < .05$). Moreover, the association between MMIND with mathematics capital was also indirectly through MSE ($\Delta\beta = .39$, $p < .05$). MTEAC accounted for indirect effects of SSS ($\Delta\beta = .01$, $p < .05$) and MMIND ($\Delta\beta = .02$, $p < .05$) as well, but more weakly than MSE. In other words, once MSE and MTEAC were included in the model, the effects of SSS and MMIND were solely through them. All told, the predictors accounted for 70.7% of the variability in mathematics capital, and the final model remained a good fit to the data ($\chi^2_{(33)} = 189.51$, $p < .05$, CFI = .98; RMSEA = .07, SRMR = .03).

Discussion and Implications

Our research aims to define and describe mathematics capital, quantitatively and qualitatively, similar to the way past researchers have developed science capital (Archer et al., 2012; Archer et al., 2014; Archer et al., 2015; Archer et al., 2024; DeWitt et al., 2016; Godec et al., 2017; Godec et al., 2024; King et al., 2015; Kontkanen et al., 2025; Moote et al., 2020; Moote et al., 2021; Nomikou et al., 2017). Thus, this research is in direct response to the call for more studies to understand the nature of the mathematics capital, how it interacts with MSE, and how it affects undergraduate students' outcomes and/or experiences in mathematics education (Anderson et al., 2025). In prior work, we noted that the strongest observed association was between MSE and mathematics capital (Anderson et al., 2025). In this study, we replicated this observation and refined our analyses to better understand the relationship between these two variables. Based on our model and setting MSE and MTEAC as prominent predictors, we note that these variables are relevant to describing differences in undergraduate students' mathematics capital. MMIND, CC, and SSS impact mathematics capital indirectly through MSE and MTEAC. These results show how different variables can predict mathematics capital, both directly and indirectly, which is critical as we continue to investigate mathematics capital as a construct in mathematics education and aim to develop interventions to impact individual undergraduate students' mathematics capital in future work.

Prior achievement in mathematics has been widely researched along with its relationship to individual student outcomes (Anthony, 2000; Ashcraft, 2002; Boaler, 2016; Cribbs et al., 2021; Darragh, 2016; Graven & Heyd-Metzuyanim, 2019; Montague et al., 1991; NCTM, 2014). Certainly, mathematics capital considers prior achievement in mathematics as a component of individual student outcomes in mathematics (Anderson & Uhing, 2024). However, mathematics capital is more dynamic and comprehensive than prior attainment (Anderson, 2024; Anderson et al., 2024; Anderson et al., 2025; Anderson & Uhing, 2025; Archer & Mendick, 2024). For these reasons, we argue that developing a deep understanding of mathematics capital is critical in learning about differences between student outcomes in mathematics. We theorize mathematics capital as a framework for mathematics educators to provide interventions in mathematics for adoption or adaptation based on individual learner's mathematics capital.

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DESCRIBING AND EXPLORING IMPLICATIONS OF MIDDLE SCHOOL STUDENTS' MULTIVARIATIONAL STRUCTURES

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Research shows middle school students can engage in multivariational reasoning by coordinating multiple changing quantities. This report extends that work with empirical evidence from a case study of two middle school students working through a faucet task sequence. We examine how two students each constructed and used two different multivariational structures. We describe the structures each student developed and how their structures influenced how they responded to follow-up prompts. We conclude with methodological and instructional implications for designing tasks that support quantitative and multivariational reasoning and using these tasks in instruction.

Keywords: Multivariation, Middle School Education, Mathematical Representations, Graphs

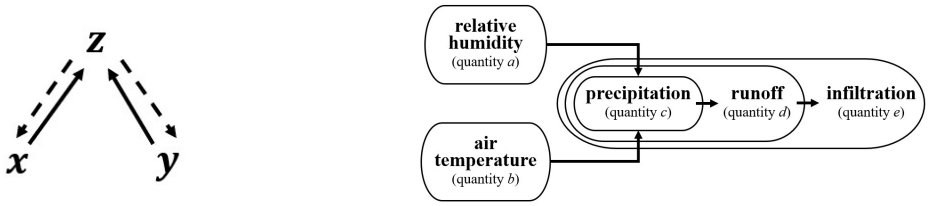
A growing body of research has emphasized how students' constructing and reasoning about quantities that change together can form a powerful basis for their mathematical development (see Thompson & Carlson (2017) for a review). Whereas much of this work has focused on students' constructing and coordinating two quantities (i.e., *co*variational reasoning; Carlson et al., 2002; Saldanha & Thompson, 1998), a growing body of research has begun to explore students' constructing and coordinating three or more quantities (i.e., *multi*variational reasoning (MV); Jones, 2022). Several researchers have shown that students as young as middle school (Ellis et al., 2015; Panorkou & Germia, 2013; Paoletti et al., 2025) can engage in MV reasoning as they construct and coordinate multiple quantities in situations that entail numerous quantities.

As students engage in covariational and MV reasoning, they construct quantitative structures that help them understand the interrelationship between quantities (Thompson, 1990, 1993). For example, Thompson (1993) described different structures students can construct when comparing the relative heights of two brother/sister pairs from different families (e.g., first comparing siblings within families and then comparing across families, versus separately comparing brothers and sisters across families and then comparing the differences). In this report, we leverage research on students' MV reasoning (Jones, 2022; Panorkou & Germia, 2023) to address the research question: *What multivariational structures do middle school students develop in the context of the faucet task, and how do they use them to address novel prompts?* Addressing this question, we characterize the different MV structures two middle-school students developed as they made sense of a faucet context across three sessions in a larger teaching experiment focused on their quantitative and covariational reasoning. Further, and going beyond extant research, we highlight how differences in their MV structures played a role in their responses to follow-up questions. We conclude with implications for researchers, teachers, and curriculum developers.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Literature Review and Theoretical Background

Extending research on students' quantitative and covariational reasoning (Thompson & Carlson, 2017), Jones (2022) defined MV reasoning as a student's reasoning about situations in which more than two quantities change in relation to one another. He distinguished different ways students relate quantities: when inputs vary separately (*independent MV*), when changes are mutually linked (*dependent* MV), and when one change feeds into another (*nested MV*). Jones (2022) described the mental actions of ten multivariable calculus students reasoning about multi-quantity situations. For example, in reasoning about $z = x^2 - y^2$, one student conceived an *independent* MV relationship and explained that x and y can each change (even at the same time), and that those changes together determine how z changes, without x and y being dependent on each other (Figure 1a).



(a) Independent MV Reasoning (b) Integrated MV Reasoning

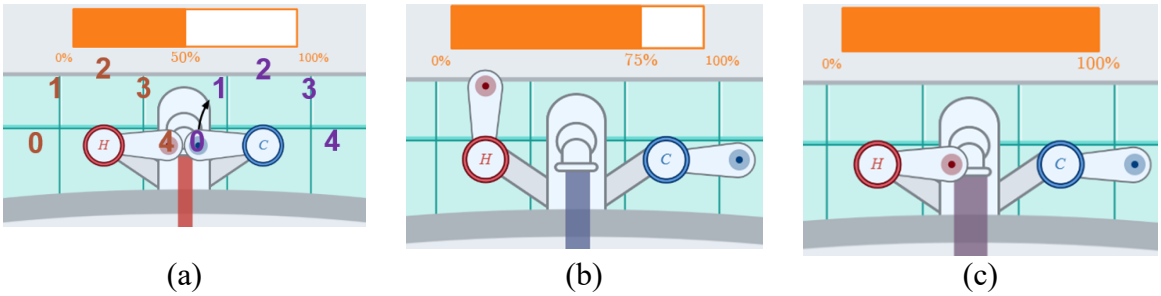
Figure 1: (a) Independent MV Reasoning (adapted from Jones, 2022, p. 6) and (b) Integrated MV Reasoning (adapted from Panorkou & Germia, 2023, p. 21)

Panorkou and Germia (2020, 2023) showed that MV reasoning is not limited to college students. They worked with sixth-grade students in whole-class design experiments using science simulations (e.g., water cycle, rock cycle, sea level rise). They found that students could coordinate complex relationships among multiple changing quantities, and identified several types of MV reasoning, including independent and nested MV, as described by Jones (2022). They also introduced *integrated* MV reasoning, in which students combine multiple MV categories. For example, in a water cycle context, a student used integrated MV reasoning by combining independent and nested MV: the student first treated relative humidity and air temperature as two independent quantities that together lead to "a lot of rain and a lot of runoff," (p. 21), then linked a nested chain by explaining that more rain produces more runoff, and increased runoff leads to more water going into the aquifers (infiltration; Figure 1b).

Whereas Jones (2022) and Panorkou and Germia (2020, 2023) described students' MV reasoning, we would characterize their descriptions (such as those in Figure 1) as MV structures (Thompson, 1993). A MV structure is a researcher's explanation of a student's meanings for relationships between quantities. Of relevance here, we focus on *independent* (Jones, 2022; Panorkou & Germia, 2023) and *integrated* (Panorkou & Germia, 2023) MV structures. We explain these structures in our task context, the faucet (e.g., Figure 2a).

In this task, we were particularly interested in students' abilities to coordinate the relationship between the amount each knob was turned and the percentage of water leaving the faucet with respect to the maximum amount possible. To support students' attention to the amount each knob is turned, we defined a full knob turn (i.e., 180 degrees from left to right) as comprised of four quarter knob turns (QKT; Figure 2a). With this numbering system, a knob pointing left (off) can be described as 0 QKT while the knob pointing right (all the way on) can be described as 4 QKT.

Leveraging the number of Hot QKT and Cold QKT, we asked students to consider how these quantities related to the total percent of water leaving the faucet (hereafter *percent of total water*), because there was a maximum of 8 QKTs, a knob at 4 QKT contributed 50% of total water. We intended the orange bar (Figure 2) to represent the percent of total water.



(a) (b) (c)

Figure 2: Faucet Task Applet with: (a) 4 Hot QKT, 0 Cold QKT, 50% of Total Water; (b) 2 Hot QKT, 4 Cold QKT, 75% of Total Water; and (c) 4 Hot QKT, 4 Cold QKT, 100% of Total Water

A student engages in independent MV reasoning when they coordinate how two quantities, which change independently of each other (i.e., input quantities), influence a third quantity (Jones, 2022; Panorkou & Germia, 2023). For example, in this context, a student may reason that the number of Hot QKT and Cold QKTs added together constitute a new quantity, Total QKT (Figure 3a). Importantly, the student treats Hot and Cold QKT as varying independently, and both contributing to the magnitude of Total QKT.

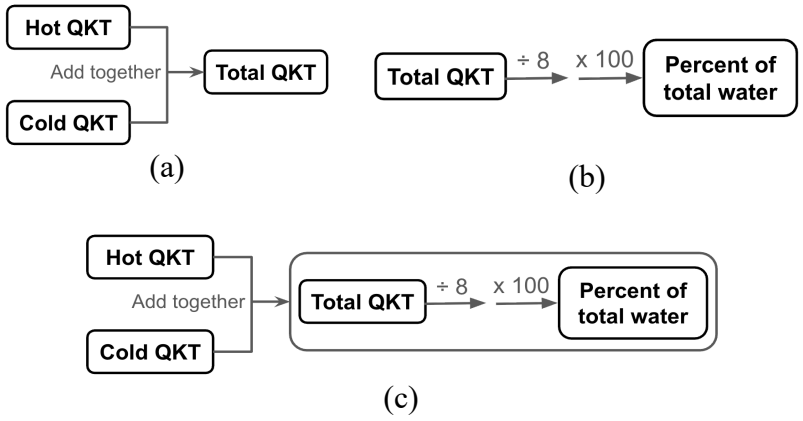


Figure 3: (a) Independent MV Structure, (b) Causal Covariational Relationship, and (c) Integrated MV Structure for the Faucet Context

A student engages in integrated MV reasoning (Panorkou & Germia, 2023) when they coordinate at least three quantities by linking quantitative structures. For example, a student may understand they can divide the Total QKT by eight and then multiply that result by 100 to obtain the percent of total water (Figure 3b). Such a relationship constitutes a causal covariational relationship, in which changes in one quantity cause changes in another quantity (Thompson & Thompson, 1996; Paoletti et al., 2015). Such reasoning is akin to Jones' (2022) nested MV

structure, although it involves only two quantities. Finally, the student can creates an integrated MV structure by linking these structures in such a way that the three focal quantities of Hot QKT, Cold QKT, and percent of total water are linked (Figure 3c). Additionally, this structure includes another quantity (Total QKT) introduced by the student in the process of establishing the integrated MV structure.

In this study, we characterize the different MV structures that two middle school students constructed in the faucet context. We also highlight how their different MV structures influenced their reasoning when addressing novel follow-up tasks in the faucet context.

Methods

In this report, we use a qualitative case study (Yin, 2018) to characterize the MV structures two middle school students developed and used in the aforementioned faucet context. We use data from a teaching experiment (Steffe & Thompson, 2000) that we conducted with two seventh-grade students with the self-selected pseudonyms Lil Man Man (LM) and Michael Jameson (MJ). The students had worked with the teacher-researcher (TR; the second author) in sixth grade to develop meanings for various types of covariational relationships (e.g., Paoletti et al., 2025). In this follow-up study, we were interested in exploring how students might reason about novel situations involving more complex relationships involving three or more quantities.

The TR conducted twelve, 35–40-minute sessions that were held approximately weekly with LM and MJ (typically paired, but at times individually). We video- and audio-recorded each session to capture their utterances and gestures. We digitized all their written work and screen-recorded their computer activity. However, for this report, we focus only on the task sequence used in Sessions 6–8 (Table 1) involving the faucet context. In Sessions 6–7, we focused on the MV structures students constructed to make sense of the faucet context. In Session 8, we explored how MJ and LM's MV structures influence their reasoning about graphs that we intended to represent all three focal quantities from the faucet context (Figure 4b).

Table 1: An Overview of the Teaching Experiment Sessions Over Two Years

Session	Participants	High-level Goal
1–5	LM, MJ	Construct the relationship between two quantities (Base length and Side length) and describe them via graphs within the context of playpens.
6–7	LM, MJ	Construct and reason about three quantities (Hot knob turns, Cold knob turns, and total percentage of water coming out) within the faucet task in Desmos.
8	LM	Make sense of the meanings for two graphs (Figure 2b and 2c; Figure 2b was introduced first) and connect them with specific situations within the context of the faucet task.
9	LM, MJ	
10–12	LM, MJ	Extend to relative percentage problems in the faucet context.

Analysis

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000), we analyzed the data using conceptual analysis, by "building models of what students actually know at some specific time and what they comprehended in specific situations" (Thompson, 2008, p. 45). Specifically, we looked for evidence of the MV structures LM and MJ were developing. We

did this by identifying instances in which each student made connections between how the three quantities in the task varied, comparing these instances over time to make inferences about the structures they were developing. We also looked for instances in which we inferred that the MV structures we identified influenced their activity in constructing or interpreting graphical objects (i.e., points and lines) representing these quantities. At times, we revised our inferences about their MV structures after observing additional activity, which supported a new model of their mathematics that was more explanatory than our original descriptions. Through this iterative process, we constructed explanatory models of each student's MV structures.

Results

LM and MJ each developed integrated MV structures, but these structures differed, and they used them in different ways across Sessions 6–7. Further, these structures influenced the way they reasoned about a novel graph representing the focal quantities in Session 8.

MJ developed the integrated MV structure discussed earlier (Figure 3c). He treated Hot QKT and Cold QKT as two quantities whose sum formed a new quantity, Total QKT. Dividing Total QKT by 8, then multiplying by 100, produced the percent of total water.

LM also developed an integrated MV structure. He understood each knob contributed up to 50% of the total water. Using this meaning, he knew he could divide the number of Hot or Cold QKTs by 4, then multiply by 100 to determine the percent of total water contributed by *that* knob (a causal covariational relationship; Figure 4a). He would then add these values to determine the percent of total water. Each percent changed independently, so this represents an independent MV structure (the "Add together" link in Figure 4b). By combining these structures, LM constructed an integrated MV structure (all of Figure 4b) that was distinct from MJ's structure. Next, we highlight key moments illustrating LM's and MJ's MV structures and how these structures influenced their reasoning in multiple instances.

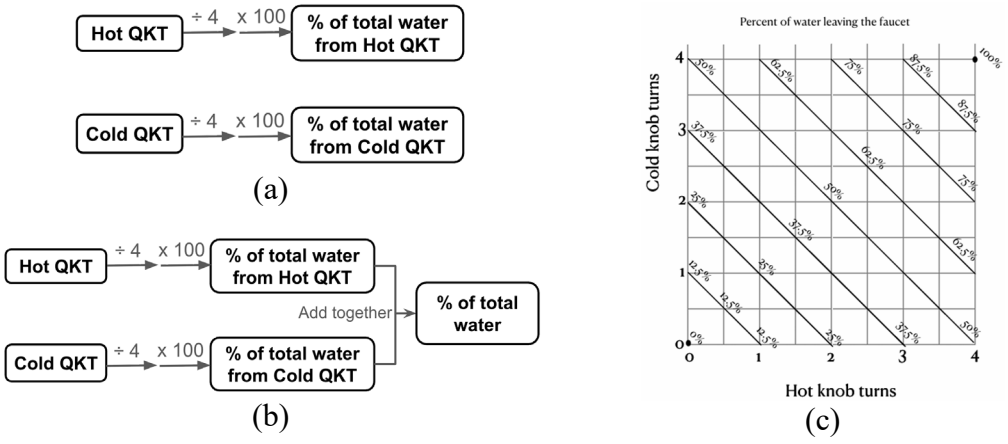


Figure 4: (a–b) LM's MV Structure and (c) a Graphical Representation Showing the Three Focal Quantities

Fixing One Knob: Different Structures Result in Different Reasoning

In Session 7, the TR asked each student to consider how the percent of total water changed as one QKT varied while the other remained fixed. MJ and LM provided different descriptions of how the two quantities changed, and we infer these differences stemmed from their distinct MV

structures. We first present each student's activity, then our analysis of how their structures influenced that activity.

When the TR prompted MJ to consider how the Cold QKT and percent of total would change with the Hot QKT fixed at 2, MJ responded, "It's going lower or higher [as the cold knob is turned off and on]." When asked for a specific percentage range as Cold QKT increased from 0 to 4, he specified, "It's gonna go higher." MJ's response provided evidence that he understood the directional relationship between the quantities (e.g., as Cold QKT increases, the percent of total water also increases), though he did not provide specific numerical values for the percentage of total water.

In contrast, when the TR asked a similar question to LM, he argued the percent of total water would change by 50%. Specifically, when asked how the percent of total water would vary as Hot QKT varied with 1.6 Cold QKT fixed, LM immediately dragged the cold knob to 1.6 QKT. However, the applet still showed 2 Hot QKT, which meant 45% of total water was shown. Thinking that the situation currently represented 1.6 Cold QKT and 0 Hot QKT, LM stated, "I think it's going to be anything between 45 and [*paused for 8 seconds*] 95." In the moment, the TR inferred that LM assumed the applet was showing 0 Hot QKT, so he asked LM to move the hot knob. As LM dragged the hot knob, he realized that with 0 Hot QKT and 1.6 Cold QKT, there was 20% of total water. LM then revised his answer, saying, "anything between 20 and 70[%]."

We infer that LM anticipated that Hot QKT changing from 0 to 4 would change the percent of total water by 50% (i.e., initially anticipating that the range will be 45–95%, then revising it to 20–70%). Based on his later activity, we infer that this meaning also informed his reasoning about how changes in Cold QKT would affect the percent of total water. We infer that LM's MV structure (Figure 4b), entailing each knob contributing a maximum of 50%, led to his conclusions. In contrast, for MJ, the task did not entail finding the sum of QKT as an intermediate step to determine the percent of total water. We infer the lack of this intermediate value could explain why MJ did not provide specific percent values, focusing instead on the directional relationship between two quantities. His MV structure (Figure 3c) required determining the Total QKT before determining percent of total water values.

Fixing the Percent of Total Water and Varying Cold and Hot Turns

We also observed differences in how the students responded to questions with a fixed percent of total water, which we infer were related to their respective MV structures. The TR asked, "Say we want the water to always be 50% coming out, but you can change the hot and the cold knobs at the same time. How are the cold and hot knobs going to change?" MJ responded first, and without moving the knobs. He said, "both [knobs] have to be two, or two turns. Or it could be three on, one turn … So, it could be four in all. So long as [it's] equal to four in all." The TR probed further, "Say I'm taking the cold knob from zero to four, what do I have to do to the hot knob?" MJ immediately responded: "Take it to zero … at the same time, it would have to go left." Across these responses, we infer that MJ understood that for 50% of total water, the number of Total QKT must sum to four (Figure 3c). He was able to leverage this meaning both for individual values (2 and 2; 3 and 1) as well as to reason covariationally; he argued that the two knobs needed to turn in opposite directions to maintain this sum of "four in all," which is consistent with his MV structure (Figure 3c).

We note that LM was able to understand MJ's arguments, and adopt them in the moment (e.g., arguing, "So long as the turns equals four. It's like four turns is like half of eight, and 50%

is half of 100."). However, LM did not spontaneously adopt this reasoning in the subsequent activity, instead relying on his own MV structure.

Determining One Quantity Given the Other Two: A Novel Graph

Given the productive but different MV structures each student exhibited in Session 7, we were interested in exploring how each student might interpret graphs representing the three quantities (Figure 4c). Hence, in Session 8(a/b) the TR worked individually with each student to explore how they may leverage their MV structures to interpret this novel graph.

During Session 8a, the TR provided LM the graph in Figure 4c and asked what percentage of total water would correspond to 2 Hot QKT and 1 Cold QKT. Instead of using the graph, LM quickly responded, "Maybe like 35[%], because like 2 [QKT] equals, like, when it's at 2 [QKT], it's 25[%]. And then 1 [QKT] is like, uh, I don't know … 15[%]? Wait, wait, I think it's like 12.5%. 25 divided by 2." Consistent with his MV structure (Figure 4a), LM described using the number of each QKT to determine the percent of total water contributed by that knob (2 QKTs contributes 25%, and 1 QKT contributes 12.5%). Using this inference, LM correctly determined the answer of 37.5% of total water by adding these two percent together. In follow-up prompts, whenever the TR provided two integer QKT values, LM would engage in similar reasoning.

In contrast to LM, MJ's response to this type of question in Session 8b showed a specific focus on the Total QKT. The TR's first asked, "What if we have 1 hot turn and 2.5 cold turns, what will the percent be?" After thinking for 10 seconds, MJ responded:

> It's right here [*marked a point in the graph in the region between 62.5% and 75%*]. I think it's up there because four full turns and two turns, it's 75%, so for 3.5 [turns] and on, I would say like right between … I would say actually near 50% because I'm saying, wait, no I changed my mind. 62.5[%], I decide that because 3.5 [turns] is near half, well near four [turns].

We infer that MJ connected 6 Total QKT to 75% of total water, then tried to relate 3.5 Total QKT to 6 Total QKT by selecting a percentage below 75%. He further refined his answer by comparing the 3.5 Total QKT with 4 Total QKT ("actually near 50%"). However, he ended up choosing a value above 50% of total water rather than approximating a value below 50% of total water. We conjecture the decimal value of Total QKT (3.5) may have created a complexity for MJ's desire to determine the exact percent. However, when later asked to determine the number of Cold QKT if there were 75% of total water and 3 Hot QKT, MJ immediately responded that there would be 3 Cold QKT. The TR asked "How did you figure that out? That was fast." MJ responded, "Because I remember, with 4 and 2 equals 6, so all I know is you have to equal 6 to equal 75%." Once again, MJ relied on his integrated MV structure (Figure 3c) to relate the percent of total water to the Total QKT, with Total QKT determined by the sum of Hot and Cold QKT.

Discussion

Addressing the first part of our RQ (What are the MV structures that middle school students develop?), we characterized two different MV structures that students constructed when conceiving a relationship between three focal quantities in a faucet context. Although both students constructed an integrated MV structure (Panorkou & Germia, 2023) that entailed both independent MV (Jones, 2022) and causal covariation (Thompson & Thompson, 1996), they did so in different ways. MJ's structure emphasized finding the number of Total QKT as the sum of the Hot and Cold QKT (independent MV structure) and then relating the Total QKT to the percent of total water (a causal covariational structure). In contrast, LM's integrated MV

structure first entailed separately determining the percent of total water contributed by the Hot QKT and the Cold QKT (two separate causal covariational structures) and then summing these to determine the percent of total water (an independent MV structure). In addition to characterizing novel MV structures, this report contributes to the literature on students' MV reasoning by exploring how these structures influenced students' activity addressing other prompts.

Addressing the second part of our RQ (how students leverage MV structures?), we highlighted how these structures influenced students' responses to tasks relevant to the faucet context; each structure had affordances and constraints when addressing certain follow-up prompts. For example, MJ's structure allowed him to reason covariationally about the relationship between Hot and Cold QKT for a fixed percentage of total water, as he understood the Total QKT would need to remain fixed. However, his structure was less productive when the TR tasked him to determine a range of percent values for one knob varying and the other fixed. LM's structure allowed him to quickly determine various relationships when given an integer number of QKT (e.g., determining percent of total water for a given number of Hot and Cold QKT; reasoning about a range of percent of total water with one knob fixed and the other varying). However, his structure was not as productive to reason about how the number of Hot and Cold QKT covary for a fixed percent of total water.

We highlight that though we presented our inferences about each student's MV structures and implications of these structures, our inferences about their structures were informed by the overall collection of their activity, which is consistent with retrospective analysis in the teaching experiment methodology (Steffe & Thompson, 2000). Their activity addressing prompts related to the graph in Figure 4c was critical for allowing us to support and refine our models of each student's MV structure. Hence, methodologically, when exploring students' developing structures in situations involving multiple quantities, it is important for researchers to pay careful attention to the ways students may construct and coordinate quantities across a variety of prompts. Different prompts can provide different insights into students' MV structures (e.g., fixing different quantities and asking how the others change; asking to determine one quantity's value for fixed values of the other quantities). Further prompts can provide students with opportunities to refine or deepen their structures; we call for future research exploring this latter possibility.

Although we only present findings from a single context, we conjecture that most situations entailing multiple quantities allow students to construct different MV structures. Although this may create complications for teachers using such situations for instructional purposes (i.e., needing to carefully attend to each student's MV structures), we contend this is an affordance of such tasks. When making sense of such situations, students can engage in purposeful sense-making in ways accessible to them, just as LM and MJ were able to do. We also conjecture that such contexts may support students in connecting mathematical ideas in ways that deepen their understanding of these ideas individually (i.e., connecting covariation, percentages, linearity, etc.). We call for future research exploring the potential productivity of engaging students in situations entailing multiple quantities that can support MV reasoning. Such research could entail creating tasks that support students' MV reasoning as a powerful extension of their quantitative and covariational reasoning (Thompson & Carlson, 2017), thereby providing a stronger basis for their mathematical development.

Acknowledgements

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EMOTIONS AS EXPLANATORY MECHANISMS IN MATHEMATICS LEARNING: A MULTILEVEL ANALYSIS OF AFFECTIVE-COGNITIVE CONFIGURATIONS

EMOCIONES COMO MECANISMOS EXPLICATIVOS EN EL APRENDIZAJE DE LAS MATEMÁTICAS: ANÁLISIS MULTINIVEL DE CONFIGURACIONES AFECTIVO-COGNITIVAS

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Using data from 650 secondary school students from 20 mathematics classes in a rural region of Mexico, the study analyzes how emotional experiences are structurally linked to students' cognitive–attitudinal orientations and mathematics achievement, and differ within gender and teachers. A quantitative, cross-sectional design was employed, combining descriptive analyses, multiple regression models, and multilevel modeling. Enjoyment, boredom, and frustration are systematically associated with cognitive orientations toward mathematics, directly related to achievement. Significant variability in achievement was observed across teachers. Gender analyses revealed differentiated affective–cognitive configurations for female and male students. These findings position emotions not as background correlates, but as dynamic processes that organize cognition, shaping students' engagement and achievement in mathematics.

Keywords: Academic Emotions, Mathematics Achievement, Affective Domain, Teacher-Level Context

Purposes and Research Questions

The purpose of this study is to examine students' emotions as explanatory components within broader affective-cognitive configurations associated with mathematics performance. Although affective variables are frequently included in studies of mathematics achievement, they are often treated as correlates, background conditions, or secondary factors within predominantly cognitive explanations of learning. Consequently, while research has consistently documented associations between emotions and mathematics achievement, less is known about how emotional experiences become linked to students' beliefs, attitudes, and achievement outcomes. This gap is particularly relevant because affective processes do not operate in isolation but are intertwined with broader cognitive and social structures that shape students' engagement with mathematics (Goldin et al., 2011).

Specifically, the study aims to (a) characterize students' affective profiles in mathematics learning, (b) examine how emotions such as enjoyment, boredom, and frustration are associated with students' cognitive-attitudinal orientations and mathematics achievement, and (c) explore whether these affective-cognitive configurations vary across gender and teacher-level instructional contexts. Together, these analyses contribute to a model in which emotions are conceptualized as components of broader affective-cognitive systems embedded in students' mathematical activity and instructional contexts. Based on these aims, the study is guided by the following research questions:

1. What affective profiles characterize students' emotional experiences in mathematics learning?
2. How are students' emotional experiences structurally related to cognitive–attitudinal orientations and mathematics achievement?

3. To what extent do these affective–cognitive mechanisms vary across gender and teacher-level instructional contexts in mathematics classrooms?

Theoretical Framework

Emotions Within the Affective Domain in Mathematics Education

Within mathematics education, the affective domain has traditionally encompassed constructs such as emotions, attitudes, and beliefs (McLeod, 1989). Although these constructs are often discussed jointly, they are not theoretically equivalent. Early scholarship emphasized the importance of distinguishing them according to their temporal stability, scope, and relationship to mathematical activity. Emotions have been conceptualized as relatively short-lived, situation-dependent processes emerging during engagement with specific mathematical tasks, whereas attitudes and beliefs tend to reflect more stable and generalized orientations toward mathematics (McLeod, 1992).

This distinction is consequential for explanatory modeling. When emotions are treated as interchangeable with attitudes or beliefs, their dynamic and situational character is obscured, and they risk being reduced to background characteristics rather than active components of learning processes (Roth & Walshaw, 2019). By contrast, conceptualizing emotions as processes that unfold during mathematical activity foregrounds their potential to influence students' interpretations of tasks, engagement, and persistence in real time.

Contemporary psychological perspectives converge on this process-oriented view. Across theoretical traditions, emotions are understood as dynamic phenomena that emerge from interactions among individuals' appraisals, contextual demands, and prior experiences (Barrett, 2017; Moors, 2017; Pekrun, 2006). Rather than fixed traits, emotions are situated, context-sensitive processes that organize how individuals respond to learning environments. This processual understanding differentiates emotions from more stable affective constructs and underscores their relevance for models that seek to explain variation in academic engagement and performance.

Bringing these insights into mathematics education reinforces the need to treat emotions as analytically distinct components of the affective domain. If emotions arise in relation to specific classroom interactions and task demands, then their explanatory role cannot be adequately captured through correlational or background-variable approaches alone. Instead, they must be modeled explicitly within frameworks that account for how students experience and engage with mathematics in situated instructional contexts.

Emotions as Explanatory Mechanisms in Mathematics Learning

A growing body of research demonstrates that emotions play a substantive role in students' learning processes, particularly through their influence on engagement, regulation, and achievement-related behaviors (Camacho-Morles et al., 2021). Emotions shape how students allocate attention, interpret difficulty, and persist in challenging tasks, thereby influencing achievement not only indirectly but also through integrated cognitive–affective pathways (Pekrun et al., 2023).

Within the literature on academic emotions, enjoyment and boredom have been consistently associated with contrasting patterns of engagement. Enjoyment is linked to sustained effort, deeper processing, and adaptive regulation strategies, whereas boredom is associated with disengagement, avoidance, and diminished investment in learning activities (Pekrun & Linnenbrink-Garcia, 2014). These findings suggest that emotions operate not merely as concomitants of achievement but as processes that organize students' engagement with academic tasks.

Moreover, academic emotions are closely tied to students' appraisals of control, value, and task difficulty (Pekrun et al., 2023). Emotions can thus be understood as mediating processes that connect students' interpretations of instructional contexts with observable learning outcomes (Camacho-Morles et al., 2021). In this sense, emotional experiences are relational and embedded in classroom practices rather than detached internal states.

Research in mathematics education provides domain-specific evidence supporting this view (Hannula, 2012; Xolocotzin Eligio, 2017). Students' emotional experiences in mathematics are intertwined with how they interpret mathematical tasks, perceive difficulty, and position themselves within classroom norms. Emotions emerge in relation to particular forms of mathematical activity and instructional expectations, shaping students' opportunities for engagement and meaning making within the discipline.

Because emotional processes are situated within classroom interactions, they are not expected to operate uniformly across learners. Prior research has documented systematic differences in affective engagement related to gender and to instructional practices (Boaler, 2002; Forgasz & Leder, 2017). In some educational contexts, female students demonstrate higher classroom achievement than male students, yet this advantage is not consistently mirrored in standardized assessments (Gómez, 2025). Such discrepancies suggest that achievement outcomes may be organized through differentiated socio-affective configurations rather than through cognitive factors alone (Gómez, 2022).

From this perspective, emotions function as explanatory mechanisms that organize how students interpret instructional demands, regulate effort, and respond to evaluation contexts. Emotional experiences such as enjoyment, frustration, and boredom are expected to shape students' cognitive–attitudinal orientations toward mathematics and, through these configurations, influence achievement. These mechanisms are embedded within specific instructional environments and may be organized differently for female and male students.

Method

Research Design

This study employed a quantitative, cross-sectional research design to examine students' emotions as explanatory mechanisms in mathematics learning. Analyses were conducted using an integrated approach combining multiple regression analyses and multilevel modeling in order to examine explanatory relationships among affective, cognitive–attitudinal, and achievement variables, while accounting for the nested structure of the data (students nested within mathematics teachers). This design allows for the examination of emotional mechanisms at the student level, as well as for the assessment of teacher-level variability in students' mathematics achievement.

Participants and Context

Participants were secondary school students and mathematics teachers from public schools located in the municipality of Halachó, Yucatán, Mexico. This predominantly rural region is characterized by socioeconomic vulnerability and limited educational resources, providing a relevant backdrop for the study. Within this setting, teachers constitute the primary contextual level of analysis, as students' achievement and affective–cognitive configurations are examined in relation to the mathematics teacher to whom they are assigned.

The student sample consisted of approximately 650 students enrolled in Grades 1 to 3 of lower secondary education (ages 12–15), including both female and male students. Participants attended five public secondary schools: one general secondary school and four telesecundarias, a distance-based educational modality designed to serve rural communities.

The teacher sample included 20 mathematics teachers responsible for mathematics instruction in the participating classrooms. Teacher data were used to characterize classroom contexts and to inform the classroom-level structure of the multilevel analyses.

Measures and Data Collection

Students' emotional experiences in mathematics were assessed using the Inventory of Emotions Experienced by Adolescents when Solving Mathematical Tasks (INETAM), a self-report instrument developed and validated in Spanish for Mexican secondary school students (Gómez et al., 2020). INETAM measures achievement-related emotions experienced during mathematical activity, including enjoyment, boredom, enthusiasm, and frustration, and conceptualizes emotions as situational and task-related processes rather than stable affective dispositions. Responses ranged from 1 (not at all like me) to 4 (exactly like me), with higher scores indicating greater endorsement of the corresponding construct.

Additional student measures assessed students' satisfaction with mathematics classes, general mood states related to school experience, and perceptions, beliefs, and attitudes toward mathematics, including views about learning mathematics and personal engagement with the subject. These measures used the same four-point response format. Mathematics achievement was operationalized using students' course grades obtained from official school records.

Teachers completed a questionnaire assessing sociodemographic and professional characteristics, as well as affective experiences related to teaching mathematics. Teachers also reported perceptions of students' motivation, learning, and performance differentiated by gender. These data were used to characterize classroom contexts and to support the specification of classroom-level components in the multilevel models.

Analytic Strategy

Preliminary analyses included data screening, inspection of missingness and outliers, and evaluation of regression and multilevel assumptions. Correlational analyses were conducted to examine associations among cognitive, affective, and achievement variables and to guide model specification.

Analyses proceeded in three stages. First, descriptive statistics and gender comparisons were estimated for key cognitive and affective variables. Second, sequential multiple regression models were estimated at the student level to examine (a) cognitive predictors of mathematics achievement and (b) affective predictors associated with each cognitive node. Additional regression analyses were conducted controlling for student sex and school type to evaluate the robustness of the proposed model. Third, multilevel modeling was employed to account for the nested structure of the data (students within teachers). A null random-intercept model was estimated to compute the intraclass correlation coefficient (ICC), followed by models including student-level predictors to examine whether cognitive effects remained significant after accounting for classroom-level variability.

All analyses were conducted using IBM SPSS Statistics (Version 26), with statistical significance evaluated at conventional alpha levels.

Results

Descriptive Statistics of Key Study Variables

Descriptive statistics were computed for mathematics achievement, students' perceptions, and emotional variables included in the analytic models. Mathematics achievement, measured through students' average mathematics grades, showed a mean of 8.08 (SD = 1.45). An independent-samples t test indicated a statistically significant difference in mathematics achievement between female and male students, with girls obtaining higher mean scores than

boys, $t_{(635)} = 3.79$, $p < .001$. On average, students reported moderate levels of positive emotions related to mathematics learning, such as enthusiasm ($M = 2.28$, $SD = 0.71$), facilitation–enjoyment ($M = 2.37$, $SD = 0.73$), and effort–enjoyment ($M = 3.12$, $SD = 0.76$). Negative emotions associated with boredom and frustration were reported at moderate levels, including frustration ($M = 2.04$, $SD = 0.64$), disengagement–boredom ($M = 2.31$, $SD = 0.75$), and discomfort–boredom ($M = 1.88$, $SD = 0.75$). Students' perceptions of mathematics reflected moderate levels of perceived interest and self-efficacy.

Preliminary Correlations Among Key Study Variables

To examine the structural relationships among achievement, cognitive–attitudinal variables, and academic emotions, Pearson correlation analyses were conducted. The purpose of this preliminary step was to identify patterns of association and internal organization among constructs prior to advancing to regression and structural modeling analyses.

Table 1: Correlations Among Mathematics Achievement and Cognitive-Affective Variables

Variable	1	2	3	4	5	6	7	8	9
1. Mathematics Achievement	—								
2. Mathematics Is Easy	.353	—							
3. Defeatist Attitude	−.381	−.588	—						
4. Rejection Attitude	−.323	−.397	.593	—					
5. Deterministic Beliefs	−.327	−.483	.713	.708	—				
6. Boredom	−.329	−.434	.640	.642	.703	—			
7. Frustration	−.244	−.397	.605	.559	.595	.752	—		
8. Enthusiasm	.266	.411	−.478	−.513	−.475	−.496	−.354	—	
9. Enjoyment	.288	.481	−.522	−.516	−.508	−.540	−.435	.840	—

Note. Pearson's r coefficients are reported. Only correlations of moderate magnitude or higher ($r \geq .25$) are presented to emphasize substantively meaningful associations (Cohen, 1988; Schober et al., 2018). All reported correlations are statistically significant at $p < .001$. Sample size ranges from 637 to 655, depending on the variable pair.

The correlational matrix (Table 1) reveals a coherent structure linking mathematics achievement, cognitive beliefs, attitudes, and emotions. Achievement aligns positively with perceived ease of mathematics and positive emotions, and negatively with defeatist and rejection attitudes, deterministic beliefs, boredom, and frustration. Perceived ease follows the same directional pattern, showing consistent alignment with positive emotional experiences and inverse associations with negative beliefs and attitudes.

Defeatist attitude, rejection attitude, and deterministic beliefs form a tightly interconnected cluster and are strongly associated with boredom and frustration. Overall, the matrix displays two clearly differentiated configurations: one grouping achievement, perceived ease, and positive emotions; and another grouping negative attitudes, deterministic beliefs, and distress-related emotions. The pattern suggests substantial shared variance within each configuration.

The emotion-only matrix (Table 2) shows strong internal coherence within valence categories. Positive emotions (enthusiasm and enjoyment) cluster together, while negative emotions (frustration and boredom) also form a tightly associated grouping. Associations between positive and negative emotions are consistently inverse but weaker than the within-valence associations. This indicates that emotional experiences are organized into two internally

consistent clusters, with substantial but not complete opposition between positive and negative affective states.

Table 2: Correlations Among Mathematics Achievement Emotions

Variable	1	2	3	4
1. Frustration	—			
2. Enthusiasm	−.354	—		
3. Enjoyment	−.435	.840	—	
4. Boredom	.752	−.496	−.540	—

Note. Pearson's r coefficients are reported. All reported correlations are statistically significant at $p < .001$. Sample size ranges from 637 to 655, depending on the variable pair.

Teacher-Level Variation in Student Achievement

A null multilevel model with students nested within teachers was first estimated to examine the extent to which variability in mathematics achievement was attributable to differences between teachers. Results revealed significant between-teacher variability in students' mathematics achievement. The intraclass correlation coefficient (ICC) indicated that approximately 17% of the total variance in achievement was attributable to teacher level, with the remaining variance attributable to differences among students within classrooms.

To determine whether teacher-level variability persisted beyond students' individual characteristics, a random-intercept model including cognitive predictors of mathematics achievement (perceived ease, attitude of defeat, and traditional beliefs) and affective variables (satisfaction with teacher, mood states, and perceptions of mathematics) was estimated. The variance component for teacher level remained statistically significant, indicating that meaningful between-teacher differences persisted after controlling for student-level predictors.

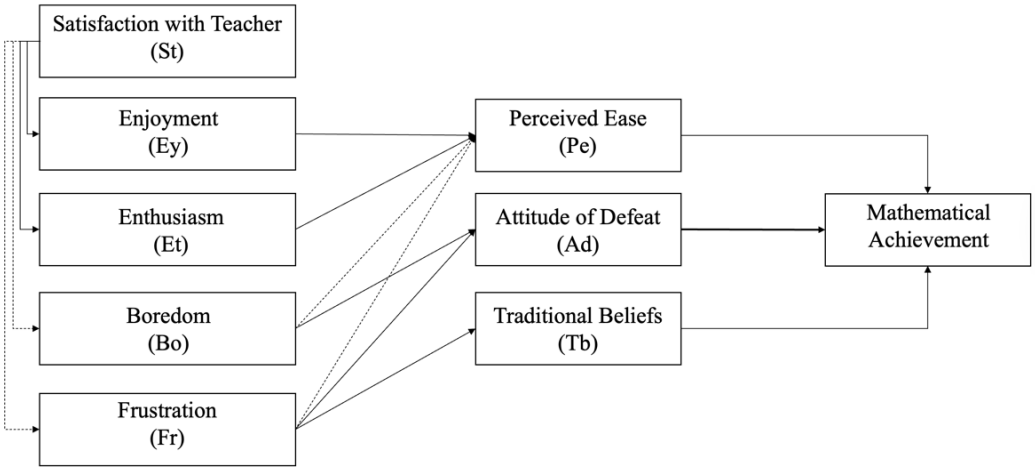
These findings suggest that mathematics achievement is shaped not only by students' affective-cognitive profiles but also by the instructional contexts in which they learn. Although satisfaction with teacher was associated with students' affective experiences, exploratory analyses revealed significant differences among teachers in mathematics achievement, $F(20, 632) = 6.25$, $p < .001$, $\eta^2 = .165$, perceived ease, $F(20, 616) = 3.12$, $p < .001$, $\eta^2 = .092$, and traditional beliefs, $F(20, 626) = 2.16$, $p = .003$, $\eta^2 = .064$. These findings suggest that teacher-level variability extends beyond students' perceptions of teacher satisfaction and may reflect broader classroom-level affective-cognitive configurations.

Cognitive and Affective Predictors of Mathematics Achievement

To examine how students' cognitive–attitudinal orientations and affective experiences relate to mathematics performance, a series of sequential multiple regression models was estimated at the student level. Model specification was informed by correlational patterns and distinguished between direct cognitive predictors of achievement and affective predictors associated with each cognitive node.

In the cognitive model, mathematics achievement was regressed on perceived ease (Pe), attitude of defeat (Ad), and traditional beliefs (Tb). The model was statistically significant and explained 23% of the variance in achievement. Perceived ease ($b = .176$) and traditional beliefs ($b = .264$) were positively associated with achievement, whereas attitude of defeat showed a negative association ($b = −.376$) and emerged as the strongest direct predictor. Affective variables were then modeled as predictors of each cognitive node. Enjoyment and enthusiasm were positively associated with perceived ease, whereas boredom and frustration were linked to

lower perceived ease and stronger negative attitudes. Boredom and frustration showed robust positive associations with attitude of defeat, and frustration was also positively associated with traditional beliefs. Satisfaction with teacher was consistently related to students' affective experiences but showed only weak direct association with achievement once cognitive and affective variables were included, suggesting an indirect role through affective–cognitive positioning. The resulting structure is summarized in Figure 1.



Note. Solid arrows represent positive associations; dotted arrows represent negative associations. Cognitive–attitudinal variables (Perceived Ease, Attitude of Defeat, and Traditional Beliefs) were entered as direct predictors of mathematics achievement. Affective variables were modeled as predictors of the cognitive nodes. Satisfaction with teacher was associated indirectly with achievement through students' affective experiences. The figure represents the sequence of regression models tested and should not be interpreted as evidence of causal or strictly unidirectional relations among variables.

Figure 1: Cognitive–Affective Configurations Associated with Mathematics Achievement

Additional analyses indicated that the cognitive model remained stable after controlling for student sex and school type. Perceived ease (b = .152), defeatist attitude (b = −.382), and traditional beliefs (b = .213) continued to emerge as significant predictors of mathematics achievement. Student sex (b = −.170) and school type (b = −.175) were also significant predictors of achievement.

Gender Differences in Affective Experiences and Cognitive–Affective Configurations

Independent-samples t tests identified gender differences in a limited subset of variables. Female students reported higher frustration and higher effort–enjoyment, whereas male students reported greater perceived pleasantness of mathematics. Mathematics achievement also differed significantly, with girls (M = 8.31, SD = 1.42) outperforming boys (M = 7.88, SD = 1.46), $t_{(636)}$ = 3.79, p < .001. No statistically significant gender differences were observed for enthusiasm, boredom dimensions, perceived difficulty, usefulness, interest, or mathematics self-efficacy.

Regression analyses showed a shared cognitive structure across sexes, with attitude of defeat negatively predicting achievement in both groups. However, magnitudes and explained variance differed. For females (N = 297), the cognitive model explained 26.6% of variance (R^2 = .266), with attitude of defeat as the strongest predictor (b = −.396, p < .001), followed by beliefs of incapacity (b = .244, p < .001) and perceived ease (b = .208, p = .002). For males (N = 314), the model explained 22.9% of variance (R^2 = .229); attitude of defeat remained strongest (b= −.329,

$p < .001$), though weaker than among females, while perceived ease ($b = .151$, $p = .012$) and self-efficacy ($b = .140$, $p = .009$) were also significant.

Affective models showed differentiated salience. For females ($N = 310$), 13.8% of variance was explained ($R^2 = .138$), with boredom emerging as the strongest predictor ($b = -.321$, $p < .001$). For males ($N = 316$), the model explained 17.8% ($R^2 = .178$); enjoyment ($b = .209$, $p = .001$) and satisfaction ($b = .156$, p = .006) were positively associated with achievement, whereas boredom showed a smaller negative association ($b = -.173$, $p = .005$). Overall, cognitive predictors accounted for more variance in both groups, but negative attitudinal weight was stronger among females, whereas positive affect showed greater salience among males.

Discussion and Conclusions

The present study examined emotions as explanatory mechanisms in mathematics achievement by integrating cognitive, affective, and contextual levels of analysis. Consistent with process-oriented perspectives on academic emotions, the findings indicate that emotions are not peripheral correlates of learning but dynamic processes that shape students' engagement with mathematics within instructional contexts (Pekrun, 2006; Pekrun et al., 2023). Across models, affective variables such as enjoyment, boredom, and frustration were systematically associated with students' cognitive orientations toward mathematics—particularly perceptions of ease and attitudes of defeat—which in turn were directly associated with achievement outcomes. Rather than supporting a strictly unidirectional pathway from emotions to cognition, the findings suggest that emotional experiences, beliefs, and attitudes operate as interconnected components of broader affective–cognitive systems that contribute to mathematics achievement.

These results align with prior work in mathematics education emphasizing the central role of affect in students' mathematical activity (Hannula, 2012; Roth & Walshaw, 2019), while extending this research by demonstrating that emotional processes operate within classroom-level structures. The significant between-teacher variability observed in the multilevel models supports the interpretation of classrooms as distinct affective–cognitive environments rather than interchangeable instructional settings, reinforcing sociocultural and context-sensitive views of mathematics learning (Boaler, 2002).

Gender-related analyses further revealed differentiated affective pathways to achievement. Female students' performance appeared more strongly linked to the regulation of negative emotions and persistence under conditions of frustration, whereas male students' achievement was more closely associated with positive engagement and perceived ease. These differentiated configurations are consistent with prior research showing that gender differences in mathematics are better understood in terms of affective experiences and engagement patterns than in terms of ability differences (Forgasz & Leder, 2017).

From a PMENA perspective, this study advances research on affect in mathematics education by moving beyond variable-centered approaches and demonstrating how emotions operate as explanatory mechanisms embedded in classroom contexts. By integrating student-level affective–cognitive configurations with teacher-level variability, the study contributes to a more process-oriented understanding of mathematics learning.

The results underscore the importance of attending to emotional processes when seeking to understand equity in mathematics education. Recognizing that students may achieve through differentiated affective pathways highlights the need for instructional approaches that are sensitive to emotional diversity and contextual demands. From this perspective, addressing affective mechanisms may be critical for understanding discrepancies between school-based achievement and performance in standardized assessments, particularly in contexts where girls'

higher classroom achievement is not consistently mirrored in large-scale testing outcomes (Gómez, 2025; Gómez, 2022).

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THE ROLE OF SELF-EFFICACY IN CALCULUS STUDENTS' ENROLLMENT AND ENGAGEMENT IN A MERIT PROGRAM

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At large universities where calculus is taught in a large lecture with an accompanying recitation, a Merit recitation program can replace the typical recitation. Merit recitations meet for twice as long, have a smaller class size, and a specifically trained graduate teaching assistant who guides students through tough problems and understanding the core concepts. We consider the role of self-efficacy and how Bandura's sources of information influence Merit students' decisions to opt into an elective Merit recitation and influence their in-recitation learning experiences. Data consists of two end-of-semester focus groups with 12 Merit participants. Through the data analysis, we provide two personas, which embody the rich variety of goals and motivations amongst undergraduate calculus students. The findings and methods can help program directors conduct student-centered program refinement.

Keywords: Calculus; Identity and Agency; Affect, Emotion, Beliefs, and Attitudes; Undergraduate Education

The 2015 Mathematical Association of America's study of successful calculus programs in the United States (Bressoud, Mesa, & Rasmussen, 2015) found that the use of active-learning strategies as the primary mode of instruction, or as a complement to lecture, was one of seven key characteristics of success. Unfortunately, in a follow up study on which of these characteristics were being implemented by graduate-degree granting mathematics departments, Rasmussen et al. (2019) found that many mathematics departments are struggling to implement active learning despite valuing this approach to instruction. As is common at many large universities in the United States, Calculus 1 at our large R1 university is taught in a large lecture recitation format. This is a newer model for our institution; in Fall 2022 we shifted from a model of many small sections taught by graduate teaching assistants (GTAs) to large lectures taught by a faculty member with supporting recitations led by GTAs. Many of our GTAs, unfamiliar with this model, lectured in recitation (despite students attending a lecture on the same content the day prior).

To spark a broad shift towards the use of active learning techniques within calculus recitations, we initiated a Merit program for Calculus 1 adapted from the Merit Program for Emerging Scholars from the University of Illinois at Urbana-Champaign and the well-known Treisman (1992) Mathematics Workshop Program (University of California Berkeley). This type of program has been implemented under a variety of names (e.g. MathExcel) at various points in time across the United States, including at Northwestern University, Northeastern University, University of Wisconsin Madison, University of Kentucky, Wayne State University, San Francisco State University, Oregon State University, and Cerritos College (University of Illinois, n.d.). The version at the University of Texas at Austin, called the Emerging Scholars Program, was shown to increase participants' odds of earning a calculus grade of an A or B and increased their preparation for Calculus 2 (Moreno et al., 1999).

Our research project emanates from a design problem: After building this alternative calculus recitation modeled after Merit, one designed to provide extra support in Calculus 1, we did not achieve full enrollment capacity (reaching ~76%). The program director wanted to understand

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

who the program was serving and what influenced their choice to enroll. Our research objective was to understand the incoming mathematical self-efficacy of program participants and how this influenced enrollment and in-program learning experiences. From this research, we supported the redesign of the program by building personas of program participants. Coming from the field of user design of digital technologies, personas are "life-like archetypes that stress users' diverse goals and motivations and embrace their corresponding needs and challenges" (Huynh et al., 2020) that are derived from the results of certain types of qualitative research. Rather than bypassing our design problem and solely focusing on the resulting research, we foreground our paper within a design process because we believe this user-centered approach to solving complex programmatic challenges will be of broader interest to the North American chapter of the International Group for the Psychology of Mathematics Education (PME-NA) community. In this paper, we share the findings of our research study and the resulting personas.

Context

A Merit recitation replaces the typical recitation and is twice as long to allow more time for the implementation of active-learning strategies, primarily group work. Merit recitations also have fewer students than standard recitations and are facilitated by a specifically trained GTA to guide students through solving problems. A central component of this change effort was the introduction of an 8-session Merit GTA professional development workshop series, which in 2024 and 2025 was attended by 9 and 11 GTAs respectively. Two GTAs were selected from each training cohort to lead a Merit recitation section the following fall; the other GTAs taught standard-type recitations where they still had the opportunity to put into practice the active learning strategies from the training, however with less in-class time than was available to their Merit GTA peers. In Summer 2024 and 2025, every student enrolled in Calculus 1 (700–900 students) was invited by email to switch from their standard-type recitation into a Merit recitation section. The email subject called this an "opportunity" with the body saying:

> You are eligible for a special section of Calculus 1, called Merit recitation! These recitation sections: have much smaller class size and a more personalized learning experience with a highly trained facilitator, and provide extra time to tackle tough problems and prepare for exams. (Earn one extra credit hour). We will fill these sections in the order I receive emails, so if you would like a slot, please email me at [email] as soon as possible!

In Fall 2024, 16 and 10 students opted into a Merit recitation section respectively. In Fall 2025, 10 and 17 students enrolled in a Merit recitation section. Time of day, either an 8 am start or a 5 pm end, partially accounts for the uneven distribution of students between the two Merit sections. In this report, we focused on the Fall 2024 Merit student cohort.

Relevant Theory & Research Question

We conceptualize mathematical self-efficacy broadly as a student's belief in their own capacity to learn and be successful—as they define it—in calculus. Mathematical self-efficacy has been shown to predict students' problem solving performance and to mediate effects of gender and prior experience on academic performance (Pajares & Miller, 1994). Psychologist Albert Bandura (1978) provided the foundation of our modern understanding of self-efficacy, positing the existence of four "sources of information" that strengthen or hinder self-efficacy: performance accomplishments, vicarious experiences, verbal persuasion, and emotional arousal. Performance accomplishments encapsulate prior success, or mastery, experiences which can increase overall self-efficacy and experiences of failure, which can negatively impact self-

efficacy. In mathematics, an experience of "getting a good grade" on an exam operates as a performance accomplishment. Vicarious experiences come from social comparison to peers and relate to witnessing or listening to the experiences of others. In our setting, a student seeing their peers struggling with a homework assignment and concluding that they will also struggle would constitute a vicarious experience operating as a source of information influencing self-efficacy. Verbal persuasion concerns the implicit or explicit external messages relayed from peers, instructors, advisors, or parents and can be both positive and negative. A recitation leader saying "I believe you can do well on this exam" could be verbal persuasion. In our study, emotional arousal concerns students' self-assessment of their anxiety or stress level. For example, a student who knows they are prone to high levels of test anxiety is more likely to have low self-efficacy with respect to completing exams.

Many researchers have explored connections between self-efficacy and various mathematical outcomes. For instance, Hammad et al. (2022) studied the relationship between self-efficacy, motivation, and achievement (grades) among undergraduates placing into foundational mathematics. Implementing a program similar to Merit, they found students self-efficacy improved throughout the semester and established a positive correlation between self-efficacy and achievement. Further, students (n = 130) described the teaching methods, group work, teacher attitude, and gamification as significant factors for increasing their self-efficacy. A study by Zientek et al. (2019) examined self-efficacy scores among students enrolling in developmental mathematics courses designed to incorporate active learning pedagogy and just-in-time supports at multiple U.S. universities. They found overall positive changes in self-efficacy and noted that sources of self-efficacy differed based on students' course grades. Further, mastery experiences and social persuasion saw the greatest gains, as was expected since group work and collaborative learning were emphasized in the course. They also reported a decrease in emotional arousal, suggesting students experienced less math anxiety. Thus, as self-efficacy influences the effort and persistence a person puts into a task (Bandura, 1978), it has been a useful construct to investigate in the context of mathematical learning.

Clemente et al. (2024) conducted a literature review on interventions designed to enhance students' mathematics self-efficacy. Findings grouped interventions into four types: (1) mastery-based, (2) peer modeling and collaboration, (3) teacher support and feedback, and (4) technology-based. Although the effectiveness of technology-based interventions was mixed, all other interventions showed consistent effectiveness for fostering positive beliefs and influencing student engagement, motivation, and mathematical achievement. Zakariya (2022) also reviewed and described sources of information in effective intervention studies designed to promote mathematical self-efficacy. Additionally, students themselves can provide insight into how and why these sources of information change their self-efficacy (White, 2025). We consider the following research question: In what ways do sources of information influence Merit students' decisions to opt into a Merit recitation and influence their in-recitation learning experiences?

Methods

A total of 12 students participated in two focus groups, of size eight and four respectively, conducted during the last week of class in Fall 2024. The gender ratios of the focus groups were skewed (this was not purposeful on our part) with the first group being primarily women (with one man) and the second being all men. The focus group protocol explored their decision-making process for enrolling in Merit in Summer 2024, overall Merit program experiences, and suggestions for improving the course. For example, under recruitment, students were asked

"What about the Merit program might have seemed appealing to you?" and "Why did you decide to sign up?"

Both focus groups were audio-recorded, transcribed by AI, cleaned by a member of our team, and coded by both authors. The data analysis involved two levels of coding. We began with an a priori approach to coding (Saldaña, 2016), independently analyzing the transcripts for examples of Bandura's four sources of information (performance accomplishments, vicarious experiences, verbal persuasion, and emotional arousal), as well as indications of the overall state of their mathematical self-efficacy (high or low). After coding less than half of the first focus group, we met to compare codes. A research log (Maxwell, 2013) documented discussions of differences in coding, resulting in a stronger codebook with examples and agreed upon definition of terms. For instance, we agreed that "vicarious experiences" included imagining scenarios or experiences such as what happens in other recitations or making predictions about student experiences in a general context (i.e. no one would pass a test learning in lecture alone). Thus, the first cycle of coding informed our research log of examples and improved our codebook for sources of information in context of calculus recitations.

We began a second cycle of coding by independently coding the entire first focus group for the four sources of information and overall assessment of self-efficacy. We met weekly to discuss codes to reach consensus. A second level of coding was added to explore the content of the passages. For example, codes for "verbal persuasion" included the source of the persuasion (email, parents, GTA, etc.) while "emotional arousal" identified the feeling or emotion expressed (humbling, fear, feeling unprepared, etc.). Second level coding included verses codes (Saldaña, 2016) as our data often included comparisons either between Merit and lecture or between Merit recitations and other recitations, and in-vivo codes (Saldaña, 2016) to maintain the students' perspective in the analysis. This process continued with focus group two. Once the coding was completed, passages containing the same codes were collapsed into code stacks. Code mapping, collapsing codes into common categories, was used for each source of information (Saldaña, 2016), with each author analyzing one focus group. For example, categories of level two codes under "performance accomplishments" included "Merit class norms," "Merit verses lecture," and "background." Memos summarizing the code mapping were drafted and reviewed by both authors, forming the basis for the development of each persona. Thus, the personas illustrate how Bandura's sources of information can interact to shape the learning experiences of students.

Findings

We will first present the overall findings and then disaggregate by focus group to present the personas. Sources of information are presented in the order of frequency and/or strength of influence, with the most frequent/strongest findings first.

Decision to Enroll in Merit

Students described all four sources of information as they discussed the appeal of the Merit recitations and what they hoped to gain from enrolling.

Verbal persuasion. The recruitment email served as the main source of verbal persuasion that encouraged students to sign up for Merit, although parents were also mentioned as a factor for some students. Students frequently pointed to the smaller class size mentioned in the email as a key appeal of the Merit recitations. For instance, one student stated: "I saw less people and I said join," and another explained "the main thing was not as many people." Others appreciated having more class time, working with a specially trained TA, being able to receive one-on-one help, and the opportunity to earn an extra credit hour. A few found the title of "Merit" special and thought the "experimental" section would be good for their resume or suggested it sounded

like an honor to enroll. In one focus group, several students mentioned parents affirming the need for Merit by suggesting their students would feel lost in a big class or needed the extra help.

Vicarious experiences. While discussing why they opted to enroll in a Merit recitation section, some students repeatedly alleged that Calculus 2 at this institution was notoriously difficult according to other students and identified this as their reason for enrolling in Merit. Many of these students explained that even though they had taken Calculus 1 in high school, they had been told by other students that "Calculus 2 is apparently really hard, so I wanted to redo it [Calculus 1]." Students also imagined they could "actually get help" in Merit based on the smaller class size, ensuring they would "know what I am doing."

Emotional arousal. Emotions were another source of information which influenced many students' decision to enroll. Students described feelings such as stress, fear (e.g. one student shared "I was just afraid" to take calculus), and nervousness about grades. Additionally, even students with high school calculus experience felt they had gaps in their knowledge or expressed a fear of enrolling in Calculus 2. In one focus group, students repeatedly emphasized their stress around being prepared for Calculus 2 with one student saying, "I kind of said earlier about the Calculus 2, just like I really wanted to, I wanted to get Calculus 1 *down*, like good, before I got to that." Accordingly, emotion helped motivate these students to elect for the extra support offered by a Merit recitation.

Performance accomplishments/entering self-efficacy. Although both focus groups included students who had taken calculus in high school (both AP Calculus and non-AP were reported and some had AP credit), one focus group also included students with no prior calculus experience. Further, the groups described differing entering levels of self-efficacy for university level mathematics. Some students explained they were typically comfortable engaging in mathematics yet did not feel prepared for Calculus 2. Others expressed low self-efficacy, stating they were behind in math or needed "all the help I can get." Examples of low mathematical self-efficacy included statements such as I am "not very good at math," or "math is not my strong suit," or I know I am "bad at math."

Experiences of Merit

Evidence of all four sources were also evident as students described their experiences in Merit and the calculus lecture.

Performance accomplishments/overall self-efficacy. Students reported that the Merit recitations helped them understand the material, built their confidence, and provided a smooth transition from high school into university mathematics. Students frequently contrasted their experiences in Merit recitation with their experiences in the large lecture and declared "you definitely learn a lot more in recitation than you did in the lecture." Students also reported "you can't ask a question" in lecture and admitted it was difficult to focus in the large lecture hall. In contrast, students described Merit recitations as "fun" and "helpful" and joked they "have no choice but to pay attention" in Merit due to the small class size and active learning design. Students reported being consistently engaged in doing mathematics during recitation and found it a safe place to ask "stupid questions." Additionally, some students formed friendships through Merit and "grew a relationship" with their GTA.

Verbal persuasion. Many students mentioned their GTA, who expressed a belief that they could learn and understand calculus, as a positive influential source of verbal persuasion throughout the semester. According to the students, GTAs offered encouragement and helpful test taking advice. One student explained "seriously, sometimes before an exam I've been like, I'm going to fail and then [GTA] has changed my mind." Further, the GTAs broke down the

material, explained content well, engaged with all students, and guaranteed groups worked together during recitation.

Vicarious experiences. Many students attended another recitation for one class when a GTA was ill and described "night and day" differences between a normal recitation and their experience in Merit. Students perceived their peers "were not learning anything" in other recitations and observed that many were on their phones. Students also spoke about common challenges all students encounter with passing calculus, noting many have to change majors due to not passing calculus.

Emotional arousal. Students overwhelmingly described positive emotions when reflecting on their experiences with Merit recitations. Many felt cared for and supported, enjoyed the fun learning environment, and were "comfortable" asking for help. Some explained how the class environment made them feel "comfortable" and able to learn. One student explained how they got to know their peers "kind of like high school in a way. That's one of the good things, you start to get actually comfortable, and it makes it a little easier to learn when you're more comfortable with the people around you." Some recalled moments that were humbling due to reviewing prerequisite material yet noted "it needed to happen" or felt embarrassed not knowing something yet were willing to receive help from peers.

Personas

The sources of information underlying students' self-efficacy for university-level calculus described above form the content of our personas. Coming from the field of user-centered design in the context of computer technology development, personas are composite characters collated from research findings that personify specific trends and patterns in the data (Dam & Teo, 2025). In our case, personas will stress students' diverse motivations behind choosing to enroll in Merit to increase their chances of success in calculus and the ways that they filter sources of information. Personas create memorable users to design for and convey information in ways that other artifacts, such as results displayed in a chart, cannot (Adlin & Pruitt, 2010, p. 1). Personas have already been used productively in a few limited cases by STEM education researchers; e.g. in understanding the unique needs of undergraduate researchers in a physics department (Huynh et al., 2020) or secondary mathematics students in Austria (Weinhandl et al., 2024). In industry contexts, one may encounter personas that are fictional or "ad hoc," that is generated from the assumptions of the design team based on their past experiences interacting with users. However, Adlin & Pruitt (2010, p. 22) argue for a much more rigorous process that creates data-driven (and more effective) personas, which we will outline with respect to our process: (1) identify ad hoc personas that allow us to articulate our pre-existing assumptions and biases, (2) analyze the data (as done above), (3) generate skeletons by creating brief bulleted lists of distinguishing data points from findings, (4) advance skeletons into fully fledged personas by adding concrete data to individualize and convey personality, (5) validate your personas against your Step 2 findings. This process aligns with our methods as described above.

After completing our analysis, we discovered that the focus groups provided sufficient distinguishing characteristics for the generation of a skeleton for each group. Accordingly, we constructed two personas: Maddie captures the motivations and mathematical self-efficacy of students in Focus Group 1 (an 88% female group) and Jacob is a composite of the male participants in the second focus group.

Note that these two personas were disseminated as a poster at the 28th Annual Conference on Research in Undergraduate Mathematics Education (Crooks Monastra & Lanius, 2026).

In the skeleton phase, we identified the following salient differences: (1) incoming self-efficacy, (2) the role of parents in deciding to enroll, (3) how the classroom environment supports asking questions, (4) nature of relationships in recitation, and (5) the role of calculus in degree attainment.

Maddie. Maddie has low mathematical self-efficacy and enrolled in Merit because she "just wants to pass calculus." When Maddie shared information about the Merit program with her parents, they strongly encouraged her to enroll, conveying that she needs additional support especially because of the large lecture setting. During the semester, Maddie valued Merit recitations not only for the time and space to ask her questions but also for creating a safe feeling space to ask "humbling" questions, particularly about material she felt she was "supposed to know." Accordingly, Maddie's relationship with the Merit GTA was more salient than her relationship to her classmates. This GTA-student relationship served as the foundation for a safe place to ask questions. Maddie imagines she would have failed Calculus 1 without a Merit recitation.

Jacob. Jacob has at least modest mathematical self-efficacy in general but had low self-efficacy with respect to passing the next course in the math sequence: Calculus 2 at our institution. He has heard a lot of stories about how our Calculus 2 is "apparently really hard." When Jacob shared information about the Merit program with his parents, they were neutral, telling him to decide for himself if he wanted to take Merit. During the semester, Jacob valued Merit because it gave time and space for asking questions, especially in contrast to large lecture and regular recitation. Whereas Jacob valued his relationship with his Merit GTA, the relationships he developed with other students in the class were particularly important to Jacob. He valued having made friends during Merit and studying with these peers outside of Merit class time. Jacob reports leaving Merit feeling prepared for Calculus 2 in the spring.

Leveraging personas to redesign. These personas guide structural adjustments that better support the wide variety of students who choose a Merit recitation. In this example, the Merit director better understood the role of parents in summer enrollment and can better market the program. Additionally, knowing what motivated those who enrolled and what their concerns were, we can be more purposeful in what we highlight in marketing materials. Finally, to better support students like Maddie, who need to feel comfortable in order to ask questions during recitation, the director can stress particular aspects of the Merit GTA training. Accordingly, the persona approach to user-centered design may be a useful tool for other providers of elective student supports in the mathematical sciences.

Discussion

We wish to note that the gender of the personas directly reflects the composition of the focus groups. We are mindful of balancing accurate reflection of the data and also not wishing to reinforce harmful stereotypes with a particular presentation of self-efficacy in relation to gender. We have chosen this gendered presentation so as to not erase the voices of these female students and to convey their unique needs. Sanabria & Penner (2017) found a gendered difference in students' responses to failing calculus, and that women in particular were substantially less likely to earn a STEM degree after failing this course. On average, women may encounter unique challenges and could be internalizing sources of information differently than the average male peers in recitation. This internalization process may impact their mathematical self-efficacy to a greater extent than their male peers.

In terms of sources of self-efficacy, mastery experiences are often considered the strongest influence on an individual's overall efficacy (Zientek et al., 2019). However, in our study, we

found vicarious experiences and verbal persuasion to be more influential on students' decision to enroll in an elective recitation. This may not be surprising, given they were entering a new learning environment (i.e. transiting from high school to university) and thus relied more heavily on others' experiences. Further, our findings of parental involvement in students' enrollment decision-making may also be contextualized in the broader literature. We note that due to the privacy rights given to students at age 18 in the Family Educational Rights and Privacy Act of 1974 (FERPA), math educators at the tertiary level may not think of parents as active participants in a students' learning of mathematics. However, as we saw here and is the case nationally, parents can play an important and highly influential role in a students' transition from high school to college (Smith et al., 2018) and may influence students course enrollment decisions.

With respect to students' perceptions of active learning, Deslauriers et al. (2019) found that students learn more with an active teaching method but feel like they learn more in a passive lecture. This is inconsistent with our findings, which is likely due to the massive disparity in classroom size. In their review of the literature, Wright et al. (2019) concluded that research consistently indicates students perceive learning more as class size decreases. This suggests future work on the role of class size in students' perceptions of learning with respect to pedagogical style.

Conclusion

In the national landscape, calculus is a common target for programming that can range from mandatory (standard recitation) to an elective enrollment (e.g. Merit recitation), and supplemental programs (tutoring, supplemental instruction, final exam bootcamps). Understanding why students choose to engage in these activities can help the field more broadly understand how to support students in calculus. Although personas are highly context driven and our exact findings will not represent the broad range of student motivations and experiences, our process for developing personas may be empowering for other departments aiming to redesign calculus supports. Additionally, although there is a large body of research investigating the outcomes of participation in optional academic programs like supplemental instruction, this research typically does not account for student "self-selection," i.e. who decides to participate and why (Bowman et al., 2021). Thus, our findings, although context dependent, do contribute some understanding to this gap in the research.

Limitations

We focused solely on the decision process of students that did enroll in the program. To more fully capture the nuance of this design challenge, future research should target students that were invited and chose not to enroll. Further, our study only includes end-of-semester student data. Future research could follow students longitudinally throughout their semester, which would provide a more complete understanding of student efficacy and how it changes over time.

We hope that this study may serve as an inspiring case for others looking to engage in student-centered redesign of their calculus offerings.

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COMPARING OPERATIONALIZATIONS OF CONCEPTUAL INSTRUCTION TO PREDICT STUDENT MATHEMATICAL SELF-BELIEFS

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This analysis examines four different operationalizations of conceptual instruction (CI) in mathematics education, derived from previous research. Using data from the High School Longitudinal Study of 2009 (HSLS:09), confirmatory factor analyses (CFAs) and ordinary least squares (OLS) regression models are used to assess the psychometric properties and investigate the predictive relationship between conceptual instruction in mathematics and student self-concept and self-efficacy. Results demonstrate that operationalizations of conceptual instruction are robust to slight variations but suggest that conceptual instruction relies heavily on mathematical reasoning, problem solving, and connecting mathematical ideas. Conceptual instruction functions as a significant predictor of students' self-beliefs in mathematics.

Keywords: Measurement, Beliefs, Mathematical Thinking

As research in mathematics education tries to understand and overcome the challenges students face in learning mathematics, recent studies have focused on improving classroom instruction. The National Council of Teachers of Mathematics (NCTM) identified one problem with instruction in the US as "too much focus is on learning procedures without any connection to meaning, understanding, or the applications that require these procedures" (NCTM, 2014, p. 3), and advocates for high-quality instruction for all students. Fauzan (2013) states that high-quality mathematics instruction should include problem-solving, reasoning, communicating, connecting, and representing mathematics, and should have students engaged in critical and creative thinking. This type of instruction, frequently called conceptual instruction, contrasts with procedural instruction which focuses on memorization and proficiency in using algorithms to solve computational problems in mathematics. Though some research suggests both types of instruction are important (Nahdi & Jatisunda, 2020), other research found that spending extra time on conceptual instruction improved student understanding more than if that extra time was used on procedural practice (Rittle-Johnson et al., 2016).

Despite the recognized need for conceptual instruction, there is much variation in how it is described and operationalized. As a result, it is difficult to know exactly what educational practices most significantly contribute to conceptual instruction. This lack of clarity makes it difficult for experienced teachers to know how to improve their practice and for new teachers to receive the appropriate training that will enable them to provide conceptual instruction for their students. Additionally, though a link between students' conceptual knowledge and mathematics performance has repeatedly been observed, little research has investigated a connection between conceptual instruction and students' self-beliefs related to mathematics. Since increasing student confidence and interest in pursuing STEM fields is a current goal of mathematics education, it is important to know if conceptual instruction can improve those outcomes. Therefore, this analysis seeks to answer the following two questions: (1) How can conceptual instruction in mathematics be operationalized/measured? (2) Is there a relationship between conceptual instruction in mathematics and students' self-beliefs, specifically self-concept and self-efficacy?

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Literature Review

Though the need for high-quality mathematics instruction is universally agreed upon, research with this focus incorporates multiple educational theories and instructional activities. Some of the theoretical frameworks cited when working to increase students' conceptual understanding include Freudenthal's (1983) Realistic Mathematics Education (RME, See Sahin & Gurbuz, 2022), Taba's (1963) Concept Based Instruction (CBI, See Ncube & Luneta, 2025) and Ben-hur's (2006) Concept-Rich Instruction (CRI, See Tashtoush et al., 2022). Each framework takes a constructivist approach, positioning students as active learners who build deep understanding through applying mathematical principles in a variety of authentic contexts, and connecting procedures to principles. Most research on the topic studies the implementation of an instructional intervention or teaching activity that is meant to increase students' conceptual understanding and then measures its influence on learning outcomes. These studies can provide insights into how to improve conceptual instruction, but they do not measure conceptual instruction as a variable. Without a way to clearly operationalize and measure conceptual instruction in a classroom, future research is limited in its ability to examine how variation in the amount or quality of conceptual instruction relates to student outcomes.

Statistical Operationalization of Conceptual Instruction

A recent body of research has begun to address this concern by using data from the High School Longitudinal Study of 2009 (HSLS:09, Ingels et al., 2011, 2014) to statistically operationalize and measure conceptual instruction. The HSLS:09 is a nationally representative study designed to track students' trajectories from the beginning of high school through postsecondary education. In the baseline year it included a supplemental teacher survey asking ninth-grade mathematics teachers to report how much emphasis they placed on various instructional objectives. These items have been used by multiple research teams to construct measures of conceptual instruction, though their operationalizations differ in important ways.

McQuillen (2016) conducted a confirmatory factor analysis on the HSLS:09 teacher survey items and identified a conceptual instruction factor which included six items describing emphases on teaching mathematical concepts, developing problem-solving skills, reasoning mathematically, connecting mathematics ideas, teaching the logical structure of mathematics, and explaining ideas effectively. This same combination was theoretically proposed and statistically verified by Miles (2024) as well. Yu and Singh (2018) used the HSLS:09 data and included the six items that McQuillen and Miles models used but added three additional items that emphasize increasing student interest in mathematics, preparing students for further study in mathematics, and teaching about the history and nature of mathematics. Ji (2020) also had a six-item factor for conceptual instruction, but his was unique as he included an emphasis on preparing students for further study in mathematics and excluded the emphasis on having students explain ideas in mathematics. Finally, Gerasimova et al. (2023) operationalized conceptual instruction with a six-item factor, but their factor differed from the McQuillen and Miles models in significant ways. First, they excluded emphases on teaching mathematical concepts and reasoning mathematically, citing measurement issues. Secondly, they included the emphasis on increasing student interest in mathematics and were the only research team to include an emphasis on teaching students how to apply mathematics in business and industry. Other researchers argued that the emphases of preparing students for further study in mathematics or for business and industry applications could be enacted using either conceptual or procedural approaches, so they should not be included in a conceptual instruction variable (Miles, 2024).

Taken together, these HSLS:09 studies suggest that it is possible to statistically operationalize conceptual instruction using teacher survey data. However, the variation in item selection highlights the lack of consensus on which specific instructional emphases indicate conceptual instruction. This underscores the need for continued study and refinement of the different operationalizations and instruments designed to capture conceptual instruction.

Self-Beliefs in Mathematics: Self-Efficacy and Self-Concept

Most research on conceptual instruction has focused on student achievement as the dependent variable. However, achievement is not the only important outcome of mathematics instruction. Students' self-beliefs—particularly self-efficacy and self-concept—are among the strongest non-cognitive predictors of academic performance (Lee & Stankov, 2018; Stankov & Lee, 2017), and they influence a range of educationally significant outcomes including persistence, course selection, and career aspirations (Bandura, 1997; Eccles, 2009). Understanding whether instructional approaches can shape these beliefs is therefore consequential for both theory and practice.

Self-efficacy is part of Social Cognitive Theory (SCT) and refers to an individual's belief in their capacity to successfully carry out specific tasks (Bandura, 1986, 1997; Britner & Pajares, 2006). It is primarily developed through mastery experiences, vicarious experiences, social persuasion, and physiological states. It describes a student's belief that they can be successful at specific tasks in the future, like doing well on a mathematics test or mastering difficult material. Self-efficacy influences behavioral choices, effort, and persistence in the face of obstacles and is positively related to performance and persistence in mathematics (Bandura, 1986; Multon et al., 1991, as cited in Yu & Singh, 2018).

Self-concept, by contrast, is a broader, domain-specific evaluation of one's abilities that is past-oriented rather than future-oriented (Bong & Skaalvik, 2003; Marsh & Craven, 2006). Whereas self-efficacy asks, "Can I do this math task?" self-concept addresses the more general question, "Am I a math person?" The formation of self-concept relies on relative comparisons of one's own performance in mathematics to their performance in other domains, as well as comparison to their peers' performance in mathematics (Arens et al., 2022; Wolff et al., 2018). Self-concept evaluations also often incorporate students' attitudes related to mathematics while self-efficacy does not (Bong & Skaalvik, 2003).

Although self-efficacy and self-concept are related, they are empirically distinguishable constructs that are operationalized differently and predict different outcomes (Parker et al., 2014; Miles, 2025). Mathematics self-concept is more strongly related to school grades, whereas mathematics self-efficacy is more strongly related to standardized test scores (Arens et al., 2022). Both constructs display a reciprocal relationship with mathematics achievement where positive self-beliefs enhance future achievement, which in turn strengthens self-beliefs (Marsh & Craven, 2006). In large-scale international assessments, self-efficacy and self-concept consistently emerge among the strongest non-cognitive predictors of mathematics achievement, showing stronger effects than socioeconomic status (Lee & Stankov, 2018; Stankov & Lee, 2017). These findings position self-beliefs as promising targets for educational intervention.

Connecting Conceptual Instruction to Self-Beliefs

There are strong theoretical reasons to expect conceptual instruction to positively influence students' self-beliefs. According to SCT, mastery experience is the most powerful contributor to self-efficacy (Bandura, 1997), and conceptual instruction may provide more meaningful mastery experiences, leading students to view their success as genuine and transferable. Yu and Singh (2018) hypothesized such a relationship but noted that few studies had directly examined links

between conceptual instruction and student motivation or beliefs. Ncube and Luneta (2025) likewise suggest that students who receive conceptual instruction may show increased confidence and come to view mathematics as "doable, accessible and enjoyable."

From a self-concept perspective (Eccles & Wigfield, 2002; Tracey et al., 2020), conceptual instruction may strengthen self-beliefs and attitudes by helping students connect ideas and apply mathematics in personally meaningful contexts. As students develop deeper understanding and see mathematics as relevant and meaningful for themselves, their mathematics self-concept may improve. Despite these theoretical rationales, most studies focus on achievement outcomes so direct empirical evidence linking conceptual instruction to self-beliefs remains limited.

In summary, the existing literature suggests that conceptual instruction can be operationalized and measured; however, uncertainty remains regarding the most appropriate operationalization. Additionally, though a large body of literature suggests that conceptual instruction positively predicts student achievement, there are fewer studies that examine its influence on students' self-beliefs. Further investigation into the relationship requires both refined measurement of conceptual instruction and direct empirical investigation of its relationship to students' self-efficacy and self-concept. Such research would advance understanding of the full range of benefits that conceptual instruction may confer and help educators design instruction that not only improves performance but also strengthens the mathematical self-beliefs that sustain long-term mathematical engagement.

Methods

To statistically evaluate alternative operationalizations for conceptual instruction and identify any relationship that exists between conceptual instruction in mathematics and students' mathematics self-efficacy and self-concept, I conducted an analysis of the publicly available HSLS:09 data (Ingels et al., 2011, 2014). The specific data for this study came from the baseline year and first follow up surveys. In the baseline year study, parents and mathematics teachers of all participants were asked to complete surveys to provide additional context. Both the student and teacher surveys are used in this study, and all analyses were run in Rstudio (version 4.3.1) using the lavaan (Yves, 2012) package.

The Variables

The following variables were included in the analysis:

1. Conceptual Instruction (CI) – Four versions of this variable align with the four different HSLS operationalizations outlined previously. The items for each variable came from the teacher instrument in the baseline year of data collection where teachers rated the amount of emphasis (on a 4-point scale) they placed on different elements of mathematics instruction. Individual factor scores were extracted from CFA models to use as measures of conceptual instruction in the regression analysis.
2. Mathematics self-efficacy (x1mtheff, x2mtheff) – the HSLS:09 team created composite variables using principal components analysis to measure student self-efficacy. The variables are comprised of four items asking students to rate how confident they are at understanding their textbook, mastering skills, and doing well on exams and assignments. Variable weights correct for missing data, and each variable is standardized with a mean of 0 and standard deviation of 1.
3. Mathematics self-concept (x1mthid, x2mthid) – the HSLS:09 team created these variables from two items at each timepoint that asked students to rate how strongly they (and others) considered themselves a "math person." Again, variable weights correct for

missing data, and each variable is standardized with a mean of 0 and standard deviation of 1.

4. Control variables: Student sex (dummy variable coded 0 for male and 1 for female), student race (dummy variable coded 1 for White/Asian and 0 for other non-white), and students' socioeconomic status (continuous HSLS:09 created variable that measures SES in relation to locale and parent occupation and educational level).

Procedure

To compare different operationalizations of conceptual instruction, I created one-factor confirmatory factor analysis (CFA) models that aligned with each of the models cited previously in this paper. I ran all models using robust standard errors and a WLSMV estimator to correct for non-normal Likert-type data. I compared the models using the *Comparative Fit Index* (CFI), the *Tucker-Lewis Index* (TLI), the *Root Mean Square Error of Approximation* (RMSEA), and the *Standardized Root Mean Square Residual* (SRMR). Table 1 compares the fit for each of the conceptual instruction variables.

Table 1: Fit Indices for Confirmatory Factor Analysis Models of Conceptual Instruction

Model	Items	CFI	TLI	RMSEA	SRMR
Yu & Singh (2018)	9	.990	.987	.041	.041
McQuillen (2016) / Miles (2024)	6	.995	.992	.042	.033
Ji (2020)	6	.992	.987	.053	.046
Gerasimova et al. (2023)	6	.992	.987	.037	.029

Note. CFI = Comparative Fit Index; TLI = Tucker-Lewis Index; RMSEA = Root Mean Square Error of Approximation; SRMR = Standardized Root Mean Square Residual. All models used robust standard errors and diagonally weighted least squares (DWLS) estimation. Sample sizes ranged from n = 12,846 to n = 12,971.

Following the CFA analysis, I ran 20 linear regression models to compare how well each of the proposed variables predicted student self-efficacy and self-concept (in mathematics) at both the baseline year and the first follow-up. Many of the baseline year surveys were completed in the fall of the students' ninth grade year. Since students may not have had much time with their ninth-grade mathematics teachers it is possible that insufficient time had passed to reveal a measurable relationship between mathematics instruction and students' self-beliefs. Including student self-beliefs two years later could help clarify or support any relationship. If the instructional emphases of a ninth-grade mathematics teacher continue to predict student self-beliefs two years later, it suggests a possible longer lasting relationship which continues to have influence even after the student moves on in their education.

Confirmatory Factor Analysis Results

Table 1 shows fit indices for the four single-factor CFA models of conceptual instruction. Each model incorporated items from the mathematics teacher survey where teachers reported their instructional emphases, with items reverse-coded such that higher values indicate greater emphasis. All models employed robust standard errors to account for the non-normality of the data, as indicated by significant multivariate skewness ($b_1p = 11.86$, $p < .001$) and kurtosis ($b_2p = 227.87$, $p < .001$) in Mardia tests.

All models demonstrated acceptable to excellent fit across conventional thresholds. The McQuillen (2016) and Miles (2024) model showed the strongest fit when measured with the CFI (.995) and TLI (.992) while the Gerasimova et al. (2023) model exhibited the lowest RMSEA

(.037) and SRMR (.029) values. Despite these variations, all models met established criteria for adequate fit (CFI > .90, TLI > .90, RMSEA < .08, SRMR < .08), suggesting that the different item selections captured the conceptual instruction construct with comparable psychometric quality. Slightly greater variation appeared in the individual item regression coefficients for each model. The McQuillen and Miles model showed strong regression coefficients for all six items ($\lambda > .55$). In the Yu and Singh (2018) model the two items on emphasizing the history of mathematics ($\lambda = .2$) and preparing students for future study in mathematics ($\lambda = .51$) did not meet this same benchmark. The Ji (2020) model had one item that did not meet this benchmark (i.e., preparing students for future study in mathematics, $\lambda = .48$), as did the Gerasimova et al. (2023) model (i.e., preparing students for business or industry applications, $\lambda = .32$). Overall, the items emphasizing problem solving, reasoning mathematically, and connecting mathematical ideas were the strongest in all models ($\lambda > .72$).

Predictive Validity: Regression Results

To assess the predictive validity of each operationalization, I estimated ordinary least squares regression models predicting four outcomes: ninth grade mathematics self-concept (*x1mthid*), ninth grade mathematics self-efficacy (*x1mtheff*), junior year mathematics self-concept (*x2mthid*), and junior year mathematics self-efficacy (*x2mtheff*). All models controlled for student sex, race (White/Asian vs. other non-White), and socioeconomic status. Factor scores for conceptual instruction were extracted using a ten Berge correction to ensure comparability across operationalizations. Table 2 presents the results for all four dependent variables.

Table 2 Regression Results: Conceptual Instruction Predicting Mathematics Self-Beliefs

Variable	Yu & Singh (2018)	McQuillen (2016) Miles (2024)	Ji (2020)	Gerasimova et al. (2023)
Ninth Grade Mathematics Self-Concept (x1mthid)				
CI	0.136*** (0.015)	0.137*** (0.015)	0.139*** (0.016)	0.104*** (0.013)
Sex (Male)	0.112*** (0.018)	0.112*** (0.018)	0.112*** (0.018)	0.113*** (0.018)
Race (W/A)	−0.006 (0.020)	−0.005 (0.019)	−0.008 (0.019)	−0.006 (0.019)
SES	0.168*** (0.012)	0.171*** (0.012)	0.171*** (0.012)	0.174*** (0.012)
R^2	.030	.030	.030	.029
n	11,910	12,000	12,020	12,020
Ninth Grade Mathematics Self-Efficacy (x1mtheff)				
CI	0.074*** (0.015)	0.071*** (0.016)	0.065*** (0.016)	0.062*** (0.014)
(Sex) Male	0.167*** (0.019)	0.167*** (0.019)	0.167*** (0.019)	0.168*** (0.019)
Race (W/A)	−0.093*** (0.020)	−0.093*** (0.020)	−0.094*** (0.020)	−0.093*** (0.020)
SES	0.198*** (0.012)	0.199*** (0.012)	0.200*** (0.012)	0.200*** (0.012)
R^2	.034	.034	.034	.034
n	10,770	10,850	10,870	10,870
Junior Year Mathematics Self-Concept (x2mthid)				
CI	0.122*** (0.016)	0.126*** (0.017)	0.127*** (0.017)	0.091*** (0.014)
Sex (Male)	0.176*** (0.020)	0.179*** (0.020)	0.179*** (0.020)	0.176*** (0.020)
Race (W/A)	−0.024 (0.022)	−0.026 (0.021)	−0.026 (0.021)	−0.024 (0.021)
SES	0.163*** (0.013)	0.165*** (0.013)	0.166*** (0.013)	0.169*** (0.013)
R^2	.030	.031	.031	.029
n	10,330	10,410	10,420	10,420
Junior Year Mathematics Self-Efficacy (x2mtheff)				
CI	0.077*** (0.016)	0.081*** (0.017)	0.077*** (0.017)	0.063*** (0.014)
Sex (Male)	0.200*** (0.020)	0.202*** (0.020)	0.201*** (0.020)	0.204*** (0.020)
Race (W/A)	−0.112*** (0.021)	−0.113*** (0.021)	−0.114*** (0.021)	−0.110*** (0.021)

SES	0.168*** (0.013)	0.169*** (0.013)	0.171*** (0.013)	0.170*** (0.013)
R^2	.031	.031	.031	.031
n	10,210	10,280	10,300	10,290

Note. CI – Conceptual Instruction. Unstandardized coefficients with standard errors in parentheses. All models control for student sex, race, and socioeconomic status (SES). All sample sizes rounded to the nearest ten per NCES requirement. *** $p < .001$.

Discussion of Findings

The results presented support two principal conclusions. First, no single operationalization of conceptual instruction emerges as definitively superior to the others. While the models exhibit minor variations in fit indices and coefficient magnitudes, these differences are substantively small and do not suggest that one measurement approach captures the construct more accurately or comprehensively. All four models appear statistically similar in their ability to predict self-efficacy but the same cannot be said when predicting self-concept. Three of the models produce nearly identical regression coefficients when predicting self-concept ($\beta = 0.136–0.139$ for ninth grade and $\beta = 0.122–0.127$ for junior year) with overlapping confidence intervals that preclude meaningful distinctions. The Gerasimova et al. (2023) model is not as strong at predicting student self-concept ($\beta = 0.104$ for ninth grade, $\beta = 0.091$ for junior year) but still shows conceptual instruction as a statistically significant predictor with similar patterns of association. This consistent pattern, despite variation in operationalizations, suggests the observed relationships reflect a genuine association with conceptual instruction that is independent of a specific measurement artifact.

The statistical similarity across the models suggests that conceptual instruction is a broad latent construct that can be achieved through a variety of instructional emphases and practices. However, since teaching the history of mathematics and preparing students for future study in mathematics, or business/industry applications had the weakest factor loadings, the inclusion of those items does not seem to be statistically necessary. In fact, three items appeared as the strongest contributors to the conceptual instruction variable. Focusing on reasoning mathematically, connecting ideas in mathematics, and developing problem solving skills consistently had the strongest factor loadings ($.7 < \lambda < .9$) suggesting that conceptual instruction may be increased by focusing solely on these three items. Further investigation should be done to verify this finding, but if conceptual instruction can be achieved through an emphasis on these three objectives, then teacher preparation and professional development programs can become more focused in giving teachers the tools necessary to improve their ability to provide conceptual instruction by engaging students in problem solving, reasoning mathematically, and connecting mathematical ideas.

The second conclusion is that conceptual instruction consistently emerges as a significant predictor of both mathematics self-concept and self-efficacy. Across all regression models the coefficient for conceptual instruction achieves statistical significance at the $p < .001$ level. This robustness holds for both self-efficacy and self-concept at the ninth grade and junior year timepoints, with the strongest association seen when predicting ninth-grade self-concept ($\beta = .139$). Specifically, depending on operationalization, a one-unit change in conceptual instruction predicts a change of .38 to .49 standard deviations in ninth-grade self-concept after controlling for sex, race, and socioeconomic status. This means that if a student with an average level of self-concept in mathematics was moved from a classroom with a minimum level of conceptual instruction to a classroom with a maximum level of conceptual instruction, their predicted self-concept would increase approximately 19 percentile points. For many students, self-beliefs in

mathematics begin to decline significantly in middle school. Improving conceptual instruction could help slow that decline and keep students more engaged and confident in learning mathematics. These results are also noteworthy as conceptual instruction shows a stronger predictive ability than either sex or race when predicting ninth-grade self-concept, suggesting that conceptual instruction may be instrumental in overcoming gender and racial gaps in mathematics.

Admittedly, these conclusions are based on correlational analyses and therefore cannot be used for causal inference. However, the robustness of these associations across multiple operationalizations is particularly noteworthy given the variation in item composition across models. Various operationalizations produce substantively similar results, suggesting that the positive relationship between conceptual instruction and student mathematics beliefs is not contingent on specific measurement choices. This consistency enhances the generalizability of findings across studies employing different operationalizations of conceptual instruction.

The fact that focusing on problem solving, reasoning mathematically, and making connections in mathematics appear to be the most influential components of conceptual instruction, suggests that teacher training and development programs which focus their attention on these three areas could result in essential improvements in providing conceptual instruction in mathematics. Improving conceptual instruction may not only increase student achievement but also bring about improvements in student's self-beliefs. Though the influence of a single teacher has its limits, as more teachers provide increased conceptual instruction, we can harness the power of the reciprocal relationship between self-beliefs and achievement to create a positive upward spiral, resulting in substantial ongoing improvements in both areas.

In sum, these results show that conceptual instruction can be effectively operationalized and that it is a consistent predictor of student self-beliefs in mathematics. Future research should continue to examine the mechanisms through which conceptual instruction influences student self-beliefs and explore whether these relationships vary across student subgroups or instructional contexts.

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PROVOKING PRE-SERVICE TEACHERS' EXPERIENCE OF INTELLECTUAL NEED FOR ANGLE MEASURE AND EXPONENTIAL GROWTH

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We present a study of pre-service mathematics teachers' engagement with mathematical tasks designed to provoke their experience of intellectual need. Grounded in the theories of Piagetian equilibration and reflecting abstraction, we analyzed three PSMTs' engagement with tasks about angle measure and exponential growth. Findings indicate that carefully engineered impasses—where existing procedural schemes proved insufficient—triggered students' need for new mathematical structures (e.g., radians and partial growth factors). Participants moved from pseudo-empirical generalizations toward structural justifications. These results offer implications for pre-service teacher education by demonstrating how mathematics educators can foster deeper conceptual coherence in future teachers.

Keywords: Problem-Based Learning, Cognition, Design Experiments, Undergraduate Education

Introduction

Problem solving has long been viewed as both an essential source and product of learning mathematics (Carpenter et al., 1993; Erickson, 1999; Hiebert et al., 1996). Students' experience with mathematics, however, is often "problem-free" in the sense that the tasks they encounter primarily consist of routine exercises soluble via the execution of rehearsed skills rather than the application of deep strategic reasoning (Fuller, Rabin, & Harel, 2011). Procedural tasks of this type frequently lack an intellectual challenge, leaving students without a clear mental image of the problem-solving process (Fuller et al., 2011). This reliance on routine exercises is prevalent in undergraduate mathematics instruction (Schoenfeld, 2016) and curricula (Lithner, 2004), which limits students' opportunities to deepen their mathematical understandings through reasoning about novel problems. In contrast, "Problem-laden" activity originates in and is sustained by students' construction of a mathematical task in such a way that they (1) recognize their current knowledge structures as insufficient to solve the problem, and (2) construct an image of particular characteristics of the understanding(s) that would enable them to progress towards a solution. Well-designed problems thus serve as critical catalysts for students to develop their reasoning and build deeper understandings.

Educational psychologists have long emphasized the influence of "desirable difficulties" and "failure-driven scaffolding" in supporting students' mathematical learning (Dewey, 1938; Kapur, 2015; Sinha et al., 2021). Brousseau (1997) argued that authentic problem engagement is necessary to stimulate the evolution of students' conceptions. Similarly, Chevellard's Anthropological Theory of the Didactic suggests that activity is fundamentally motivated by the resolution of experienced difficulties (Bosch & Gascón, 2014). This aligns with Piagetian theory (1971, 1985), which identifies disequilibrium or cognitive perturbation as a vital precondition for the modification of cognitive structures (Tallman & O'Bryan, 2024). Empirical studies support these theoretical frameworks, demonstrating that cognitive incongruities (Graesser et al., 2005), solving problems prior to instruction (Kapur & Rummel, 2012), and

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navigating impasses (VanLehn, 1988; Kapur, 2014) correlate with constructive reasoning and improved learning outcomes.

Consistent with this broad theoretical tradition of impasse-driven catalysts for cognitive development, Harel (1998) introduced the *necessity principle*, proposing, "For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need" (p. 501). Since its inception, intellectual need has emerged as a cornerstone of instructional and curriculum design (Burger & Markin, 2016; Caglayan, 2015; Foster & de Villars, 2015; Koichu, 2012; Leatham et al., 2015; Rabin et al., 2013; Zazkis & Kontorovich, 2016), reflecting a broad consensus that it serves as a necessary catalyst for learning mathematics.

While the literature frequently calls for educators to induce these experiences, there remains a significant gap in practical guidance regarding the design and implementation of *intellectual need-provoking tasks* (IN-PTs). The few recommendations for developing intellectual need-provoking tasks reported in the literature lack detail about how an instructor might design and implement mathematical tasks that support students in experiencing a perturbational state while simultaneously orienting them to extend, modify, or create new knowledge structures. Our study seeks to address this gap.

The central aim of the research reported in this paper is to investigate undergraduate mathematics students' collaborative engagement with IN-PTs. We focused our analysis on the characteristics of methodically designed IN-PTs and instructors' pedagogical facilitation of them that promoted or limited students' experiences of intellectual need. The primary goal of this research is to inform actionable recommendations for the design and implementation of IN-PTs in mathematics teaching. Specifically, our study was guided by the following research question: *What intellectual need did students appear to construct through their collaborative engagement in IN-PTs for angle measure and exponential growth?*

Theoretical Framing

Harel developed and refined the concept of intellectual need in the radical constructivist tradition of impasse-driven catalysts for cognitive development. Compatible with Piaget's theory of the *equilibration of cognitive structures* (Piaget, 1977), a student experiences intellectual need when they enter a state of disequilibrium and feel a sense of curiosity that compels them to develop cognitive resources to resolve their perturbation. Intellectual need is a psychological state with cognitive and affective entailments that results from an individual's conception of a situation as problematic, and their awareness that resolving the situation requires an extension, modification, or creation of knowledge structures. Distinct from other mental states such as confusion or bewilderment, intellectual need results from one's conception of a problematic situation that orients them to engage in conceptual activity intentionally directed toward resolving the situation. Intellectual need is therefore constructive in that it stimulates, rather than impedes, generative reasoning.

Intellectual need is *not* a state that paralyzes reasoning and induces an affective state of interest, which would result only in a student being receptive to *absorbing* the mathematical concepts communicated to them, or the skills demonstrated for them, by a knowledgeable expert (e.g., "I don't know how to do this. I'm ready for someone to show me."). Intellectual need cannot, on the one hand, be conceptualized in a manner consistent with Piaget's idea of equilibration of cognitive structures—and constructivist epistemology generally—while on the other hand promote an exclusively affective state where learning is a product of perception—a premise of empiricist epistemology. Theoretical coherence requires that both the inducement of

intellectual need and the learning afforded by its experience be a product of the actions, operations, abstractions, and generalizations *of the learner*.

For this study, we define intellectual need as an exigence a learner experiences to engage in the mental actions that, through the process of *reflecting abstraction*, form the substrate of cognitive schemes. *Reflecting abstraction* is an abstraction of *actions*, dissociated from the effect they produce and projected to the level of thought (Ellis, Paoletti, & Lockwood, 2024; Piaget, 2001; Tallman & O'Bryan, 2024). This is distinct from *pseudo-empirical abstraction*, which is an abstraction of the properties of objects that have been modified by or created through an individual's actions (Piaget, 1977).

Methods

Experimental Context

We recruited three students (pseudonyms Abbie, Tyra, and Kent) from a large public research university in the Midsouth United States to participate in the study. All participants were mathematics majors with an emphasis in secondary teaching. The participants were enrolled in the same mathematics content course for pre-service secondary teachers taught by the first author for the duration of data collection. Students were invited to participate based on their willingness to collaborate with peers to solve challenging problems and ability to clearly communicate their mathematical reasoning.

Experimental Methods

The research team collected video data during class sessions of participating students' collaborative work IN-PTs addressing the following topics: (1) *constant rate of change*, (2) *angle measure*, (3) *partial growth factors of exponential functions*, and (4) *symbolic representations of exponential functions*. We also collected and scanned students' written work.

The research team also conducted and video recorded individual semi-structured clinical interviews with participating students within one or two days after they had worked in class on a particular IN-PT but before the next class, unless scheduling conflicts occurred. The central focus of these interviews was to further explore each participant's interpretation of the task, their conception of what contributed to its difficulty, and their image of the mathematical meaning(s) that one must leverage to reason productively about the task. In addition to a series of general questions that addressed these foci, we presented each student with strategically-selected video excerpts from their in-class engagement with the task. Presenting these video excerpts during the clinical interviews stimulated students' recall of the emotions they experienced while working on the IN-PTs, the difficulties they encountered, and the problem-solving strategies they employed.

Analytical Methods

We uploaded all video data to Vosaic, a password-protected, strongly encrypted web-based video analysis program. The research team developed a provisional coding scheme that was iteratively extended and refined through several cycles independent coding and discussion, resulting in stable definitions of our codes and subcodes. The research team independently analyzed all video data using our refined coding framework and generated analytical notes attached to each code instance. After this initial analysis, we met to discuss the codes and resolve any discrepancies. To clean the data, we aggregated any temporally-overlapping codes and concatenated the related analytical notes.

We then used the default instance of Microsoft 365 Copilot (running Azure OpenAI Service and Anthropic large language models) to search for patterns in the analytical notes. Copilot created Python code to translate the CSV codes, subcodes, and timestamps into a Pandas data frame, search for patterns among the codes, and augment them with patterns from the comments

based on keywords in the code descriptions. The resulting description included sequences of codes and descriptors along with related comments from the analytical notes. We read these descriptions and compared them with our own notes to construct analytical chronology of the students' collaborative engagement with the IN-PTs.

Intellectual Need-Provoking Tasks

In this report, we include only the results of our analysis of students' engagement with the IN-PTs related to angle measure and partial growth factors of exponential functions. We present these two IN-PTs, describe our design rationale, and discuss our expectations concerning their potential to provoke intellectual need.

Angle measure IN-PT. Tallman and Frank described an essential characteristic of a quantitative conception of angle measure:

> Achieving a quantitative understanding of angle measure involves identifying an attribute of a geometric object to measure and conceptualizing a unit with which—and process by which—to measure it. … One might conceptualize quantifying the "openness" of an angle as measuring the length of the arc of a circle centered at the angle's vertex that the angle subtends. For the measure of the angle to be independent of the size of this circle, the subtended arc must be measured in a unit whose magnitude is proportional to the magnitude of the subtended arc and, by extension, the circumference of the circle that contains it. (Tallman & Frank, 2020, p. 73)

To engender students' experience of intellectual need for a quantitative conception of angle measure, we modified a task initially proposed by Thompson, Carlson, and Silverman (2007). Our IN-PT for angle measure prompted students to sketch a graph of the composite trigonometric function $f(\theta) = \sin(2\cos(\theta))$. We intended students' engagement in this task to provoke their consideration of how $2\cos(\theta)$ could make sense as the argument of the sine function. We expected that students would be familiar with associating angle measures with the input quantity of basic trigonometric functions. However, we anticipated that students would struggle to conceive of $2\cos(\theta)$ as a directed magnitude (measured in units of a circle's radius), which could correspond to the length of an arc an angle subtends, and thus the measure of an angle in *radians*. Figure 1 displays the relationship between the input (θ), argument ($2\cos(\theta)$) and output ($\sin(2\cos(\theta))$) of f.

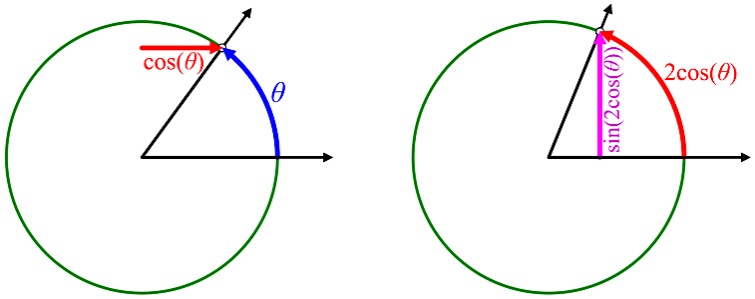


Figure 1: Relationship Between the Input, Argument, and Output of $f(\theta) = \sin(2\cos(\theta))$

Partial growth factor IN-PT. An essential characteristic of exponential functions is that their output values vary proportionally for corresponding inputs that maintain a constant additive relationship (i.e., for an exponential function f, $f(x + \Delta x) = k \cdot f(x)$ for some constant k and fixed Δx). The constant of proportionality relating $f(x)$ and $f(x + \Delta x)$ is defined in terms of the fixed

but arbitrary value of the parameter Δx. Students' understanding of exponential functions is often based on their generalization of the process by which terms in a geometric sequence are generated. Conceptualizing exponential functions as a generalization of repeated multiplication—where the general form of an exponential function ($f(x) = ab^x$) results from representing individual terms in a geometric sequence as a function of their position in the sequence (x), and then affording x with the status of a continuous variable—does not support productive reasoning about the continuity of exponential functions or the relationship between function values for non-integral values of x.

We designed the task below to engender students' experience of intellectual need for a property of exponential function values that is maintained under the continuous variation of these values.

> A biologist first measures the mass of bacteria in an experiment to be 50 micrograms (μg). After 7 hours the mass grows to 100 μg. This pattern of doubling mass occurs every 7 hours following a continuous exponential model. Compute the mass of the bacteria culture (in μg) for the following values of time elapsed since the initial measurement of 50 μg: (i) 7 hours, (ii)14 hours, (iii) 28 hours, (iv) 3.5 hours, (v) 3 hours, (vi) 10 hours, (vii) 47 minutes, (viii) 4.875 hours, and (ix) N hours.

Results

We present our study's results in two sections that respectively describe the students' engagement with each of the IN-PTs.

IN-PT for Angle Measure

Abbie, Tyra, and Kent's initial strategy to construct a graph of $f(\theta) = \sin(2\cos(\theta))$ was to consider how the "2" in the expression $\sin(2\cos(\theta))$ influences the shape of the graph. The students attempted to recall associations between graphical transformations and parameters in the symbolic representation of trigonometric functions.

Tyra proposed that the "2" in the expression $\sin(2\cos(\theta))$ affects the amplitude of the function: "You should be going between two and negative two, right?" In response, Abbie expressed uncertainty when she considered that, although the multiple of two affects the amplitude of $\cos(\theta)$, it might not affect the amplitude of sine since the multiple of two is "inside" the sine function. Abbie proposed, "It could make sine longer," suggesting Abbie's attention to the period of f, not its amplitude.

Students abandoned their effort to apply heuristics for graphical transformations and instead evaluated $f(\theta)$ for integer multiples of $\pi/2$. They expected that plotting these points would reveal enough of the graph's behavior that they could sketch a continuous, differentiable curve connecting these points. However, this approach quickly stalled when the students had to compute $\sin(2)$ and $\sin(-2)$.

After this second impasse, the instructor asked the students to verbalize why this task is difficult. Tyra responded, "there is just, like, multiple things that we have to juggle that makes this relationship unique." Kent described "having cosine inside of sine" as a challenge and explained that they can easily evaluate $\cos(\theta)$ for integer multiples of $\pi/2$, but get stuck trying to evaluate $\sin(2)$ and $\sin(-2)$, which result from evaluating $\cos(\theta)$ for integer multiples of π. Discussing the group's efforts to evaluate $\sin(2\cos(\theta))$ for convenient values of θ, Abbie offered the following reflection:

> We put our radian angles in for theta, in cosine, and then once you get that for cosine, um, it's still in radian measure but without the pi in it. It comes out to two and ends up being slightly what we're not used to. … When you take a radian measure and put it through a trig function, will you get a radian measure out still? [This] is the thing that my brain is trying to figure out if it's true or not. Because then that helps me reason that if what I'm putting into sine makes sense. … The thing is, when you're graphing these [sine and cosine functions], I realized that the one and negative that you're oscillating between, like, is that in radian measure? The *y*-value that we get out? … We know this [θ] is radians but what is this [$2\cos(\theta)$]? … So then whatever we're putting into sine doesn't make sense to put into sine.

The instructor asked the students to clarify why it is a concern for values of $2\cos(\theta)$ to be inputs of sine. Tyra offered the following explanation:

> Like [Abbie] was saying, … when you do cosine of zero and you get one as the output, that one corresponds to, like, your length of your radius of your circle. … And so you wouldn't plug a radius unit back into sine. You need to plug in radians. So you would have to convert that.

We interpret the students' consideration of the units associated with the input (θ) and argument ($2\cos(\theta)$) of f to signify an important advance in their experience of intellectual need. The primary perturbation expressed by students in Group A throughout their engagement in the angle measure IN-PT was how to interpret "cosine inside of sine." We earlier cited Tallman and Frank's (2020) criteria of a quantitative conception of angle measure, which includes, "identifying an attribute of a geometric object to measure and conceptualizing a unit with which—and process by which—to measure it" (p. 73) Abbie, Tyra, and Kent's collaborative work on the IN-PT culminated in their grappling with determining the *quantity* (i.e., measurable attribute) of which θ is a measure and $2\cos(\theta)$ is a measure, and whether $2\cos(\theta)$ could represent a value in *radians*.

IN-PT for Partial Growth Factors

Tyra immediately associated the growth rate with the expression 2^n, where one-unit variations in n correspond to seven-hour intervals of elapsed time. All three students quickly computed the mass of the bacteria culture after seven hours, 14 hours, and 28 hours by repeatedly doubling the initial mass of 50 micrograms for each successive change of seven-hours of time elapsed. When considering how to determine the mass of the bacteria culture after three hours, Tyra proposed to compute $50\left(\frac{3}{7}\right)(2)$ (i.e., "Fifty of three-sevenths of two"). She reasoned that since three is three-sevenths of seven, then the mass of the bacteria culture after three hours is computed by multiplying the initial 50 micrograms by three-sevenths of two (the seven-unit growth factor). The students computed $50\left(\frac{3}{7}\right)(2) \approx 42.8571$, and upon realizing that this value is less than the initial 50 micrograms, decided to add this value to the initial mass. Tyra noticed that the resulting sum (92.8571μg) is close to the mass of the bacteria after seven hours (100 μg), and concluded that this cannot make sense as a value for the mass of the bacteria after only three hours—a concern echoed by Abbie and Kent.

Tyra then suggested that the expression $50(2)^{\frac{3}{7}}$ might represent the mass of the bacteria culture after three hours, asking, "Could it be fifty times two to the three-sevenths? Is that where my three-sevenths comes in instead?" She reasoned that since 2^n models the growth of the bacteria culture, and since n represents increments of seven hours, it makes sense for 3/7 to be the exponent of 2: "Earlier how I said it was fifty times two to the n for whatever increments you

have of seven. So if we have three out of seven there, that should be two to the three-sevenths instead of three-sevenths times two." Abbie described 3/7 in the exponent as representing "a fraction of the doubling," but then admitted, "I don't think that makes sense." The three students computed $50(2)^{\frac{3}{7}} \approx 67.3$ and concluded that this solution is reasonable in the context of the situation. They similarly computed the mass of the bacteria culture after 47 minutes by evaluating $50(2)^{\frac{47}{420}}$. To this point in the students' engagement in the task, they did not seem to reason about the relationship between *n*-unit growth factors and partial growth factors. Instead they appeared to have pseudo-empirically abstracted from their prior activity the symbolic form [Initial Mass]$(2)^{[N/(\text{Doubling Interval})]}$, where *N* represents the elapsed time (in hours) for which the expression represents the mass of the bacteria culture. After having engaged in this pseudo-empirical abstraction, all three students quickly and correctly computed the mass of the bacteria culture for each value of time listed in the task and correctly represented the mass of the bacteria culture after *N* hours.

The instructor prompted the students to justify why three-sevenths is an exponent of two instead of a multiple. Tyra explained, "It was fifty times two to the three-sevenths because it was an exponential relationship." We interpret Tyra to have reasoned that since the mass of the bacteria culture grows *exponentially*, seven-hour intervals of elapsed time must be represented in the *exponent*, as $50(2)^n$ represents the mass of the bacteria culture after $7n$ hours. Tyra's response again indicates that her understanding of the expression $50(2)^{\frac{t}{7}}$, where *t* represents time elapsed in hours, was not based on an abstraction of the relationship between an *n*-unit growth factor and partial growth factors. To support this abstraction, the instructor asked, "How is the mass of the bacteria changing every hour?"

Tyra computed $50(2)^{\frac{1}{7}} - 50 \approx 5.2$ and claimed, "It changes by 5.2 micrograms per hour." Although having successfully defined the mass of the bacteria culture as a function of the number of hours elapsed, Tyra's comment reflects an image of a constant growth rate (fixed changes in mass for uniform successive changes in time elapsed), providing further evidence that the expression she defined to represent the mass of the bacteria after *N* hours, $50(2)^{\frac{N}{7}}$, was based on a pseudo-empirical abstraction of her reasoning about computing the mass corresponding to integer multiples of seven hours of time elapsed. Tyra paused for several seconds and said, "I don't think that's right." The three students sat quietly in thought for 30 seconds. Then Tyra proposed, "I think it would change by two to the *x* over seven per hour, or just two to the one over seven." Despite Abbie's observation that "It's growing a little faster every hour," Tyra and Kent continued to consider how to represent additive changes in the mass of the bacteria for successive one-hour changes in elapsed time.

The three students described the meaning of $2^{\frac{1}{7}}$ as, "The amount of change in mass of the bacteria in one hour." It was not clear whether the students conceptualized $2^{\frac{1}{7}}$ as a growth *factor* (i.e., a constant of proportionality relating the mass of the bacteria culture at some moment and the mass seven hours later), or as representing an *additive change* in mass, so the instructor prompted them to specify how they conceptualized "amount of change." Referring to the meaning of the expression $2^{\frac{2}{7}}$, Tyra responded, "Like the difference from the initial measurement to the two hours elapsed, in the bacteria. There's a difference in what the original mass was versus this." The instructor then proposed that the function $y = f(t)$ gives the mass of the bacteria culture (in micrograms) as a function of time elapsed (in hours). He asked the students to state whether the equation $f(2) - f(0) = 2^{\frac{2}{7}}$ accurately represents their thinking. Tyra responded,

"Yes" while Abbie and Kent nodded in agreement. The students also agreed that the equation $f(4) - f(2) = 2^{\frac{2}{7}}$ is correct. The instructor finally prompted the students to consider whether $f(n+2) - f(n) = 2^{\frac{2}{7}}$. After a moment of thought, all three students agreed that the equation is correct.

The equation $f(n+2) - f(n) = 2^{\frac{2}{7}}$ is not correct for an exponential function f. The instructor attempted to support students' multiplicative reasoning about the relationship between function values $f(n+2)$ and $f(n)$ by asking, "If you know the mass of the bacteria after 26 hours, how would you compute the mass of the bacteria after 28 hours?" All three students spent a few minutes independently testing on their calculators different possible ways to calculate the mass of the bacteria culture after 28 hours given the mass of the bacteria culture after 26 hours. That they needed to do this demonstrates that until this moment, the students likely had not conceptualized the relationship between measures of the mass of the bacteria culture corresponding to different values of time elapsed. Tyra then clarified her thinking: "So then, when we said earlier, when we were trying to calculate differences, and it was two to the two-sevenths, whatever, between $f(4)$ and $f(2)$, that wouldn't necessarily– because it's by a factor of it; it's not additive!" This reflection prompted Tyra and Abbie to conclude that the mass of the bacteria culture increases continuously within seven-hour intervals of elapsed time, and that this increase occurs at a non-constant rate.

The students' lengthy consideration of whether expressions of the form $2^{\frac{N}{7}}$ represent *differences* in the mass of the bacteria culture, or *factors* relating measures of the mass corresponding to different values of elapsed time, contributed to a key shift in their thinking. When asked what $2^{\frac{N}{7}}$ represents in the context of the situation, Tyra claimed, "That is the factor at which the mass of the bacteria culture changes per hour" and Abbie clarified the meaning of N, "with the N being the number of hours." The students later expressed that $2^{\frac{N}{7}}$ represents the constant ratio $\frac{f(t+N)}{f(t)}$ where t represents a varying time elapsed in hours and N represents a fixed but arbitrary change in time (in hours).

Discussion and Conclusion

Throughout the students' collaborative engagement in both IN-PTs, the impasses they experienced were not side-stories—these impasses *organized the trajectory* of the students' conceptual activity and cognitive reorganization. The students' confusion sustained their inquiry into foundational meanings (e.g., growth rates relating to differences versus factors); their curiosity stimulated productive algebraic generalization; their surprise prompted reconsideration of basic assumptions and critical evaluation of their prior knowledge. Our findings demonstrate that students' experience of disequilibrium does not represent a "failure" but can be a vital precondition for their conceptual learning of mathematics.

The results of our study suggest that instructional designs for undergraduate mathematics courses for pre-service secondary teachers should include opportunities for students to experience very specific impasses that facilitate their progression through an explicit learning trajectory. The IN-PTs we designed for this research serve as models of tasks for which students' uncritical application of rehearsed procedures yields contradictions that can provoke students' experience of intellectual need.

We encourage researchers to extend the findings we report by examining the educative value of students' experience of intellectual need as they participate in classroom instruction related to the mathematical concepts necessitated through students' engagement with an IN-PT.

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EXPERIENCES SHAPING MOTIVATIONS FOR GRADUATE STUDY IN MATHEMATICS: A QUALITATIVE STUDY OF PERSONAL STATEMENTS

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Concerns about the leaking pipeline from undergraduate to graduate study in mathematical sciences raise important questions about what shapes students' decisions to pursue advanced degrees. Understanding the experiences and motivations of prospective graduate students is essential for supporting them and sustaining a robust mathematical science workforce. This qualitative study analyzes personal statements of 29 applicants to mathematics and statistics graduate programs, focusing on the experiences they describe and how they position themselves in relation to them. Using the lens of pivotal experiences and personal agency, we find that applicants rarely frame a single experience as pivotal; instead, they emphasize coursework, research, and teaching as motivational factors. The applicants also tend to portray themselves as active or collaborative participants, actively shaping their decisions to pursue graduate study.

Keywords: Graduate Study in Mathematics, Pivotal Experiences, Agency, Motivation

The decision to pursue graduate education in the mathematical sciences—including pure mathematics, applied mathematics, mathematics education, and statistics—is a significant milestone in an individual's academic and professional trajectory. Although many students complete undergraduate degrees in mathematics or related fields, only a subset ultimately pursue graduate study. Understanding why some individuals persist in the mathematical sciences is important for strengthening graduate programs and supporting prospective graduate students, as well as for sustaining a robust mathematical sciences workforce (National Academies of Sciences, Engineering, and Medicine [NASEM], 2016).

A growing body of literature has examined motivations for graduate study across disciplines. Researchers identify both internal factors, such as intellectual curiosity and the desire to contribute to knowledge or society (Incikabi, 2013; Kowalczuk-Waledziak et al., 2017), as well as external factors, including career advancement and societal pressures (Hinkle et al., 2014; Ivankova & Stick, 2007). In mathematics education research, emerging work has begun to explore these motivations in more discipline-specific ways (e.g., McEldowney et al., 2024; Towle & Buchbinder, 2026).

While prior studies document a range of motivational factors (e.g., Towle & Buchbinder, 2026), less is known about the experiences and forms of agency individuals interpret as shaping their interest in and desire to apply to graduate study in mathematics. This study addresses this gap by examining personal statements in graduate admissions applications to explore how applicants construct narratives of personal experiences and agency they highlight as affecting their pathways to graduate study. Specifically, we ask: *What types of experiences do applicants to graduate programs in the mathematical sciences describe as influential in their mathematical journey and decisions to apply to graduate school?*

Literature Review

Research on graduate school intentions, not necessarily specific to mathematics, highlights that students' educational decisions are shaped by a combination of internal and external motivations (Gibbons & Schoffner, 2004; London et al., 2014; Teowkul et al., 2009). External

motivations typically center on professional gain, career advancement, job security, and higher salaries (Hinkle et al., 2014; Ivankova & Stick, 2007; Teowkul et al., 2009). Advanced degrees are often required in academia and in many industry positions, making alignment with career goals a strong predictor (Basalla & Debelius, 2014; Towle & Buchbinder, 2026). External influences, such as mentor and faculty encouragement and social influences from family and peers, can also shape students' awareness of and motivations for particular fields, programs, or research opportunities, leading individuals to consider graduate studies or career paths they had not previously envisioned (Brailsford, 2010; Sherman, 2021).

While external influences play an important role, internal motivations often become central as students develop clearer academic and career goals. Internal motivations, such as enjoyment of research, seeking intellectual challenges, and drive for academic achievement, are linked to feelings of personal satisfaction, self-efficacy, and a sense of purpose (Kowalczuk-Waledziak et al., 2017; Teowkul et al., 2009). In mathematics specifically, the pursuit of knowledge, research interests, and appreciation for the discipline itself (i.e., mathematics, statistics) have been shown to motivate students to pursue advanced study (Towle & Buchbinder, 2026).

Beyond identifying the types of motivational factors, considerably less is known about the experiences applicants describe as influential in shaping their interest in graduate study. Some studies suggest that internal motivations are shaped by personal experiences that help individuals discover their passions (e.g., Løvoll et al., 2017; Valas & Søvik, 1993). In mathematics, such experiences may include coursework, research involvement, or meaningful interactions with faculty. However, little is known about how specific experiences shape students' motivations to pursue graduate study. This study aims to address this gap.

Conceptual Framework

To address our research question, we draw on two complementary constructs: pivotal experiences and personal agency. Together, these constructs support an analytic approach attending to both *what* applicants describe in their personal statements and *how* they position themselves in relation to those experiences. Some experiences described in personal essays appear to be more influential in shaping applicants' trajectories. We refer to such experiences as *pivotal* and define them as events that fundamentally redirect an individual's life course. Central to such experiences is a pivotal moment—often an "aha" realization—through which an individual gains a new perspective that meaningfully alters their trajectory (Giorgi, 2011; Maclennan, 2015). Pivotal experiences are characterized by a clear "before" and "after" state, often initiating lasting changes in priorities, beliefs, or circumstances, and functioning as a defining moment in one's life (Giorgi, 2011).

The second analytic dimension concerns how individuals describe their relationship to the experiences they recount. To examine this dimension, we draw on the concept of *agency*. Following Abele et al. (2008), we define agency as "a person's striving to be independent, to control one's environment, and to assert, protect and expand one's self" (p. 1204). This is commonly understood as the experience of generating one's own actions and outcomes (Haggard & Tsakiris, 2009). Agency extends beyond overt action to include the ability to set goals, make decisions, and act in ways that help achieve them (Markus & Kitayama, 2003). Importantly, agency is not limited to an action initiated by an individual. It can emerge in response to a situation where a person must act and navigate in their environment (Holland et al., 2003). The degree and form of agency may vary across situations, reflecting differing levels of choice and opportunity (e.g., Hansen, 2022). Therefore, we conceptualize agency along a continuum: high agency, shared agency, and limited (or responsive) agency (Adler et al., 2008; Hansen, 2022).

Taken together, pivotal experiences and agency provide a framework for analyzing both the significance of experiences and the degree of agency applicants assume in those experiences. The six cells of our analytic framework (Table 1) describe the intersection of various forms of agency (high, shared, and limited/responsive) with whether an experience is portrayed as pivotal.

Table 1: Theoretical Framework for Applicants' Experiences

	High Agency	Shared Agency	Limited / Responsive Agency
Pivotal Experience	Applicant initiates and pursues a transformative experience, presenting a clear turning point driven by their own decisions.	Experience becomes pivotal through collaboration or external encouragements; the applicant participates meaningfully.	Life-altering event outside of applicants control that produces a clear turning point; the applicant's response shapes their path.
Non-Pivotal Experience	Applicant intentionally seeks a meaningful experience that supports growth without redirecting their trajectory.	Experience is suggested or facilitated by others and supports development without a major turning point.	Experience, not initiated by applicant, that subtly shapes their perspective without a dramatic shift.

This integrated framework enables us to systematically interpret how applicants make sense of their experiences. Specifically, it distinguishes experiences that applicants actively initiate and frame as transformative, those that become pivotal through collaboration or guidance, and those that arise outside their control but elicit meaningful responses. Importantly, we consider *all* experiences described in personal statements meaningful and influential. Although only those that produce a clear "before-and-after" shift were deemed pivotal, non-pivotal experiences are still thought to contribute to applicants' development and motivation to pursue graduate study in mathematics. Additionally, in this study, agency serves as a lens for interpreting how applicants present their experiences in personal statements—how they portray their roles in initiating, shaping, or responding to those events. Attending to agency thus provides insight into how applicants' narrative self-positioning relates to their motivations and aspirations for advanced study in mathematics. Taken together, this approach provides a nuanced lens for understanding how applicants portray their role in shaping, responding to, or being shaped by their experiences.

Methods

Data Sources and Collection

The data for this study are drawn from personal statement essays submitted by applicants to graduate programs in mathematics and statistics at a mid-sized public research university located in the Northeastern United States. These essays are a central component of the application process and require applicants to describe their academic interests, professional aspirations, prior research or work experience, and the motivations for pursuing graduate study. As such, the statements provide rich qualitative data and have been used in general higher education research to provide insight into how applicants frame their educational trajectories and future goals (e.g., Forister et al., 2011; Kuncel et al., 2020; Wouters et al., 2014). In mathematics education,

personal statements remain a rarely utilized data source despite offering a unique lens into the motivations and experiences of prospective graduate students in the mathematical sciences.

Personal statements are *purposeful narratives*, meaning that the experiences included in these essays are those that applicants likely consider important in their decision to pursue graduate mathematics or statistics, or they felt would strengthen their applications. However, personal statements are written for a specific audience and purpose, in which applicants strategically frame their experiences to present themselves as viable candidates for the graduate program. These potential effects of social desirability should be considered when interpreting the findings.

Participants

All applicants from 2015 to 2024 were invited to participate. For consenting participants, the second author, a graduate admissions committee member with access to application materials, compiled personal statements and background information, including gender, application year, intended program, and citizenship status. Race and ethnicity were excluded because this data was optional and unavailable for many participants.

The sample analyzed in this paper, consists of 29 admitted applicants to eight graduate programs in the Mathematics and Statistics Department, collectively referred to as the "mathematical sciences." The sample includes 20 doctoral students and 9 master's students. Doctoral programs include Applied Mathematics (n = 3), Statistics (n = 8), Pure Mathematics (n = 6), and Mathematics Education (n =3). Master's-level programs include Applied Mathematics (n = 4), Statistics (n = 3), Pure Mathematics (n = 1), and a Master of Science for Teachers (MST; n = 1). Fourteen participants identified as male and 15 as female; 20 reported U.S. citizenship status, seven non-U.S., and two reported dual citizenship (including U.S.).

Data Analysis

The unit of analysis was a *distinct narrated experience* in each personal statement, ranging from one sentence to several paragraphs. Experiences were defined by how applicants structured their stories, rather than by an attempt to catalog their entire academic careers. In some cases, applicants described multiple events leading up to a single decision or turning point. When these events were mentioned as a background for a larger experience, they were not coded separately. Instead, coding focused on the experience that the applicant positioned as central or meaningful within the story (e.g., a decision to change majors). This approach prioritized the applicant's narrative emphasis over accounting for every single experience discussed.

In line with the aims of the study—to examine the types of experiences applicants describe as influential, and how they position themselves in relation to those experiences—each identified experience was coded along the two dimensions of our analytical framework: (1) experience type and (2) degree of agency. Table 2 shows examples of each of the six codes.

Table 2: Examples of Coding Experiences in Personal Statements

Experience Type	Degree of Agency	Example
Pivotal Experience	High Agency	During my MS semester studying fashion … My work excelled and the professors were encouraging, saying that I had talent, but I was filled with an unreasonable dissatisfaction … My love of calculus won over my love of fashion, and I switched my focus to mathematics. (Applied Mathematics MS)

	Shared Agency	I have been interested in proving mathematical theorems ever since high school … This started in my sophomore year when I had an amazing Math teacher. I credit [this teacher] with instilling in me the excitement I have about mathematics … The enthusiasm that was shared in this class has never diminished. (Mathematics PhD)
	Limited / Responsive Agency	My passion for statistics has stemmed from my arduous years searching for a diagnosis for a rare medical condition from which I suffered in college. (Statistics PhD)
Non-Pivotal Experience	High Agency	I wanted to learn upper-level mathematics at a university and contribute research in a mathematical field. At [university], I started to do just that. I took classes in Algebra, Analysis, Topology, and applied math. (Mathematics Education PhD)
	Shared Agency	Early in my sophomore year, I became a research assistant working in dendrochronology with one of [faculty member's] doctoral students. (Statistics MS)
	Limited / Responsive Agency	My biggest prize [was] to see the pride in my mother's eyes when she talks about how I managed to graduate from the best university in the state and when she talks about my academic achievements … This undoubtedly fills me with joy and pride. (Statistics PhD)

Note that an experience was identified as pivotal when the author clearly articulated a "before-to-after" transition. Thus, the identification of pivotal experiences relies on the text's narrative construction and the presence of an explicit or strongly implied shift in direction.

Results

Across the 29 personal statements, applicants described a diverse range of experiences, averaging five distinct experiences per statement. Overall, 142 experiences were coded. Below, we present the results according to the six categories of our analytic framework.

Pivotal Experiences with High Agency

Experiences coded as pivotal and high agency counted for 4.9% of instances, characterized by a clear "before-and-after" shift that applicants positioned as redirecting their career or life trajectories. This was driven by self-recognition leading to deliberate action, such as switching majors, recommitting to mathematics, or discovering a deep personal affinity for the subject. For example, one applicant described how realizing their love for mathematics prompted them to shift their major, saying, "I switched my focus to mathematics" (see Table 2). Another applicant described rediscovering their love for mathematics after leaving it, writing, "I made the mistake of leaving the world of mathematics behind once […] it is not a mistake I intend to make again."

A few applicants also described early self-driven experiences with mathematics. For instance, one wrote, "When I was in elementary school, I strove to keep up with my older brother's math homework […] Due to this, I came to love being constantly challenged mathematically at a young age." Across these accounts, applicants present themselves as the ones who recognized the misfit and chose to redirect their trajectory toward mathematics.

Pivotal Experiences with Shared Agency

Pivotal experiences with shared agency accounted for 6.3% of experiences. These involved transformative "before-and-after" moments facilitated by others, often faculty, through courses or advising. Some experiences centered on encounters with mathematical content or instruction.

For instance, one applicant who described taking an Abstract Algebra course stated, "I knew it was different than the other math classes I took." This applicant described how discovering a new understanding of proofs eventually led them to pursue research in this area. We coded this as an instance of shared agency because it was strongly implied that the course was required for the major, rather than chosen by the individual, yet the applicant actively engaged with and built on it to the point that it became a pivotal experience for them.

Other applicants highlighted influential teachers whose enthusiasm sparked lasting interest in mathematics, as one applicant wrote, "I have been interested in proving mathematical theorems ever since high school in trigonometry…when I had an amazing Math teacher." This individual goes on to say, "I credit [this teacher] with instilling in me the excitement I have about mathematics […] The enthusiasm that was shared in this class has never diminished but has expanded to more topics within mathematics." In these narratives, the initial opportunity was provided by someone else, but applicants actively engaged with it and incorporated it into their trajectories, demonstrating shared agency in pivotal experiences.

Pivotal Experiences with Limited or Responsive Agency

Pivotal experiences with limited or responsive agency accounted for 5.6% of coded instances. They describe major life events outside applicants' control that nonetheless reshaped their educational trajectories. Half of these experiences involved health-related issues, often causing individuals to pause their education and, consequently, realize how much they missed mathematics, as one wrote, "taking a year off from learning mathematics made me feel like I was missing something in my life." Another applicant described how a prolonged medical challenge shaped their academic interests: "My passion for statistics has stemmed from my arduous years searching for a diagnosis for a rare medical condition from which I suffered in college."

Other experiences in this category involved family loss or formative early-life events. One applicant wrote, "The hardest puzzle I have ever faced was my father's death," explaining that this initiated a lifelong "quest to ask questions and search for answers," ultimately drawing them toward mathematics. Another described being pulled out of class in elementary school, reflecting: "My interest in math education ironically began the moment my father pulled me out of my math classroom." In these cases, applicants do not claim agency in the event; instead, they frame their *responses* as central to developing motivation and engagement in mathematics.

Non-Pivotal Experiences with High Agency

Non-pivotal experiences with high agency accounted for 43.7% of all coded experiences, making this the most frequent category. These narratives describe experiences in which applicants intentionally pursued and strengthened existing interests, skills, or commitments as part of an ongoing developmental trajectory, rather than through a single dramatic turning point. Nearly half of the experiences in this category involved mathematical content or coursework (22.5%) and teaching (21.0%). For example, one applicant wrote,

> The passion I currently have in algebra began at [university] when I took Abstract Algebra. This course seemed to come more naturally to me. … As a result, when I became a graduate student I made every effort to take as many algebra courses as possible while still satisfying my degree requirements. I managed to take separate courses in Ring Theory, Group Theory, and Field Theory, and thus, my foundation in algebra was immensely strengthened.

Here, the applicant emphasizes deliberate, self-initiated actions ("I made every effort") that strengthen an emerging interest in algebra that originated in their undergraduate coursework. Applicants similarly described engaging in additional advanced coursework, pursuing independent learning, and reflecting on their teaching practice. For example, one applicant to the

Mathematics Education PhD program wrote, "I have worked honing the craft of teaching as a high school math teacher at a small private boarding school."

Across these accounts, the applicants position themselves not simply as students progressing through coursework, but as mathematical doers and teachers who actively develop, apply, and share their expertise. Their narratives depict ongoing participation in the practice of mathematics rather than passive movement through degree requirements. Together, these experiences illustrate sustained, intentional engagement with mathematics beyond minimal expectations. Applicants frame their trajectories as self-directed and purposeful, highlighting the choices they made to deepen understanding and reinforce their commitment to the discipline over time.

Non-Pivotal Experiences with Shared Agency

Experiences in this category accounted for 38.0% of all codes and included non-pivotal experiences facilitated by others. The most common themes were research (40.7%), followed by coursework (22.2%) and teaching (16.7%). While similar to the previous category, they differed in the level of agency. Applicants frequently referenced working "under" or "with" a professor or as part of a group, indicating shared responsibility and guidance. One applicant wrote, "I worked as a research assistant under [faculty member]," while another described, "My group's research explored the inner workings of the quasi-random Polya Board."

Applicants sometimes attributed their developing interests in specific mathematical areas to professors. For example, one applicant wrote,

> At undergraduate level, my interests became more focused … Analysis and its applications and Mathematical Modeling truly captured my interest. I was fortunate to have innovative and distinguished professors, and I performed well in all these subjects. Therefore, I have chosen to specialize in Applied Mathematics and Computational in Mathematics.

In this quote, the applicant emphasizes gratitude for their professors ("I was fortunate to have") whose instructional practices helped refine their interests, highlighting a collaborative developmental process. Teaching experiences in this code typically involved a teaching assistant position, for which the applicant stated they were given this opportunity, sometimes by a professor. For example, one applicant wrote, "I assumed the role of teaching assistant […] coordinated by [faculty member] in my junior year." Across these narratives, applicants frame mentors and instructors as central facilitators in these experiences, highlighting how guidance and collaboration supported their learning and preparation for graduate study.

Non-Pivotal Experiences with Limited or Responsive Agency

Experiences in this category only accounted for 1.4% of all codes (two instances). They were non-pivotal and marked by limited or responsive agency, in which applicants described reacting to external circumstances rather than initiating them. One applicant reflected on the pride their mother expressed in their academic achievements, emphasizing their emotional response, stating, "This undoubtedly fills me with joy and pride." Another applicant described being approved to study statistics at their undergraduate institution, framing entry into the academic environment as something that happened to them ("I started my contact with the academic environment"). These few instances highlight experiences that, without being framed as agentic or pivotal, were still portrayed by the applicants as influential in pursuing graduate study.

Summary and Discussion

Overall, relatively few experiences were categorized as pivotal (accounting for 16.9% of all codes, regardless of agency level) in the applicants' essays. When pivotal experiences did occur, they most often involved realizations of a deep affinity for mathematics, mentor-initiated

opportunities that reshaped goals, or major life events, all of which prompted self-reflection and redirection toward graduate studies. Some pivotal experiences were deeply personal and emotional. The fact that individuals chose to include them in their personal statements (and consented to have these statements analyzed) may suggest that the applicants viewed their decision to pursue graduate programs in mathematics as a crucial part of their life journey, intrinsically tied to other significant life events, rather than a simple career choice.

Most experiences in the personal essays were non-pivotal yet meaningful (accounting for 83.1%). These experiences commonly involved coursework, research, and teaching and were frequently narrated with either high agency, emphasizing personal initiative, or shared agency, emphasizing the role of mentors and structured opportunities. This aligns with prior work showing that continued involvement in coursework and research helps students develop interest and intentions to pursue graduate study (Ivanka & Stick, 2007; Tennant, 2004). The relatively low frequency of pivotal experiences suggests that applicants rarely frame their pathway to graduate study as the result of a single dramatic turning point. Instead, their narratives portray graduate aspirations as emerging through the accumulation of multiple experiences over time. This concurs with prior research indicating that individuals hold multiple, interacting motivations for pursuing graduate study (e.g., Teowkul et al., 2009; Towle & Buchbinder, 2026).

Agency emerged as central across narratives (high agency experiences accounting for 48.6% and shared agency experiences accounting for 44.4%), meaning that applicants positioned themselves as active participants in shaping their educational trajectories, while also often acknowledging the importance and influence of others in facilitating opportunities. This reflects a relational view of agency: it can arise both from an individual's actions and from interactions or opportunities provided by others (Holland et al., 2003). Additionally, this aligns with previous work that identifies external influences such as mentoring and encouragement as central to graduate school aspirations (Incikabi, 2013; Kowalczuk-Waledziak et al., 2017; Sherman, 2021).

It is important to note the limitations of this study. The sample is drawn from a single mid-sized public research university in the Northeastern United States, limiting the generalizability of this work. Additionally, personal statements are self-presentational documents written for an admission audience. While they provide insight into how applicants frame and interpret their experiences, applicants may selectively emphasize experiences and personal qualities they perceive as desirable, meaning these narratives may not fully capture all factors that influenced their decision-making to pursue graduate study. Thus, the experiences analyzed here reflect constructed accounts shaped by the admissions context and strategic self-presentation. Furthermore, although our broader dataset includes 75 applicants, both admitted and non-admitted, this study focuses on 29 admitted individuals to examine experiences associated with successful entry into graduate programs. Future work will extend this analysis to provide a more comprehensive understanding of experiences shaping graduate study decisions.

This study contributes to the field in multiple ways. The analysis of personal statements provides insight into applicants to mathematical sciences programs, addressing a knowledge gap. Findings suggest that graduate aspirations often emerge through accumulated experiences, underscoring the importance of sustained access to authentic mathematical experiences, such as advanced coursework, research and teaching—all of which support the development of students' skills, interests, and motivation. Future research should include multiple institutions and program types to explore how contexts shape applicants' experiences and motivations for graduate study in mathematics. Collectively, results of these studies can help researchers and educators address the leaking pipeline from undergraduate to graduate study in mathematical sciences.

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STUDENTS' GOALS, BELIEFS, AND ENGAGEMENT PATTERNS IN AN INTRODUCTION TO PROOF COURSE

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The shift to proof-based mathematics in US universities often takes place in an introduction to proof (ITP) course. Existing literature describes the nature of these courses and the difficulties students experience in the course, but little is known about how students' goals, beliefs, and conceptions of proof develop throughout an ITP course (David & Zazkis, 2020). Five students participated in a longitudinal study examining how ITP students' goals, beliefs, and engagement patterns evolved during an ITP course. Students' goal orientations (Dweck & Leggett, 1988) were seen to be malleable throughout the course with most students starting the semester with learning goals. Students adopting performance goals engaged in maladaptive behaviors such as relying on AI. Student beliefs about the purpose of mathematical proof varied; all students believed verification and explanation to be key purposes of proof.

Keywords: Undergraduate Education, Reasoning and Proof, Advanced Mathematical Thinking

The comprehension and construction of mathematical proof is central to undergraduate mathematics education in the United States. However, many students struggle to transition from primarily computational mathematics to proof-centered mathematics (Moore, 1994). To support students in this transition, many US universities designate an introduction to proof (ITP) course focused on enculturating students into the community of proof-based mathematics (David & Zazkis, 2020). We conducted a longitudinal study examining students' goals, beliefs, and engagement patterns during an ITP course. In this report we summarize the results of the research questions:

1. How do students' goals and beliefs about mathematics and mathematical proof interact with the way they engage in an ITP course?
2. What are ITP students' goals and beliefs about mathematics and mathematical proof? How do these goals and beliefs evolve during an ITP course?

Literature Review

The purpose of ITP courses is to teach logic rules, proof structures, set theory, and, in some cases, basic content from one or more branches of mathematics (David & Zazkis, 2020). David and Zazkis categorized ITP courses in the US into three categories: standard only, standard + topic, and topic-based. Standard only ITP courses, which accounted for 41% of their data, focus on teaching skills necessary for proof comprehension and production such as understanding symbolic logic, quantifiers, set theory, and truth tables. While mathematical content is necessarily used in these classes, it is not the course's aim to introduce new mathematical content to students. Standard + topic courses (40% of ITP courses surveyed) teach the same topics as the standard only course and include an exploration of a new mathematical topic, frequently discrete mathematics. The exploration of new mathematical content often occurs at the end of the term. Topics-based courses (19% of ITP courses surveyed) focus on teaching a specific mathematical topic, most commonly analysis, and include the development of mathematical argumentation as a

goal of the course. The data for this study was gathered in a standard + topic course where the topic was abstract algebra.

Mathematics education researchers have struggled to produce a definition of proof that draws an indisputable line between proof and non-proof (Stylianides et al., 2017). This problem is sometimes called the "hard boundary problem" (Czocher & Weber, 2020). For example, there is debate about whether non-syntactic proof, such as computer-generated proofs, picture proofs, or probabilistic proofs qualify as proof (Aberdein, 2009). A definition of proof should provide justification for the inclusion or exclusion of such arguments. The lack of clarity about a definition of proof may cause confusion for newcomers to the community.

Many definitions of proof proposed by researchers are couched in satisfying one of the many purposes a proof can fulfill. Proof can serve a variety of purposes for mathematicians including verification, explanation, systematization, discovery, communication, construction, exploration, and incorporation (Hanna, 2000). Hanna suggests that students entering the mathematical community should primarily engage with proofs serving the two fundamental purposes of verification and explanation. Instruction in introductory courses should reflect these two primary purposes of proof.

While the norms of proof writing, and consequently the definition of proof, held by distinct communities may differ, the values that underly those norms are likely similar (Dawkins & Weber, 2017). For example, most communities engaged with mathematical proof would likely agree that proof should increase mathematical understanding. A proof submitted for publication may be held to the norm that routine calculations or obvious justifications be omitted from the proof; however, in an ITP course a student may be required to show such calculations and justifications. It is critical to understand the norms established within the community from which a proof originates. Instructors of the same course at the same university may even have differing goals and beliefs about norms that should be adopted by students. We suspect that while practical norms differ between communities, the underlying values of what it means to prove something are relatively consistent across mathematical communities. For example, a value held by most communities engaged with mathematical proof is that a proof should have explanatory power. What constitutes the best way to explain differs between communities. When presented with arguments during a classroom discussion about why the sum of two odd numbers is even, a group of pre-service teachers valued a well-explained empirical example over a valid deductive argument (Morris, 2007). The empirical example gave evidence that the student author was attending to the general nature of the argument by referring to making pairs, leftover pieces, and fitting the leftover pieces together to form a new pair. We suspect that many mathematicians would find a deductive argument more helpful in explaining that mathematical claim. The two communities may adopt different norms to uphold the same underlying value that proofs should explain. Even subtle changes to setting such as having a different instructor or sitting next to a different classmate can influence the norms a student chooses to adopt in proof construction.

Theoretical Framework

Two theoretical frameworks underpin this study: the zone of proximal development (Vygotsky, 1987) and motivation theory (Dweck & Leggett, 1988). The zone of proximal development (ZPD) describes the interplay between intra-psychological and inter-psychological domains that enables the emergence of more advanced cognitive activity. A common interpretation of ZPD is the interplay between what an individual can do on their own and what they can do with the help of a more knowledgeable other, thus locating the content and form of

productive instruction. Vygotsky also described ZPD as the dynamic interplay between spontaneous concepts and scientific concepts.

A *spontaneous concept* develops through everyday experience, usually without the aid of formal instruction (Vygotsky, 1987). Ideas within a spontaneous concept are specific and often lack generalization. For example, a child learning their native language is developing a spontaneous concept because the child knows many specific words but may not be able to generalize about categories such as "noun" or "fruit" through direct interaction with the associated objects, properties, effects, etc. Ideas within a spontaneous concept are specific and lack conscious awareness, volitional control, and generalization. Even if a student entering an ITP course has never been introduced to formal proof, they likely have significant spontaneous conceptions of what it means to prove a statement based on everyday interpretations of inference and justification. While these spontaneous conceptions may proport the same purpose of proof, they are likely inadequate to fully equip students to comprehend or construct valid mathematical proofs. A student's spontaneous conception of proof provides important grounding to their emerging scientific conception of proof. That is, as students learn the nuanced rhetoric of mathematical proof, their spontaneous conception of proof grounds them in the purpose of the practice, which is to explain, justify, convince, etc.

Scientific concepts are those learned through mediated interaction with their meanings (e.g. instruction; Vygotsky, 1987). They are more general and systematized but lack the accessibility and applicability of spontaneous concepts. For example, a child learning a second language is developing a scientific concept because the child is usually first introduced to general ideas like "verbs" or "clothing items" and then learns a list of vocabulary words that fall into each category. The child's verbalizations in the second language will be calculated and deliberate. As a student continuously engages in an ITP course, they will develop a scientific conception of proof that equips them with structures and vocabulary to comprehend and construct mathematical proof, even if it is initially experienced as abstract and disconnected from mathematical concepts they understand more conceptually. As students begin to develop a scientific conception of proof, they may necessarily focus on details and lose track of purposes of proof such as understanding or explaining. Spontaneous conceptions are the foundation upon which students may link their scientific conceptions of proof to meaningful ideas and ways of reasoning as they engage in course material and experience formal instruction about mathematical proof. Conversely, scientific concepts enable students to give structure to and generalize their spontaneous conceptions.

As the student develops both spontaneous and scientific concepts, the complementary strengths of each support the development of the other: scientific concepts become grounded and spontaneous concepts become refined. Initially, a student's spontaneous conception of proof may rely on the principles of proving things in familiar settings such as being provided an explanation by a parent or convincing a friend. This conception may also be shaped by some early school instruction, as when justifying an answer to a classmate or explaining a solution to a teacher, possibly with implicit or explicit feedback about what constitutes an acceptable product. As the student develops a scientific conception of proof by learning about logical structures, they will become more proficient constructing arguments to convince, justify, or explain what they constructed when they had only a spontaneous conception of proof. As the student participates in formal instruction and develops their scientific conception of proof, they may reach roadblocks where instruction is not compatible with their spontaneous conception of proof. In these

moments of incongruence, the student must adjust their spontaneous or scientific conceptions of proof.

An important component of a students' conception of mathematical proof is their belief about the purpose proof fulfills. Hanna suggested eight purposes a proof may fulfill, including verification and explanation (2000). Students' spontaneous conceptions of mathematical proof include the purposes they believe their own proofs fulfill while scientific conceptions include the purposes they are told a proof may fulfill. Once again, it is the interplay between these two conceptions that supports the development of operational conceptions of proof consistent with the mathematical community. For example, a student may be instructed that proofs can verify the truth of a statement. After writing a proof on a homework assignment the student may reflect on the argument they have constructed and recognize they have proved participated in the verification of a statement.

The goals students set during a mathematics course influence the types of activities they engage in and, consequently, the conceptions of proof they develop. Intelligence theory and goal orientation have been used to investigate the nature of student goals in mathematics classrooms (Dweck & Leggett, 1988). Dweck and Leggett distinguished between performance goals and learning goals. Individuals who adopt performance goals in a particular context are concerned with maintaining or gaining favorable judgements of ability, often from external sources. This may lead them to avoid challenging problems, which is typically seen as a maladaptive behavior since many worthwhile pursuits in both academic and non-academic life come only after confronting and resolving a challenge. Individuals who adopt learning goals in a particular context are concerned with increasing their ability. Individuals may adopt different types of goals when they are in different settings or learning different subjects. Selection of performance and learning goals are sensitive to outside factors such as task difficulty and task consistency (Steele-Johnson et al., 2000). In this study we consider student goal orientation in the context of mathematics courses with task difficulty and content consistent with that required by the course. A student's goal orientation is influenced by motivation, personality, willingness to persist through challenges, and personal beliefs about intelligence (Dweck & Leggett, 1988). Students who believe intelligence to be a fixed entity (sometimes referred to as having a fixed mindset) are prone to adopt performance goals; students who view intelligence as malleable (sometimes referred to as having a growth mindset) are prone to adopt learning goals.

A student's goal orientation, mindset about intelligence, and mathematical experience have the potential to reflexively influence one another. When a student with a growth mindset and learning goals is confronted with a challenging task, they may persevere, believing themselves capable of learning what is necessary to successfully complete the task. Alternatively, a student with a fixed mindset and learning goals may interpret failure as a sign of their personal limitations. This may lead to task avoidance or other counterproductive behaviors. A student who experiences repeated public failure in a mathematics classroom may feel shame about their abilities and begin to feel that intelligence is fixed and they are incapable of performing required tasks. This belief may lead the student to set goals to perform well in front of peers or authority figures. A student who consistently experiences the thrill of problem solving may set learning goals to stretch themselves to solve new and challenging problems. This pattern may be support or be supported by a growth mindset about intelligence.

Methods

Five undergraduate students were recruited during the first week of their ITP course in Fall 2025. Data collection consisted of two activities: weekly self-reflections and periodic clinical

interviews. At the end of each week, students completed a brief self-reflection via Google Forms about their goals, beliefs, and engagement behaviors in the course. The primary purpose of the reflections was to inform clinical interviews. The interviewer reviewed weekly reflections before each interview and asked targeted questions about specific goals, beliefs, and engagement behaviors mentioned in the reflections. For example, if a student's reflection mentioned a helpful engagement activity (e.g. taking notes in class), the interviewer asked the student to elaborate on the activity and its benefits during the interview.

The primary source of data collection was clinical interviews occurring at weeks 1, 4, 7, 12, and 16 of the semester. Phillip only participated in the first three interviews; we do not know whether he dropped the course at that time or simply withdrew participation in the study. Luke withdrew from the course between the third and fourth interview; he participated in an exit interview discussing his experience in the class and reasons for withdrawal. During each interview, students were asked to describe their current goals and ways they engaged with the class and material to achieve those goals. Students were also asked to construct at least one proof in each interview and to describe what elements of their constructed argument constituted a proof (or not). These conditions provided insight into how students define and evaluate proof. The research team then developed models of the scientific and spontaneous conceptions of mathematical proof for each student to be tested by subsequent data collection and analysis.

Interviews were transcribed and underwent open, axial, and selective coding (Corbin & Strauss, 1990). The research team conducted open coding on all five of Vivian's interviews. These initial codes were organized into categories and subcategories during axial coding, which lead to a second draft of initial codes. This codebook was then applied to other students' interviews. We employed a constant comparison method to ensure any adjustments to the codebook were retroactively applied to interviews that had already been coded. Codes were condensed and expanded to adequately represent students' experiences. Summaries describing the evolution of each student's goals and beliefs about mathematical proof were written and underwent selective coding to identify themes that persisted across multiple students. Coding occurred across three main categories: (1) the goals students expressed explicitly and implicitly, (2) students' beliefs about mathematics and proof, including their conception of mathematical proof and its purpose, and (3) the ways students engaged in the course.

Results

Goal Orientation and its Interaction with Learning Proof

Data from the first interview suggest most student participants set learning goals for the semester. For example, Luke said:

> I think approaching it initially it was more so just get an A because it's not relevant to my major, but I find it really interesting so far. So I think my opinion on that's going to change. I'd rather understand the subject at the—at its level.

We infer that Luke's proclivity to set learning goals was motivated by his newfound interest in the subject. Some students continued to express primarily learning goals throughout the semester. For example, in the fourth interview (Week 12) Max said he may not be able to achieve the high A he was hoping for, but believed it was "still very possible to like still understand everything if I go back and review everything that like I got wrong and stuff like that." We infer that Max was still orienting his behavior with the intention of learning all the material in the course. Other students, like Vivian, adopted performance goals as the semester progressed. External pressures of GPA, extracurricular activities, and parental expectations

influenced her shift to focus primarily on getting a particular grade on her transcript. In the third interview (Week 7) she said, "But I can't get a good grade without actually learning and taking something away from the course. So they go hand in hand. But to me, I just want a good letter grade on my transcript." Vivian recognized that learning would enable her to perform at a higher level, but at the time of the third interview she prioritized performance over learning.

Phillip was one of the few students who expressed primarily performance goals during the first interview and beyond. Students operating with learning goals engaged with proof in different ways than students operating with performance goals. For example, Phillip was a senior mathematics major who perceived his future career plans did not involve interacting with mathematical proofs. Of his ITP course he said it was "going to not benefit me with anything I want to go do. This is just one of the classes. I'm just trying to get—just slide by in the class." We infer he saw little value in learning proofs and repeatedly emphasized throughout his three interviews that his main goal was to pass the class and move on. By the time of his second interview Phillip was becoming more confident in his ability to construct proof but claimed he still didn't understand them. He said:

> I can go do those types of problems. Like, I know how to do them. It just doesn't make sense. But like, I'm not sitting there questioning myself if it's right or if it's wrong. I just know that it's right.

We infer that by the second interview he was developing a scientific conception of proof that included some procedural skills about how to construct a proof and that his spontaneous logical reasoning was not yet compatible with his new scientific conception. During Interview 2 there seemed to be a disconnect between his scientific and spontaneous conception of logical argumentation. In the third interview he said, "Yeah, like these ones and all like the, we started doing the uniqueness and existence. Those ones I was able to do really well too, I felt like. Those ones started to make a lot more sense." We infer that some of his scientific and spontaneous reasoning about logic were beginning to interact in productive ways, particularly in the context of writing uniqueness and existence proofs. When talking about proofs presented by the instructor during class, Phillip said, "I don't think I'll ever understand it." We infer that Phillip had a fixed mindset about learning mathematical proof.

Vivian began the semester with primarily learning goals about implementing her mathematical knowledge in the context of proof writing and building a strong foundation for future mathematics classes. However, as the semester progressed her goals seemed to shift to primarily performance goals. Vivian used artificial intelligence to help her on her homework. In the first interview she said she had recognized her tendency to give up on problems too early. She set a goal to not rely heavily on AI. In each interview Vivian discussed how she was using AI to supplement her learning in class. For example, she requested AI generate a practice exam for her based on the exam review guide provided by the instructor, used a proof generated by AI as an outline for a proof on her homework assignment, and asked AI which proof technique would be best for a particular statement. We do not believe the use of AI is necessarily maladaptive but note that many of the activities AI completed for Vivian precluded her from engaging in and thus potentially further developing strategic knowledge about mathematical proof (Weber, 2001). Perhaps in the case of Vivian, the use of AI was a maladaptive behavior engaged in with the hope that it would improve her performance in the class.

The Purpose of Mathematical Proof

In this section we focus on how students' beliefs about the purpose of proof, as described by Hanna (2000), emerged in the interviews. Students had differing views about the purpose of

mathematical proof in general and the purpose of mathematical proofs they produced themselves. For some students, the purpose of proof in general was believed to be the same as proofs generated by students, which we interpret as scientific conceptions about the purpose of proof being grounded in spontaneous conceptions; other students saw the activity they engaged in as distinct in purpose from the activity of proving in general, which we interpret as discordance between scientific and spontaneous conceptions of the purpose of proof. Phillip saw two primary purposes of proof: verify a statement is true and explain why it is true. He gave evidence that he believed these to be the purposes for both the proofs he constructed and proofs in general. Max, however, seemed to view the purpose of proof in general to be different from the primary purpose of proofs he constructed. In four interviews he made statements indicating he believed the purpose of proof in general to be explanatory; in two interviews he made claims that in general a proof should verify the truth of a statement. In three interviews, he expressed that the purpose of proofs he wrote was primarily to practice proof writing skills. In his final interview he said:

> I think it's just like good practice exercise. Other than that, I can't really think of much else just because I feel like it's very like specific thing and can't really think of a time where I would like use this fact really.

We infer that he sees the proof writing activity he engaged in as useful primarily for pedagogical purposes and not helpful in other contexts. In his final interview he included verification and explanation as additional purposes his proofs could fulfill.

Students differed in the number of distinct purposes they believed proof could fulfill. Max saw proof as potentially fulfilling three purposes (verification, explanation, and practice); Chase saw proof as potentially fulfilling seven purposes (verification, explanation, convincing, systematization, discovery, practice, and assessment). Chase saw proof in general and proofs he generated as potentially fulfilling each of these seven purposes, with the exception of assessment (only applied to student-generated proofs) and discovery (only applied to proof in general). We infer his scientific conceptions of the purpose of proof are grounded in corresponding spontaneous conceptions. Chase noted that not all proofs fulfilled the same purpose. For example, in the second interview he constructed a proof about the solution to a particular rational function then discussed what purpose the proof might fulfill. He said, "Like the core math idea it's trying to prove it's, because it's just algebra, because I've done it before a whole bunch of times, I already understand it. So it doesn't help me understand." He was then asked to construct a proof about set containment. Again the interviewer asked him what purpose the proof might fulfill. He said:

> I think it is different than the other one because, I mean, prior to this class, I hadn't really done anything with sets, so it is helpful to understand. It is helpful to understand how, I guess, maybe the properties of sets work?

We infer that Chase differentiated between the purpose of two different proofs and noticed that different proofs contribute different benefits to an author or a reader.

Discussion

Goal orientations were malleable throughout the semester. Most students started with primarily learning goals. Some maintained a learning goal orientation throughout the semester while others shifted to adopt more performance goals as the semester progressed. Learning goal orientations have been linked to more adaptive behaviors while performance goals have been

linked to maladaptive behaviors in learners (Dweck & Leggett, 1988). It is possible that the performance goals set by Vivian and Phillip limited their developing conceptions of proof. Phillip's desire to pass the class without feeling the need to understand the material may have led him to focus on the procedural components of proof at the beginning of the semester. Phillip's fixed mindset may also have impacted his goal orientation; he seemed to feel neither capable nor motivated to learn the content, so he focused his efforts on achieving a passing grade. He seemed to adopt some surface level aspects of proof related to language norms perhaps at the expense of trying to understand the logical nature of proof. It is not surprising that Phillip chose to focus on the new structures and techniques he was learning. Indeed, when individuals begin to learn scientific concepts they often experience a momentary decrease in their overall performance until scientific and spontaneous concepts begin to coalesce. We suggest that the interaction of spontaneous and scientific concepts may have been delayed by Phillip's goal orientation. Vivian's performance goals perhaps motivated her to turn to AI earlier in the problem-solving process than is desirable. Her focus on achieving a certain letter grade for her transcript may have influenced her to turn to AI, which she seemed to view as a readily accessible source of mathematical authority. We note that performance goals are not inherently negative, but they have been seen to be linked to maladaptive behaviors such as those adopted by Vivian and Phillip.

Student beliefs about the purpose of mathematical proof varied with some seeing proof as fulfilling only a few purposes while others viewing proof as fulfilling myriad purposes. Some students believed the purpose of student-generated proofs was distinct from the purpose of proof in general. We note that four of the five student participants saw explanation and verification as roles proof can play. Hanna viewed these two purposes of proof as fundamental and the most important for students entering the world of mathematics (2000). Explanatory proofs are of particular importance in educational settings.

> But in the classroom, the fundamental question that proof must address is surely "why?" In the educational domain, then, it is only natural to view proof first and foremost as explanation, and in consequence to value most highly those proofs which best help to explain. (p. 8)

We consider it productive that students saw proof fulfilling the two fundamental roles of proof. We also note that some students who view the proofs they construct as distinct from mathematician-generated proofs may recognize more specialized purposes of proof like discovering new results or organizing results into a system of axioms and results (Hanna, 2000). Alternatively, these students may feel they are separated from this aspect of participation the mathematical community.

Future Research

Data collection and analysis for this study is ongoing. Reflection and interview data is currently being gathered from students enrolled in an introduction to analysis course, which is typically the course taken immediately after the ITP course at the institution where the data were gathered. Two of the participants from this study are also participants in this second phase of data collection. We are investigating how goals, beliefs, and engagement patterns persist or evolve through extended study of mathematical proof and changes in instructor and content.

Most students talked about using AI in many ways to support their learning of mathematical proof even though the interviewer did not ask targeted questions about AI. From the perspective of the research team, some of the AI usage could have promoted productive ways of thinking about mathematical proof while others may have limited students' understanding and learning

about mathematical proof. We anticipate the availability and use of AI by students in proof-based mathematics courses to significantly increase and evolve in nature throughout the coming years. Future research could investigate ways students use AI in the learning of proof, their motives for using AI, and the perceived affordances and drawbacks. Researchers could also investigate instructors' views of AI use in proof centered mathematics classes.

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AN ECOLOGICAL SYSTEMS THEORY PERSPECTIVE ON HOME MATHEMATICS ENVIRONMENT AND YOUNG CHILDREN'S FUNDS OF IDENTITY

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As we strive to understand how young children develop productive mathematical identities, we turn to ecological systems theory to understand the nuances of family mathematics engagement. Within this framework, little is known about the lens that children and their families use to define mathematics and how this may connect to their funds of identity. Therefore, the purpose of this interpretive instrumental case study is to understand how first-grade children and their families define mathematics by analyzing the products produced as part of a Math Strong Project and then determine whether and how these lenses connect to funds of identity.

Keywords: Early Childhood Education, Identity, Ecological Systems Theory

Research suggests that children's abilities to count (i.e., sequence, one-to-one correspondence, cardinality), recognize and understand values associated with written numbers, and compare and order numbers predict concurrent and later mathematics achievement (e.g., Amland et al., 2025; Devlin et al., 2022; Lau et al., 2021; Liu et al., 2025; Wilkins et al., 2021). Research also has identified facets of the home environment and family interactions that are associated with mathematics learning. This includes the types of activities that families engage children in early math learning (Blevins-Knabe, 2016; Elliott & Bachman, 2018a, 2018b; Sonnenschein & Dowling, 2019), their mathematics beliefs and attitudes (Elliott & Bachman, 2018a; Levine et al., 2019), and their school involvement and support for learning related to children's math outcomes (Blevins-Knabe, 2016; Sonnernschein & Dowling, 2019). Yet, little is known about the lens that children and their families use to define mathematics and how this may connect to their funds of identity (i.e., the bodies of knowledge, skills, and resources that one generates through their interactions with others and their environment; Esteban-Guitart, 2016; Esteban-Guitart & Moll, 2014). Therefore, the purpose of this interpretive instrumental case study (Stake, 1995) is to understand how first-grade children and their families define mathematics by analyzing the products produced as part of a *Math Strong Project* and then determine whether and how these lenses connect to funds of identity.

Theoretical Framework

Defining Mathematics in Mathematics Education

Defining mathematics is not an easy task, and because of this, there is an ongoing debate in the field of mathematics education. To clarify this construct, we draw upon Thanheiser's (2023) work using framing (Lakoff, 2006; Louie, 2017) to explore three frames of mathematics and the implications of each frame. The three frames are as follows: (a) mathematics is an abstract body of knowledge organized into systems and structures, (b) mathematics is contextual and helps us to make sense of the world, and (c) mathematics is a human activity and part of one's identity.

Mathematics is an abstract body of language. The first frame considers mathematics as a noun. Oxford Languages (2026) defines mathematics as the science of number, quantity, and space. While others (Steen, 1988) argue that the growth of computing, statistical science, core mathematics, and applied mathematics has broadened the relationship between mathematics and science so that mathematics is not just the study of number, quantity, and space, but also has

become the science of patterns. Within the context of K-12 mathematics education, mathematics is simply viewed as a list of national standards (e.g., National Governors Association, 2010) where school mathematics becomes a space to learn about abstract rules and procedures (Adiredja, 2019). Yet, research suggests that transferring abstract knowledge of mathematics from school to the real world is difficult for children (Boaler, 1993) and adults (Lave, 1988).

Mathematics is contextual. Ascher and D'Ambrosio (1994) note that mathematics grew out of context through abstraction. In other words, mathematics is contextual. Critical scholars such as Freire (1970, 1994) and Gutstein (2006) note the differences between reading the word and reading the world. In the field of mathematics, this means mathematics can transcend procedural and conceptual understanding and be used to understand power relationships, inequities, and disparate opportunities between different social groups. Specifically, mathematics can be used to "dissect and deconstruct media and other forms of representation" in order to understand the world (Gutstein, 2006, p. 26). Further, Ani (2021) states that mathematics "isn't some arbitrary collection of skills to memorize. It is a prism for looking at the world, for analyzing how it works and reimagining how it could" (p. 11). In other words, mathematics is more than a collection of skills; it is a way of thinking.

Mathematics is a human activity. The final frame considers mathematics as a verb or action, thereby positioning mathematics as something people *do* rather than something people discover (Gutiérrez, 2018). Because of this, mathematics is more than an abstract body of language because, at its core, mathematics is a human activity and part of one's identity. Scholars (e.g., Aguirre et al., 2013, 2024; Cheng, 2022; Martin, 2009; National Research Council [NRC], 2001; Woods, 2023) define mathematical identity as how one sees themselves as a doer of mathematics. This means that children need to have opportunities to learn mathematics "in light of who they are and the diverse gifts that they bring to their experiences every day" (Aguirre et al., 2013, p. 10). To do this, teachers must work with children to become authors of their own mathematical understanding in ways that transcend procedural and conceptual understanding so that they are empowered to change the course of their lives.

So what? Recall that literature has identified that frequency and types of activities (e.g., Sonnenschein & Dowling, 2019), families' beliefs and attitudes towards mathematics and use of mathematical language are contributors to early mathematics achievement (Levine et al., 2019). Further, these three frames of mathematics may define the activities, attitudes, and use of mathematical language in home mathematics environments. Because of this, we wonder how children and their families think about mathematics in their daily lives. To guide us in our investigation, we turn to Bronfenbrenner's (1977, 1979) ecological framework to consider how humans are impacted by the changing environments in which they live and grow.

Conceptual Framework for Family Mathematics Engagement

In his work, Bronfenbrenner (1977, 1979) argues that the ecology of human development is a nested arrangement of structures describing the environment in which one lives. These structures are the microsystem (direct environment), mesosystem (connections), exosystem (indirect environment), macrosystem (culture), and chronosystem (time). These five interconnected structures shape one's growth and development. When considering the *home mathematics environment* (HME) or the role of families in early mathematics learning, scholars have used this ecological system framework to identify and assess potential moderators of mathematics learning and development (e.g., Hart et al., 2016).

Building upon the HME work, Eason and colleagues (2022) expanded the conceptual ecological framework for family mathematics engagement by combining the Integrative Model

for the Study of Developmental Competencies in Minority Children (García Coll et al., 1996), Cultural Microsystem Model (Vélez-Agosto et al., 2017), and the concept of M(ai)crosystem (Rogers et al., 2021). The intent of this new framework was to show the integration of "family, community, and societal factors that characterize children's math[ematics] experiences to provide a more comprehensive, developmental picture" (p. 4).

In our work, we leverage Eason and colleagues' (2022) framework and consider how funds of identity (FoI: i.e., a person's resources or tools that have been historically accumulated, culturally developed for self-definition, self-expression, and self-understanding; Esteban-Guitart, 2016) fit within the ecology of human development (see Figure 1). We argue that FoI allows us to contextualize how prior knowledge, experiences, and agents in and out of school may impact mathematical identity development. This framework also highlights how the three frames of mathematics may impact the breadth and depth of mathematics learning for young children.

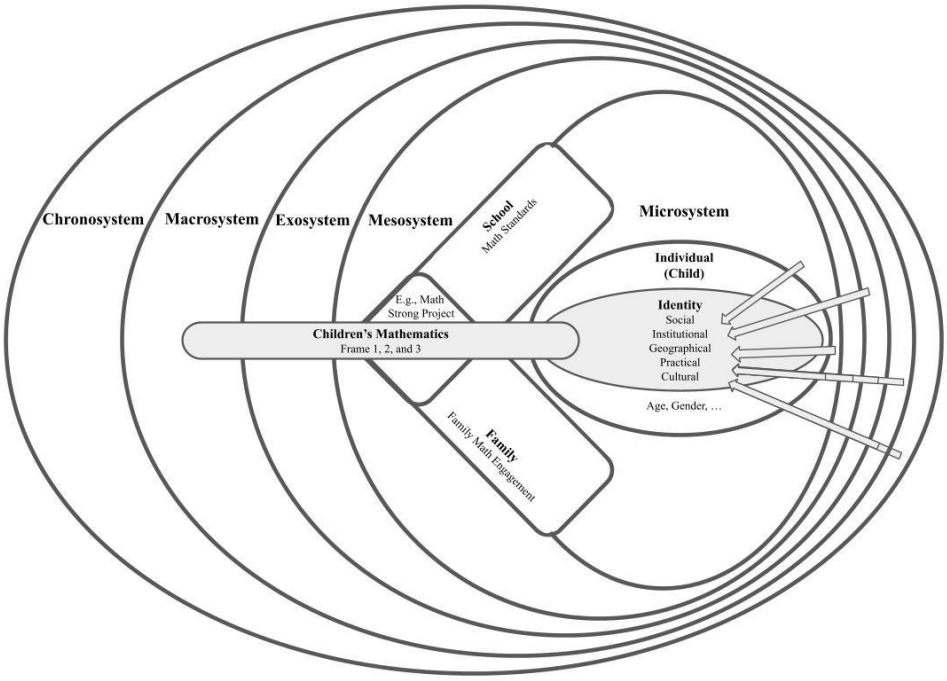


Figure 1: Funds of Mathematical Identity within an Expanded Conceptual Framework for Family Mathematics

To understand children's HME, specifically the lenses that children and their families use to define mathematics, and how this lens may connect to FoI, this study investigates two research questions. First, how do first-grade children and their families define mathematics? Second, how do these lenses connect to funds of identity?

Methods

This study was situated in a five-year longitudinal participatory research project (Vaughn & Jacquez, 2020) following kindergarten through third-grade children to understand how mathematical identity development changed over time as they participated in number talks (i.e., ten-to-fifteen-minute math discussions where children mentally solve mathematics problems and then come together as a class to share their mathematical reasoning; Parrish, 2010/2014). During

the second year of the project, our work was in first-grade classrooms, and we wondered how families defined mathematics and how it related to their child's identity development. As part of this work, teachers designed a family project to understand how families integrated mathematics into daily lives and then leveraged this to design number talks with an eye on mathematical identity development. To understand the family projects collected, we adopted an interpretative instrumental case study design to serve as a tool to guide us in understanding the relationships between families' lenses of mathematics and children's/families' FoI, rather than understanding individual children and families (Stake, 1995). This study design allowed us to obtain a more well-rounded sense of how mathematics was connected to lived experiences outside of school. Grounding this study within the literature helped us to develop a trustworthy interpretation as we collected a variety of data (triangulation), ensured adequate time in collecting data (adequate engagement), and shared findings with participants (member checks) (Stark, 2020).

Participants and Study Context

The setting was at Green Elementary School, situated in a large, public, suburban district in the Midwest. Two certified teachers, Ms. Roberts and Ms. Collins, participated in this study. (The school and participants have been provided with pseudonyms.) Ms. Roberts had 36 years of teaching experience in kindergarten and first grade, while Ms. Collins had 15 years in fifth/sixth grade and four years in first grade. Along with the two teachers, 37 (out of 43) young children who returned parent permission forms were included in the study. Among the 37 participating children, boys slightly outnumbered girls (56.8% vs. 43.2%) and most ethnicities were not Hispanic or Latino (91.9%). White constituted the largest racial group at 81.1%, followed by Native Hawaiian or Other Pacific Islander at 13.5%, and Black or African American at 5.4%. Regarding home languages, 83.8% of children spoke English, 5.4% spoke Spanish, and 10.8% spoke other languages (including Chinese, Telugu, Arabic, Albanian). For special education services, 13.5% of children had Speech Individualized Education Programs, and 5.4% had Section 504 plans.

Data Collection

One source of data was collected and analyzed in this study: *Math Strong Projects*. The goal of the *Math Strong Project* was to make a connection to children's families and the communities in which they live. We wanted to find connections between the microsystem, mesosystem, ecosystem, and macrosystems. Families had two choices in how they could participate in the project. Their first choice was an interview project where the child could interview a family or community member about how they used mathematics in their daily lives (e.g., Yeh et al., 2024). The second choice was to take a math walk around their neighborhood and take photographs or video about their mathematical discoveries (Woods, 2026). Since children and their families had the freedom to choose which activity to partake in, 13 families conducted interviews, while 24 undertook Math Walks.

Analysis

Analysis began with the authors annotating the videos and photographs collected from the *Math Strong Project* to familiarize themselves with the data (Miles et al., 2014). During the annotation process, we observed that different individuals held varying definitions of mathematics, and after turning to literature on the definitions of mathematics in mathematics education, we adopted the three frames of mathematics proposed by Thanheiser (2023) as a coding framework. During the first round, we generated open codes for systems and structures highlighted in the projects; generated open codes to identify real-world elements; and coded the five categories of FoI. As we reflected on the first round of coding these data, we turned to

literature to better understand the codes, connections to literature, and ultimately the data. To better understand Frame 1, we realized that we needed to include K-6 mathematics concepts (e.g., National Governors Association, 2010) since family members brought a sophisticated lens to the procedural and conceptual knowledge of mathematics. To better understand Frame 2, we considered if the contexts presented in the projects were purely observing mathematics or used mathematics to think critically about the world in daily life (Ani, 2021; Freire, 1970, 1994; Gutstein, 2006). In addition to coding the five categories of FoI (Esteban-Guitart, 2016), we coded if mathematics was portrayed as a noun or a verb. Table 1 explains the coding scheme, including examples from the data corpus.

Table 1: Codes, Descriptions, and Examples of Three Frames of Mathematics

Code	Brief Description	Examples
Frame 1: Mathematics is an abstract body of knowledge organized into systems and structures (e.g., National Governors Association, 2010; Thanheiser, 2023).		
Counting & Cardinality	Know number names, count sequence, and cardinality.	A child silently points and counts a total of 64 ducks on the lake.
Operations & Algebraic Thinking	Understand and use four operations, including their properties, relationships, and patterns, to represent and solve problems.	A child uses leaves and branches to represent the equation $2 + 4 = 6$.
Number & Operations	Read, write, understand, compare, and perform operations with whole numbers, rational numbers, decimals, and fractions.	A mom described how said she teaches fifth-grade children about fractions.
Measurement & Data	Describe, compare, and use measurement and estimation of different attributes in various units to solve problems. Also, represent and use data.	A child uses the words "bigger and smaller" to describe two seashells.
Geometry	Identify, describe, analyze, compare, create, compose, and classify shapes based on their attributes and properties. Solve real-world problems.	A child states that he saw a square window with four sides in the picture.
Ratios & Proportional Relationships	Understand ratio concepts and use ratio reasoning to solve problems.	A mom explains how she used mathematics to figure out percentages.
Frame 2: Mathematics is contextual and helps us to make sense of the world (Ani, 2021; Thanheiser, 2023).		
Looking At Mathematics	Mathematics as an object of observation.	A child counts the number of mailboxes on her street.
Looking With Mathematics	Mathematics is a lens for thinking critically about the world.	No examples from our data.
Frame 3: Mathematics is a human activity and part of one's identity (e.g., Esteban-Guitart, 2016; Esteban-Guitart et al., 2023; Gutiérrez, 2018; Thanheiser, 2023).		

Noun	Math is something that people discover.	A photo of 6 swings.
Verb	Math is something people do.	A child showing the number of leaves on his fingers.
Social	People who are important in one's lived experience (e.g., friends, family members).	A mom helping her child estimate and then measure the weight of a bag of sweet potatoes.
Institutional	Structures and mechanisms of society that manage behavior within a community or group (e.g., school, religion, laws).	A photo of a stop sign.
Geographical	Places and landscapes of people environments (e.g., community)	A photo of play structures inside a park.
Practical	Significant activities, interests, or hobbies (e.g., sports, exercise, work, shopping, cooking, music).	A mom explaining how she doubled a recipe to cook a favorite recipe.
Cultural	Symbols, tools, social categories that characterize, define, and mediate human experience, including money, time, religious symbols/beliefs (e.g., cross, Christmas tree), gender (e.g., hair, earrings, necklace), pop-culture (e.g., Halloween, superhero, rainbows), and technology (e.g., smartphone).	A child counting all the Halloween decorations in the yard.

Findings

Across these data, we realized that each *Math Strong Project* represented interconnected relationships between families' lenses of mathematics, providing insight into the ecological perspective of the HME. In what follows, we describe how first-grade children and their families define mathematics and how these lenses connect to funds of identity based on the descriptive statistics presented in Table 2.

Frame 1: Mathematics is an Abstract Body of Language

Frame 1 provided insight into our first research question, which is how do first-grade children and their families define mathematics. During the analysis, we realized that families prioritized mathematical concepts connected to school mathematics. Across the data, we found that the *Math Strong Projects* related to Counting and Cardinality (about 90%), followed by Measurement and Data (around 75%), Geometry (over 60%), and Operations and Algebraic Thinking (almost 60%). In contrast, Number and Operations, as well as Ratios and Proportional Relationships, were rarely presented (both lower than 10%). While the lack of Ratios and Proportional Relationships was expected since it was part of sixth-grade mathematics standards, we were surprised by the small presence of Number and Operations. We realized that this could be because it was difficult to capture place value understanding within a photograph or video, as well as limited exposure to fraction concepts.

Frame 2: Mathematics is Contextual

Analysis connected to Frame 2 also provided insight into how first-grade children and their families defined mathematics. In this data, we found that every project was situated within a real-

world context (i.e., home, park, grocery store, at work). Yet, 0% of the participants presented mathematics as a lens for critically examining the world. We found that mathematics was primarily practiced by families to address microsystem-level issues, such as supporting parents' work needs of pricing goods and managing systems. However, the findings indicated that mathematics had not yet been employed by families to understand macrosystem-level issues, such as understanding power relationships, inequities, and disparate opportunities faced by different social groups.

Table 2: Child-Level Percentages of Codes' Presentation in Interview, Math Walk, and Combined Works (the Whole Math Strong Project)

Codes by Frame	In Interview	In Math Walk	In Total
Frame 1			
Counting & Cardinality	76.9%	95.8%	89.2%
Operations & Algebraic Thinking	76.9%	50.0%	59.5%
Number & Operations	23.1%	0.0%	8.1%
Measurement & Data	76.9%	75.0%	75.7%
Geometry	30.8%	79.2%	62.2%
Ratios & Proportional Relationships	7.7%	0.0%	2.7%
Frame 2			
Looking At Mathematics	100.0%	100.0%	100.0%
Looking With Mathematics	0.0%	0.0%	0.0%
Frame 3			
Mathematics Portrayed as Noun	84.6%	29.2%	48.6%
Mathematics Portrayed as Verb	15.4%	70.8%	51.4%
Social	100.0%	54.2%	70.3%
Institutional	0.0%	12.5%	8.1%
Geographical	100.0%	87.5%	91.9%
Practical	0.0%	20.8%	13.5%
Cultural	15.4%	54.2%	40.5%
Social	100.0%		100.0%
Institutional	15.4%		15.4%
Geographical	100.0%		100.0%
Practical	92.3%		92.3%
Cultural	46.2%		46.2%

Note. C represents the children and F represents other family members.

Frame 3: Mathematics is a Human Activity

Frame 3 provided insight into how first-grade families defined mathematics. Here, children and their families portrayed mathematics as a noun (i.e., something people discover) or as a verb (i.e., something people do) with roughly equal frequency. Yet, we noticed that the math walk activity provided a better opportunity for families to do mathematics together.

Also in these data, we noticed connections to children and other family members' FoI. We observed that children and their families frequently presented geographical FoI, both accounting for over 90% of cases, through scenes such as parks, courtyards, and houses. Additionally, since the *Math Strong Project* was designed to encourage family mathematics engagement, there's

also a frequent presence of social FoI via family members' interactions. The presentation rate of social FoI among family members reached 100%, while it was approximately 70% among children (partly because the identities of individuals present cannot be determined when people are not visible in the photos or videos).

Across these data, children presented their practical FoI less frequently, at only about one-seventh the rate of other family members. This may suggest that family members are great resources and create a positive environment to help children develop practical FoI. Further, this may suggest that children were still learning to connect mathematics with their own significant activities, interests, or hobbies.

Interestingly, there was a low frequency of institutional code for both children and their families. Only 8.1% for children and 15.4% for family members showed a representation of institutional FoI. This seems to be consistent across the data sent, especially when noticing that mathematics was not used as a tool to critically think about the world at the institutional/systemic level (i.e., Frame 2).

Interpretations and Implications

Scholars (Freire 1970, 1994; Gutstein, 2006) noted the difference between reading the word and reading the world. This means that mathematics should be a tool to look at the world, to analyze how it works, and then reimagine how it could be better for all (Ani, 2021). With this in mind, we turn to the two key takeaways from this study. First, this study confirms that children and families define mathematics as an abstract body of language as they prioritize procedural knowledge over conceptual understanding when doing mathematics in connection to the *Math Strong Project*. Yet, in every project, children demonstrated this understanding in connection to real-world contexts that were a part of the children's and family members' identities. Further, mathematics was portrayed as a noun (i.e., something people discover) or as a verb (i.e., something people do) with roughly equal frequency. This means that roughly 50% of the children had the opportunity to do mathematics in ways that made sense to them, and we argue that these opportunities to use real-world contexts inspired mathematical identity development.

Although there were connections to real-world contexts that were a part of children's and families' identities, we found that mathematics was used as a tool to focus on the *word*, but not as a tool for critically examining the *world*. This finding was triangulated in the low frequency of the FoI institutional code for both children and their families. As it is becoming increasingly important for families, and society at large, to use mathematics to change the course of their lives, this research helped shed light on the HME. Additionally, this research also sparked the wonder of how we can support children and their families to use mathematics as a prism for looking at the *world* in ways that can better the human experience.

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A SENSEMAKING EPISTEMIC GAME: THE CASE OF HAZEL AND LAYLA AND THE LAUNDRY TASK

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In this study, we investigated how two pre-service teachers (PSTs), Hazel and Layla, engaged in sensemaking as they explored bacterial growth through a laundry task. Particularly, drawing on the sensemaking epistemic game framework (Odden & Russ, 2018), we examined Hazel and Layla used mathematics to represent a realistic situation. To support their sensemaking game, the PSTs engaged in two sub-epistemic games, list-making and categorization, and multiple epistemic moves. Although the PSTs were not able to attain their target epistemic form within the sense making game, they were able to make progress in mathematically representing the laundry situation. We end with discussing the implications of drawing on the epistemic games and forms framework, largely used in science education, in analyzing students' mathematics as they use it to address scientific inquiries.

Keywords: Pre-Service Teachers, Epistemic Games and Forms, Epistemic Moves, Sensemaking Games

Scientific inquiries in STEM education often involve using mathematical and statistical analysis to address the problems posed. Engaging in scientific inquiry is epistemic in nature; scientific knowledge is constructed, justified, evaluated, and refined. Within the mathematics education community, this epistemic process of using mathematics to address scientific questions has been studied from a mathematical modeling perspective (e.g., Lesh & Doerr, 2003). Within this perspective, usually, addressing scientific inquiry stems from an epistemological viewpoint that addressing questions about complex real-world problems follows certain processes that turn a non-problematized situation into a situation that can be solved and described using mathematics. For example, a scientist needs to understand the situation, make simplified assumptions about the situation, formulate a mathematical model, perform mathematical and computational analysis, evaluate the results, and interpret the results in terms of the real-world situation (Anhalt & Cortez, 2016; Blum & Leiss, 2007; Niss, 2004). While this perspective has shed light on the advancement of mathematical process-based learning, it has been limited in addressing questions such as *what do learners do as they construct and refine scientific knowledge using mathematical reasoning*, where learners' actions are taken in service of knowing rather than a set of processes they follow. In light of this, in this paper, we present a case study that examines an epistemological perspective on how mathematics is used to organize scientific problems.

Empirical and Theoretical Background: Epistemic Games and Sensemaking

Epistemic forms and games provide a powerful theoretical lens for understanding how students, the players of the game, structure their knowledge as they work on a scientific inquiry. This framework was originally proposed by Collins and Ferguson (1993). An *epistemic form* is the target structure that guides the game; this is what the players aim to attain (Collins & Ferguson, 1993; Sherry & Trigg, 1996). The *epistemic game* is the set of rules, actions (or moves), and strategies employed to develop an epistemic form (Collins & Ferguson, 1993). Therefore, each epistemic game has a *target epistemic form*.

An epistemic game consists of the following features: *entry conditions, moves, transfers, constraints,* and *target epistemic form* (Collins & Ferguson, 1993; Oden & Russ, 2018). The *entry condition* is when the game starts for the player—when the player organizes sufficient background knowledge to begin the game. The *moves* are the actions that are allowed within the game to help construct the target epistemic form. For example, in the list making game—the simplest type of game and is often used early in the inquiry process (Collins & Ferguson, 1993)—the typical moves when playing the list-making game include adding, deleting, combining, splitting, or changing items on a list (Sherry & Trigg, 1996; Collins & Ferguson, 1993). Differently, in the hierarchy game—commonly referred to as *categorizing or classifying* (Collins & Ferguson, 1993)—moves include adding items to the group and defining the categories. *Constraints* are the rules that govern the game. *Transfer* is when the player changes from one game to another, upon deciding that the current game no longer supports them in attaining their target form. For instance, students might transfer from a list-making game to a categorizing game when they observe patterns or similarities amongst the listed items (Collins & Ferguson, 1993). Lastly, the *target epistemic form* is the "goal" or knowledge structure one creates by "playing" the game (Oden & Russ, 2018, p. 3). For example, in the tic-tac-toe game, the *form* consists of the game's 3×3 matrix, *constrained* by attaining three-in-a-row of the same kind, and completing each cell with *x*'s or *o*'s to attain the constrained form are the *moves* of the game (Collins & Ferguson, 1993).

Throughout the last several decades, science education research has developed frameworks for understanding how students' reason during problem solving. For instance, in physics education research, scholars have closely examined students' as they reasoned in the moment, using mathematical symbols and coordinating them with conceptual schema (Sherin, 2001). Similarly, Tuminaro & Redish (2007) showcased how their students engaged in epistemic games and forms as the students organized their equations through conceptual resources. More recent research has further articulated sensemaking as a distinct epistemic game, emphasizing students' efforts to resolve gaps in their understanding (Odden & Russ, 2018). Parallel efforts in chemistry education have adapted and extended these frameworks to examine how students engage in sensemaking during chemical problem solving (e.g., Hunter et al., 2021; Rodriguez et al., 2020). For example, Hunter et al. (2021) used a sensemaking epistemic game to show how collaborative gas law activities can prompt students to recognize gaps in understanding and build explanations together. Collectively, this literature helps us to see students' knowledge construction as an evolving complex process rather than being linear.

Odden and Russ introduced *sensemaking* game when students are trying to figure something out using their prior experiences or knowledge (Odden & Russ, 2018, p. 2). Like any epistemic game, the sensemaking game has an entry condition, constraints, moves, transfers, and a target epistemic form. For instance, when one of their students, namely Emma, *noticed a gap* in her understanding of how an electric current works. This recognition set off the sensemaking game, and that led to a sequence of moves that helped Emma with her understanding. She made a move by suggesting that current might circulate rather than be used up. Accordingly, the target form of a sensemaking game is the resolution of a gap in students' understanding, which may or may not result in a resolved explanation (Odden & Russ, 2018); the game is afforded by the moves that help the learners resolve the gap; the constraints help define the permissible rules in the sensemaking game. With this in mind, we address the following question: *How did the PSTs enter into and sustain the sensemaking game while working on the laundry task?*

Methods

Data for this exploratory analysis were drawn from a larger study that consisted of two pairs of K-8 PSTs working on twelve modeling tasks. The PSTs were recruited from a mathematics content course required for K-8 pre-service teachers at a large public university in the western United States. The participants had completed prerequisite coursework in college algebra. The sessions were held on campus and were video- and audio-recorded. The research team met with each pair for 60 to 90 minutes, for a total of ten sessions. The PSTs were free to conceive of the laundry task however they wanted, as it was an open-ended task. For this paper, we focus on how one pair of PSTs (pseudonyms Layla and Hazel) played the sensemaking game as they worked on the laundry task. We selected Layla and Hazel's work on the laundry task because they provided an explicit episode of reasoning that allowed us to trace their entry conditions, moves, and transfers within an epistemic games framework. The abridged version of the task is provided below in Figure 1.

> On January 10, 2023, TikTok user Allison Delperdang sparked a lively debate when she shared a video about wearing the same pajamas multiple times before washing them. Dermatologist Dr. Anthony Rossi emphasized that our cultural norms often lead to "over-washing," even though not all clothes need frequent cleaning. At the same time, wearing hygiene clothes can cause skin infections due to bacterial overgrowth in the unwashed clothes. Research indicates that bacterial loads exceeding 1000 colony-forming units per square centimeter ($\boldsymbol{CFU/cm^2}$) can pose a risk, particularly for pathogenic bacteria like Staphylococcus aureus. Develop a mathematical model for determining how long clothing can be safely worn before washing?

Figure 1: The Laundry Task (abridged version). Task description taken from Smith (2023)

Data Analysis

The data was transcribed and analyzed using MAXQDA, a qualitative data analysis software, to support the identification of PSTs' sensemaking game. The analysis proceeded in three steps. First, the first author watched the video in its entirety and documented any salient epistemic episodes. During this review of the video, characteristics of a sensemaking game were noted. For example, an entry point was first captured when Hazel stated, "I'm not sure how we're supposed to work that in[referring to the 1000 CFU/cm^2]." Hazel recognized that a quantity had been provided but did not yet see how it fit into an explanation of what was happening. This moment reflected *noticing a gap* in understanding, which Odden and Russ (2018) describe as the entry condition for the sensemaking game. Hazel articulated an explanation focused on how bacteria form and multiply and attempted to connect this process to the given quantity of 1000 CFU/cm^2. Rather than using a formula or definition to construct a model, she engaged in explanatory reasoning to make sense of how the bacteria could reach that quantity.

Next, the video was subsequently revisited through the lens of a sensemaking game framework (Odden and Russ, 2018). Like any epistemic game, the sensemaking game has an entry condition, transfer, moves, constraints, and an epistemic form. For instance, the PSTs' introduction of the parameter 100 CFU/cm^2 per day was coded as a move. This was identified as a move because it helped the PSTs make sense of how many times someone could wear their clothing before the bacterial load reached 1000 CFU/cm^2 a level that could pose a risk.

Finally, the video was revisited to identify other features of epistemic games and forms, using Collins & Ferguson's (1993) framework. For example, when the PSTs started talking about their own experience regarding laundry, this instance was coded as an entry condition, as

this was the information the PSTs knew coming into the game. Similarly, the task objectives were identified as the constraint of the game since the task required the PSTs to create a mathematical model for the scenario, governing how the game could be played.

Findings

In this section, we illustrate how the sensemaking game unfolded over time. We summarize our findings in Figure 2, which presents the entry conditions, constraints, moves, transfers, and target form of the game and serves as a roadmap for the findings that follow. We describe these elements in the order in which they emerged as they worked through the laundry task.

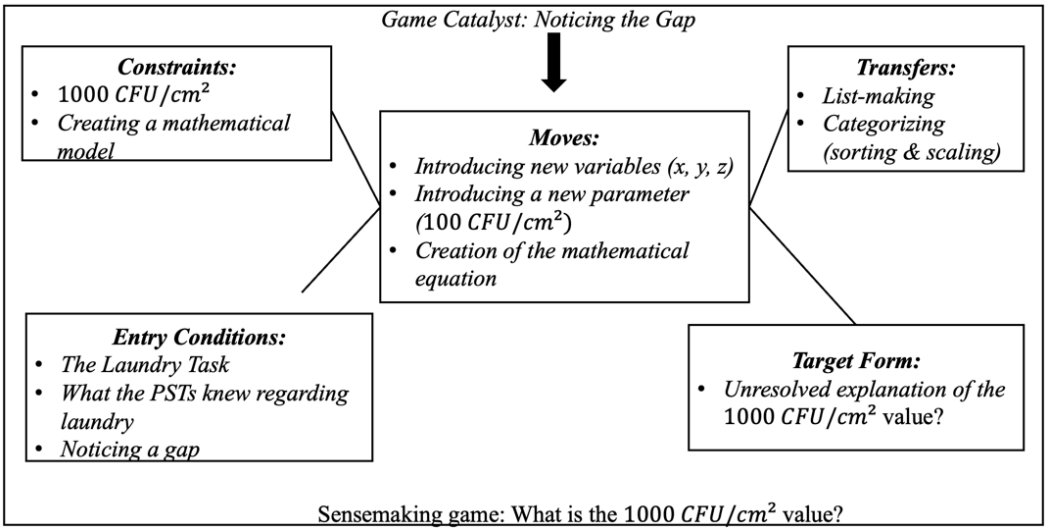


Figure 2: Overview of Hazel and Layla's Sensemaking Game

Entry Conditions

When the PSTs were given the laundry task, they started to brainstorm on how often they do their own laundry. They discussed how laundry habits vary, as it depends on factors such as clothing type and level of activity. They noted, for example, that jeans are not washed after every wear because frequent washing can cause them to wear out more quickly.

The entry condition for the sensemaking game emerged as a gap in the PSTs' understanding of how bacterial growth on clothing relates to the 1000 CFU/cm^2. As Layla and Hazel reflected on their own experiences, they referred to the statement, "Bacterial loads exceeding 1000 CFU/cm^2 can pose a risk," provided in the task. At this point, the PSTs recognized uncertainty about how bacteria accumulate on clothing and how this quantity could be incorporated into a model. They continued to discuss reasons for washing clothes; it's because of the stains on the clothes or due to hygiene. Hazel reread the prompt and still felt uncertain about how to think of bacteria growing on clothes and how to incorporate the 1000 CFU/cm^2 number into their model. Hazel says, "I'm doing my laundry… Like I've worn this three times, there's probably a lot of bacteria on here." Layla responds, "Right. Probably more than 1000 … bacteria on this person. Am I going to start thinking about that?"

Based on the conversation, we see that the PSTs drew on their prior knowledge of laundry and everyday experiences to reason about bacterial growth on their clothes.

Constraints

Within the laundry task, as the PSTs reasoned, the task was governed by rules in a few ways. First, the fixed threshold 1000 CFU/cm^2 restricted the PSTs from interpreting bacterial growth and only allowed them to work within the given threshold. This constraint shaped the allowable moves in the game as it directed attention to whether bacterial levels exceeded the stated threshold. A second constraint was the requirement that PSTs construct a mathematical model, as specified in the task. This requirement restricted how the PSTs could represent their reasoning and compelled them to translate their everyday understanding of laundry and hygiene into a mathematical model.

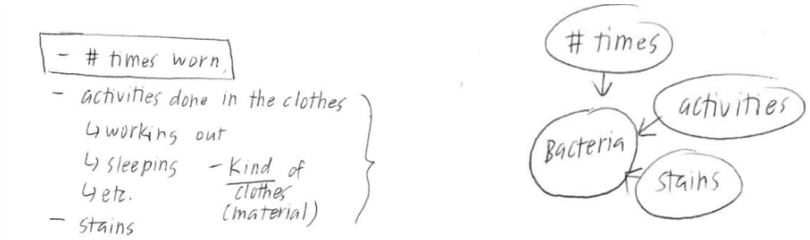
Moves and Transfers

We present the moves and transfers in the order in which they unfolded. As the PSTs were trying to make sense of 1000 CFU/cm^2 , the interviewer prompted them to document any ideas that came to their minds. In response, they began to create a list of reasons why they would wash their clothes. This shift marked a *transfer* from the sensemaking game to a list-making game.

List-making game [transfer]. They began to list factors that could potentially influence how frequently clothes should be washed, as shown in Figure 3: (i) the number of times worn, (ii) activities done in the clothes (e.g., working out, sleeping), (iii) stains, and (iv) the kind of clothing (material). This represents a list-making epistemic game**,** with the list serving as the epistemic form. The list-making game helped the PSTs begin making progress in their sensemaking by externalizing and organizing their ideas.

After the PSTs listed these factors shown in Figure 3a, Hazel expressed that she's having a hard time incorporating the listed items into the provided threshold of 1000 CFU/cm^2.

Hazel:	Yeah. I do wonder what it takes to get to this point, the 1000 [referring to the 1000 CFU/cm^2].
Interviewer:	What do you mean?
Hazel:	Like what sort of conditions or sort of numbers on these things [referring to the list in Figure 3a] would lead to this amount? I'm wondering if this is like you haven't washed this clothing item in months, or if this is average, like by the time you do your laundry every two weeks … causing a dangerous risk. I really have no idea, though, because I don't know anything about bacteria.



(a) (b)

Figure 3: (a) Listing-making Game, (b) Contribution Towards Bacteria

We can observe that Hazel is still uncertain about how the 1000 CFU/cm^2 value was determined. She speculates whether it results from extreme conditions, such as not washing clothing for an extended period, or if it represents an average accumulation.

This uncertainty continues to illustrate the gap in Hazel's understanding of how bacteria accumulate, which in turn hinders her ability to progress in developing the model. At this point, the PSTs transferred back to the sensemaking game from the listing-making game. They then start making *moves*, such as introducing a new parameter and a new variable, to help with their progression of making sense of the 1000 CFU/cm^2 threshold.

Introducing a new parameter [move]. After the conversation above, Layla suggested coming up with numbers, making a move by introducing a *new parameter*, which was 100 CFU/cm^2 per day. This is a move as the PSTs wanted to understand the rate at which bacteria can be accumulated and for how long clothes could be worn to reach the 1000 CFU/cm^2 threshold. Layla wrote that on day two, it becomes 200 CFU/cm^2, on day three, it becomes 300 CFU/cm^2, and so forth. Here, for the PSTs, 100 CFU/cm^2 per day is a daily rate that, in this specific scenario, is treated as a constant parameter. They used this parameter to reason about bacterial growth, which can change over time. The PSTs reasoned that changes to the value may occur in coordination with other quantities, such as activities done on the clothing or the presence of stains.

Introducing a new variable [move]. Right after they introduced the parameter, the PSTs introduced a new variable, $\boldsymbol{x}$, to represent the number of times worn. They expressed this relationship as $\boldsymbol{x \times 100}$ *units = how many units formed on the clothing*. They reasoned that on an average day, clothing could be worn up to ten times before reaching $\boldsymbol{1000\ CFU/cm^2}$. They explained that when you wear clothes one time, you get $\boldsymbol{100\ CFU/cm^2}$, so wearing the clothes ten times would lead to $\boldsymbol{1000\ CFU/cm^2}$**.** The introduction of the variable, *x*, supported the emergence of their representation and enabled the PSTs to reason about the change in bacterial load over repeated wears.

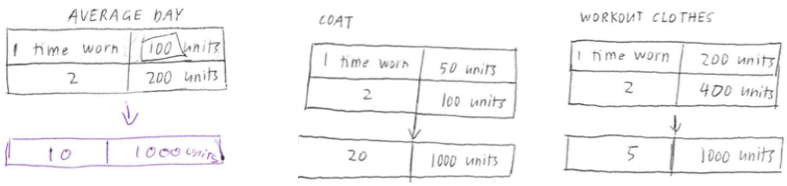
Categorizing game [transfer]. After the PSTs made the two moves of introducing a new parameter and variable, they started to reason about the dependency of bacteria accumulation on the factors listed in Figure 3a. At this moment, the students *transferred* from the sensemaking game to the categorizing game. Within the categorizing game, the PSTs employed two types of categorizations: (a) sorting and (b) scaling.

Sorting. When PSTs listed the factors that influence the number of bacteria accumulated, Hazel realized that they needed to sort out the factors they listed in Figure 3a into different categories because activities like exercising, being sick, and carrying out a normal day will all influence the growth of bacteria differently. As for Figure 3b, the PSTs drew a presentation showing that the type of activities done in the clothing, the number of times worn, and stains on the clothing all contribute towards the rate of bacteria.

Hazel said, "I mean, you could wear something for 30 minutes, while you're doing yoga, and then change out of it. I don't know how much these [referring to Figure 3b] are affecting this first table [referring to Figure 4a], because I think it's sort of a single input-output sort of situation." For the PSTs, an average day is when they are not doing anything physically demanding. The PSTs then sorted items based on similarities in how bacterial formation will occur. As shown in Figure 4(a, b) below, they sorted the amount of bacteria formed based on the activity done or the clothing type. They were sorted as an average day, workout clothes, and coats, with 100, 200, and 50 CFU/cm^2, respectively, for each wear.

Scaling. Another instance in which we observed the PSTs utilize categorizing occurred when they employed a scaling approach to organize physical activity and type of material by assigning values from 1 to 10. The idea of scaling emerged when the interviewer asked how the factors shown in Figure 3b influence the 100 CFU/cm^2. This scaling approach was proposed because it

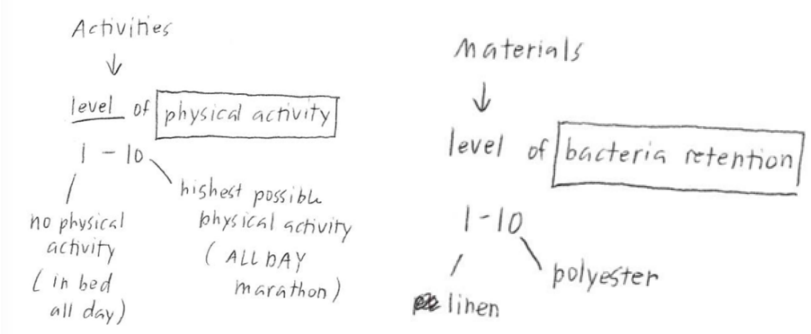
was easier for the PSTs to interpret and apply how the activity level influences bacterial accumulation on clothing.



(a) (b)

Figure 4: PSTs' Sorting

The PSTs applied the same scaling approach to clothing material, informed by online sources. Using this scaling approach, as shown in Figure 5a, a value of 1 represented the lowest level of bacterial retention, which they associated with linen, while a value of 10 indicated the highest retention, which they attributed to polyester, as shown in Figure 5b. Hazel further explained that an "average day" may involve minimal physical activity, such as walking to class, whereas some individuals exercise once or twice a day. She therefore suggested representing physical activity using a numerical scale from 1 to 10. On this scale, a value of 1 corresponds to no movement at all (e.g., staying in bed all day), while a value of 10 represents the highest level of exertion, such as hiking or running a marathon.



(a) (b)

Figure 5: PSTs Scaling

Creation of the mathematical equation [move]. After sorting and scaling, the PSTs introduced two additional variables y and z that represented the level of physical activity and bacterial retention, respectively. They proposed the model $x = units \div (y \times z)$, where units refer to 1000 CFU/cm^2. Layla critiqued the model by testing it against an extreme but plausible scenario, noting that under the model, someone could run multiple all-day marathons in polyester without risk-posing bacterial buildup, which did not make sense to her. Hazel agreed with Layla and further emphasized her uncertainty about how quickly bacteria accumulate, stating that 1000 CFU/cm^2 seemed like a large number without a clear basis.

Target Epistemic Form

While the structure of the mathematical model created by the PSTs allowed them to reason quantitatively about clothing type, activity level, and bacterial buildup, they remained uncertain

about how the value of 1000 CFU/cm^2 was determined. Although the PSTs made progress in structuring their reasoning, their sensemaking remained unresolved. The PSTs also developed representations such as lists, categories, and a model; these artifacts reveal the ways in which students approached the problem, highlighting their reasoning processes rather than emphasizing a final correct result.

Discussion

Our findings indicate that the PSTs used multiple epistemic sub-games, namely list-making and categorizing games, to make progress in their sensemaking game. This suggests that the flexibility and fluency to transfer between epistemic sub-games support the gradual attainment of a mathematical model for the scenario, despite being unable to resolve their sensemaking of 1000 CFU/cm^2. This insight raises important questions for the field: (1) what roles do epistemic games play in students' construction of a mathematical representation for a realistic scenario? and (2) how can we design learning experiences that facilitate the emergence of epistemic games and moves among students? In our case, the list-making game helped the PSTs think about why someone would wash their clothes, leading them to identify various contributing factors. On the other hand, the categorizing game helped the PSTs organize these factors based on how bacterial accumulation depends on them. Had the PSTs already known the formation of bacteria, we might not have been able to observe the PSTs' sensemaking game; instead, what we might have observed is an "answer-making" game. Thus, we hypothesize that the tasks, learning environment, and students' prior knowledge can inform the game a student plays.

Furthermore, in an era where artificial intelligence can readily provide answers to scientific prompts, the focus of instruction should not merely depend on having students generate answers to problems. Instead, learning experiences should focus on how students can connect their past knowledge in meaningful ways to construct new knowledge (Schoenfeld, 1992). In this view, the final epistemic form that students construct becomes subordinate to the *ways in which* (i.e., epistemic moves and actions) they constructed the product. In our case, Hazel and Layla made progress in the ways they reasoned as to how bacteria can accumulate, through their epistemic actions. These actions were instrumental for them to move beyond their knowledge gap and attend to the goals of the task. Thus, educators should provide learners with opportunities to develop tools that enable students to advance their knowledge construction even in the presence of perturbations and inconsistencies, rather than abandoning the process.

We acknowledge certain limitations of our study. One key limitation is that our findings may only hold true in the case illustrated here. That is, we demonstrated the feasibility of using the epistemic forms and games framework to analyze Hazel and Layla's construction of mathematical representation for the laundry scenario. Looking through a sensemaking epistemic game lens, different students working on different tasks may yield different outcomes. Thus, while our findings are locally rich, they do not propose a principled way for documenting learners' diverse sensemaking strategies as they engage in scientific inquiry.

That said, a substantial body of research across other disciplines, such as chemistry, physics, and life sciences, have been examining how students construct knowledge (e.g., Tuminaro & Redish, 2007; Hunter et al., 2021; Rodriguez et al., 2020). Collectively, these works have resulted in robust frameworks, such as symbolic forms (Sherin, 2001) and resources (Hammer, 2000), that have been influential in analyzing students' scientific knowledge construction. For mathematics to be considered as a *vehicle* for addressing scientific inquiries, in addition to a domain of inquiry, frameworks that enable the explanation of such knowledge, such as epistemic forms and games, should be adapted and tested for feasibility. We conclude by stating that our

study contributes to this effort by showcasing the feasibility of such an approach through a detailed case example.

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EXPLORING STUDENTS' PERCEPTIONS OF MATHEMATICS INSTRUCTORS AND THEIR IMPACTS ON THEIR ACHIEVEMENT

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This study aims to explore 8th-grade students' perceptions of mathematics instructors and their impacts on achievement in TIMSS 2015 and 2023. The publicly available TIMSS 2015 and 2023 data were used to explore students' perceptions of mathematics lessons and their achievement. The data were analyzed in IBM SPSS 29 to examine students' perceptions of mathematics instructors, using mean and standard deviation, and linear multiple regression was performed to examine the effect of these perceptions on mathematics achievement. The results showed that students had positive perceptions of their mathematics lessons in both 2015 and 2023. However, their effects on achievement were inconsistent across both years. These differences in the impact of students' perceptions of mathematics lessons on achievement indicate that other intervening variables may influence perceptions of lessons and mathematics achievement over time.

Keywords: Student Perceptions, Mathematics Lessons, Achievement, TIMSS

Students' perceptions of mathematics instructors may determine the quality of their learning and may affect their achievement. Past studies have reported that students' achievement in mathematics is influenced by a variety of contributing factors, such as teacher-related factors (Appiah et al., 2022), school factors (Chowa et al., 2015; Wardat et al., 2022), student factors (Wardat et al., 2023), and home background (Oginni, 2018; Tomul et al., 2021). Past studies also focused on students' perceptions of mathematics, mathematics learning, and their influence on achievement (Ampadu, 2012; Arthur et al., 2017; Daud et al., 2020; Hagan et al., 2020). However, there is a scarcity of literature that focuses on students' perceptions of mathematics instructors and their influence on achievement. This study is conducted to fill the gap in the literature. The objective of this study is to explore eighth-grade students' perceptions of mathematics instructors and their impacts on achievement from TIMSS 2015 and 2023 data. The research questions for the study were: How do U.S. eighth-grade students perceive their mathematics lessons, and how do these perceptions influence their achievement in TIMSS 2015 and 2023?

Materials and Methods

The data for the study were retrieved from the IEA TIMSS and PIRLS at Boston College (link: https://timssandpirls.bc.edu/index.html) for both 2015 and 2023 US data. The data is available in the public domain for research purposes. The TIMSS 2019 data were not used in this analysis because the items on students' perceptions of mathematics instructors were slightly reframed and the data was gathered during 2020, which was heavily impacted by the Covid-19 Pandemic. There are ten items for students' perceptions of mathematics. These items are given in Table 1.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Table 1: Eighth-Grade Students' Perceptions of Mathematics Lessons in TIMSS 2015 and 2023

Sl. No.	Items of students' perceptions of mathematics lessons	Agree a lot	Agree a little	Disagree a little	Disagree a lot
1	I know what my teacher expects me to do.				
2	My teacher is easy to understand.				
3	I am interested in what my teacher says.				
4	My teacher gives me interesting things to do.				
5	My teacher has clear answers to my questions.				
6	My teacher is good at explaining mathematics.				
7	My teacher lets me show what I have learned.				
8	My teacher does a variety of things to help us learn.				
9	My teacher tells me how to do better when I make a mistake.				
10	My teacher listens to what I have to say.				

The original data coded as: Agree a lot = 1, Agree a little = 2, Disagree a little = 3, and Disagree a lot = 4. This coding was reversed to assign the highest value of 4 to "Agree a lot" and the lowest value of 1 to "Disagree a lot" for easier comparison of the degree of agreement and its association with mathematics achievement. Average and standard deviations for each item for students' perceptions of mathematics were computed for both 2015 and 2023. Then, a linear multiple regression analysis was performed to examine the impact of each perceptual item on students' mathematics achievement in 2015 and 2023. TIMSS uses item response theory to construct five plausible values for student achievement. This study used only one plausible value (the first one) as the dependent variable for the regression analysis because there were no statistically significant outcomes while using all five plausible values. The ten perceptual items in Table 1 were used as independent variables. The residual analysis for the dependent variable (achievement data) indicated normality, and hence, regression analysis was deemed valid.

Results

The results of the descriptive analysis of students' perceptions of mathematics lessons in 2015 and 2023 were positive across all 10 items, with average values greater than 3 (Figure 1).

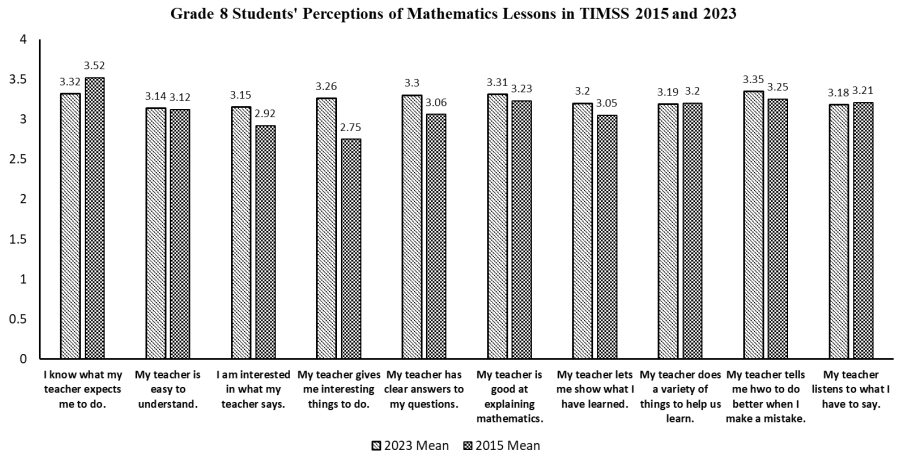


Figure 1: Eighth Grade Students' Perceptions of Mathematics Instructors in 2015 and 2023

The results of the linear multiple regression showed that students' mathematics achievement was influenced by only one item from their perceptions of mathematics lessons in 2015 and 2023. For example, "my teacher is easy to understand" with regression coefficients of 11.64 and 8.195, both statistically significant ($p < .001$). Another item, "my teacher gives me interesting things to do," had a positive influence on achievement in 2023, whereas it had a negative effect in 2015. Likewise, another item, "my teacher tells me how to do better when I make a mistake," is positively associated with achievement in 2023, whereas it was negatively associated with achievement in 2015. However, the item "my teacher lets me show what I have learned" was negatively associated with achievement in 2023, though it was positively associated in 2015. In a similar vein, the item "my teacher listens to what I have to say" shows a negative association with achievement in 2023, though it was positive in 2015 (See Table 2).

Table 2: Regression Coefficients and Their Significance for Students' Perceptions of Mathematics Lessons (Independent Variables) and Achievement (Dependent Variables)

SN.	Independent Variables	B (2023)	Sig. (2023)	B (2015)	Sig. (2015)
0	Constant	449.894	< .001	465.411	< .001
1	I know what my teacher expects me to do.	–1.402	.486	8.666	< .001
2	My teacher is easy to understand.	11.640	< .001	8.195	< .001
3	I am interested in what my teacher says.	–2.010	.347	8.329	< .001
4	My teacher gives me interesting things to do.	11.803	< .001	–7.695	< .001
5	My teacher has clear answers to my questions.	–2.396	.255	0.783	.586
6	My teacher is good at explaining mathematics.	1.289	.525	7.463	< .001
7	My teacher lets me show what I have learned.	–13.916	< .001	5.663	< .001
8	My teacher does a variety of things to help us learn.	0.684	.703	–12.679	< .001
9	My teacher tells me how to do better when I make a mistake.	14.325	< .001	–9.659	< .001
10	My teacher listens to what I have to say.	–6.206	< .001	6.627	< .001

Dependent Variable: Achievement (Plausible Value 1 for TIMSS US 20215 and 2023)
Note. Collinearity was examined using Variance Inflation Factors (VIFs), and low collinearity was observed among the independent variables (VIF < 5).

Discussion and Conclusion

Students' achievement in mathematics is affected by a variety of factors, such as school, teacher, classroom environment, family background, and students' personal perceptions of the subject, their learning of mathematics, and classroom lessons. This study examined eighth-grade students' perceptions of mathematics instructors and how these perceptions influence their mathematics achievement in the TIMSS 2015 and 2023 US. The results showed that eighth graders generally had positive perceptions of mathematics lessons in both 2015 and 2023. However, these perceptions had differing effects on mathematics achievements. Most of the items measuring perceptions of mathematics lessons show a significant association with achievement, except one item, 'my mathematics teacher has clear answers to my questions', in 2015. However, the very same perceptual items that had a significant influence on mathematics achievement in 2025 were ineffective in 2023. For example, the item "I know what my teacher expects me to do" had no statistically significant effect on achievement in 2023, though it had a significant positive effect on achievement in 2015. Similar is the case for items "I am interested in what my teacher says," "my teacher is good at explaining mathematics," and "my teacher does

a variety of things to help us learn." One possible reason for these changes is students' ability to learn mathematics more independently, supported by different technological tools in general, and their greater self-awareness and higher self-esteem in mathematics learning, rather than relying on teachers. Long school closures led to dramatic drops in math skills, making students more dependent than ever on qualified teachers for remediation. The sequential nature of math made distance learning difficult, creating compounding learning gaps that teachers had to fix upon return to the classroom. (Sattem, 2022). Further study is needed to confirm what prompted for students' independence in mathematics learning and the less impact of their perceptions of classroom instructors on achievement. One of the major limitations of the study was that only one plausible value was used for the analysis out of five plausible values for students' achievement.

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MATH IS HARD, WE ACCEPT THE CHALLENGE! SIX-WORD STORIES FROM INFORMAL LEARNING ENVIRONMENTS

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STARMath (STM) is a near-peer, mathematics mentoring and tutoring program that has served hundreds of marginalized middle and high school students in the Midwestern United States over the past decade. Built on the strength and resilience of their communities, this after-school program has supported mathematics instruction and learning in several urban settings serving predominantly Black, Brown, and Asian students. Mentutors (mentor + tutor) in this program are undergraduates from the local state university, providing mathematical expertise and peer advice that sustains this community. With the goal of centering the voices of program stakeholders, this research report examines the six-word stories of mentutors from the program's inception through the present day. Findings suggest that academics, personal validation, and community are key intersections within a supportive math learning environment.

Keywords: Community-Based Praxis; Identity and Agency; Culture and Asset-Based Pedagogies; Equity, Inclusion, and Diversity

Since the 1990s, many states have responded to the national Algebra for All initiative — a program aimed at ensuring equal access to eighth-grade algebra or Algebra 1 by ninth grade (Loveless, 2008; Carolan, 2014). However, urban districts nationwide have continued to face alarming algebra failure rates over the past two decades (Loveless, 2008; Heppen et al., 2016). Given this challenge, the STARMath (STM) program integrates near-peer mathematics tutoring, community building, and family collaboration to enhance students' content knowledge in algebra and foster their mathematics self-concept (Hattie, 2014), which is critical to their future success. "Many authors consider that mathematics self-concept plays a key role in mathematics teaching and is related to the student's attitudes, mathematics world perspective and social identity" (Ansuategui & Miravet, 2017, p. 1188). As mentutors are key to the success of this program, this paper examines their six-word stories and what the STM program means to them, over the course of the program's 13 years. As a unit of analysis, the six-word story focuses our attention on mentutors' most salient thoughts and ideas, as they are restricted to brevity to tell their story.

Purpose

STM uses a unique near-peer tutoring and mentoring model that supports the study of algebra in small learning communities. Each week, middle and high school algebra students sit together with an undergraduate who is also studying a STEM (Science, Technology, Engineering, and Mathematics) discipline. The high school student tutors the middle schooler (with the supervision of the undergraduate), and then the undergraduate "mentutor" (the combined roles of mentoring and tutoring) supports the high school student with their homework and preparation for the ACT. Weekly, groups of 6–8 students gather around tables (named after underrepresented mathematicians and scientists) to work on mathematics homework and also share informal

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conversations during a healthy meal. This intentional structure facilitates authentic conversations and opportunities for informal mentoring, interrupting and dismantling power dynamics. Moreover, this program develops a shared capacity for collective dreaming and change-making where participants share stories connected to and moving beyond their educational wellbeing. STM's primary purpose was to support the mathematics experiences and outcomes of middle and high school students in order to better position them to compete in STEM-centered academics and careers. While the impact and outcomes for the middle and high school students were anticipated, the larger project from which this data is drawn, uncovered unanticipated outcomes for the mentutors. This sub-study moves forward with the following research question in order to examine the other stakeholders in STM. "How do mentutors describe shifts in their identities as a result of participating in an urban mathematics tutoring program?"

Theoretical Framework

Because our study takes place in a social and academic mathematics learning setting (i.e. the tutoring space), and we are interested in program participants' stories they tell from within this space, we come to this work from a lens of identity formation within educational spaces. Mead (1913/2011) proposes that identity is an action situated within an interactive social environment, making it multifaceted and complex (pp. 374–380 as cited by Darragh, 2016). In education research, identity has been an essential concept as it bridges learning with the cultural and social context (e.g., Sfard & Prusak, 2005; Graven & Heyd-Metzuyanim, 2019) and provides a lens for analyzing power, access, and equity issues (Radišić et al., 2024). Sfard and Prusak (2005) name identity as a "collection of stories about persons, or more specifically, those narratives about individuals that are reifying, endorsable and significant" (p. 16). This study employs a sociocultural lens to explore how mentutors construct their social identity in mathematics through six-word stories.

Methods

This study captures the concise stories of STM mentutors and seeks to understand how their mathematics identities developed through their participation in the program. While identity as described above is a complex inner and outer sociocultural phenomenon, we argue that the six-word story, though brief in nature, provides powerful insight to the most salient themes within a person's identity, as they themself choose to tell. As an analogy, climate and nature are complex systems, but how do environmental scientists uncover clues about atmospheric conditions? The series of concentric tree rings reveals insights and narratives about human impacts on the environment (Norris, 2024). While dendrochronology examines various patterns and contours of tree rings, the six-word story methodology functions as a "social tree circle" (Norris, 2024, p. 325), revealing the patterns of participants' thoughts, emotions, and experiences, as well as their interactions with the mathematical environment and the impacts they leave behind for the future. The six-word story serves as a "living portal" that encapsulates intricate human experiences related to teaching and learning mathematics, intertwining with culture, self-identity, belonging, and more in a nuanced and succinct manner (Norris, 2024, p. 325). We capture six-word stories that span time and space, yet stories of all kinds demonstrate common patterns, and "patterns themselves tell a story" (Norris, 2024, p. 326).

Six-word stories challenge the writer to tell their story in only six words. Brevity forces the writer to choose their words carefully while capturing the totality of their thoughts. This "methodology" is used in this study in order to understand the effects of the STARMath program through short, carefully selected words forming a sentence that provides insight into the most

important and meaningful experiences from their program participation. Additionally, participants expanded on the narratives within their six-word stories by responding to a follow-up question that probed them on their word choice.

Data Sources and Evidences

Six-word stories and the story meanings were collected from current and previous mentutors from over the past 13 years. The majority of mentutors were in their first year of service to the program and were STEM or education undergraduates at the local, midwestern division one research institute. There were 21 total six-word stories collected. We were not interested in using a traditional methodology to examine the effects of a non-traditional program. Rather, we employed the six-word story as a contemporary form to investigate the phenomenon of word choice, tone, and content. In essence, this study employs a phenomenological approach to understand the impact of STM on the mentutors. Each member of the research team read the six-word stories and identified three to four themes individually that encapsulated most of the stories. The research team met to compare themes and negotiate the themes into three overarching themes. During this process, we realized that the themes overlapped for many of the six-word stories, motivating the creation of a conceptual framework that captures these themes.

Findings

We identified three themes that appeared within the six-word stories: academics, personal growth, and community. These themes were not independent and overlapped in notable ways, thus forming a conceptual framework (Figure 1) that captures the mentutors' experiences and shifts in identity that resulted from participating in STARMath. The intersection of each category offered insights into how mentutors grew and developed through this programming. The intersection of academics and personal growth produced *math confidence*. *Joy* was cultivated from the overlap of personal growth and community while academics and community was identified as a *sense of belonging*. The following discussion examines each of the intersections (subthemes) from the framework.

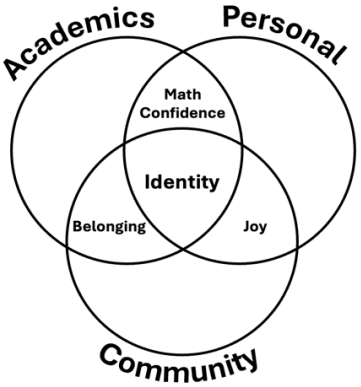


Figure 1: Conceptual Framework Showing the Intersection of Academics, Personal Growth, and Community

Mathematics Confidence: The Intersection of Academics and Personal Growth

When mentutors' descriptions of academic content and their personal development intersected, we called this *mathematics confidence*. For example, one mentutor shared, "One plus one does equal two" and further described:

> …when I was working with the high school students I was reminded about how much foundational math matters in the long scheme of things. If one does not know that 1 + 1 = 2, how can you possibly be prepared for things to get more difficult?

This mentutor described both the importance of foundational mathematics and how this basis prepared them for future challenges. This story highlights how, whether in mathematics or in life, learning is a continuum that builds on itself. This was an important lesson that students and mentutors learned from participating in the STARMath program.

Sense of Belonging: The Intersection of Academics and Community

We identified a *sense of belonging* as the intersection of academics and community. One mentutor shared, "Community empowers students and their learning" and elaborated:

> I found that being in an environment where everyone had something to learn from one another (even from the students to the tutors—whether it was about the math or it was about how to be a good listener/communicator), a strong community was built where students felt comfortable. It empowered them to be able to be themselves … which led them to a better understanding of the material.

This mentutor highlighted the reciprocal nature of the STM program. Mentutors were there to mentor and tutor high school and middle school students, however many of them found that they developed a mutual exchange of learning. By feeling a sense of shared belonging, students felt empowered to engage in both academic and social learning.

Joy: The Intersection of Personal Growth and Community

Finally, we used *joy* to describe when personal growth and community intersected. Several six-word stories indicated this shared joy, for example: "Succeeding in life—a community endeavor" and "Got me looking forward for Wednesdays." These six-word stories demonstrate the joy mentutors experienced from participating in a community environment and how these experiences positively influenced their time in the program. One mentutor described this joy:

> The support provided by [STM] did more than tutor kids … it was the sincere interest and gratitude in one another's presence. All these are easy to lose once you leave school or have to be independent, so it shines as a bright example of how to weave a community together.

Creating a program space where participants had opportunities to demonstrate this shared gratitude, allowed mentutors to grow both personally and as a community.

Discussion

STM was developed to support algebra learning for marginalized middle and high school students, who have always been the primary stakeholders of programming and research. The undergraduate and graduate students who served as mentutors were an integral part of the program delivery, and their six-word stories provide evidence of identity shifts. We found that their participation in STM contributed to shifts in their mathematical identities in ways similar to the shifts experienced by primary stakeholders. We argue that mathematics confidence, a sense of belonging, and joy were foundational to developing the identity of the mentutors in connection to the STM program. They were there to support students, but in doing so, they were helping themselves develop as a tutor, as a mentor, and as a math learner. The implications for the mentutor's role do not lie only in the content delivery and mentorship, but raises up the change in identity, levels of joy and fulfillment, and increased sense of belonging for the mentutor as well.

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STUDENT-RELATED FACTORS THAT INFLUENCE THEIR INTEREST IN LEARNING MATHEMATICS AT THE SENIOR HIGH SCHOOL LEVEL

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This study investigates student-related factors affecting interest in mathematics at the senior high school (SHS) level in Cape Coast, Ghana. Using a concurrent mixed-methods design, data were collected from 450 students and six teachers through surveys and interviews. Analysis through chi-square tests revealed no significant association between interest and gender or age, but a strong correlation with the program of study. Key drivers of interest included a willingness to exert effort and perceived real-world relevance. Current literature suggests that interest acts as a significant mediator between perception and performance. Furthermore, clear teacher explanations and adaptive pedagogies are vital for sustaining long-term engagement. By identifying these levers, the study supports the "Educating for Change," advocating for pedagogical shifts that dismantle mathematics as a gatekeeper and address systemic injustices within academic tracks.

Keywords: High School Education; Advanced Mathematical Thinking; Learning Trajectories and Progressions; Affect, Emotion, Beliefs, and Attitudes

Background to the Study

Mathematics serves as a critical gatekeeper in the global educational landscape, dictating access to advanced STEM opportunities and professional efficacy. Despite its importance, consistent underachievement remains a persistent challenge; in Ghana, pass rates for standardized mathematics examinations have shown a concerning decline, dropping from 65.3% in 2019 to 54.11% in 2021. While traditional research often focuses on external variables, current scholarship identifies student-level interest as the primary determinant of mathematics achievement. This study specifically investigates student-related factors impacting interest in mathematics at the senior high school (SHS) level within the Cape Coast Metropolis. By exploring these dynamics, the research contributes to the objective of "Changing Mathematics Education by Educating for Change" by identifying how internal student dispositions can be leveraged to overcome instructional stagnation and promote a more mathematically literate society.

Objectives and Purposes of the Study

Mathematics serves as a critical gatekeeper in the global educational landscape, dictating access to advanced STEM opportunities and professional efficacy. However, consistent underachievement remains a persistent challenge; in Ghana, pass rates for standardized mathematics examinations have shown a concerning decline, dropping from 65.3% in 2019 to 54.11% in 2021. While traditional research often focuses on external variables, current

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

scholarship identifies student-level interest as the primary determinant of mathematics achievement (Wijaya et al., 2022).

This study specifically investigates student-related factors impacting interest in mathematics at the senior high school (SHS) level within the Cape Coast Metropolis, Ghana. The research aims to address:

1. The association between students' demographic factors (gender, age, and program of study) and their interest in learning mathematics.
2. Specific student-related factors, such as self-efficacy and perceived utility, that influence interest.

By exploring these dynamics, the study contributes to the PME-NA 2026 objective of "Changing Mathematics Education by Educating for Change" by identifying how internal student dispositions can be leveraged to overcome instructional stagnation and promote a more mathematically literate society.

Perspective and Theoretical Framework

The study is grounded in cognitive constructivism, which posits that knowledge is actively constructed through pre-existing cognitive structures (GSI Teaching and Resource Centre, 2015). It also draws upon *Situated Expectancy-Value Theory (SEVT)*, suggesting that students' beliefs about their competence (expectancy) and the significance ascribed to the subject (value) are primary drivers of motivation (Eccles & Wigfield, 2002). Recent work extends this by examining the "cost" dimension—the perceived negative consequences of engagement, such as effort and ego costs.

Current research highlights that student interest partially mediates the relationship between perception of the subject and final performance, with a significant indirect effect. Furthermore, individual psychological factors such as mathematics anxiety can cause students to avoid engagement, leading to performance deficits (Ashcraft & Krause, 2007). Conversely, intrinsic motivation—derived from personal satisfaction—can elevate learners into a state of "flow" (Csikszentmihalyi, 1990). Modern frameworks like the Technological Knowledge of Algebra for Teaching (T-KAT) also suggest that teachers' ability to "unzip" content using digital tools is essential for making ideas learnable in the 21st century.

Methods and Modes of Inquiry

A concurrent mixed-methods approach was employed to provide a multidimensional analysis of student engagement.

Participants and Sampling

The target population included all public SHS students in the Cape Coast Metropolis. A multi-stage sampling procedure was used to select 450 second-year students from three co-educational schools (150 from each school). The sample represented General Science, General Arts, and Home Economics programs. Additionally, six mathematics teachers were purposively selected for semi-structured interviews.

Data Collection and Analysis

Quantitative data were gathered using a self-developed 5-point Likert scale questionnaire (reliability coefficients: $\alpha = 0.821$ for student factors; $\alpha = 0.850$ for interest). Qualitative data were collected through interviews to capture teacher perspectives. Analysis involved descriptive

statistics and chi-square tests of association (x^2) performed through SPSS version 26 at a 0.05 level of significance.

Results

The findings provide insight into the demographic and psychological barriers to mathematical interest.

Demographic Associations

Statistical analysis showed no significant association between gender and interest $x^2 = 5.150$, $p = 0.076$) or age and interest ($x^2 = 5.391$, $p = 0.495$). This contradicts traditional gender stereotypes suggested by Hyde and Mertz (2009). However, a highly significant association was found for the *program of study* ($x^2 = 34.561$, $p = 0.000$). General Science students reported the highest interest (30.89%), while Home Economics students reported the lowest (21.78%).

Student-Related Psychological Factors

The highest-rated factor was a willingness to exert effort and persevere ($M = 4.08$, $SD = 1.070$), which aligns with Bandura's (1997) construct of self-efficacy. This was followed by the belief that mathematics is relevant to real-life situations ($M = 3.93$, $SD = 1.111$), affirming Ciftci's (2015) findings. Other significant drivers included previous positive experiences ($M = 3.73$) and intrinsic motivation/curiosity ($M = 3.36$), which is identified in the literature as a forceful determinant of learning outcomes.

Teacher Perspectives

Teachers reported that students who excel often form study groups and view challenges as growth opportunities. Conversely, a lack of interest was often tied to the perceived irrelevance of the subject to future career goals, particularly in non-science programs. Teachers also noted that clear explanations and the use of real-world applications are critical factors in reducing learner anxiety.

Discussion and Conclusions

The findings suggest that student interest is malleable and heavily dependent on perceived relevance. The lack of a gender gap indicates that instructional shifts or shifting societal norms may be closing traditional divides (Bakar & Tahir, 2009).

Educating for Change and Epistemic Justice

In alignment with PME-NA 2026, the significant impact of the "program of study" highlights an "epistemic injustice" where students in certain academic tracks feel systemically excluded from the perceived utility of high-level mathematics. To "educate for change," mathematics must move beyond rigid curriculum delivery to become an inclusive experience that empowers all learners, regardless of their career track. Current research emphasizes that providing adaptive formative assessment strategies across varying mathematical competency demands is essential for this transition.

Recommendations

1. *Contextualized pedagogy*: Teachers should adapt methods to emphasize practical applications in diverse fields like economics and art to engage students across all programs (Ainley & Ainley, 2011; Tomé, 2025).
2. *Fostering self-efficacy*: Interventions should focus on building confidence and rewarding perseverance, leveraging socio-constructivist teaching methods.

3. *Mitigating anxiety*: Institutions should strengthen technical and emotional support services to reduce mathematics-related stress.

Implications of the Findings to the Teaching and Learning of Mathematics as Well as Policy Makers

Although the study has limitations, the findings show that students' interest in mathematics is not significantly associated with gender or age, but is significantly related to program of study. In addition, psychological factors such as willingness to exert effort, perceived relevance of mathematics, prior positive experiences, and intrinsic motivation strongly shape students' interest. Teachers also observed that students' engagement is influenced by perceived usefulness of mathematics for future careers and by instructional clarity and real-life applications.

For the teaching and learning of mathematics, this implies the need to strengthen instructional approaches that enhance perceived relevance and engagement across all student groups. Teachers should emphasize real-life applications of mathematics in diverse contexts and support students' willingness to persist in learning through positive reinforcement and clear explanations. Attention should also be given to creating supportive classroom environments that reduce anxiety and build on students' prior positive experiences to enhance interest and engagement.

For policy makers, the significant differences in interest across programs of study suggest the need to ensure that mathematics curricula are meaningfully relevant across all academic tracks, not only science-oriented programs. Curriculum planning should therefore promote the applicability of mathematics in varied fields to sustain interest among all learners. In addition, policy support for teacher professional development is necessary to strengthen the use of real-life examples, clear instructional strategies, and supportive feedback practices that foster student engagement in mathematics.

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MAKING INTERPRETIVE PROCESSES VISIBLE: A UNITS-IN-MATHEMATIZING LENS ON YOUNG STUDENTS' MATHEMATIZING IN CT-MATH CONTEXTS

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Mathematizing is central to math education, yet how young students structure real-world situations remains underexamined. This study addresses this gap by proposing the Units-In-Mathematizing (UIM) lens, focusing on the referents students attend to and coordinate when making sense of situations. We analyzed first-grade students' mathematizing in a coding-math context using the UIM lens. Findings reveal that tensions in coordinating units such as count of moves and directional change influenced students' choice of operations and equations in their mathematizing process. These results suggest that students' interpretive structuring of a situation shape the direction of their mathematizing. We conclude with implications for instruction, emphasizing making students' referential choices explicit in integrated STEM contexts.

Keywords: Cognition, Computational Thinking, Elementary School Education, Modeling

Mathematizing is essential for constructing mathematical meaning (Freudenthal, 1968; Schoenfeld, 2017). While modeling research (Blum, 2011; Kaiser & Sriraman, 2006) defines mathematizing as transforming real models into mathematical ones (Blum & Leiß, 2007), the initial cognitive process of interpreting a situation to form a real model remains underexamined. Additionally, existing studies predominantly focus on older students and text-based problems (Doerr & English, 2003; English, 2006; Kenderov, 2018). Thus, a gap exists in understanding how young students interpret real, non-textual situations through their interactions with them. This study addresses this gap by proposing a *Units-in-Mathematizing* (UIM) lens. Drawing on previous literature, this framework analyzes how young students cognitively organize and coordinate *units* within integrated coding and mathematics contexts. By illustrating this lens through empirical data, we demonstrate its utility in revealing the nuances of students' interpretive processes. The study concludes with implications for mathematics instruction and research on mathematizing.

Literature Review

Units-In-Mathematizing (UIM)

Unitizing has primarily been studied within multiplicative thinking (Clark & Kamii, 1996; Lamon, 1994; Zwanch & Wilkins, 2021) and fraction learning (Hackenberg, 2007; Norton et al., 2015; Steffe & Olive, 2010), emphasizing how students coordinate units of one. Further research on quantitative reasoning (Tasova & Moore, 2025; Thompson, 2011) and model-eliciting activities (Chamberlin & Moon, 2005) highlights the importance of determining which quantities and relationships to prioritize. Extending the scope of units from units of one to referents that play a role in mathematical reasoning, we define *Units-In-Mathematizing (UIM)* as *any aspects or factors of a situation that students take up and use as units of their interpretation of the situation*. The focus of the UIM lens is not every referent that students notice in a situation;

rather, it attends to the referents that students identify, coordinate, and use as anchors for their mathematizing. In this sense, UIM extends prior work on unitizing and quantitative reasoning by broadening the scope of units beyond units of one, quantities with measurable attributes, or relationships among quantities, to include any referent that students take up as a *mathematizing unit* in constructing their interpretation of a situation (hereafter, referred to as *UIM)*. By adopting this lens, we can examine students' mathematizing more broadly, tracing how they organize their reasoning around the referents they take up as meaningful units in a situation.

CT-Math Integration as a Productive Context for Various Units

Computational Thinking (CT) research on decomposition (Bers, 2020; Relkin et al., 2021), breaking complex goals into manageable parts (e.g., Murcia & Tang, 2019), suggests that learners must decide what constitutes a unit of action or subtask. From this perspective, decomposition creates opportunities for multiple aspects of a context to become candidate units. Unlike word-problem contexts where units are often pre-determined by the text (Galbraith & Stillman, 2006), CT-math integrated environments require students to actively designate and coordinate these candidate units for coding while mathematizing them through mathematical representations. In this sense, CT-math tasks foreground not only how students break a goal into parts, but also how they mathematize the whole context by attending to specific aspects as *UIM*. Accordingly, these hybrid settings provide a productive environment for examining how students navigate and coordinate diverse *UIM* within complex, non-textual situations.

Research Question

This study investigates young students' mathematizing in a CT–math integrated setting through the following question: *In what ways does the UIM lens provide insights into the cognitive analysis of young students' mathematizing within an integrated CT–mathematics context?*

Methods

Data Collection

Setting and participants. This study was conducted in a first-grade duallanguage immersion classroom in the western U.S. From a larger project called SPARC-Math (Phase 1 of NSF Grant #2300357; N=68), we purposively selected a focal group of four students (Sarah, Cindy, Ron, and Riley (pseudonyms)). This group was chosen because their interactions during lessons involving forward and backward codes captured diverse and contrasting ways of mathematizing the same coding tasks, providing rich data for a UIM analysis.

Lesson design. We implemented four 30-minute integrated lessons using Robot Mouse and a custom number line. The first two lessons introduced the coding system (forward/backward moves and rotations). The final two lessons, facilitated by the first author, involved group activities where students solved coding sequences and represented them as mathematical equations.

Data Analysis

We analyzed 60 minutes of video from the final two lessons, focusing on three episodes where students' mathematizing varied significantly. An episode was defined as a coding problem followed by the task of writing a corresponding equation. First, we transcribed verbal explanations and documented gestures and written work. Guided by the UIM lens, we identified moments where students produced mathematical representations and examined their underlying *UIM*s (e.g., number of moves, starting/ending positions). Finally, we traced whether these *UIM*s were maintained, shifted, or conflated, and how these dynamics shaped their equations and justifications.

Results

Mathematizing a Robot's Movement on the Number Line Using a Math Equation

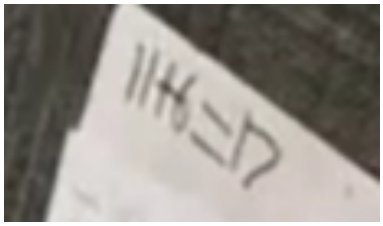
Throughout the two lessons, students were asked to write a mathematical equation representing the robot's movement on the number line. As they mathematized different coding problems using equations, diverse *UIM*s emerged in their reasoning.

Episode 1, Lesson 3: Six Forward Codes and + 6.

In the third lesson, the first task required students to create a code that would move the robot from 11 to 17. All four students used a sequence of six forward arrow cards (Figure 1). When asked to write a mathematical equation representing their code, all students wrote 11 + 6 = 17 (Figure 1). When prompted to explain their reasoning, Riley responded, "*I did it because that's how much the robot went in the math, so I wrote the equation*."



(Code)



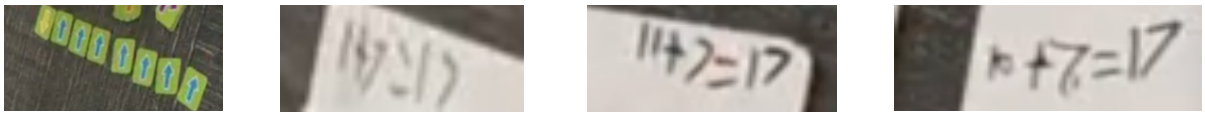
(11 + 6 = 17)

Figure 1: Six Forward Codes and Students' Corresponding Mathematics Equations

Using *UIM* lens, we interpret that students attended to the robot's net change in position (from 11 to 17) as *UIM*, which aligns with the six forward moves in their code. They mathematized the situation using addition, coordinating the starting position (11), the movement (6 forward movements), and the ending position (17) in a single equation. Notably, this episode suggests that a forward command does not inherently entail addition; rather, addition becomes appropriate when forward movement is interpreted as an increase in position.

Episode 2, Lesson 3: One Backward Code and Seven Forward Codes

In the second problem where the robot moved from 11 to 17 using 1 backward and 7 forward moves, students' equations varied based on their *UIM*. Cindy wrote an equation (1 + 7 = 17 in Figure 2) linking movement-based quantities (1 backward, 7 forward) directly to the endpoint (17). When explaining her reasoning, Cindy said, "*One means, it went backward one and it went forward seven and it landed on 17*." We interpret this as a misalignment where she attended to the count of code cards as a *UIM* without taking up directionality or net change on the number line as units in her mathematizing process. Cindy's equation makes visible a *UIM* distinction that remained implicit in Case 1, where total movement and net change coincided. In contrast, Sarah and Rob (11 + 7 = 17) took up the starting and ending positions as *UIM* as well as the total count of cards (7), rather than the net displacement. Riley (10 + 7 = 17) coordinated the total moves with the endpoint by introducing an intermediate number (10), representing the robot's position after the initial backward move. Taken together, these equations illustrate that even in the same coding situation, students' mathematizing can differ substantially depending on which aspects of the robot's movement they take up as *UIM* in their interpretations.



(Code) (Cindy : 1 + 7 = 17) (Sarah & Rob: 11 + 7 = 17) (Riley: 10 + 7 = 17)

Figure 2: 1 Backward, 7 Forward Code and the Math Equations

Episode 3, Lesson 4: Three Backward Codes and One Forward Code

For the second task in Lesson 4, the robot started at 5 and landed on 3. After initially using two backward codes, students then created a program using three backward codes and one forward code (Figure 4). When asked to write down a mathematics equation, Ron stopped writing a math equation after writing down 5–3 while Sarah wrote down 5 – 2 = 3, and Cindy and Riley did not write down anything. The facilitator intervened, asking Ron what the 5 and 3 represented. Ron responded, "*it started at 5 and took away 3*." The facilitator then pointed to the one forward card on the floor and asked, "What do we have here?" Ron replied, "one?" and then completed his math equation. Following this, Cindy copied Ron's equation onto her board. The facilitator then engaged Sarah and Riley, asking them to describe the meaning of each number in the equation. The lesson concluded with a discussion on interpreting the movement and numbers in their mathematical representations.



(Code) (Ron) (Sarah) (Ron's complete equation)

Figure 4: Three Backward Codes and One Forward Code with Corresponding Equations

The UIM lens suggests a referential tension in Ron's pause after writing 5−3: the "3" could refer either to the positive count of backward cards or to the directional change (−3) in position. Interpreting the backward moves as "taking away 3," Ron coordinated the starting position (5) and the intermediate position (2) as *UIM* in his equation. Notably, this episode illustrates that the operation used in an equation (addition or subtraction) is not determined by a command label (forward/backward) alone, but by which aspects of the activity students take up as *UIM*. Meanwhile, Sarah (5 – 2 = 3) focused on the start (5), end (3), and net change (2) as *UIM* without incorporating the full arrow-card sequence in her representation.

Discussion and Limitation

The UIM lens reveals that young students' mathematizing in CT-integrated contexts begins with a process of negotiating *units-in-mathematizing* (UIM) rather than simple translation. This suggests that the way students interpret and organize the physical situation, specifically, how they determine what counts as *UIM*, directly shapes the direction and structure of their mathematizing. As seen in the tensions between representing "count of moves" and "directional change," the choice of mathematical operations is not a direct mapping of coding labels, rather it emerges from how students cognitively structure the situation. These findings highlight that CT-

math environments function as productive spaces where students explicitly coordinate diverse *UIM* to construct mathematical interpretations of situations. For instruction, these findings suggest that teachers should support students in articulating their referential choices before moving directly to equations or code. Rather than treating mathematizing as immediate symbolic translation, teachers can ask students what they are taking as *UIM* in the situation, how those units are represented numerically, and why those units call for particular operations or representations. This preliminary study is limited by its small focal group and the specific affordances of the Robot Mouse. Future research should investigate how these *UIM* patterns generalize across diverse student populations and different CT tools to better understand the stability of students' interpretive mathematizing processes.

Acknowledgements

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SHADES OF MISCONCEPTION: HIERARCHIZING IMPROPER FRACTION ERRORS VIA UNITS COORDINATION

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The theory of scheme and units coordination has been established as a pivotal analytical framework for investigating students' mathematical knowledge. Employing the theoretical lens, the present study explores the cognitive mechanisms underlying students' errors on an improper fraction task. By analyzing responses from 1,472 students, we hierarchize misconceptions into three distinct tiers based on the nature of activated schemes. Results indicate that their errors reflect varying levels of units coordination: students who engaged in reversible reasoning but failed to use three levels of units as given; those who assimilated the problem solely with a unidirectional scheme; and those not invoking an improper fraction scheme. We argue that these error patterns evidence developmental trajectories, providing a foundation for differentiated instruction and promoting equity by legitimating the rationality of marginalized learners.

Keywords: Assessment, Cognition, Learning Theory, Number Concepts and Operations

Introduction

The study of student errors and misconceptions in mathematics education has a long-standing history, evolving alongside shifts in educational philosophy and psychological trends. Early research, particularly in the early 20th century, was heavily influenced by behaviorism in the United States. During this period, researchers focused on listing types of errors or analyzing their frequency in arithmetic, treating errors primarily as targets for elimination (Radatz, 1979, 1980). However, with the emergence of cognitive psychology in the 1970s, errors came to be understood not as mere mistakes, but as meaningful indicators of how students process information based on systematic rules (Ginsburg, 1977).

A constructivist paradigm, in conjunction with the cognitive approach, had elucidated a more nuanced understanding of students' misconceptions. From a constructivist standpoint, misconceptions were re-evaluated as inevitable and productive resources that emerge when knowledge valid in limited contexts is extended to new domains (Smith et al., 1993). The focus of research has shifted toward illuminating the unique cognitive rationality inherent in students' reasoning. Empirical studies in various domains, such as algebra (Clement, 1982) and probability (Shaughnessy, 1977), have substantiated that these errors are products of students' own systematic thinking. To date, research has endeavored to characterize students' unique cognitive frameworks by scrutinizing their diverse problem-solving strategies and responses and classifying them into specific typologies (e.g., Mathaba et al., 2024; Säfström et al., 2024).

However, we argue that treating student errors as discrete, mutually exclusive categories and merely enumerating them in parallel is insufficient to fully account for the concomitant cognitive mechanisms and the interconnected developmental trajectories of these errors. While the pathways to correct solutions are relatively canonical, the manifestations of misconceptions might exhibit extraordinary heterogeneity; consequently, treating them as mere isolated entities constrains the capacity to holistically explain the cognitive processes engaged in specific conceptual domains. In order to harness misconceptions as genuinely productive pedagogical

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

resources, it is imperative to delineate the 'interrelationships' or 'hierarchies' inherent among them. If it is established that certain erroneous strategies are cognitively more sophisticated or possess greater potential toward the target concept than others, error analysis would remain as a promising methodological approach for addressing fundamental issues in mathematics learning (Radatz, 1980) and serve as an essential guideline for teachers to make pedagogical decisions (Simon, 1995).

This paper reports the initial results of our data analysis, categorizing students' diverse errors through the lens of *scheme* and *units coordination* when solving an improper fraction problem. We aim to identify the varying cognitive abilities inherent in these *marginalized learners* – whose cognitively valid reasoning is routinely undetected and dismissed merely as 'incorrect' in traditional assessments – and provide tangible implications that facilitate differentiated pedagogical interventions tailored to each level of (mis)understanding.

Theoretical Perspectives

Scheme and Units Coordination

von Glasersfeld (1995) defined a scheme as a cognitive structure comprised of three parts: an 'experiential situation,' an 'activity,' and an 'anticipated result.' Steffe (2010) incorporated a 'generated goal' into this framework, emphasizing child's intrinsic motivation as the pivotal mechanism orchestrating the overall functioning of the scheme.[1]

Units coordination refers to the organizations of a sequence of mental operations using units—such as *partitioning* a given referent unit, *disembedding* a part, and *iterating* it multiple times—to achieve a specific goal (Steffe & Olive, 2010). Investigating these units-coordinating activities allows for an assessment of quantitative reasoning that surpasses computational proficiency; specifically, such investigation reveals the extent to which a learner assimilates a problem situation into a structure of *nested* units and utilizes this structure flexibly *prior to activity* (Shin & Lee, 2024; Thompson, 1988). Drawing upon scheme theory, this study defines '4/3 scheme' and 'reversible 4/3 scheme' and employs them as analytical tools to examine students' fraction problem-solving activities.

4/3 Scheme and Reversible 4/3 Scheme[2]

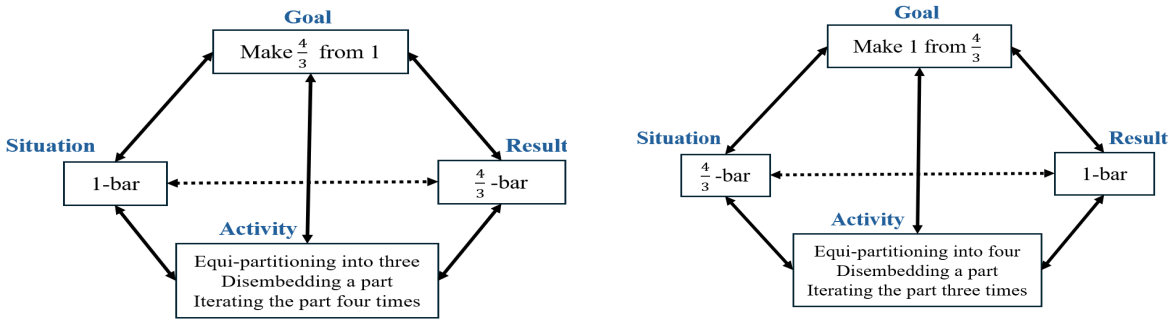


Figure 1. 4/3 Scheme (Left) and Reversible 4/3 Scheme (Right)

[1] For a detailed history of the development of scheme theory, refer to Tillema & Gatza (2024).

[2] Although this study builds upon the foundational concepts of the Iterative Fraction Scheme (Steffe & Olive, 2010), we operationally employ the term 'reversible 4/3 scheme' to denote the specific goal-directed operations elicited by the assessment task, acknowledging the constraints of a large-scale cross-sectional design.

A 4/3 scheme is activated with the goal of constructing an improper fractional quantity, 4/3 from a referent unit of 1. Given a single bar, a student partitions it into three to generate a unit fractional part, 1/3, and iterates the part four times to construct 4/3, with which the activated scheme is closed (Figure 1, Left). While this demonstrates that the student has constructed the improper fraction 4/3, comprising a *three-level units structure* of [1/3, 1(3/3), 4/3] through the units-coordinating activity, it does not necessarily imply that she used the three-level structure *as a given prior to* the activity, nor does it warrant the persistence of the structure once the task is completed.

Meanwhile, whether a student has assimilated 4/3 with the three-level units structure can be corroborated by the execution of a reversible 4/3 scheme. The concept of reversibility entails reconstituting the initial referent unit, 1, from the result, 4/3, of the 4/3 scheme. As shown in Figure 1 (Right), the reversible 4/3 scheme aims to construct the original unit bar of 1 when 4/3 of the bar is given. To accomplish it, the student performs a units-coordinating activity of partitioning the given 4/3-bar into four to construct a unit fraction of 1/3 and iterating it three times to construct the referent unit of 1. The manifestation of such reversibility implies that she conceives the nested composite unit of 4/3 comprising four 1/3s without losing the referent unit of 1 in which the fractional unit of 1/3 is embedded. It serves as definitive evidence that the student has developed to a cognitively higher level, constructing improper fractions as 'numbers in their own right' that are measurable at any time (Hackenberg, 2010).

Methods

Data Collection and Participants

The data used in this paper were drawn from part of the data collected for the ASFK_AI (Assessing Students' Fraction Knowledge with AI) project, which aims to develop an AI-driven constructed-response assessment model for fraction knowledge. Drawing upon prior research, the research team developed six items designed to diagnose students' stages of units coordination and construction of relevant schemes within fractional contexts. The assessment was administered to a total of 1,472 students in South Korea (318 fifth graders, 657 sixth graders, and 497 seventh graders). This paper focuses on analyzing students' solutions to the task devised to assess construction of 'reversible 4/3 scheme' (Figure 2).

The length of Bar A is 4/3 of that of Jinsu's bar. Draw Jinsu's bar using Bar A and explain how you draw it.	**Draw your bar here. Write your explanation in the box below (Bar A)**
	(Explanation)

Figure 2. Task Designed to Assess Reversible 4/3 Scheme[3]

Data Analysis

Among the 1,472 participants, 713 (48.4%) provided correct answers, whereas 759 (51.6%) were deemed incorrect; the latter group consisted of 611 (41.5%) overt errors and 148 (10.1%) indeterminate responses, including no responses. In contrast to previous research that primarily relied on students' drawings for analysis (e.g., Norton & Wilkins, 2009, 2012), written explanations were incorporated as one of the evaluative components. We sought to strengthen

[3] This task has been translated from the original Korean for this publication.

the validity of our inferences regarding students' units-coordinating operations by corroborating the evidence found in their drawings with the reasoning manifested in their written accounts.

Preliminary Results and Discussion

Students who provided overt-incorrect responses (611, 41.5%) were categorized into three primary groups—W1, W2, and W3— with each group further divided into several subgroups (Table 1). This report presents the initial result of analysis, focusing on the distinctive characteristics of each group within the incorrect category. Due to space limitations, this paper primarily outlines the overarching characteristics of each tier. To provide full transparency regarding how specific cognitive inferences and categorizations were made, our conference presentation will explicitly feature representative excerpts and student drawings, illustrating the nuanced decisions behind subgroups such as W2-3.

Table 1: Classification of Students' Incorrect Answers Based on Activated Schemes

Group	Activated Scheme	Subgroup(N=1472)	Drawing
W1	Reversible 4/3 Scheme	W1-1 (30, 2%)	Divide Bar A into 4 and mark 1 part.
		W1-2 (34, 2.3%)	Divide Bar A into 3 and mark 2 parts.
W2	4/3 Scheme	W2-1 (315, 21.4%)	Draw 4/3 of Bar A.
		W2-2 (17, 1.2%)	Draw 4/3 of 1/2 (or 1/3) of Bar A.
		W2-3 (9, 0.6%)	Draw 4/3 of Bar A and mark 1 part OR Divide Bar A into 3 and mark 1 part.
		W2-4 (29, 2%)	Draw 4/3 of Bar A and mark 3 parts.
W3	No Scheme Identified	W3-1 (27, 1.8%)	Divide Bar A into 3 (Others).
		W3-2 (38, 2.6%)	Divide Bar A into 4 (Others).
		W3-3 (69, 4.7%)	Draw a bar longer or shorter than Bar A with no partitioning.
		W3-4 (13, 0.9%)	Divide Bar A into a number other than 3 or 4.
		W3-5 (30, 2%)	Draw multiple copies of Bar A.

We established a hierarchical relationship among the three groups (W1 > W2 > W3) according to the schemes implemented. The highest tier, W1 students activated 'reversible 4/3 scheme'; they demonstrated reversibility by recognizing the given bar as 4/3 and aiming to reconstruct the initial unit of 1. However, they conducted units-coordinating operations without assimilating 4/3 with a three-level units structure composed of four 1/3s —for instance, partitioning the bar into four and marking one part. The subsequent group, W2, activated '4/3 scheme' without reversibility; despite the problem stating bar A represented 4/3, these students treated it as a reference unit of 1 and attempted to construct 4/3 of it. The lowest tier, W3, showed no activation of either scheme, lacking evidence of goal-directed operations related to improper fraction construction (e.g., drawing three additional bars of the same size as bar A). This classification reveals that responses, though all incorrect, stem from varying levels of units-coordinating ability. Such findings provide a concrete framework for utilizing misconceptions as 'productive resources' to facilitate differentiated instruction (Hackenberg et al., 2021).

It is also noteworthy that, except for subgroup W2-1, each of the remaining subgroups accounts for less than 5%. The low prevalence of these diverse atypical responses suggests a potential risk of them being overlooked or dismissed as 'unidentifiable' errors in mathematics classrooms. However, the result of our study demonstrates that even these minority students are striving to understand and solve problems grounded in their own cognitive rationality; they are

actively navigating distinct pathways toward the construction of fraction concepts envisioned by researchers and teachers. Elucidating and legitimating the mathematical thinking of these marginalized students is not only an act of profound respect for their mathematics but also an essential process for promoting *equity* in mathematics classrooms (Hackenberg et al., 2024).

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BRIDGING COLLABORATION AND COMMUNITY: TEAM-BASED LEARNING FOSTERS BELONGING

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This sequential explanatory mixed-methods study examines how Team-Based Learning (TBL) supports belonging in seventh-grade honors mathematics (n = 13 students). Using social network analysis (SNA), we identified patterns of belonging over a six-week unit. SNA revealed four distinct patterns of network integration associated with belonging: peripheral to central, remains central, remains peripheral, and central to peripheral. Findings suggest fostering belonging requires attention to individual positioning within collaborative structures and ongoing negotiation of participatory norms. This study contributes a novel SNA approach to examining belonging and highlights implications for designing collaborative mathematics instruction.

Keywords: Affect, Emotion, Beliefs, and Attitudes

Introduction

A sense of belonging plays an important role in shaping students' motivation, engagement, and achievement in mathematics classrooms (Barbieri & Miller-Cotto, 2021; Good et al., 2012; Ryan & Deci, 2002). In mathematics education, belonging is more than feeling accepted by others (Good et al., 2012), it means being recognized as a real contributor to the shared process of doing mathematics (Barbieri & Miller-Cotto, 2021; Darragh, 2013; Hamm & Faircloth, 2005). Research shows that belonging can be supported when teachers design tasks that promote interdependence, highlight diverse ideas, and build mutual respect among students (Anderman, 2003). Collaborative learning (CL) is one effective way to foster a sense of belonging by encouraging teamwork, problem solving, and mutual support. CL helps students connect with classmates and recognize the value of their ideas (Cohen & Lotan, 2014). In this study, we utilize, Team-Based Learning (TBL), a structured CL model (Michaelsen & Sweet, 2011) that creates opportunities for students to develop belonging through four elements: 1) strategically formed, permanent teams, 2) readiness assurance process, 3) application activity, and 4) peer feedback. TBL offers teachers meaningful ways to monitor and support students' sense of belonging such as iterative rounds of instructor-screened peer feedback. Drawing on Watson's (2023) conceptualization of belonging, we operationalized one aspect of belonging—students' integration into the mathematics class collaboration network—through social network analysis. We do not claim that network position fully captures belonging; rather, collaboration network integration provides one observable indicator of students' opportunities to experience belonging within the mathematics classroom. To contribute to the research on how CL may support students' belonging in mathematics over time, we sought to answer the research question: *How is*

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students' sense of belonging reflected in their participation within the collaboration network of a middle school mathematics class, and how does this participation change over time?

Theoretical Perspectives

Through a sociocultural perspective, students' sense of belonging includes their affective responses (Cobb & Hodge, 2002) such as feeling "accepted, respected, included and supported by others" (Goodenow, 1993, p.80) as well as their participation in the mathematical community. Watson (2023) identifies five factors of belonging in mathematics: 1) *You see me*, 2) *I see myself*, 3) *Others see me*, 4) *I am valued*, and 5) *I can grow*. In other words, students experience belonging when their ideas are acknowledged (*You see me*), when their thinking shapes class discussions (*Others see me* and *I am valued*), and when they see themselves as members (*I see myself*) of a mathematical community (Morales-Chicas & Graham, 2021). Classrooms that promote students' sense of belonging balance academic challenge with relational support (*I can grow*) by encouraging them to take risks (SanGiovanni et al., 2020), explain their reasoning (Thanheiser et al., 2025), and engage in collaborative problem solving (Slavin, 1991). Thus, belonging is relational and participatory. Our study provides a novel approach of leveraging social networking analysis (SNA) to visualize and quantify the connectedness of the mathematics class collaboration network as well as to describe students' belonging over time using student-level descriptive statistics. This approach demonstrates the factor of belonging, *Others see me,* through recognition of mathematical collaboration.

Methods

This mixed-methods study follows a sequential explanatory design (Creswell & Plano Clark, 2018) to identify and describe emerging patterns of belonging within the seventh-grade honors mathematics class collaboration network.

Data Collection

The research team for this study includes $n = 4$ mathematics education researchers with K-12 mathematics teaching experience as well as $n = 2$ in-service mathematics teachers one of which we implemented the six-week unit in their seventh-grade honors mathematics class. We implemented TBL in a 14 person seventh-grade honors mathematics class at a rural, Title I school with a mathematics teacher with $n = 3$ years of teaching experience. In the class, 92.86% of the students (female $n = 9$, male $n = 4$) participated in the study, with one student not assenting. During the six-week unit, TBL was implemented once per week as a follow-up to direct instruction the day before.

SNA data regarding students' collaborative relationships in the seventh-grade honors mathematics class were collected weekly after the application activity during the TBL unit. Qualitative data to explain the SNA results were collected through weekly peer feedback (Sanders et al., 2024) and post TBL open-ended reflections.

Data Analysis

For the SNA data transformation, collaborative connections (i.e., ties) between students in the seventh-grade honors mathematics class were modeled using undirected ties by creating a symmetrized collaboration matrix for each time point (i.e., week 1 = time point 1) where all ties were reciprocal (Borgatti et al., 2022). This approach uses the "OR" or "union" rule (Krackhardt, 1987) as well as accounts for missing data (Borgatti et al., 2022; Žnidaršič et al., 2012) through the assumption of mutual ties (i.e., if one student selects another there is connection between them). For each time point we calculated network-level descriptives (i.e., density, connectedness, percent homophilous ties) and a node-level measure of belonging, normalized eigenvector

centrality. We selected this measure of centrality since it describes the degree to which a student collaborates with peers who are themselves well-connected, reflecting our definition of belonging as students' ongoing integration into the mathematics class collaboration network grounded in Watson's (2023) factors of belonging, *Others see me* and *I am valued*. While network centrality cannot directly measure students' internal feelings of belonging, it provides a useful indicator of how students are positioned within the classroom collaboration structure and the extent to which their participation is recognized by others.

Next, SNA results were used to identify dynamic patterns of belonging: 1) peripheral to central, 2) remains central, 3) remains peripheral, and 4) central to peripheral. The emergent patterns in the collaboration network were identified using students' normalized eigenvector centralities and network visualization (Borgatti et al., 2022). To establish a threshold distinguishing central from peripheral positions, we used the second quartile (Q2 = 0.08) of all normalized eigenvector centrality scores across students and time points (n = 78 total observations, M = 0.08, SD = 0.07, Min = 0.00, Max = 0.27). The quartile-based cut point reflects the properties of the sample rather than an a priori cut point. Patterns of belonging in the collaboration network were distinguished using a students' initial (Week 1) and final (Week 6) normalized eigenvector centrality as well as visual inspection of the network plots.

Following the identification of patterns of belonging, the SNA results and qualitative data were blended to describe the four emergent patterns of belonging (Creamer, 2018). In this phase, the SNA results were integrated and interpreted through the discussion of interwoven qualitative data. Qualitative data from students' peer feedback and post-survey open-ended responses are triangulated (Creswell & Plano Clark, 2018) to provide inter- and intra-belonging pattern comparisons using Watson's (2023) factors of belonging. Of note, we focus on the factors *Others see me, I see myself, I am valued*, and *I can grow* since we utilized peer feedback and student self-reflection as the qualitative data sources.

Findings

Across the six-week unit, the mathematics classroom collaboration network became increasingly cohesive (see Table 1). This shift in the network was not linear but fluctuated between fragmentation and cohesion. Within these fluctuations, students experienced differing opportunities for integration into the network. Our discussion of patterns of network integration associated with belonging highlights the experiences of Alicia (peripheral to central), Crystel (remains central), Derek (remains peripheral), and Jessica (central to peripheral).

Table 1: Collaboration Network Descriptive Statistics

	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6
Network Density	0.12	0.14	0.18	0.18	0.12	0.19
Connectedness	0.37	0.71	0.85	0.85	0.31	0.85
Percent Homophilous	100.00	81.82	78.57	57.14	100.00	93.33

For some students, belonging developed through increasing integration into the collaboration network across the six-week unit. These students moved from more peripheral positions toward greater participation and recognition within the mathematics collaboration network. Alicia's experience in the collaboration network illustrates how belonging developed over time. Alicia moved from the margins of the network (Week 1 centrality = 0.03) toward the core (Week 6 =

0.13). Early feedback recognized her kindness while Greg requested she "speak a little more." By Week 3, Crystel described her as "really smart" and noted she "helps the team figure out the answer," signaling growing peer uptake of her contributions. By week 6, Eli called her "a good leader" and Greg appreciated that she "helped me when I was stuck." The progression from "speak a little more" in week 1 to "good leader" in week 6 reflects increasing participation and recognition within the network over time and represents *I can grow*.

While some students experienced increasing integration into the collaboration network, other students maintained relatively stable access to belonging throughout the unit. Crystel held a central position throughout (Week 1 = 0.15, Week 6 = 0.14), beginning as a broker connecting the two network components. Her peers consistently recognized both competence and a collaborative orientation: Brittany described her as "always wanting to listen to others." The stability of her position and the recurring affirmation in her feedback reflect sustained access to *Others see* me and *I am valued*. However, even among students with sustained belonging, participation was not entirely stable.

For several students, belonging became fragile, unstable, or remained persistently out of reach despite participating in the same collaborative learning environment. Jessica represents the clearest example of movement from central to peripheral participation. She began with the second-highest normalized eigenvector centrality in the class (Week 1 = 0.20) and ended at complete isolation (Week 6 = 0.00). Her peer feedback highlighted her efforts in helping the team despite missing several days of class leading up to the team activities. Examining what her team members requested of her though surfaces that within her efforts students recognized the need for her to "be more serious" and to "recheck her work, because she was somewhat rushing." Her feedback in week 2 (i.e., "slow down"), echoes the week 1 requests demonstrating that Jessica is detached from the factor of belonging; *I can grow*. In the remains peripheral pattern, Derek stayed below the Q2 threshold in both Week 1 (0.00) and Week 6 (0.03), despite a temporary mid-unit peak. Peers acknowledged his ability (e.g., "he is smart") but repeatedly requested he "stay on task" and "not goof off" across the unit, indicating that his mathematical presence was recognized while his collaborative participation was not taken up as consequential.

Discussion and Conclusions

The four patterns demonstrate that engaging in CL through TBL did not guarantee students developing or maintaining belonging in the network. Students moved in and out of central participation, illustrating Faircloth and Hamm's (2011) description of dynamic adolescent peer networks. Further, integration into the collaboration network was determined less by perceived mathematical competence than on how students' peers took up students' contributions as legitimate (Kotsopoulos, 2014; Lave & Wenger, 1991). These findings highlight the need to investigate individual students' positioning within the collaboration network and how to support the needs of all learners for developing belonging.

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AFFORDANCES AND LIMITATIONS OF DUAL ENROLLMENT MATHEMATICS STUDENTS' EXPERIENCES

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Dual enrollment (DE) programs, in which students take college courses while in high school, have been utilized in the US to enhance students' access to and success in college. Existing research has mainly focused on the effects of DE programs through quantitative, outcome-oriented frameworks; few studies utilize qualitative, sociocultural approaches to examine DE students' experiences based on their own voices. Through a case study, we address this need by exploring how three high school DE students taking a college mathematics course on a college campus describe their DE experiences. We use theme analysis to identify the emerging affordances and limitations of DE students' experiences based on their responses to semi-structured interviews. We then discuss the implications of our findings for research and practice.

Keywords: Dual Enrollment, Student Perspectives, High School Education, Undergraduate Education

Dual enrollment (DE) programs, in which high school students take college courses, have been utilized in the US to enhance students' access to higher education, college readiness, and interest in pursuing STEM degrees (An & Taylor, 2019; Cevallos et al., 2016; Hoffman, 2003; Jamieson et al., 2022). Existing research mainly uses outcome-based approaches and quantitative measures (e.g., academic performance through course and college grades; college enrollment, retention, and graduation rates) to examine the effectiveness of DE programs (e.g., Allen & Dadgar, 2012; An & Taylor, 2019; Blankenberger et al., 2017; Henneberger et al., 2022). While some studies explored DE students' self-perceptions as college students and adaptation to college (e.g., Cevallos et al., 2016; Kanny, 2015; Karp 2007; Lile et al., 2018), few studies examine DE students' experiences through qualitative and sociocultural approaches that prioritize students' own perspectives and voices (McGowan Bucci & Simpson, 2022; Williams, 2015). Our work focuses on this area needing more attention by addressing the following research question: How do three high school students taking their DE courses on a college campus describe their DE experiences? Exploring DE students' experiences through their own voices, particularly when taking courses in a college setting, can help examine the extent to which their experiences align with the goals of DE programs in easing the transition from high school to college and provide insights on how to improve DE programs and models.

Theoretical Framework

Using a sociocultural perspective, we view DE students as operating in and transitioning between different communities of practice (Lave & Wenger, 1991). Although high school and college communities have similarities (e.g., academic and educational focus), they also differ in terms of epistemological, institutional, and cultural presuppositions, expectations, norms, and practices to which participants need to adjust (Clark & Lovric, 2008; Di Martino et al., 2023; Gueudet, 2008). Consistent with our sociocultural framing, we purposefully focus on students' *descriptions* of their experiences in their narratives rather than framing our work via beliefs. *Belief* is a construct through which researchers have examined students' views about their experiences by framing beliefs as internal and stable cognitive entities that can be used as

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predictors of behavior (Cobb, 2007; Lerman, 2002; Sfard, 2008). Treating beliefs as detached from social and cultural contexts fails to capture the social, discursive, and situated nature of students' practices and experiences (Cobb, 2007; Lerman, 2002). We argue that students' views and positions about their experiences occur within social and cultural contexts and are participatory rather than merely cognitive (Lave & Wenger, 1991; Lerman, 2002; Sfard et al., 2001). By privileging student *descriptions* over *beliefs*, we treat talk as a unit of analysis in itself (Edwards, 1993) and view what students say as the situated accounts of their evolving narratives through which they construct meaning instead of viewing student descriptions as windows to their inner states such as beliefs or perceptions.

Methodology

We used a case study approach to examine DE students' descriptions of their experiences (Merriam, 1998). DE programs have been implemented in the US through a variety of models including students taking DE coursework only at their high schools (either with a college or qualified high school instructor); both at their high schools and a college campus; and only at a college campus, where they take courses with regular college students or in special sections composed solely of DE students (An & Taylor, 2019; Andrews, 2001). Our case study focuses on the experiences of a group of DE students taking college courses on a college campus together with college students because this model has the most potential in providing students with authentic college experiences (Edwards et al., 2011; Karp, 2007), and a key motivation for students to take DE courses is to gain experiences with college (Kanny, 2015; Lile et al., 2017; McGowan Bucci & Simpson, 2022).

The participants were three high school students, who, at the time of the study, were taking a calculus course at a 4-year university called Tech College (pseudonym) located in northeastern US. Our participants were Catherine, Billy, and Richard (pseudonyms), who all self-identified as Black/African American and attended Eastern High School (pseudonym), where 68% of the students were designated as low-income and consisted primarily of students of color. Eastern High School utilized DE to provide career-oriented coursework for their students in computer science, engineering, and health sciences. All DE students were in the engineering pathway and calculus was the first mathematics course they were taking on the college campus, whereas they took some college engineering courses at Tech College prior to the study.

The data we report here is based on two semi-structured interviews conducted with the students individually. Each interview was recorded and conducted via video-conferencing technologies to allow the greatest flexibility for the students to participate in the study. The first interview elicited general information about the students and their DE program, courses, and overall experiences. The second interview elaborated on each participant's responses in the first interview to provide more in-depth information about their experiences and allowed us to ask additional questions to each participant based on other participant interviews. For example, one of the questions in the first interview was "Can you describe your experiences in your dual enrollment courses?" In the second interview, depending on each participant's responses, we asked questions such as:

> In our last interview, you mentioned that you could not attend your professor's office hours due to your high school schedule. How do you navigate not having access to this additional assistance? What do you usually do when you need help?

Since the second interview was informed by the first interview, data analysis occurred simultaneously with data collection (Merriam & Tisdell, 2015). We transcribed each interview

and used theme analysis (Guest et al., 2012) involving deductive coding derived from our interview questions as well as inductive coding derived from the emerging themes in students' descriptions and narratives. We classified the emerging themes and subthemes as affordances and limitations of DE students' experiences depending on whether they supported or hindered ease of transition from high school to college.

Findings

Affordances of DE Students' Experiences

Deeper conceptual understanding of mathematics. The students reported that taking DE courses assisted them in developing deeper conceptional understanding of mathematics. Through her experiences in her DE calculus course, Catherine made a distinction between two types of mathematical skills: procedural "when you see just like an integration sign and a function it's automatic about what you have to do," and conceptual, where:

> They don't just give you these numbers, they give you … a situation and you have different variables to it … [in] optimization and related rates … you have to paint a literal picture and consider this and this…and you actually have to think about what you have to do.

Although Catherine mentioned that she struggled with optimization and related rates, she also said she appreciated being exposed to them since it supported her overall understanding of calculus. Billy expressed that, while taking his college mathematics courses, he started to understand how "a certain concept is the way it is, and not just how to do it … and how to get from A to B and why I'm getting from A to B in this certain way." Billy further elaborated that his DE mathematics courses allowed him to go "beyond the world of [routine] problem solving and really seeing … why you do things specifically this way and why and how, if you do it another way, then you know it's not gonna work." Richard mentioned how he performed calculations more easily in his DE engineering courses because of the connections he made between engineering and mathematics concepts: "[In] engineering graphics class, they will give you measurements … [but other] measurements you have to calculate it … and I could just look at it and already tell what the measurement was based on the geometry, everything was connected at times." Richard further explained that these moments "uplifted my knowledge or what I thought … of math," indicating enhanced understanding through making connections. The student responses suggested that their DE experiences helped them learn how to solve non-procedural mathematical problems (Catherine, Billy), why a mathematical concept is the way it is (Billy), and make connections between mathematical and other concepts (Richard), which are components of conceptual mathematical understanding (Kilpatrick et al., 2001).

Learning about applications of mathematical concepts. The participants mentioned learning about applications of mathematical concepts in real life and to other disciplines as an aspect of their DE experiences. Catherine explained how the inclusion of real-life applications in her college calculus course helped her see the usefulness of mathematics.

> Catherine: The examples and the problems that we're made to solve in calculus … they were … dealing with planes and bridges and we were calculating different things that people, especially engineers, use a lot. It was good to see math in a practical way and see that it actually is used pretty often.

Billy commented that taking the college calculus course allowed him to "better understand" the "fundamentals" of mathematics, which included "not knowing just how to solve an equation but knowing how to apply it to the real world." Billy also noted that "calculus ties into basically

everything related to engineering," suggesting that his DE experiences fostered understanding of applications of mathematics in engineering and the real world. Richard noted "I see math as already in my future solidified, since I want to do engineering," indicating that he viewed mathematics as applicable to engineering. Richard also mentioned that "math is everywhere. I think a lot of people say that but it's just the truth. You need math for everything … math is pretty big in society," suggesting a connection he made between mathematics and real life. The students' DE experiences appeared to have assisted them in making connections that give mathematics meaning through the relationships between mathematical ideas and external contexts (Kilpatrick et al., 2001) and participating in meaningful mathematical practices to help them view mathematics as a tool for interpreting the world (Cobb, 1994).

Development of agentic practices. All students stated that their DE experiences facilitated the development of various practices in support of their agency.

> Catherine: I think that having that type of [college] coursework go along with your normal workload, a balancing those two, I think adds a skill, a time managing skill to the students that take it … Now I am better able to handle a lot of coursework and balance it better with my outside of school life than I was able to before.

Catherine also explained that, to balance her DE and high school coursework, she prioritized her studies each weeknight and utilized her weekends to spend time with family and friends or "catch up on any missed sleep" during the weekdays. Billy similarly stated that he had to alter his study habits when taking DE courses.

> Billy: There's more expected of you because you're a college student and you're approaching higher level education. So, there's no time to really constantly keep having those bad habits and procrastinating a little bit too much … even now, I still have to make sure I don't slack off or anything.

Billy also noted starting to work at a part-time job the previous semester and stated he would have to "finish all my [school)]work before … I go to work. Because I'm not able to do anything at work. So, it's better to get all the stuff I have to do out of the way." Richard, who also worked part-time on the weekends, said "I would always … prioritize early college work over everything" to balance his multiple roles and responsibilities. Based on the student responses, their DE experiences helped them develop practices in support of their agency, where agency refers to managing their participation in different communities of practice and across multiple contexts (Wenger, 1998). From a sociocultural perspective, practices like time management, balancing high school, college, and personal lives, prioritizing responsibilities, and developing productive practices (e.g., avoiding procrastination) are expressions of student agency as students organize and manage their participation and navigate the demands of socially structured systems (Emirbayer & Mische, 1998; Holland et al., 1998).

Enhanced sense of community within DE students. Since all students were in the engineering pathway of their DE program, they took many of their high school and college courses together with the same group of students. As Billy shared, for the last two years, "we've always took the same classes, it's very rare that we were separated … I've pretty much connected with them throughout the years." This sentiment of connection and the formation of a supportive community within the group of DE students was also echoed by Richard, who said "the little bubble we had, it was … like a safe spot for all of us if we needed any help, or we were kind of anxious about the work, just lend a hand and whatnot, just like a community." Similarly, Catherine mentioned feeling a sense of comfort and camaraderie working with her fellow DE

peers as “we know what each other's been through, we know the traumas and other times.” Catherine also detailed how being a part of the DE student community helped her feel more willing to take intellectual risks, such as asking for help, in her DE courses.

> Catherine: When … you're the only one that doesn't know something, it can feel really uncomfortable to ask the teacher for help or ask a random person for help. But if we're all in there together and we're struggling, it's less sad, it's more like funny, we laugh about it, and one of us will maybe get the courage to ask for help.

Catherine mentioned that the students typically had “a free hour” together and would “take that time to do homework and also projects” because each person was “good at different things.” Billy corroborated that the DE students would “always do our homework together” and “would always collaborate.” Richard also shared that when given group projects in their DE courses, the DE students “didn't have to work with college students, we could actually work with each other.”

The student responses indicated that their DE experiences fostered their sense of connection, comfort, and camaraderie with one another; created a safe space for them to take risks and develop trust; and has allowed them to work collaboratively and navigate challenges as they worked towards a common goal, which are components of developing a sense of community (Lave & Wenger, 1991; Wenger, 1998). However, student descriptions of enhanced sense of community formed within their DE peer group contrasted with their feelings of discomfort and alienation when working with college students, which will be discussed later.

Limitations of DE Students’ Experiences

Difficulties with meeting multiple demands. All DE students mentioned struggling with taking high school and college courses concurrently and highlighted challenges in balancing the expectations of all their courses with the limited time they had. Richard pointed out how the students’ schooldays typically ended in evening hours due to their class schedules: “You have to take the bus to go to Tech College … [it is] a time commitment because … we literally got at Tech College at like two [p.m.] and then [class] would end at five [p.m.].” Richard further commented on the limited time he had outside of his coursework, stating “it was kind of crazy, honestly … We don’t have that much time as high school students, because we have a whole day’s worth of work we have to do.” Billy commented on the time DE students needed outside of class to study saying, “be prepared to set a lot of time aside after school and when you get home to do your college work and do your high school work.” Based on her own experience, Catherine discussed what a DE student’s schedule could look like after coming home.

> Catherine: I would get home at six [p.m.], I had three AP [advanced high school classes], and two regular [high school] classes, and the [college] class, and to get all the homework done from six [p.m.] to the time I have to sleep and eat and take a shower and do everything else in the middle of it … I was getting like five hours of sleep.

Billy expressed that, in addition to their full schedules, DE students needed additional time to understand new material they were not yet familiar with in their college courses.

> Billy: I was putting in so much work and putting a lot of time outside of class … to understand what’s actually going on … It was something that was new to me so I didn’t know how to exactly go about approaching the [college] class and the material.

Despite the agentic and productive practices students developed to better manage their time and meet multiple demands they face, the interview responses indicated that the students had ongoing struggles balancing the requirements of high school and college, which, when combined with the demands of their jobs and everyday lives, can lead to stress, exhaustion, and potential burnout in the absence of consistent support structures (Nagy & Hokstok, 2026).

Challenges having an authentic college experience. The participants took classes on the Tech College campus, but they reported not being able to participate in various aspects of college life, which indicated that they could not have authentic college experiences. By authentic college experience, we refer to experiences that make it possible for DE students to fully have access to and participate in (Lave & Wenger, 1991) college activities, practices, and resources. Three subthemes emerged under this theme based on student responses: limited access to general college activities and resources, lack of access to tutoring center and college instructor office hours, and limited sense of community with college students.

Limited access to general college activities and resources. Catherine and Richard commented that their DE schedules allowed for limited time on the Tech College campus and, due to their busy schedules, they could not attend campus events and activities.

> Catherine: Because we have traditional [high school] classes too, it was hard to get that college experience. We were taking the [college] classes, but we couldn't do any of the fun parts of college … So I think we definitely missed out on a lot of the what the other college students actually had to take advantage of.
>
> Richard: There would be events at Tech College that we couldn't go to because we would be in class or it would end before we even got to Tech College … [and] we didn't get to do anything with like the Tech College clubs.

Billy also mentioned the barriers the DE student schedules posed and how this issue prevented him from attending college events by commenting "obviously we're not getting the full college experience." Richard commented on the challenges he experienced utilizing various resources at Tech College. He said that he could not use the gym in the Tech College because DE students did not meet the college's "health insurance requirements." The participants' lack of access to getting "the full college experience" (Billy) including the "fun parts of college" (Catherine) such as participating in college clubs and utilizing college resources (Richard) does not only limit having authentic college experiences but can also hinder DE students' opportunities to socialize and interact with their college peers as well as participate in college communities (Lave & Wenger, 1991; Wenger, 1998).

Lack of access to tutoring center and college instructor office hours. All participants mentioned not being able to attend their college instructors' office hours due to their high school schedules. Billy stated that "office hours, we can't even attend those because it's during high school." Catherine said she could not attend office hours "because they're always at 11 a.m. on a Thursday and I'm in [high] school then." Richard said limited time on campus meant he "never attended tutoring" at the college tutoring center and his college instructor's office hours. He mentioned developing strategies to overcome these issues by working on his assignments early and asking questions in the class.

> Richard: I would just wait until I had class again and then asked my teacher about it in class to get the additional help. Because I would always prioritize early college work over everything … So I'd always start on [assignments] early, so I had time to ask [the instructor] in class about the work.

Similarly, Billy commented that he would "use resources outside of class to understand [the course material] and maybe watch some math videos" to get extra help with his college calculus course. Catherine highlighted how the DE students "definitely use each other a lot for support" in completing their college assignments. While the DE students developed some strategies to get help with their college coursework, lack of access to the college tutoring center and college instructors' office hours limited their opportunities to have direct interactions with university educators, which are opportunities available to typical college students. Lack of access to academic support is a barrier that can impact DE students' academic success in college courses (Barnett & Stamm, 2010; Karp & Hughes, 2008).

Limited sense of community with college students. The participants reported discomfort and alienation while interacting with the college students in their DE classes.

> Billy: They're [college students] just doing their own thing. They have their own school and life and there's not really a chance or too much time to really connect on a deep level … it's not as encouraged as it is in high school … I guess it's uncomfortable in a way.

Richard referred to some interactions with his college-aged peers as "scary", which he attributed to being an "introvert" and "shy." Catherine reported feeling alienated from her college peers, especially in her college engineering courses.

> Catherine: They [college engineering students] would hear you struggling, but would never offer any help, even if they know they can do something … it just felt very separated. It was my high school group all together, struggling together, and then there was them finishing their work in 10 minutes and then leaving the class … [In] my first year engineering classes … the teacher would give the instruction to pair up and they [college students] would just sit there and … not say a word to you … they don't make you feel comfortable … I knew they knew more than me, but they were using that as kind of leverage above me … not putting themselves in a supportive mindset, not making themselves feel like someone that I can ask for help because I feel they would judge me a little, call me stupid.

Catherine also observed a difference in how college engineering students interacted with her compared to male DE students: "they were much more willing to work with him [a DE male student] rather than me, you could see the energy change when they were working with them [male DE students] versus me." Although Catherine added, "it didn't really bother me, but like, it's crazy," her comments suggested experiencing gender-based differences in college engineering courses, which might be why she thought, "you just have to get away from the engineering people"—an alarming view from a Black, female student on the engineering track of her DE program. On the other hand, Catherine spoke fondly of her interactions with the college students in her calculus course and referred to them as "cool people."

The DE students' descriptions of sense of community with each other contrasted with how they described their interactions with college students. The DE students shared feeling discomfort (Billy and Catherine), not connected (Billy), separated (Catherine), judged (Catherine), and scared (Richard) when interacting with the college students in some DE courses. They viewed some of their interactions with college students as indicating potential unwillingness or lack of effort to make deeper connections (Billy); indifference or refusal to offer help and interact (Catherine); and gender-based differences in favor of male DE students (Catherine). The DE students' discomfort interacting and collaborating with college students can

limit their academic learning opportunities and sense of belonging in college communities (Giani et al., 2023; Lave & Wenger, 1991).

Discussion

Our results indicate that the DE students' experiences afforded them some opportunities as well as presented challenges. Taking DE courses on the college campus gave students access to deeper conceptual understanding of mathematics and learning about applications of mathematical concepts. It also fostered students' agentic practices and sense of community with other DE students. The students faced challenges in aspects of college life and reported difficulties in meeting the demands of their high school and college work. Although they found ways to manage their time and prioritize college coursework, they all seemed stressed in the process of balancing high school, college, and everyday life, which can lead to exhaustion and burnout (Nagy & Hokstok, 2026). Students' lack of access to the tutoring center, office hours, and other college activities were tied to their time-related challenges, impacting the authenticity of their college experiences as well as academic and socialization opportunities (Barnett & Stamm, 2010; Karp & Hughes, 2008). The DE students struggled forming communities with college students and experienced discomfort, alienation, and at times, marginalization when interacting with college students, which can lead to fragmentation of their experiences and impair their sense of belonging (Johnson, 2022; Karp, 2007). While some discomfort is expected since the DE students are newcomers entering a college community with different norms and expectations than high school (Clark & Lovric, 2008; Di Martino et al., 2023; Gueudet, 2008), lack of access to college activities and resources except scheduled college coursework can limit DE students' opportunities to form social connections with college students and influence whether they can fully participate or remain peripheral in college communities (Lave & Wenger, 1991; Wenger, 1998).

Research indicates that DE students benefit most from having experiences that closely mirror the college environment such as taking courses on the college campus (Burns & Lewis, 2000; Edwards et al., 2011; Karp, 2007), which is aligned with the main motivation for high school students to take DE courses: gaining college experience (Kanny, 2015; Lile et al., 2017; McGowan Bucci & Simpson, 2022). The student responses in our work suggest that, despite having access to DE courses on a college campus, DE students can continue facing challenges in terms of access, legitimacy, and participation in many aspects of college life (Lave & Wenger, 1991). These barriers can be particularly problematic in relation to the goals of existing DE programs in aiding high school to college transition for students who, like our participants, come from historically marginalized communities and can limit opportunities for engagement, identity development, and belonging, which in turn can influence students' academic learning and participation (Giani et al., 2023; Lave & Wenger, 1991; Wenger, 1998). High schools and post-secondary partners need to coordinate efforts and provide support structures to ensure that DE students have more opportunities to have access to and participate in college communities while also taking into account the time limitations and demands of DE student coursework and schedules (Barnett & Stamm, 2010; Hofmann & Voloch, 2012). Support structures that promote academic as well as non-academic opportunities in college can play key roles in the social integration of DE students into college communities and in cultivating DE students' sense of belonging through opportunities to interact with, expand social networks, and develop supportive relationships within college communities (Garrett, 2022). More research utilizing sociocultural perspectives that also prioritize DE students' own voices can be useful in gaining more insights

on the specific needs of DE students and providing insights on how to improve DE programs and models based on the needs of the students.

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EXAMINING JULIA'S FRACTIONAL THINKING IN SMALL GROUP PROBLEM SOLVING DISCUSSIONS ON EQUAL SHARING TASKS

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This study examines how a bilingual child, Julia, develops her fractional mathematical thinking through problem-solving instruction over five weeks. Using a qualitative case study, we used the constant comparison method of analysis to trace Julia's evolving fractional reasoning strategies, drawing on her gestures, drawings, and verbal descriptions. Initially, she relied on rudimentary partitioning and non-anticipatory strategies, but gradually Julia demonstrated emergent anticipatory strategies. Initially, she relied on simple partitioning strategies (e.g., dividing in halves) but gradually transitioned to more complex partitioning strategies like partitioning by the number of sharers. These findings offer insights into how responsive teacher moves and norms of choice could support bilingual children's developing fractional understanding.

Keywords: Bilingual Learners, Fractional Mathematical Thinking, Problem Solving

Most research in special education on fractions with children identified with learning difficulties primarily emphasizes behaviorist models that focus on performance versus conceptual understanding (Lambert & Tan, 2020). Behaviorist studies typically emphasize pre- and post-assessments that capture children's knowledge at isolated points in time (e.g., Namkung & Fuchs, 2019), rather than examining their conceptual development longitudinally. Furthermore, many of these studies assume that the measured performance is inherently valid and free from bias. Performance-based research studies may have limited impact on advancing children's mathematical knowledge and, in some cases, could contribute to the marginalization of children of color by constraining opportunities for problem-solving learning (Lambert, 2015; Tan et al., 2022). Few of these studies have examined fractional interventions grounded in constructivist approaches such as teacher-responsive moves designed to support bilingual children's mathematical thinking (e.g., Lambert, 2015; Hunt et al., 2023; Silva et al., 2023). In this paper, we add to the literature by examining how a bilingual child with identified mathematics difficulties engaged with and constructed fractional knowledge. Our research question was: How did Julia, a child with identified mathematical difficulties, demonstrate her fractional thinking during problem-solving instruction when supported through responsive teacher moves and norms of choice?

Theoretical Framework

This study drew on Jacobs and Empson's (2016) framework of responsive teaching to support students' engagement in mathematical reasoning. Specifically, we incorporated eliciting (e.g., "I noticed you added four each time. Why did you do that?," clarifying (e.g., "What is the problem asking you to find?"), and extending (e.g., "What if there were five brownies instead of two? How would you share them?") prompts to support children in sharing their mathematical thinking in ways that were meaningful to them. Our norms were centered on choice and children's funds of knowledge (Silva, 2018, 2021). The sessions were designed to position children as sense-makers, with teacher support to help them exhibit mathematical agency. Children had opportunities to solve mathematical tasks using a strategy of their choice, share and represent their mathematical thinking in multiple ways, and communicate ideas in English or

Spanish while using gestures. Centering norms around choice helped humanize participation and broaden how children engaged in solving and discussing mathematical ideas. To investigate children's part-whole understandings of fractions, we utilized equal sharing tasks as described by Empson and Levi (2011). We identified equal sharing tasks as appropriate, as they are well documented in the literature as great entry points into children's fractional reasoning (Streefland, 1993). Consistent with Hunt and Empson (2015), we sought to document the development of children's fraction thinking over time.

Methods

This study took place at an elementary school in an urban city in the southwestern United States. The first author conducted and taught 12 lessons during small group instruction held within the school day. Three bilingual children participated in the small group, and each session lasted approximately one hour. Instruction occurred in a conference room equipped with a large table, manipulatives (i.e., unifix cubes), paper, pencils, and a whiteboard. This paper presents preliminary data on five problem-solving sessions on the mathematical thinking of one bilingual child, Julia (pseudonym). Julia was purposefully selected because she was a bilingual Latina third-grade student identified as an English Language Learner (ELL) and as having mathematics difficulties. Her mathematical difficulties were identified by the school through the response to intervention (RTI) model (Fuchs & Fuchs, 2007). Under this model, Julia was designated as Tier 3, indicating that she received intensive, individualized intervention. The assistant principal identified Julia as experiencing mathematics difficulties based on midyear and end-of-year mathematics benchmark assessments, and she was undergoing evaluation for a potential mathematics learning disability.

Our sessions were taught using a problem-solving model (Stein et al., 2008). Children were encouraged to solve problems using strategies that made sense to them; they were allowed to represent their mathematical strategies visually and verbally through Spanish or English, work individually or with partners, and share their ideas in ways that best supported their understanding.

Data Collection

To document Julia's mathematical thinking across sessions, we collected video recordings of each problem-solving session, her written work, and researcher-teacher field notes. Each video was transcribed verbatim, with attention to Julia's gestures, verbal explanations, and utterances. We analyzed video transcripts for her fractional reasoning, strategy drawings, and gestures to examine the development of her fractional partitioning strategies over time.

Analysis

To examine Julia's development of fractional understanding, we used retrospective analysis (Leech & Onwuegbuzie, 2007) to identify patterns of her mathematical ideas and conceptual growth across fractional tasks. We also employed content analysis (Saldaña, 2013) to identify and categorize the fraction strategies she used. The first author initially reviewed Julia's strategies from the first session using video recordings, transcripts, and written work. These data were then independently coded by the second author through analysis of the videos and transcripts. To identify and interpret Julia's fractional strategies, we drew on the initial codes established by Hunt and Empson (2015), which broadly characterize children's fractional strategies as non-coordination, non-anticipatory, and emergent anticipatory. We also developed additional partitioning codes to capture nuances within and across these categories (see Table 1 for final codes). All transcripts were transferred and analyzed using MAXQDA 2024 software, resulting in the identification of 10 analytic episodes across the five sessions. We defined

episodes as interactions in which Julia explored and discussed her fractional reasoning strategies during each equal-sharing task. Once the episodes were identified, we independently wrote analytic memos for each episode, addressing questions about what Julia's fraction understanding revealed and how she demonstrated her fractional thinking, while also noting changes across sessions. Next, we met to refine the initial codebook by collaboratively coding the first two episodes, revising and clarifying code definitions until we reached 100% agreement. We then coded the remaining episodes independently and subsequently met to discuss discrepancies and reach consensus. The final interrater agreement was approximately 96%.

Table 1: Fractional Codes of Partitioning

Category	Codes	Definitions of Codes
No Coordination	Partitioning leftovers by correspondence	Sees the whole as divisible. Links sharers to a plan for sharing after the fact (perhaps on final piece/pieces) as they distribute wholes items first and then see the need to partition leftovers. Child begins to partition leftover pieces by doubling/halving, but the partitioning seems ineffective for the child. Does not mind having additional pieces leftover and has remaining pieces that are not shared or the share for each person is not distributed equally.
No Coordination	Partitioning all wholes by half and leftovers by correspondence	Sees the whole as divisible. Child partitions all wholes by half, the remaining leftovers were partitions by correspondence. Child is bothered by having additional pieces leftover and or remaining pieces that are not shared or distributed equally among the sharers.
Non-Anticipatory	Partitioning all wholes by half and leftovers by number of sharers	Sees the whole as divisible. Partitioning all wholes by half, then the remaining leftovers were partitions by the number of sharers. Child seems to evidence an increased need for equal pieces through partitioning and distributes all the shares to sharers equally.
Non-Anticipatory	Partitioning leftovers by number of sharers	Sees the whole as divisible. Child links sharers to a plan for sharing after the fact (perhaps on the final pieces) as they distribute wholes items first and then see the need to partition leftovers. Uses number of sharers as a prior plan to partition leftover wholes. Child seems to evidence an increased need for equal pieces through partitioning and distributes all the shares to sharers equally.
Emergent Anticipatory	Partitioning all wholes by number of sharers	Links sharers to a plan for sharing before activity. She may use the number of sharers as a partitioning strategy for all the wholes (e.g., 5 share 3 cakes, says cut all 3 cakes into 5 pieces). The strategy was formed in activity with correspondence, yet its use becomes implicit as the child abstracts his/her activity.

Results

Julia made shifts within her fractional partitioning strategies. Initially, she relied on partitioning strategies by correspondence (see Table 2) and gradually transitioned to more complex strategies, including partitioning by the number of sharers. Julia adapted her strategies based on the type of representation and whether the share was greater than or less than one. When the share exceeded one, she first distributed a whole to each person and then partitioned the leftover pieces by the number of sharers. When the share was less than one, she began by partitioning the whole in half rather than dividing directly by the number of sharers. A particularly significant shift occurred between sessions nine and 12 (see Figure 1). In session

nine, Julia attempted to partition the leftovers by the number of sharers, though only after several attempts. By session 12, she consistently partitioned the leftovers by the number of sharers. Teacher questions to clarify and elicit mathematical thinking helped Julia reflect upon her strategies and carefully examine her partitioning strategies over time. Notably, in Session 11, Julia partitioned all wholes by the number of sharers rather than beginning by distributing whole pieces. When prompted to explain how many pieces she planned to make, Julia responded, "seven in each," indicating that she intended to create seven parts per candy bar before she enacted her strategy. This marked an important developmental shift, as her approach was beginning to be pre-planned and demonstrated an emergent anticipatory strategy for the first time.

Table 2: Julia's Fractional Strategies

Session	Task	Partitioning Strategy Code	Fractional Code
8	8 chocolate bars shared equally among 3 kids	Partitioning leftovers by correspondence	No-coordination
9	3 cakes shared equally among 5 kids	Partitioning all pieces by half, partitions leftovers by correspondence	No-coordination
10	10 cookies shared equally among 4 kids	Partitioning leftovers by number of sharers	Non-anticipatory
11	11 candies shared equally among 7 kids	Partitioning all wholes by number of sharers	Emergent-anticipatory
12	4 brownies shared equally among 6 kids	Partitioning all pieces by half, partitions leftovers by number of sharers	Non-anticipatory

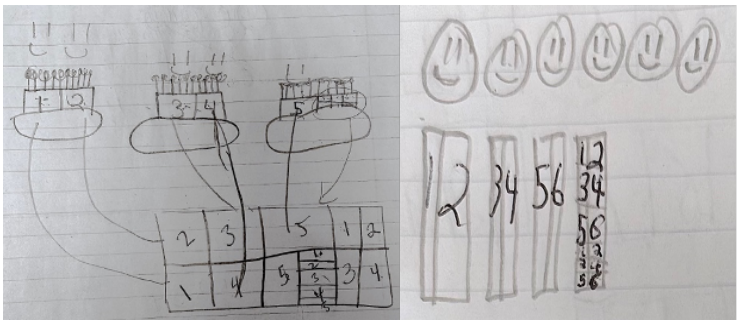


Figure 1: Julia's Written Solutions to Session 9 (right) and 12 (left)

Conclusion

Mathematics and special education teachers who work with bilingual children should critically examine their problem-solving instruction. Instruction that focuses only on standardized testing and rigid curricula reduces creativity and curiosity. In contrast, participating in mathematical teacher practices supports children's conceptual development and increases understanding in the classroom (Bodovski & Farkas, 2007). The current study examined how responsive teaching and problem-solving instruction using equal sharing tasks can help foster a bilingual child's understanding of fractions. This study demonstrates the crucial role of teacher responsive moves in enhancing students' mathematical thinking of fractions during problem-solving. Moreover, it shows the importance that teaching norms of choice can shape bilingual students' opportunities to execute problem-solving strategies that develop deeper understanding

of mathematical concepts. This study illustrates how future studies of bilingual children with math difficulties can participate and learn fraction concepts during problem-solving instruction.

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FACTORS CONTRIBUTING TO UNDERGRADUATE STUDENTS' DISLIKE OF SCHOOL MATHEMATICS

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This qualitative study examines undergraduate students' reflections on their school mathematics experiences and the development of their dislike of mathematics. Fifteen undergraduates enrolled in a college Quantitative Reasoning (QR) course at a midsize university in the Midwestern United States participated in the study. Participants were purposefully selected based on their self-reported dislike of mathematics. Semi-structured interviews conducted at the beginning of the semester were transcribed, coded using NVivo, and thematically analyzed. Findings identified six factors contributing to students' dislike of school mathematics: irrelevant content, poor instructor-student relationships and support, inflexible assessment practices, learning disabilities, parental pressure, and media influence.

Keywords: Dislike of School Mathematics, Factors Contributing to Dislike of Mathematics, Attitude Toward Mathematics

Purpose of Study

Dislike of mathematics is expressed in emotional, psychological, and social forms that can lead individuals to distance themselves from mathematics and its application in both academic and everyday life (Thapa, 2016). Aguilar (2021) explored high school students' dislike of mathematics, identifying factors that contribute to its development. News media have also reported widespread dislike of mathematics. For example, Cook (2015), in *U.S. News*, highlighted research on young students' disinterest in mathematics that often extends into high school. Similarly, a *Washington Post* article (2016) described math anxiety as a cultural phenomenon, and Strauss (2011) argued that the issue stems less from ineffective teachers and more from the content taught in mathematics classrooms. This widespread dislike affects not only academic performance (Wachsmuth et al., 2017) but also students' personal and professional development. Dislike of mathematics can lead to avoidance, reduced confidence, and limited participation in mathematics-intensive fields (Ernest, 2016). Therefore, addressing dislike of mathematics requires identifying its root causes—including teaching practices, curriculum content, and societal influences (Sam & Ernest, 2000; Taylor, 1996; Levy-Warren, 2017). Accordingly, this study is guided by the following research question: What are undergraduate students' perspectives on the factors that contribute to their dislike of school mathematics?

Literature Review

The word *like* is a transitive verb, defined by the Merriam-Webster dictionary (n.d.) as "to feel attraction toward or take pleasure in" (para 1). In mathematics, liking mathematics means that a student is attracted to or takes pleasure in learning it, whereas disliking reflects the opposite. These responses are shaped by students' experiences and environments (De Houwer & Hughes, 2020) and represent stable affective responses rather than temporary emotions (McLeod, 1992).

Attitudes toward mathematics consist of beliefs about oneself and mathematics, together with emotions associated with the subject (Di Martino & Zan, 2010). Self-concept refers to an

individual's beliefs about their ability to handle a situation (Skaalvik & Skaalvik, 2006), while positive self-concept and self-efficacy support achievement and performance (Van der Beek et al., 2017; Liu & Koirala, 2009).

Mathematics is often portrayed as abstract, rigid, and accessible only to a few, shaping students' identities, engagement, and sense of belonging (Ernest, 2003, 2016; Sam & Ernest, 2000). Students' liking or disliking of mathematics is also influenced by aptitude, instruction, and the social-psychological environment (Mzomwe et al., 2019).

Mathematics anxiety, characterized by fear and tension during mathematical tasks (Field et al., 2019), is associated with lower performance, reduced engagement, and avoidance (Núñez-Peña et al., 2013; Liu & Koirala, 2009). It disrupts cognitive processing (Ashcraft, 2002) and is influenced by classroom practices emphasizing speed, correctness, and evaluation, as well as teachers' own anxieties (Finlayson, 2014; Pieronkiewicz, 2017; Bates et al., 2013; Swars et al., 2006).

Negative attitudes can emerge in elementary school and are linked to viewing mathematics as procedural and evaluative (Larkin & Jorgensen, 2016; Di Martino & Zan, 2010). Teaching mathematics as a process rather than a product may reduce these negative dispositions (Hannula, 2012).

Although research documents students' dislike of mathematics, fewer studies examine students' perspectives on why they dislike mathematics. This study addresses that gap by exploring student perspectives on their experience with learning school mathematics and identifying factors that contributed to their dislike of mathematics.

Methodology

Qualitative research is particularly well-suited for studies involving subjective and in-depth exploration of human perspectives of their lived experience (Wilson, 2015); this approach enables a deeper understanding of complex and dynamic variables (Denzin & Lincoln, 2018; Leavy, 2020), making it appropriate for this study, which explores students' reflections on their school mathematics learning experiences and the development of their dislike of mathematics. Fifteen undergraduate students enrolled in a college-level quantitative reasoning course at a mid-sized university in the Midwestern United States in Spring 2023 participated in the study. Six of the seven sections of the QR course were invited to participate in the study. The seventh section was excluded because it was taught by the principal researcher, thereby minimizing the potential for researcher bias and perceived coercion (Creswell & Poth, 2018). The QR class was purposefully selected as this was a foundational general education course, and generally, students in non-math majors and with math aversion enrolled in it (Hennrich & Lee, 2011; Foley et al., 2023). Participants were selected based on their self-reported dislike of school mathematics and were 18 years of age or older. Each participant completed a semi-structured interview lasting approximately one hour at the beginning of the semester. The semi-structured format allowed flexibility in participants' responses (Merriam & Tisdell, 2015). Participants were given the option of in-person or virtual interviews, and the interviews were audio- and audio-video-recorded, then transcribed and coded using NVivo software.

To ensure quality and rigor, I employed thick description, peer debriefing, member checking, pseudonyms, verbatim participant quotes, and thematic saturation to support confidentiality, trustworthiness, credibility, and transferability (Patton, 2015; Brinkmann & Kvale, 2015).

Findings

Factors Contributing to Dislike of Mathematics

The findings identified six factors contributing to students' dislike of mathematics: irrelevant content, poor instructor-student relationships and support, inflexible assessment practices, learning disabilities, parental pressure, and media influence. Of these, the first three factors appeared most frequently across interviews, whereas the remaining three were mentioned by fewer participants.

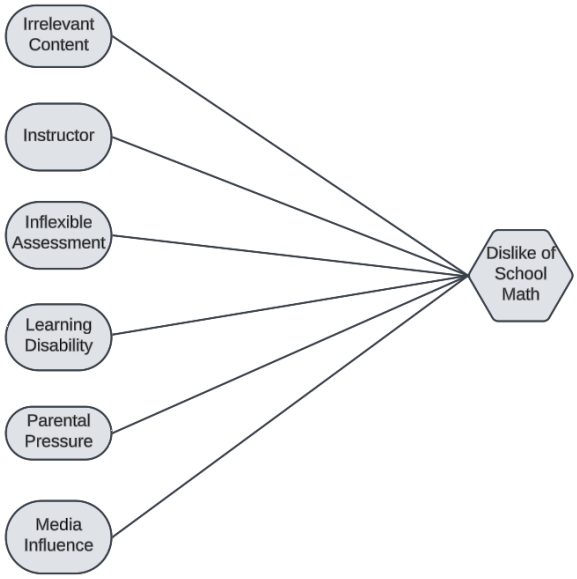


Figure 1: Factors Contributing to Dislike of Mathematics

Lack of real-world connection and relevance. Most students expressed a lack of interest in mathematics because the content and the way it was taught were not relevant to or connected with the real world. Some students stated, "*I don't recall any real-world connections in those classes.*" Another student commented, "*I know in school you never really learned much about how things are taxed. I feel like if there was a way for it to benefit my future, then I think that would have helped me a lot.*" A third student remarked, "*If the course content involved real-life units, questions, and examples, it makes a lot more sense to me. And I would enjoy doing those a lot more than random formulas that I don't even know if I'll personally use.*"

Lack of student-centered teaching and support. Most students expressed dissatisfaction with teacher-centered approaches focusing primarily on textbooks. One participant said, "*She would just pass out the papers at the beginning of class. And have you turned them in by the end? There was no personal connection, no walk-through, or anything. … I thought it was awful.*" Another participant said, "*It was mostly because there wasn't a very good teacher-student connection. … And, so, I didn't really like to ask questions. And, so, that made me kind of fall behind.*" A third participant said, "*He just has the reputation because he just does not work with kids. He's just like a straight, intimidating guy.*" And a fourth participant said, "*There have been instances where I would ask my teachers for help, and they wouldn't be very nice about it, or they wouldn't really have anything to offer me.*"

Inflexible assessment practices. Most participants expressed that inflexible assessment practices contributed to their struggle with mathematics and dislike of it. More specifically, they highlighted the timed exams and the exams focused on memorization of facts from texts, causing anxiety and fear. One participant said, "*There were a couple of times where I was like just so nervous for a math test that I would have to leave the room and go to the nurse's office. ... the thought of failing a test was terrifying*." Another participant said, "*I think a lot of the anxiety does come from one being timed … that was my biggest downfall was it was timed*." A third participant said, "*Definitely to every single math test. … I would never go in like super confident … it was painful going through a math test*."

Learning disabilities, parental pressure, and media influence. Out of the 15 participants, six mentioned that they had some form of learning disability, such as ADHD, dyscalculia, or dyslexia. For many, these disabilities were identified only later, and even then, their schools provided little support beyond extending due dates or providing extra time. For example, one participant said, "*Typically, to get additional time, I'd have to have some sort of medical documentation … And I didn't realize I had any sort of learning disability or something that would affect my learning until I was in 12th grade.*"

Similarly, some students recalled parental pressure. Although their parents encouraged them and intended to be supportive, many were unable to provide adequate academic or other support. As one participant explained, "*They definitely pushed me a lot. I did tutoring sessions outside of school for math specifically. And, I had to do online courses at some point. … It definitely made me hate it more.*"

Likewise, some participants recalled that media influenced their self-concept of mathematics ability and perceptions of mathematics, with cartoon programs and other social media reinforcing negative beliefs. For example, one participant said, "*social media or stuff like that kind of feeds into the hatred of math. It's just seeing so many other students who might also hate math talk about their hate of math … you'll be like, 'Oh, I hate it too.*'"

Discussion and Conclusions

The findings indicate that students' dislike of mathematics begins early, often in elementary grades, when foundational concepts are taught in rushed, textbook-centered, and procedural ways without meaningful real-world connections. As students progress, abstract content, fast-paced instruction, and heavy reliance on timed, standardized assessments intensify anxiety, avoidance, and negative self-concept. Learning disabilities, insufficient accommodations, parental pressure, and negative media messages further compound these experiences. Students consistently reported low enjoyment, limited perceived utility, minimal motivation, and poor performance in school mathematics, reinforcing students' dislike of mathematics.

Acknowledgments

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THE INTEGER PRINCIPLES: AN ARGUMENT FOR AN EARLY COHERENT NUMBER SYSTEM

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The transition from arithmetic to algebra is often hindered by a reliance on whole-number rules that "expire" in secondary mathematics. Paralleling the early algebra movement, we push for a similar early integer movement. This theoretical research report centers on the integer principles for addition and subtraction, a framework designed to expand K–5 numerical understandings toward the signed number system. By prioritizing symbolic flexibility, absolute and linear value, and translation models, the framework establishes a robust foundation for algebraic success. Establishing these principles early ensures mathematical coherence across number domains. Ultimately, this work reimagines learning progressions to foster the numerical, symbolic, and operational foundation required for advanced mathematics, like algebra.

Keywords: Algebra and Algebraic Thinking, Elementary School Education, Number Concepts and Operations

Algebra remains a significant gatekeeper to advanced mathematics and STEM careers (e.g., Barroso, 2020; Herscovics & Linchevski 1994; Kieran, 1992), leading to a broad consensus that students require earlier access to algebraic thinking (National Mathematics Advisory Panel, 2008; Moses & Cobb, 2001). The early algebra movement successfully addressed earlier access by promoting developmentally appropriate preparation in the early grades, such as shifting student understanding of the equal sign from an operational to a relational view (Blanton et al., 2007; Cai & Knuth, 2005). However, while students are becoming better equipped for algebraic generalization, a significant hurdle remains: the transition from the restricted domain of whole numbers to the broader system of integers (e.g., Bishop et al., 2014; Vlassis, 2004, 2008).

Despite the successes of early algebra, accounts of student struggle with the "arithmetic-to-algebra" transition remain plentiful (e.g., Cao et al., 2025; Barroso, 2020). This difficulty is due, in part, to a foundational reliance on whole-number rules that prove insufficient when negative integers surface in secondary mathematics (Bofferding, 2014; Karp et al., 2014). In traditional K-5 curricula, students are immersed in a whole-number-only environment that reinforces procedural rules that expire. For example, procedural shortcuts like "you cannot subtract a larger number from a smaller one" or "subtraction always makes a number smaller" (Karp et al., 2014) are not true from the transition from whole numbers to integers. While these are localized truths for positive whole numbers (or natural numbers), they create a limited understand of number that leads to immediate frustration and a decreased sense of competence when students encounter integers in middle school (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Vlassis, 2004).

To mitigate these difficulties, mathematics education must pivot toward a robust framework that exposes negative integers to students in the early grades. Students need the conceptual tools to move beyond binary operations toward the multiple meanings of the minus sign, like interpreting it as a unary sign and a directed distance (Vlassis, 2008). Without this foundation, structural misconceptions persist into algebra; for example, students often incorrectly extend commutativity to subtraction, transforming $a - x = b$ into $x - a = b$ (Gallardo & Rojano, 1994).

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By shifting focus from procedural shortcuts to the underlying structures of the number system—including the symmetry of the number line (e.g., Tsang et al., 2015) and directed distance—we can better prepare students to interpret addition and subtraction flexibly across all algebraic situations.

An Argument for Early Integer Introduction

The early algebra movement gained traction by arguing that integrating algebraic thinking in elementary school, rather than treating it as a separate post-arithmetic stage, is a key factor for later success (Katz, 2007; Gardiner et al., 2016). Just as early algebra focuses on promoting generalizations and noticing structures like the identity and inverse properties (Blanton et al., 2007; Cai & Knuth, 2005), an early integer approach allows students to notice the structure of the number system, rather than as a series of disconnected rules. Previous accounts illustrate that even children in kindergarten and first grade can productively reason about negative numbers (Aze, 1989; Behrend & Mohs, 2005–2006; Bofferding, 2014; Wilcox, 2008). Therefore, we argue against a strictly linear "whole numbers-then-integers" approach in favor of appropriately integrating integers into the early grades and reimagining learning progressions to promote mathematical coherence.

The goal is not to push secondary curriculum down, but to further prepare students for algebraic thinking in developmentally appropriate ways (Blanton et al., 2007). How can we introduce negative integers in developmentally appropriate ways? One way may be by introducing integers through playful mediums like number talks or games. These mediums should provide students with opportunities to explore negative integer principles, such as understanding the meanings of opposite or that the commutative property works for addition, but not subtraction, within a conceptually oriented and accessible framework (Bruner, 1963; Wessman-Enzinger & Bofferding, 2014). This early exposure ensures that when students eventually encounter algebra, they are not "unlearning" whole-number principles, but are instead extending a robust, coherent understanding of the number system. We acknowledge that similar issues may exist with the transition from whole numbers to rational numbers, but we restrict our focus here to integers. Before numerical learning progressions are reimagined or games are designed to incorporate integers, we first need to envision the types of number principles that are affected by expansion of experiencing mathematics with only whole numbers to mathematics that include negative integers.

The Integer Principles

We present the idea of the *integer principles* as a framework for thinking about what expanding the number system from whole number to negative integers in elementary schools might entail. The integer principles are the mathematical generalizations that allow for a coherent understanding of the signed number system. They extend whole-number reasoning by introducing the concepts of directionality, symmetry, and additive inverses, ensuring that mathematical rules remain consistent across all real numbers. Table 1 details whole number principles compared to integer principles. The whole number principles are ways children may conceptualize numbers if limited to only positive integers, or whole numbers. The integer principles are ways children may conceptualize numbers if their number system is expanded beyond positive integers to negative integers.

Table 1: Whole Number Principles Compared to Integer Principles

	Whole Number Principles	Integer Principles
A	Numbers are represented by numerals. Addition is represented by the plus sign. Subtraction is represented by the minus sign.	Numbers are represented by numerals and signs. The plus sign can designate an addition or a positive number. The minus sign can designate subtraction, negative, or opposite (Vlassis, 2004).
B	Numbers can be ordered by their position in the count list [starting at 0 or 1] (Steffe, 1983; Vosniadou et al., 2008).	Numbers exist on both sides of zero (Bishop et al., 2016). Numbers can be ordered by their position *and* direction from zero (Russo, 2019; Schindler & Huβmann, 2013).
C	Numbers can be embodied naturally with physical objects (Schindler & Huβmann, 2013).	Negative numbers can only be embodied abstractly with physical objects (e.g., mapping -1 or +1 to a chip; Lakoff & Nuñez, 2000; Martínez, 2006); they are relative numbers (Gallardo, 2002).
D	Numbers further from zero have larger values (Vosniadou et al., 2008).	Numbers have absolute values (numbers further from zero have larger absolute values) and linear values (positive > negative, and negative numbers closer to zero have larger linear values than negative numbers further from zero (Ball, 1993; Bofferding, 2014; Wessman-Enzinger, 2018).
E	Numbers can be decomposed into numbers with smaller values.	Numbers can be decomposed into infinite combinations of numbers that could include numbers with larger values than themselves.
F	Adding or subtracting zero does not change the value *(Identity with Addition and Subtraction)*.	Adding or subtracting zero does not change the absolute or linear value *(Identity with Addition and Subtraction)*.
G	Adding can be done on numbers in any order and still have the same answer *(Commutative Property*).	Adding numbers can be done in any order and still have the same answer *(Commutative Property*), but switching numbers in subtraction leads to different answers (3 – 2 = 1; 2 – 3 = -1, *Non-commutative Property*, Bofferding, 2010).
H	Subtracting a number from itself equals zero *(Subtracting Same*). Subtraction starts with the bigger number (Karp et al., 2014).	Subtracting a number from itself equals zero *(Subtracting Same)*. Adding a number to its inverse equals zero (*Additive Inverse*, Dickman & Bofferding, 2016-2017).

I	Addition results in bigger values (Karp et al., 2014; Vosniadou et al., 2008). Subtraction results in smaller values (Karp et al., 2014; Vosniadou et al., 2008).	Addition results in absolute and linear values that are either smaller or bigger than the original value. Subtraction results in absolute and linear values that are either smaller or bigger than the original value.
J	Adding can be represented as a translation to the right on a number line. Subtracting can be represented as a translation to the left on a number line.	Adding can be represented as a translation to the right or left on a number line (i.e., a movement more positive or more negative). Subtracting can be represented as a translation to the left or right on a number line (i.e., a movement less positive or less negative; Bofferding & Farmer, 2019; Marthe, 1979; Selter et al., 2012; Wessman-Enzinger, 2019; Wessman-Enzinger & Tobias, 2020).
K	The sum of two translations is a longer translation. Subtraction can be represented as the difference or distance between two states.	The sum of two translations is the directed distance from zero to the final position of the translation. Subtraction can be the directed difference or directed distance between two states (Marthe, 1979; Wessman-Enzinger, 2019; Wessman-Enzinger & Tobias, 2020).

The integer principles summarized in Table 1 are the fundamental structures that could naturally evolve from the whole numbers in developmentally appropriate ways during the elementary years and also provide insight into future instructional or game design. The integer principles represent the conceptual extension a student must make to move beyond "counting objects" to "imagining number beyond zero" (Bishop et al., 2016; Wessman-Enzinger, 2025). Next, we dig into the integer principles more.

Symbolic and Structural Principles (Principles A, B, C)

Principles A, B, and C (see Table 1) define what a number is and how it is represented. Principle A directly addresses the multi-faceted use of the negative sign. In a whole-number world, a minus sign only means "take away." With negative integers, it can be a binary operation (subtraction), a unary sign (negative), or a relational direction (opposite) (Vlassis, 2004, 2008). Principles B and C highlight that numbers are relative and can be placed with symmetry around zero on a number line (Russo, 2019; Schindler & Huβerman, 2013; Tsang et al., 2015). Zero is no longer just "nothing" or the starting point; it is the center of a system where number exist in two directions (e.g., Bofferding, 2019; Vosniadou et al., 2008).

Symbolic and structural principles parallel to the early algebra movement. Just as the early algebra research rejects the idea that students must master all of arithmetic before being introduced to algebraic structures, an early integer approach rejects the traditional linear progression that delays negative numbers until middle school. By introducing integer principles in the early grades, we provide a "structural" rather than a "computational" foundation, mirroring how early algebra encourages students to see the equal sign as a symbol of equivalence rather than a command to compute (Knuth et al., 2006). In both movements, the goal is not to accelerate secondary content into the elementary classroom, but to foster an early habit of mind that prioritizes mathematical generalizations and the identification of underlying structures (Cai & Knuth, 2011).

Symbolic and structural principles' impact on algebraic thinking and learning. Establishing these integer principles early makes a profound difference in a student's ability to navigate the formal abstractions of algebra. When students transition to algebra without a robust understanding of directed numbers, they often treat variables as "whole number placeholders," leading to significant errors when those variables take on negative values (Gallardo & Rojano, 1994). By mastering the multi-faceted use of the minus sign (Principle A) and the symmetry of the number line (Principles B and C) early on, students develop the "symbolic flexibility" required to manipulate algebraic expressions where signs are relational rather than just operational (Vlassis, 2008). Ultimately, this early conceptual expansion ensures that the "rules" of mathematics remain coherent; students are not forced to unlearn whole-number restricted reasoning but are instead prepared to apply generalized principles to the abstract landscapes of functions and equations.

Order and Value Principles (Principles D, E, I)

Principles D, E, and I (see Table 1) redefine how we determine which number is "larger" or "smaller." Specifically, these integer principles introduce the idea that a number has two types of value: absolute value and linear value. When students compare only positive integers, or whole numbers, linear and absolute value are the same so distinguishing between them is unnecessary (Vosniadou et al., 2008**)**. Absolute value (distance from zero) and its linear value (its relative position from left to right on a number line) do matter when the concept of number is extended beyond positive numbers to include negative integers (Ball, 1993; Bofferding, 2014; Wessman-Enzinger, 2018). This explains why –10 is "larger" in absolute value than –2, but "smaller" in its linear value. These principles also highlight that numbers can be broken down into an infinite combination of others, including numbers larger than themselves (e.g., $2 = 5 + -3$).

Order and value principles parallel to the early algebra movement. The distinction between absolute and linear value mirrors a core tenet of the early algebra movement: the shift from viewing mathematics as a series of discrete computations to a study of relationships and structures. Early algebra research argues that students must move beyond "arithmetic" results to understand the underlying properties of numbers (Blanton et al., 2011; Kaput, 2008). Similarly, introducing the dual nature of number values—where a number's distance from zero is distinct from its relative position—mirrors how early algebraists teach the equal sign as a relational symbol of balance rather than a signal to calculate (Knuth et al., 2006). In both cases, the goal is to expand the student's mathematical ontology. Just as early algebra avoids the "arithmetic-then-algebra" trap by integrating structural thinking early (Gardiner et al., 2016), introducing the absolute-value (or magnitude) distinction in the elementary grades prevents students from becoming tethered to the whole-number misconception that "larger" always means "further from zero" or always means "further to the right."

Order and value principles' impact on algebraic thinking and learning. Mastering these order and value principles is essential for developing the numerical flexibility required to handle algebraic variables and inequalities. When students do not distinguish between absolute and linear value, they struggle to conceptualize expressions like $|x| > 5$, often failing to realize that a "smaller" linear value (such as –10) can possess a "larger" absolute value (Ball, 1993; Wessman-Enzinger, 2018). Knowing that integers have these two types of values and can be decomposed into infinite combinations is a direct prerequisite for algebraic manipulation and working with equations. This conceptual shift allows students to view integers as vectors or directed distances rather than just as counting numbers. Without this foundation, students often perceive algebra as

a collection of arbitrary "tricks;" but with it, they gain a coherent system for thinking that allows them to predict and verify the behavior of negative terms in complex equations.

Operational Principles (Principles F, G, H, I)

Principles F, G, H, and I (see Table 1) reimagine how addition and subtraction are defined when negative integers extend a prior number system with only whole numbers. A notable example of this is the non-commutative nature of subtraction. Addition is commutative (e.g., $5 + -2 = -2 + 5$), and when students only operate with positive integers, they extend the commutative property (e.g., $5 + 2 = 2 + 5$) to subtraction, which does not hold. The integer principles solidify that subtraction is not commutative. Reversing the order of numbers within subtraction results in the additive inverse (e.g., $5 - 3 = 2$, but $3 - 5 = -2$). When using only whole numbers, addition always increases value. Yet, when using negative integers, adding a negative decreases linear value, and subtracting a negative increases it (Karp et al., 2014; Vosniadou et al., 2008).

Operational principles parallel to the early algebra movement. This redefinition of operations mirrors the early algebra movement's emphasis on helping students notice and generalize mathematical structures, such as the identity and inverse properties (Blanton et al., 2007; Cai & Knuth, 2005). The early algebra movement argues that arithmetic should be taught as a way for exploring these properties rather than just as a method for finding answers. Similarly, an early focus on operational integer principles moves students away from computing a result toward understanding an operation. Just as early algebraists teach that the equal sign represents a relationship of balance, these integer principles teach that operations like subtraction represent a specific relationship between two points on a directed number line (Gardiner et al., 2016). By rejecting a "whole numbers-then-integers" approach, we align with the early algebra rejection of "arithmetic-then-algebra," fostering a unified understanding of how operations behave across all numerical domains.

Operational principles' impact on algebraic thinking and learning. Establishing these operational principles early prevents the cognitive shock students often experience when they first encounter algebraic expressions containing negative terms. In algebra, students must frequently manipulate expressions where they must distinguish between the minus sign as an operator and as a sign of the number (Vlassis, 2004). If students have already mastered the principle of the additive inverse and the non-commutative nature of subtraction, they are far less likely to make common algebraic errors, such as incorrectly rearranging $a - b$ as $b - a$ (Gallardo & Rojano, 1994). Furthermore, understanding that adding numbers can result in numbers with a smaller value prepares students for the conceptual demands of combining like terms and solving linear equations. This foundation transforms algebra from a series of counter-intuitive rules into a logical extension of a robust, pre-existing numerical system.

Translation Principles (Principles J, K)

Principles J and K (see Table 1) redefine how numbers are conceptualized through addition and subtraction (often on a number line). Subtraction is not just "taking away" objects or "removing" something but can be seen as a translation or directed distance between two points (Marthe, 1979; Wessman-Enzinger, 2019). Addition no longer means "move right." It means translating a specific direction based on the sign of the integer (Bofferding & Farmer, 2019; Wessman-Enzinger & Tobias, 2020).

Translation principles parallel to the early algebra movement. This shift from a counting model to a translation model parallels the early algebra movement's focus on promoting mathematical coherence through visualization and spatial reasoning (e.g., Blanton et al., 2007;

Pinialla, 2024; Venkat et al., 2020). Early algebra research emphasizes that students must move beyond the computational view of the number line to a structural view, where they can notice patterns and generalize properties (Blanton et al., 2007). Just as early algebraists use number lines and balance scales to represent algebraic equivalence visually (Knuth et al., 2006), these translation principles use the number line as a geometric tool to represent signed numbers. By introducing these translations in the early grades, we provide students with a spatial "map" of mathematics that mirrors the early algebra goal of preparing students for algebraic thinking in developmentally appropriate, conceptually-oriented ways (Cai & Knuth, 2011).

Translation principles' impact on algebraic thinking and learning. Mastering translation principles provides the vital scaffolding needed for students to grasp the concept of vectors and slope in secondary mathematics. When students view numbers as translations rather than just static quantities, they are better prepared to understand algebraic functions as transformations. For example, the concept of a directed distance is essential when students begin graphing linear equations or calculating the difference between two variables in the form of $x_2 - x_1$. Without this translational foundation, students often struggle with the negative space of a coordinate plane (e.g., Kieran, 2007; Sarama & Clements, 2009; Skordoullis et al., 2009); however, students who have explored these principles in elementary school develop a robust mental model that accommodates both positive and negative movement (Wessman-Enzinger & Tobias, 2020). This spatial fluency reduces the cognitive load during algebra, allowing students to focus on solving complex equations rather than struggling to navigate the directionality of the number system.

Future Directions for Research and Teaching

The integer principles framework serves as a roadmap for reimagining for both instructional design and pedagogical research in the elementary grades. For teachers, this matters because it provides a language to move beyond procedural tips toward structural mathematics. For example, instead of teaching students that subtraction always results in a smaller value, educators can facilitate explorations of directed distance and linear value. For researchers, this framework opens new avenues to investigate the longitudinal impact of early integer exposure on students' later success in formal algebra. Future studies should examine how specific playful mediums, such as the design of integer-based board games or targeted number talks, can be used to explicitly highlight these principles. By focusing research on how children navigate the transition from counting objects to directed quantities, the field can continue to refine learning progressions that prevent the development of deep-seated misconceptions before they take root.

Concluding Remarks

Mathematical success should not be a matter of unlearning whole number rules that expire (Karp et al., 2014) but should be the result of a coherent, cumulative expansion of mathematical understanding. By introducing the integer principles in the early grades, we provide students with a robust numerical foundation that remains consistent as they progress into more abstract domains. This shift from a whole-number-only approach to a generalized system of signed numbers addresses the persistent "arithmetic-to-algebra" hurdle by fostering symbolic flexibility and spatial reasoning from the start. As the early algebra movement has successfully argued for the integration of algebraic thinking into elementary school, so too must we advocate for the integration of the full integer system. In doing so, we move toward a mathematics education that is conceptually oriented, developmentally appropriate, and (most importantly) mathematically coherent.

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WHEN RIGHT IS WRONG: MISAPPLIED PROCEDURAL PATTERNS THAT PRODUCE CORRECT ANSWERS IN FRACTION ARITHMETIC

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Proficiency with fractions is essential for learning more advanced mathematical skills, such as algebra (Booth & Newton, 2012). However, developing a solid understanding of fractions has been a challenge for many students, especially students with mathematics difficulties (Newton et al., 2014). The present study examined how students with MD sometimes apply incorrect fraction arithmetic procedures in ways that still produce correct answers.

The following research questions guided our study: 1) What misapplied procedural patterns result in correct answers when sixth-grade students with MD solve fraction addition, subtraction, and multiplication problems? 2) What percentages of students used each of the identified procedural patterns?

Procedural and conceptual knowledge exist along a continuum and are often intertwined, yet the two ends represent different forms of understanding (Rittle-Johnson et al., 2001). Students often experience difficulties in solving mathematics not just because of limited knowledge, but because of a disconnect between their conceptual understanding and procedural skills (Hiebert & Wearne, 2013).

The present data are from a larger study assessing the effectiveness of a fraction intervention for sixth-grade students ($N = 251$) with MD. The students performed at least two years below grade level in mathematics. We used a deductive coding approach to analyze students' procedures (Braun & Clarke, 2006). Students' assessment booklets were coded to determine the strategies they used in solving the fraction arithmetic problems. The assessment was researcher-designed with an internal reliability of $\alpha = .79$.

Overall, our study revealed that sixth-grade students with MD often followed a misapplied procedural pattern when solving fraction arithmetic problems, which sometimes produced the correct answers for some of the fraction arithmetic problems. For instance, cross-operating produced the correct answer for $1/2 + 1/2 = 3/3 = 1$. Furthermore, these students tend to misapply procedures without considering the sensibility of their answers. Approximately 64% of the students operated across both the numerators and denominators when solving fraction addition, subtraction, and multiplication problems with unlike denominators. Although operating across the numerators and denominators is the correct procedure for fraction multiplication, but the students extended this same procedure to fraction addition and subtraction problems.

These findings show that correct answers do not necessarily reflect correct application of standard fraction procedures. By examining students' written work, teachers can identify instances where students are overgeneralizing procedures and provide instruction that emphasizes conceptual understanding in addition to procedural accuracy to better support students with MD in making sense of fraction concepts.

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LLC VS NON-LLC STUDENTS' RESILIENCY, BELONGING, AND CONFIDENCE WHILE LEARNING AN UNDERGRADUATE MATHEMATICS COURSE

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Student success in foundational mathematics remains a persistent challenge, particularly for first year college students who are academically underprepared. Living and Learning Communities (LLCs) purposefully integrate academic coursework with residential life to promote engagement and holistic student development; however, despite documented benefits across disciplines (Spanierman et al., 2013), LLC's impact within foundational mathematics courses and first-year experience programs remains underexamined. This gap is significant: academically underprepared students enrolled in developmental or foundational mathematics courses are especially vulnerable to attrition, and understanding how residential-academic structures may mediate their confidence, resilience, and belonging carries direct implications for retention and institutional practice. This study addresses the research question: *How do the perceptions and lived experiences of LLC students differ from those of non-LLC students while learning a Mathematical Thinking course?*

A qualitative multiple-case study design (Yin, 2018) was used to explore the experiences of first-year LLC and non-LLC students taking Mathematical Thinking course in Fall 2025. Both sections were taught by the same instructor with no instructional intervention. Seventeen students (eight LLC, nine Non-LLC) completed three surveys—the Math Confidence (Hendy et al., 2014), the Academic Resilience ARS-30 (Cassidy, 2016), and Sense of Belonging Scale Hoffman et al.'s (2002) Sense of Belonging—and composite scores were used to identify interview participants across low, moderate, and high ranges. Twelve students participated in 30-minute semistructured interviews conducted in person or via Zoom. Data were analyzed using thematic analysis (Braun & Clarke, 2006; Merriam & Tisdell, 2016), incorporating open coding and cross-case analysis to identify convergent and divergent patterns between groups.

Cross-case analysis showed four thematic patterns. First, regarding *peer collaboration*, LLC students described peers as actively involved in learning—"Somebody else in the group would teach them, and I think that's the good part about it"—while non-LLC students emphasized independent study: "I mostly just do it on my own." Second, concerning *instructor support*, LLC participants used relational and affective language ("It gave me belief in myself"), whereas non-LLC participants referenced instructors primarily in the context of academic difficulty. Third, with respect to *classroom participation*, LLC students expressed comfort with intellectual risk ("I feel like I can be wrong, and it's okay"), while non-LLC students tied participation to certainty ("I would only participate if I knew what I was learning"). Fourth, in *responses to academic difficulty*, LLC students described structured review and peer-supported coping, while non-LLC students described individualized coping without collaborative support.

These findings suggest that LLC creates a supportive sociocultural environment that may enhance belonging, resilience, and mathematics confidence. Collaborative engagement and relational instructor support among LLC students align with sociocultural learning theory and belonging research (Walton & Cohen, 2007), while non-LLC environment reflects more individualized learning. These results extend the LLC literature into foundational mathematics education and offer implications for designing high-impact first-year programs that integrate residential and academic structures to support underprepared students' mathematical success.

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YOUTH INSIGHTS INTO NARRATIVES ABOUT THE USEFULNESS OF MATHEMATICS

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Middle School Education

Youth frequently question the usefulness of mathematics, and prior research has illustrated a range of benefits of perceiving mathematics as useful (e.g., Hulleman et al., 2010). And yet, the voices of early adolescents are typically not represented in this research despite middle school being a time during which attitudes towards mathematics often decline and youth frequently question the usefulness of mathematics (e.g., Dobie, 2019). Adults often offer perspectives that align with common narratives about mathematical utility—e.g., for getting a job or making our country more globally competitive (e.g., The President's Council of Advisors on Science and Technology [PCAST], 2010). Questions remain about youth perceptions of these narratives and potential disconnects with their own values (Dobie, 2019). Guided by expectancy-value theory (Eccles & Wigfield, 2002), research on adolescents' prosocial values (e.g., Eisenberg et al., 2009), and emphases on connectedness and harmonious relationships in Hispanic/Latine cultures (Ruiz, 2005), this study asks, "How do early adolescents perceive common individualistic and relational narratives about the usefulness of mathematics?"

Solo or pair interviews with 17 youth ages 12–15 (majority Hispanic/Latine) were conducted in after-school programs in the Mountain West. Interviews involved a card-sorting task in which youth were presented with narratives about the usefulness of mathematics and asked to explain whether and why they agreed or disagreed with each statement (see Table 1 for statements and frequencies of agreement). Inductive and deductive approaches (Fine, 2004) were used to develop a coding scheme to identify applications of mathematics emphasized by youth (e.g., money, jobs, school). Two coders coded all verbal responses and resolved any disagreements.

Table 1: Youth Agreement with Narratives About Mathematical Utility

	Agree	Unsure	Disagree
You will need math so you know how to manage your money.	12	0	0
Math will help you to be a better critical thinker.	11	0	1
Math is useful for helping people.	9	3	0
You should learn as much math as you can because you don't know what might be useful to you in the future.	9	2	1
You will need math for your job someday.	8	4	0
Math is important because we use it in everyday life.	8	4	0
Math is useful for making sense of the world.	6	4	2
You will need to be good at math to get into college.	3	8	1
You can use math to help make the world a better place.	2	9	1

Findings illustrated that youth (1) emphasized uses of math for money, jobs, and critical thinking; (2) conceptualized critical thinking as speed and smartness; (3) were skeptical about whether math is needed to get into college; (4) emphasized individual uses of math over relational uses; and (5) saw math's potential for helping individuals but not society at large, with

concerns about how math can be used "for bad" (e.g., making weapons). Future research should explore impacts of youth learning varied ways in which mathematics can be used to improve the world.

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ROOTED IN RELEVANCE: PLACE-BASED MATHEMATICS AND RURAL STUDENTS' PERCEPTIONS OF MATHEMATICAL VALUE

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Purpose

Rural students often experience school mathematics as abstract and disconnected from their lived worlds, positioning mathematics as a pathway away from community rather than a resource for participation within it (Corbett, 2011; Howley et al., 2011). Place-based mathematics education addresses this disconnect by grounding instruction in local contexts (Smith & Sobel, 2010). This study examines whether a place-based algebra modeling project using local agricultural data influences rural students' perceptions of mathematical relevance.

Theoretical Framework

Grounded in Self-Determination Theory, this study conceptualizes mathematical relevance through autonomy, relatedness, competence, and identity positioning (Ryan & Deci, 2020). Relevance was measured through four dimensions: everyday usefulness, community connection, engagement/sense-making, and future orientation (Hardré, 2011).

Method

This quantitative pre-post survey study involved 18 rural Algebra students in Montana. Participants completed a 19-item relevance survey before and after participating in a place-based modeling project. Students used CODAP to analyze local spring wheat field-treatment data.

Analysis

Mean survey scores ranged from 3.33 to 4.50, indicating generally positive perceptions of mathematical relevance. Students reported the strongest agreement on future-oriented items ($M = 4.50$) and lower agreement on conceptual relevance items ($M = 3.33$). Internal consistency reliability was high ($\alpha = .96$). Cluster analysis identified high-, moderate-, and low-relevance student profiles. ANOVA results showed significant differences across clusters, particularly for engagement and future relevance items, including Q12 ($F = 31.28$, $p < .001$) and Q15 ($F = 27.03$, $p < .001$). Post-survey responses suggested increased perceptions of community relevance and future usefulness.

Significance

This study contributes quantitative evidence on place-based mathematics education in rural secondary settings. Findings suggest contextualized modeling strengthens perceptions of mathematics as meaningful and community-connected. These findings inform equity-oriented rural mathematics pedagogy and future qualitative research.

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EXPLORATORY POST-HOC ANALYSIS ON PERCEIVED NON-LEARNING FROM METACOGNITION PRACTICES AMONGST COLLEGE ALGEBRA STUDENTS

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College Algebra enrolls many academically vulnerable students and has high failure and withdrawal rates (Flanders, 2017; Tunstall, 2018). Metacognitive strategies can support students' planning, monitoring, and evaluation of learning (Flavell, 1979; Lovett, 2011; Tanner, 2012; Verschaffel et al., 2017).

This study examined metacognition journals from 1118 students in a redesigned College Algebra course (Farthing et al., 2026). Journal responses were inductively coded through collaborative open coding and thematic analysis (Merriam & Tisdell, 2016; Saldaña, 2016; Glesne, 2018). Analyses first compared students who reported learning something with those who did not (n = 72), then compared students who viewed the journals as productive with 475 students who rated them unhelpful, unused, or reported learning nothing. Groups were compared in R using survey measures, grades, GPA, and demographic data.

Students who learned something from the journals (LS) rated good grades and content mastery more highly than those who did not learn anything (DNLA) from the journals (5.14% higher with p = .00141); DNLA students also showed a larger decline in performance goals (8.67% with p < .00001).

By then broadening the inclusion data, students were then classified as Metacognition Journals Productive Use (MPU) or Metacognition Journals Not Productive (MNP). MNP students earned higher average GPAs than MPU students (3.08 vs. 2.74) and outperformed them on 20 of 36 academic measures (Figure 1).

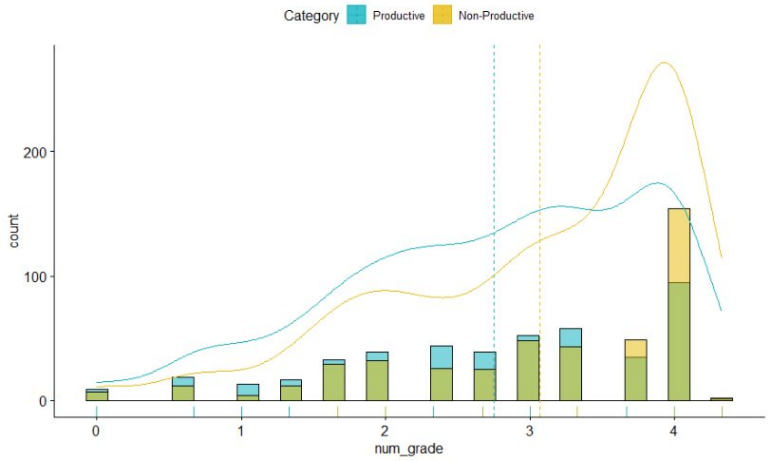


Figure 1: Distribution of Final Course GPA Between MPU and MNP

These exploratory findings suggest that stronger students may already possess effective learning strategies, reducing the journals' perceived usefulness for them. Further research should examine preparation, placement, and later mathematics outcomes.

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DOMAIN-GENERAL AND DOMAIN-SPECIFIC PREDICTORS OF EARLY MATHEMATICS ACHIEVEMENT: THE MODERATING ROLE OF GENDER

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Early mathematics achievement is linked to both domain-specific numerical competencies and domain-general cognitive abilities (Geary et al., 2017). Numerical cognition studies emphasize the role of early number sense and approximate number system (ANS) acuity as foundational to later math learning (Szkudlarek et al., 2017). Other studies suggest that more general cognitive abilities, such as nonverbal reasoning and cognitive control, may more strongly predict math achievement (review in Krajcsi et al., 2024). Gender differences in math have also been reported (Reilly et al., 2017). Based on the developmental framework that suggests early math performance emerges from the interaction of both core numerical systems and broader cognitive abilities (Geary, 2013), this study examines unique contributions of early general cognitive abilities (e.g., nonverbal intelligence and inhibitory control) and domain-specific numerical abilities (e.g., ANS performance) for children's math achievement, by gender.

A sample of 65 children (34 girls) aged 3–5 years participated longitudinally (predominantly White (77%), from middle-upper middle SES backgrounds). Math achievement, nonverbal intelligence, and inhibitory control abilities were measured using the TEMA-3 (Ginsburg & Baroody, 2003), PTONI (Ehrler & McGhee, 2008), and K-CPT (Conners, 2015). Then, they completed 3 computer tasks measuring their non-symbolic, symbolic, and arithmetic number sense. Hierarchical linear regression was used for analysis.

Boys (M = 106.80, SD = 15.36) demonstrated higher math achievement scores than girls (M = 101.96, SD = 11.95). Nonverbal reasoning robustly predicted math achievement across models (bs = 0.35 to 0.40, ps < .001). In contrast, the relation between domain-specific numerical abilities and math achievement varied by gender. A composite Balanced Integration Score (BIS; z-accuracy − z-response time) was associated with math achievement differently for girls and boys, such that BIS showed a stronger positive association with math achievement among girls than boys (interaction bs = –5.16 to –5.63, ps = .03 to .05). These effects remained significant after controlling for inhibitory control and age. Inhibitory control (reversely scored) was also independently associated with math achievement (bs = –3.14 to –3.37, ps = .035 to .053).

Thus, early math achievement may reflect both domain-general and domain-specific influences, but the contribution of domain-specific abilities may differ by gender before formal schooling. Understanding such pathways may inform developmental models and target support for children's early math learning. Future studies should include larger, more diverse samples and assess numerical abilities beyond the computer tasks used here.

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FROM PROCEDURE TO POSSIBILITIES: TRANSFORMING SECONDARY MATHEMATICS EXPERIENCES WITH UNSOLVED PROBLEMS

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Students often perceive mathematics as a rigid discipline focused solely on solving equations and finding numeric answers. This narrow view can discourage creativity and curiosity, leaving students disengaged and disconnected from the true discipline of mathematics (Latterell & Wilson, 2012). To combat this narrow view of mathematics, we created an elective course on exploring accessible but unsolved problems in mathematics (e.g., DeLong, 1971; Devlin, 2003; O'Dell, 2018; Freiman &Ts Abramovich, 2021) and implemented it with a group of middle and high school students. By treating students as active participants in the mathematical community rather than passive learners, we used a seminar-style approach for introducing approachable yet unresolved challenges (e.g., Halmos, 1980). Class sessions consisted of hands-on activities, collaborative discussions, and an exploration of unsolved problems. Thus, students were exposed to mathematics as a dynamic, creative, and sometimes open-ended field. The goal of this project is to see how exploring unsolved problems changes students' perceptions of mathematics.

A pre- and post-survey method captured students' views of mathematics before and after the course. We had 20 students in the class and 10 completed the pre-survey and post-survey. Participants included 7th–11th graders at a small private school in the Pacific Northwest. The course began in Fall 2025 and ended in Spring 2026. The survey consists of open-ended questions around what students' views of mathematics are and task value and critical thinking items from the Motivated Strategies for Learning Questionnaire ([MSLQ], Duncan & McKeachie, 2005). The MSLQ items were on a 1–7 Likert scale. This poster shares data on students' perceptions of mathematics, task value, and critical thinking.

The results on Task Value and Critical Thinking scores changed across the class averages. There was a small increase in students' Task Value from an average of 4.93 to 5 but Critical Thinking decreased from 4.46 to 4.12. To better understand these results, we also examined some of the open-ended questions. The pre-survey open-ended questions indicate participants primarily view mathematics as solving problems through use of formulas, numbers and calculations. Students held views of mathematics consistent with the literature of mathematics as solving problems. However, in the post-test, some students understanding of mathematics broadened, defining it as "Mathematics is working with numbers to discover new perspectives and ideas about them and using them to your advantage." Similarly, there were shifts in students answer to the question if mathematics problems always have an answer. Although only 4 of the students switched their yes or no answer portion to this question, the students that had the same yes or no, had changes in their justification. More examples and themes from students will be shared focusing on shifts in perspectives of mathematics as a closed versed open-ended field. The poster will also share ideas for revisions to the course for the next iteration of the course to help encourage more growth in task value and critical thinking.

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MATH CONNECTIONS THEATER: A STORY ABOUT THE HAPPY MARRIAGE BETWEEN MATHEMATICS AND THE PERFORMING ARTS

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Across cultures and historical periods, humans have relied on performance (e.g., storytelling, drama, ritual, and dance) to transmit knowledge across generations. Performance has become an educational medium, shaping how complex ideas are experienced, remembered, and rendered meaningful.

Mathematics, though often positioned as abstract and symbolic, is likewise a human embodied practice (Nathan, 2021). However, with rare exceptions (Calderazzo, 2025; Kazzazi, 2020), performative dimensions of mathematical activity are rarely explored in mathematics education. At the same time, prior literature indicates that systematic integration of theatrical strategies positively affects student learning (Inoa et al., 2014). Taken together, this suggests that mathematical theater may represent a uniquely promising and underexplored site for student engagement and meaning making.

Math Connections Theater (MCT) is a sequence of original comedy sketches that span topics across the undergraduate mathematics curriculum, from pre-calculus to topology and proofs, offering a fresh embodied approach to experiencing and communicating mathematics. Housed at a large public university in the Mid-Atlantic region, the production brings together undergraduate STEM majors and members of a local sketch comedy group in sustained interdisciplinary collaboration.

The beauty of MCT is multifaceted, and its broader impact is felt by everyone involved—from the audience, who are invited to engage with mathematics in a non-traditional and humorous way, to the non-math major comedians, who are challenged to think deeply about mathematical concepts as they bring them to life on stage. By its very nature, MCT is a quintessential embodiment of mathematics—performed, acted out, and physically explored on stage.

Drawing on data from two MCT performances (October 2025 and April 2026), this poster introduces an emerging research program organized around two guiding research questions:

RQ 1: What is the nature of the MCT phenomenon?

RQ 2: What kind of impact does it have on its audience, actors, and the production team?

Data sources include surveys, semi-structured interviews with audience members and the production team, and task-based interviews with actors. Analysis is ongoing. During the poster presentation, we will share preliminary findings and discuss how STEAM initiatives like this can inform collaborations between mathematics education and the performing arts.

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MAPPING AT THE MUSEUM: CHILDREN'S EXPERIENCES WITH MAP DRAWING AND VISUOSPATIAL REASONING

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There continues to be a need to support children's spatial conceptions (Clements et al., 2003). Visuospatial reasoning considers visual imagery and the dynamics of space in ways that are rooted in learners' lived experiences (Nuguid, 2026). Mathematics education literature has considered terms such as visuospatial reasoning (Goldenberg et al., 1998; Owens, 2015), visuospatial thinking (Hegarty, 2010), and visuospatial skills (Brock et al., 2018; Carlson et al., 2013). Commonalities among these three terms are spatial manipulation and visualization. This proposed study will focus on several aspects of visuospatial reasoning: (1) recognizing shapes, combinations, and their properties; (2) symmetry thinking; (3) comparing quantities and using ratios; (4) spatial capabilities; (5) locating; (6) intention and attention; and (7) space and place (Owens, 2020). Mathematics education scholars have used mapping tasks to study visuospatial reasoning (Nuguid et al., 2024; Richardson et al, 2024; Suazo-Flores et al., 2023). Mapping is a visuospatial task that will be the focus of this study. I argue that children's experience with mapping contextualizes their lived experience in a reframed structure that has mathematical elements.

The objective of this study is to investigate children's experiences during a series of mapping tasks, where the research question is: "How do children experience mapping?" Children will be tasked with drawing a series of maps using the following prompts: (1) Draw a map from home to school (Cohrssen & Pearn, 2021), (2) draw a map of your choice, and (3) draw a map of three of your favorite places at a children's museum (e.g., dinosaur dig site). A case study methodology (Merriam & Tisdell, 2015) will be used to investigate the cases of four to six children's experiences with mapping. The Mosaic Approach will guide data generation by positioning children as experts in sharing their experiences (Clark & Moss, 2017). Participants and I will use Owens' (2020) framework to identify aspects of visuospatial reasoning via procedural analysis (Saldaña, 2021). For example, I will ask participants to place Post-it® notes to identify shapes in their map, aligning with Owens' (2020) first aspect of visuospatial reasoning about recognizing shapes and their combinations. Cultural-historical activity theory (CHAT) (Engeström, 2000; Yamagata-Lynch, 2007) will serve as the primary analytical framework for examining how children's mapping experiences are mediated by several components that form mediation triangles (e.g., subject, object, tool, rules, community, division of labor, and outcome). Theories of experience will inform the interpretation of children's storytelling about their maps (Bruner, 1986; Dewey, 1938; Glasersfeld & Ackermann, 2011). For example, Bruner (1986) suggests that storytelling is one way that experiences become a reality. This proposed study aims to celebrate the various ways in which children know and do mathematics through mapping, echoing ethnomathematics (D'Ambrosio, 1985).

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TEACHING PRACTICE AND CLASSROOM ACTIVITY

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

BUILDING CLASSROOM CONNECTEDNESS THROUGH DISCOURSE

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This paper explores an interpersonal and relational aspect of mathematics classrooms—what I describe as classroom connectedness—using social network analysis. Classroom connectedness is the extent to which a classroom collectively values, encourages, and cares for individuals within the group. In connected classrooms, group members make contributions to shared goals, purposes, and activities while also depending on one another to achieve those goals (Goodenow, 1993; Strayhorn, 2019). Using networks, I map whole-class mathematics discourse in one middle-grades classroom as a case study to investigate how classroom connectedness is built through dialogue. Results show that certain types of interaction including caring interactions and diverse perspectives help build connectedness; that students play important roles developing connectedness; and that non-mathematical talk helps support mathematical talk.

Keywords: Classroom Discourse, Identity and Agency, Instructional Activities and Practices

> [This year I have learned] to try new things. Not just to think that one way would work all the time. Not to give up so easily. … When we share and other people show their strategies, I noticed that different ways would work. Everyone's not judging everybody else, and they accept the ways you're thinking. And help you think about other things. … Like I said before, I used to give up a lot. I was insecure I wouldn't get it right. And I have to admit I was a crybaby. I got frustrated a lot. (Aujane, Grade 5)
>
> A successful math student in our class listens to everyone. If you hear someone say something that might not make sense or might be wrong, like you don't understand how they got it, you would ask, "What do you mean by that?" And then sometimes you end up correcting them. But sometimes you realize that you were wrong. … You help other people, but you also allow yourself to be helped by other people. (Erin, Grade 5)

Erin and Aujane were students in the same fifth-grade classroom. What was it about their classroom that enabled them to feel this way? What supported this change in Aujane where she used to cry and give up but now sees herself as capable and persistent? In this paper, I focus on a feature of mathematics classroom discourse—classroom connectedness—that I believe is instrumental to productive, mathematics interaction but which may be, at times, peripheral to the mathematics under consideration. As an example, consider the following excerpt from Erin and Aujane's classroom.

> Teacher: This morning I want you to think about your 100% smarts. Turn and look at your plate right now. [On the wall are paper plates students partitioned into different smarts based on their relative strengths or smarts.] They all look different. Why?
> Macy: They all look different because we all have different sorts of smarts. Like we could be all big in Body smarts, but we're not all the same percentage big. …
> Bella: They're all different because we're all different. So if we mix them all up we can like, people can help each other. Like if I was struggling with math and Macy had like really high math smarts, I could ask her for some help. And she could build up mine a little bit more. …

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Paris: If we're all the same, then nobody can help us with our struggles.
Aujane: I wanted to add to Bella's because I think she was like saying that if you have a smart and then you help somebody to make your smart grow with their smart. Then they help you in their biggest strength and you grow in their biggest strength if that wasn't your biggest one.
Teacher: Right. We've got a few more people who would like to add on.

In previous work, I have focused more on the mathematical aspects of classroom discourse and, as a result, excluded these kinds of interactions from analysis because they did not clearly address mathematics content. But I began to wonder if part of the reason why Erin and Aujane's classroom was able to have productive mathematics conversations was precisely because they also had conversations about their "100% smarts." In this study, I explore an interpersonal and relational aspect of mathematics classrooms—what I call classroom connectedness—using social network analysis. Classroom connectedness is the extent to which a classroom collectively values, encourages, and cares for individuals within the group. In connected classrooms, group members make contributions to shared goals, purposes, and activities while also depending on one another to achieve those goals (Goodenow, 1993; Strayhorn, 2019). To understand how mathematics classroom connectedness is built through dialogue, I mapped whole-class mathematics conversations using networks to visualize who contributes and in what ways.

Theoretical Perspectives & Related Research

Psychologists argue that connection, or belonging, is a core human need (Maslow, 1943; Ryan & Deci, 2017), and sociolinguists like Halliday (1978) claim that personhood is predicated on membership in a social group, with language being the primary means through which this process occurs. Whether using the term caring (Hackenberg, 2010; Noddings, 1984), sense of belonging (Goodenow, 1993; Strayhorn, 2019), community (Boyer, 1990), or connectedness (Maloney & Matthews, 2020), scholars across disciplines and using various methods of inquiry recognize the importance of forming strong connections to peers and social groups. In fact, Noddings (1984) argues that we are irrevocably linked to others and "this fundamental relatedness is at the very heart of our being" (p. 47). Moreover, research has shown that sense of belonging or connectedness is positively related to other desirable outcomes like retention and persistence in course taking (Good, 2012; Strayhorn, 2019); motivation and engagement (Goodenow, 1993; Middleton et al., 2017; Strayhorn, 2019); and academic achievement (Battey et al., 2016; Good et al., 2012; Noddings, 1984; Strayhorn, 2019; Wilson, 2022).

Across the literature, some frame connectedness as an individual trait of students measured through Likert-scale surveys (Good et al., 2012; Maloney & Matthews, 2020; Strayhorn, 2019), whereas others frame connectedness as a dyadic relationship between two people—typically a teacher and student (Hackenberg, 2010; Noddings, 1984), and still others frame connectedness as a component of teachers' instructional practice or particular teaching moves (e.g., Battey et al., 2016; Matthews, 2020; Wilson, 2022). The relational features and instructional practices focus primarily on *teachers* and how they support classroom connectedness. I wish to extend this work by incorporating what *students* say, do, and contribute to classroom connectedness. Thus, I conceptualize classroom connectedness as a social and malleable, group-level construct comprised of moment-by-moment interactions, aggregated over time and individuals, and operationalized using networks. Discursive interactions, accumulated across classroom members and over time, constitute the larger community to which students belong. However, some discursive exchanges are more important than others in creating classroom connectedness

because they have the opportunity to communicate to students that others care for them, want them to be successful, value their thinking, and will act to help further their learning goals (Noddings, 1984). Table 1 synthesizes the extant literature into five components of mathematics classroom connectedness that research suggests are important to building and promoting connectedness: contributing and being included, caring and feeling valued, emotional support, explicit norms and expectations, and diversity of perspectives. I use the components in Table 1 as both a conceptual and analytic tool for the research presented here. The research questions I sought to answer are: (1) How is classroom connectedness built through mathematics discourse? and (2) How can social network analysis be applied to mathematics classroom discourse to model mathematical and social relationships to measure classroom connectedness?

Table 1: Components of Mathematics Classroom Connectedness

Component	Description	Supporting Literature
Contributing and Being Included	To what extent are classroom members noticed, included, and contributing to the social group/classroom?	Strayhorn, 2019; Noddings, 1984; Goodenow, 1993; Wilson, 2022; Wilson et al, 2019
Caring and Feeling Valued	To what extent are classroom members cared for and valued? To what extent do community members seek to understand one another's ideas and support each other's (learning) goals?	Goodenow, 1993; Hackenberg, 2010; Noddings, 1984; Matthews, 2020; Strayhorn, 2019
Emotional Support	To what extent are classroom members' emotions and affect recognized, affirmed, and supported in classroom dialogue? To what extent is one known outside of math?	Hackenberg, 2010; Jaber & Hammer, 2016; Maloney & Matthews, 2020; Noddings, 1984; Rolon-Dow, 2005; Wilson, 2022; Wilson et al, 2019
Explicit Classroom Norms and Expectations	To what extent does the classroom explicitly state normative values, beliefs, and expectations?	Battey et al., 2016; Cobb et al., 2009; Kazemi & Stipek, 2001; Jaber & Hammer, 2016; Ware, 2006; Wilson, 2022; Wilson et al., 2019
Diversity of Perspectives	To what extent are multiple, different ideas and perspectives encouraged?	Strayhorn 2008, 2019

Methods

Participants and Data

To answer the research questions, I used a case study approach, investigating one fifth-grade classroom across a single academic year. I purposefully selected the case to maximize the likelihood I collected data in a classroom with high levels of classroom connectedness. Thus, I selected an experienced teacher widely recognized by her colleagues, administrators, and other researchers as an excellent teacher (across all subject areas including mathematics). Teacher K had over 30 years of teaching experience, participated in many research studies (including the original Cognitively Guided Instruction study), and routinely facilitated mathematics professional development. Teacher K's class was a self-contained, inclusion classroom in a diverse school located in the northern US (40% economically disadvantaged, 26% English

learners, 38% White, 10% Hispanic, 20% Black). The participants were Teacher K and her 20 fifth-grade students. Data was comprised of video recordings and transcripts of three algebra and nine fractions lessons, totaling 17 hours of instruction across the academic year. I chose the topics of algebra and fractions because of their significance in secondary curricula and overall academic success. Four lessons were filmed in early November, four in mid-March, and the final four in May. As secondary data sources, I conducted interviews with Teacher K and a subset of students to explore what math class was like from their perspective.

Methodological Approach: Operationalizing Connectedness With Networks

I used networks, or graphs, to document and visualize who participated in whole-class mathematics instruction and in what ways. Social network analysis considers both group and individual characteristics when explaining activity or behavior but its underlying assumption is that it is the relationships that have a profound influence on social life (Borgatti et al., 2018; Carolan, 2014). Thus, its emphasis is on social ties and emergent, coordinated activity reflected in network structure and patterns. To represent classroom discourse as a network, a vertex was assigned to every classroom member (including the teacher). I created a Class vertex for instances when the class was addressed collectively and a Student node for unattributable student utterances. For each whole-class turn of talk, an edge and its directionality were assigned on the basis of who (source vertex) addressed whom (target vertex). Additionally, I also created edges when a speaker explicitly addressed a classmate's idea or contribution in their turn of talk, but was not talking to that classmate. (See Example 2 in Table 2 below.) I included these situations to capture instances in which students have made mathematical contributions that extend beyond the specific utterance. Their idea adds to the collective understanding as others refer back to that idea. These situations reflect that someone has made a significant contribution to the mathematical goals of the classroom. By incorporating the previous ideas referenced as an edge, this coding decision helps measure whose ideas are taken up and which students may play important connective roles in linking, clarifying, or challenging their peers' ideas, which are critical components of connectedness. Because I was unable to collect data from all small groups during data collection, I could not represent all of the discourse during a given lesson using networks. Thus, the networks shared in this paper are restricted to whole-class interaction during dedicated mathematics instruction time, but include both mathematical and non-mathematical turns of talk (i.e., social, behavioral, managerial) that occurred during the math lesson.

Table 2: Operationalizing Edges as the Unit of Analysis

Ex	Turn of Talk	Source	Target	Rationale for Edge
1	*Aarya*: Actually, our strategies are the same Erin. We just wrote it down different.	Aarya	Erin	Aarya addresses Erin
2	*Teacher*: Aujane, what do you think she [Bella] is missing?	Teacher	Aujane	Teacher addresses Aujane
		Teacher	Bella	Bella's expression is focus of teacher request

For each turn, a directed edge is identified by linking a source node to its corresponding target node(s). The creation/inclusion of an edge is one way I am capturing the extent to which classroom members are noticed, included in, and contributing to the classroom dialogue. I used this analytic process to create networks for each lesson. Figure 1 (see "Results") is a network

created from one 93-minute lesson in Classroom K (the second of 12 total lessons). In this lesson there were 295 edges across 20 vertices in 46 minutes of whole-class math instruction. Note that multiple edges between a pair of vertices indicates multiple, distinct directed turns of talk between two classroom members during the lesson. The size of the vertex was scaled to reflect its degree, or the total number of incoming and outgoing edges. The vertices' degrees capture members' involvement in the lesson; thus, the larger the vertex, the more that person has contributed to the lesson. However, as described in the literature review, I identified specific features of interaction related to connectedness. Because some turns of talk appeared to be more critical to creating classroom connectedness, I further classified each edge based on the nature of the utterance. For each edge, I determined if the turn of talk could be categorized into one of the connectedness components listed in Table 3. These categories are not mutually exclusive; meaning that a single turn of talk can address more than one connectedness component.

Table 3: Analytic Framework for Mathematics Classroom Connectedness

Component & Operationalization	Examples
Contributing: One is included when they make a contribution, have a contribution directed to them, and/or their ideas contribute to the math goals of the class.	This component is captured with the creation of an edge.
Caring: Identify utterances in which (a) one seeks to understand another (Noddings's [1984] engrossment) or there is evidence one does/has understood another (e.g., revoice), and (b) support another's goal (Noddings's [1984] motivational displacement)	*Grace*: I knew that there was ten pies, so I wrote a little ten here to remind myself. And then so I kind of thought of it like Charles but I just didn't draw my picture, because I had it in my head. I just drew an equation but Charles had the picture. … [evidence one has understood another]
	Teacher K: Can I put Charles's work up here [on doc camera] so you can say this is the parts, this is the ten pies? [supporting another's (Grace) goal]
Emotional Support: Statements acknowledging, recognizing, and affirming community members' emotions and lives outside of math including: Verbal/nonverbal praise or affirmations; statements of support; statements of uncertainty; Sharing out-of-school experiences, hobbies, interests, or family.	*Teacher K*: I realized last night that you guys are like, why is Ms. K even thinking about weight measures? Because most of the recipes we use are in cup measures. … This is my grandmother's cookbook from 1913. And this is the problem. Look at my recipe. How much flour does it use? Three-fourths pound. But I cook in cups. How would I ever know what ¾ pounds is? Do you notice that I have to do some mathematics? So I thought I would give you the reason why. Maybe it builds a little context for you.
Explicit Norms & Expectations: Explicit statements reflecting beliefs about math, math ability/success, expectations for engagement and behavior, and	*Teacher K*: I want you to think about what math is to you. What is math and what is it comprised of and how can you be stronger in one area of math than another area of math? Think about things that are easy for you, challenging for you, or frustrating. … And I wonder, do you think it's true we'll find out that everyone is math smart?

acknowledging emotions and feelings with respect to math.	
Diversity of Perspectives: The extent to which multiple, different ideas/perspectives are encouraged and observed as indicated by disagreements, requests for an alternative idea or strategy, and meta-level comments about multiple solutions or differences.	*Morgan*: But we're all unique and our families are all unique and that's what makes us who we are and are special. [meta-level comment]
	Teacher K: Anybody else have a way that's different than, than Drew's?

Results

To visualize connectedness, I created networks to document who participated in whole-class math interaction. The network in Figure 1 (and all networks in subsequent figures) is a representation of one lesson in Classroom K. The network shows that the teacher receives and directs more talk than anyone else (as seen in the relative size of the vertices which are scaled based on degree). However, some students make significant contributions to the class (Charles, Bella, Aujane, Elias and Elijah). Visual features of the network in Figure 1 such the number of isolated vertices (there is one, AJ), the number of student-student edges, and the size of student vertices suggest Classroom K is a connected classroom. Network measures such as density—the proportion of existing edges out of the maximum number of possible edges—confirm this. (Network density for the graph in Figure 1 is .776, meaning 78% of all possible edges in a complete graph exist. Density accounts for the presence of an edge between pairs of vertices, not the number of edges between pairs.) Visualizing mathematics classroom discourse using graphs reveals features of connectedness that might not otherwise be noticed (e.g., isolated vertices, timing of student contributions in the lesson) and provides new ways of visualizing which community members contribute (or not), how much they contribute, the extent to which their contributions are taken up, and who directs the conversation.

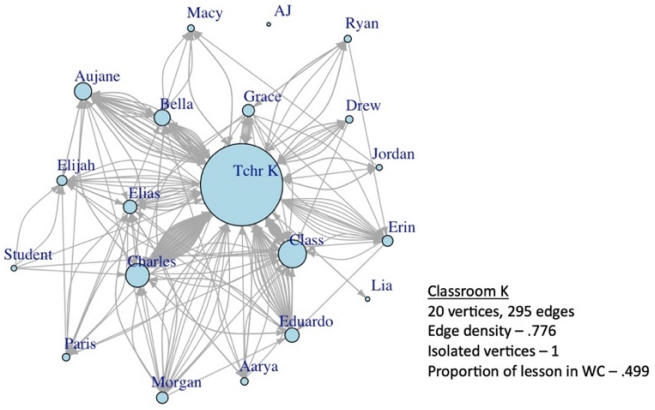


Figure 1: Network For All Turns of Talk in Classroom K

Although participating in the classroom and making discursive contributions is a necessary aspect of classroom connectedness, it seems that certain types of interaction, like Caring interactions for example, may be particularly important for creating community. Caring

interactions focus on mathematics but do so in a way that communicates to students their class values them, is invested in their success, and will act to help further their learning. Moreover, caring interactions are one way connectedness is built over time. Consider the graph of Classroom K in Figure 2 which reflects only edges coded as a Caring interaction. In this graph, multiple directed edges are combined into a single edge whose thickness reflects the number of directed turns between a given pair, and the size of the vertex reflects the total degree. Perhaps one of the more striking features of this graph is the number of Caring interactions (90 out of 295 total edges, or 31%, are caring interactions) and the large amount of student-student caring.

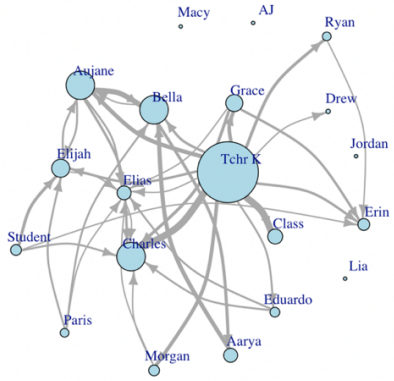


Figure 2: Network for Caring Turns of Talk in Classroom K

Further, consider who receives and who initiates Caring interactions which is shown in the networks in Figure 3. The vertex size for the graph on the left reflects in-degree (the number of caring comments directed toward, or received, by a community member) and the vertex size for the graph on the right reflects out-degree (the number of caring comments a member directs to another). Students whose vertices are relatively large in the left graph (Charles, Bella, Aujane, Elijah, Erin, Elias) may feel particularly valued and cared for, whereas students with larger vertices in the right graph (Aujane, Grace, Aarya, Paris) may play an especially important role in helping others feel cared for. In fact, student initiations of caring interactions, as reflected in the right graph, are an important indicator of classroom connectedness because they reflect the adoption of shared values, expectations, and norms. These students believe they learn better together which is seen in Erin and Aujane's quotes at the beginning of the paper.

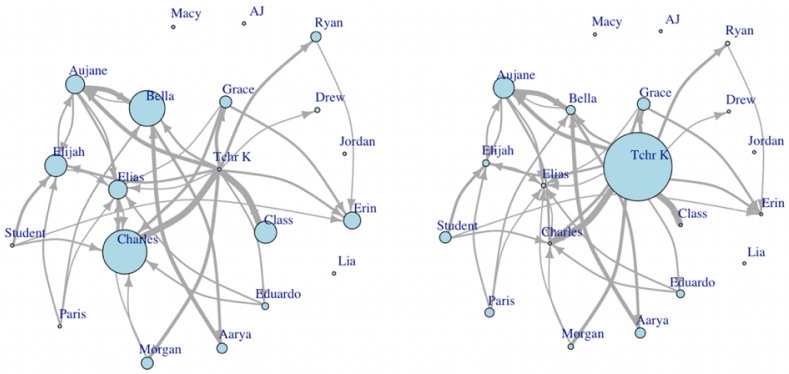


Figure 3: Classroom K Caring Networks: Vertex Scaled by In-Degree and Out-Degree

Another result from this analysis is that non-mathematical talk helps support the quality of the mathematical talk. In Figure 4, edge color is used to represent the different components of connected classrooms in the analytic framework. Though many of these interactions are not explicitly mathematical (e.g., purple edges for emotional support or orange edges for making norms explicit), this kind of interaction seems to provide the foundation upon which students feel comfortable taking intellectual risks, sharing their ideas and mistakes, monitoring their own epistemic affect (Jaber & Hammer, 2016), and responding to peers' ideas. And finally, a defining feature of each network shared is the prevalence of student-student edges. Classroom connectedness was not the sole responsibility of the teacher; students actively initiated Caring interactions (38 of 90 total instances of caring were student initiated), provided emotional support (6 of 9 total instances of emotional support were student generated), and encouraged diverse perspectives (18 of 22 instances were student initiated). Thus, students play an important role in mathematics classroom connectedness.

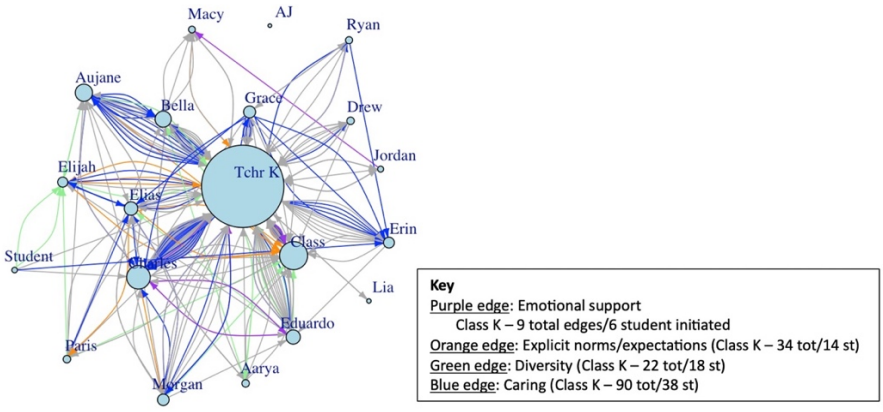


Figure 4: Classroom K Network: All Turns of Talk and Connectedness Components

Conclusion & Discussion

This study contributes to the field by providing a new methodological approach to investigate classroom connectedness and conceptualizing connectedness as a malleable, social construct emerging through moment-to-moment classroom discourse. Using networks and the components described in our conceptual and analytic frameworks, I mapped whole-class mathematics discourse in one middle-grades classroom to investigate how classroom connectedness is built through dialogue. Results of our network analyses show: (1) networks are a promising methodological tool to represent social and academic relationships; (2) certain types of interaction including the presence and prevalence of turns of talk reflecting caring interactions are particularly important to building connectedness; (3) students play critical roles in developing connectedness by taking responsibility for their own and others' learning when they initiate interaction with peers; and (4) non-mathematical talk helps support mathematical talk. In this study, I found that although some components of classroom connectedness may not focus on content, they provide the foundation on which students feel comfortable taking risks, sharing ideas, and responding to their peers' ideas. If, in Nell Noddings words, "the student is infinitely more important than the subject matter," then perhaps spending time to create a connected classroom rather than covering one more section is time well spent (1984, p. 176).

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EXAMINING SYMBOLOGY IN INTRODUCTORY STATISTICS THROUGH A UNIVERSAL DESIGN FOR LEARNING (UDL) LENS

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Mathematical symbology can pose barriers in increasingly diverse introductory statistics classrooms. This qualitative study examines instructional practices that support students' developing symbol sense through a Universal Design for Learning (UDL) lens, focusing on UDL Considerations 2.1 and 2.2. Data include interviews with 39 students, surveys from 86 students, and video-stimulated recall interviews with five instructors. Findings indicate that students view symbols as essential yet initially intimidating, and that symbol sense develops through intentional instructional mediation. Effective practices included explicit attention to symbol morphology, translation between symbols and natural language, structured reference tools, revisiting symbols, and highlighting relationships among symbols. These strategies appeared to support students' developing sense of symbols and promoted more equitable access to statistical meaning-making.

Keywords: Communication, Instructional Activities and Practices, Undergraduate Education

In recent decades, there has been increased attention to conditions that support students' access to postsecondary Science, Technology, Engineering, and Mathematics (STEM) courses and programs. As the world becomes increasingly shaped by data, quantitative reasoning and statistical literacy have been a keen area of focus for postsecondary educators and researchers. Effectively communicating data findings clearly, concisely, and efficiently requires the use of symbols e.g., (Bardini & Pierce, 2015). However, research on instructional strategies that develop students' symbol sense is nascent (e.g., Begg & Pierce, 2020; Esuong et al., 2022; Mutodi & Mosimege, 2021; Oyoo, 2022).

Today's postsecondary instructor is confronted with increasing diversity in the mathematical experiences that their students have had in secondary schools (Han et al., 2023), with at least eight distinct course-taking typologies. Whereas there had traditionally been a standard pathway of algebra I, geometry, algebra II, advanced mathematics, now only 14% of students complete that sequence, calling into question whether there is a shared foundational understanding amongst incoming postsecondary mathematics students. Additionally, students with diagnosed learning disabilities continue to grow as a segment of the overall higher education student population (National Center for Education Statistics, 2019), suggesting that the postsecondary mathematics classroom is rich in diversity.

Universal Design for Learning (UDL) is an empirically-supported framework for designing instruction to reduce barriers and increase success for all learners with a focus on those who have historically been excluded from learning opportunities (CAST, 2024). Rooted in design principles, UDL empowers educators to identify and mitigate barriers to students' learning that exist in the educational environment. UDL outlines general recommendations for multiple means of student *engagement*, content *representation*, and *action and expression*. Descriptive guidelines for content representation include several that address symbology: 2.1 Clarify vocabulary, symbols, and language structures; and 2.2 Support decoding of text, mathematical notation, and symbols. Providing specific and detailed descriptions of context-specific applications of UDL may help instructors enact the design principles of this framework in their

own practices. In this qualitative study, interviews with students and instructors lend insight into the instructional practices in introductory statistics classes that support students' developing symbol sense. This study asks: What instructional practices do introductory statistics instructors employ to support students' developing symbol sense, and how do students experience those practices?

Literature Review

Mathematical symbols are central to precise and efficient communication of concepts, relationships, and ideas in quantitative disciplines (e.g., mathematics, statistics, physics, natural sciences). Yet, for many students, the price of this conciseness is that ideas are presented abstractly, which presents challenges in the teaching and learning environment. Instructors are reminded that "it is too easy to assume that our students are thinking what we are thinking as we use and read symbolic expressions" (Begg & Pierce, 2020, p. 7). Explicit attention to symbology in the mathematics and statistics classroom is needed to ensure shared understanding and to build students' symbol sense as they become more confident communicators of quantitative ideas.

Universal Design for Learning calls instructors' attention to differences that may exist in students' ability to access information as it is presented with specialized vocabulary and symbols. Consideration 2.1 reminds instructors that "the semantic elements through which information is presented—the words, symbols, numbers, icons, and language structures—are differentially accessible to learners with varying backgrounds, languages, and lexical knowledge." Instructors must confront any taken-for-granted assumptions that students share the same understanding gleaned from symbols as they do.

Much of the existing research on mathematics symbology is grounded in the secondary school context. These findings highlight the density of symbols in the fields of mathematics and statistics while presenting several instructional suggestions. There is a need for instructors to recognize the volume of symbols that students encounter (e.g., Oyoo, 2022), that they exist with different alphabets, cases, and fonts (Begg & Pierce, 2020), and that some are polysemantic or are homonyms (Mutodi & Mosimege, 2021). Students' prior experience with individual symbols interacts with how they make sense of them in novel situations (Chin, 2013). Students' prior experience with individual symbols may be productive when a new meaning or context extends what is already understood; unproductive, such as when a symbol homonym appears in a new context with a distinct meaning; or nonexistent, as in the case when students are exposed to a symbol for the first time. The use of symbols increases students' cognitive load, especially as they are becoming more fluent with them but over time can help students accommodate the cognitive demands of mathematical thought (Mutodi & Mosimege, 2021; Oyoo, 2022; Sfard, 2008). Students' affective responses to symbols (including anxiety, frustration, and intimidation) represent an additional dimension of this challenge. Prior research on mathematics anxiety suggests that negative emotional reactions to symbolic notation can compound cognitive difficulties, particularly for students who have had limited prior exposure to formal mathematical symbols (Ashcraft & Krause, 2007).

There seems to be consensus in the research literature that teacher mediation of symbols is necessary for students to develop their symbol sense. Arcavi (1994) coined the notion of symbol sense to describe the intuitive and flexible feel for symbols and symbolic forms, including an understanding of when and how to use them meaningfully. Esuong and colleagues (2022) advocate for word-to-word translation and interpretation, finding that a specific mathematics symbol instruction treatment promoted students' algebraic word problem solving ability. Other researchers agree that alerting students to symbols and their meaning supported

students' symbol sense development (e.g., Mutodi & Mosimege, 2021; Oyoo, 2022). As Sfard (2008) describes, there is a "process of proper naming (baptizing)" (p. 169).

Calls for increased attention to symbol instruction exist throughout the research literature. While Mutodi and Mosimege (2021) lament that "teachers do not reflect and devise teaching approaches that promote conceptual understanding and representational efficiencies," it is not clear that this is a general truth (p. 1195). Examining instructional practices that students and instructors identify will help to contextualize the ways that symbol sense development engages students in "action [...] on their part to develop a deep level of mathematical understanding" (Mutodi & Mosimege, 2021, p. 1185).

Methodology

This inquiry is one component of a larger project that is supported by a National Science Foundation Improving Undergraduate STEM Education grant (NSF IUSE #2314358). Quantitative analyses identified five instructors of introductory statistics at Curry College whose students have demonstrated improvements in pre-/post-test scores on statistical concepts (Callis et al., 2026). All five instructors also reported familiarity with UDL and had participated in professional development related to inclusive instructional design, making them well positioned to reflect on how their practices addressed learner variability. Students (n = 39) of these five instructors participated in semi-structured interviews to share their perceptions of the instructional practices that their instructors used, and an instructional practices survey was administered to 165 students and returned by 86. Additionally, these five instructors participated in video-stimulated recall interviews (n = 8) in which a video-recorded classroom session of the interviewee was viewed alongside a member of the research team, with questions posed about instructional decisions evidenced in the classroom episode and interviewees' rationale behind them. Clips for the video-stimulated recall interviews were selected by the research team prior to each interview to highlight episodes representing a range of instructional decisions, including moments where students appeared to struggle with or engage symbology. Interview questions invited instructors to describe their reasoning (e.g., "What were you thinking when you introduced that notation?" and "What were you hoping students would take away from this?"). Ethical review and clearance processes for research involving human subjects (instructors and students) were conducted by the Curry College Institutional Review Board. All names in this article are pseudonyms.

The interview transcripts and open-ended survey responses were read and reread by several members of the research team and deductively coded to identify alignment to specific UDL considerations. Axial coding followed, in which the utterances clustered by UDL considerations were explored for patterns, connections, and relationships (Corbin & Strauss, 2015). Initial codes were derived from the UDL Guidelines themselves (e.g., "clarifying vocabulary," "supporting decoding of notation"), and were applied independently by two members of the research team before being reconciled through discussion. As analysis progressed, codes were refined to capture more nuanced instructional moves (e.g., "morphology attention," "symbol translation," "structural relationships"), and memos were written to document analytic decisions and emerging interpretations.

Results

Analyzing the data revealed insight into the value students and instructors attach to symbols in the introductory statistics classroom, and what instructional practices seem to be supportive of students' developing symbol sense. These themes are detailed below.

What Value Do Students and Instructors Assign to Symbols?

Students appear to recognize the importance of symbology to communicate ideas in the introductory statistics class. They spoke of statistics vocabulary and symbols as "important" and requiring effort to learn. Students characterized the symbols as "fancy," "weird," and "different from regular math," which indicates that they may have limited prior experience with the symbols and the way they were used.

As alluded to in the research base, the context and syntax in which symbols were used interacted with the meaning and appropriate use of specific symbols. Students seemed to notice this, recognizing that their efforts to understand the meaning of symbols could be stymied by underappreciating the context. As Avery shared, "the way the professor explains them might be different from the way, if you Google it, it's explained … I learned that the hard way. But once I realized that was important, I was like, 'OK, I need to actually go to class.' and then it was fine." Students seem to be aware that symbols can carry multiple meanings, and that making sense of them depends on context.

While students seem to appreciate the value of symbols as tools to communicate ideas, they are daunted by them. A survey respondent shared, "sometimes I get lost in the symbols and long equations," and several interviewees expressed their discomfort with encountering symbols in their classwork and homework. When asked to give advice to future statistics students, Madison said, "Don't be scared by equations. Like, try to break everything down and definitely go to tutoring." Ryan, a student in a different class, seemed to agree when he said, "It's intimidating. You need to look at it and not get overwhelmed by the equations because there's Greek letters, there's symbols, but it's doable." Students seemed to have an initial fearful reaction to symbology that they felt they could overcome. As Ryan added, "it's like a puzzle. You have to start it and then once you get it done, it's not that difficult." Students find symbols to be challenging but important, and demystifying them is achievable.

Instructors have a similar regard for the importance of symbols. As Dr. Landry shared, "I think I really understand why we as mathematicians use symbols. I feel like I am saying a lot of words and I'm catching myself not being perfectly precise … and you can see patterns with symbols that you can't see with words." She values the use of symbols to communicate a thought precisely and efficiently, but recognizes that students are still in the process of learning all that symbols encode. She wistfully remarks, "maybe over time I'll just be able to write that on the board and they'll understand it!" All of the instructors referred to how they attended to symbology in the lesson that was the subject of the video-stimulated recall interview.

Developing students' symbol sense was something these instructors viewed as an ongoing process. Mr. Jackson commented about how he makes symbols and their meaning explicit "regularly" and that "it works to do a lot of reinforcement." Dr. Landry explained that she "feels like we have to spiral the vocabulary and the symbols," revisiting symbols and their meanings again and again over the course of the semester. These instructors seemed to view symbology as an instructional focus that requires ongoing attention.

Instructional Strategies to Support Students' Development of Symbol Sense

Developing students' symbol sense was something instructors and students recognized required conscious attention. For Dr. Scott, symbology was one of the two instructional objectives for the class that was observed for the video-stimulated recall interview. He explained that in a previous class, students had not "used symbology or critically talked about" the structure of hypotheses for a test of a single mean, and that it was a focus for that class session. For Ms. Thompson's class, translating statistics concepts into statements using symbols

was likewise an instructional focus, with students building from statements written in words and translating them into concise symbolic expressions of hypotheses.

The conscious attention that symbol instruction required was multi-dimensional. Symbols were introduced by attending to their morphology and meaning. Dr. Landry's classroom episode included a demonstration where she wrote the symbol μ (mu) on the board and explained that it may look different from how μ (mu) is presented in the textbook. In her interview, she explained that one of her students told her he did not recognize that what she wrote on the board was the symbol μ (mu), "Even when I write it carefully, he said it looks like an M." It made her reflect on an article she had read about a student with dyslexia experiencing mathematics and building a personal lexicon for symbols (Lewis & Lynn, 2018), and she concluded, "You keep making it explicit. Every single lesson."

While morphology is important for students to recognize symbols in the various styles they can be written, conscious effort is also needed for students to glean meaning from those symbols. Dr. Scott's student Madison appreciated his word-for-word translation symbols:

> He has a notes app that he screenshares to the board. I like being able to see him write out equations and explain them and write them in plain English so I can understand it. It has been really helpful because at times I won't understand the fancy stats words and seeing it in plain English has really helped me process it. Like, "OK, that's what I need to think."

Later in the interview, Madison shared her process for translating a novel equation into plain English using Google image search and Photomath, mimicking the word-by-word process that she experienced in Dr. Scott's class.

Students appreciated collecting symbols and their meanings on a reference list. In some classes this was a poster displayed on the wall of the classroom and in others students kept personal glossaries of key vocabulary words and symbols. Avery appreciated having a dedicated place to record this information, "instead of me scrambling through my notes." Her instructor, Dr. Young, used a Frayer model graphic organizer for students to record the symbol, its meaning, examples, and nonexamples (Frayer et al., 1969).

An interesting episode in Dr. Landry's video-recorded lesson involved her developing a matrix to notice patterns and features about the structure of how symbols are assigned to ideas in statistics. She drew a 2-by-2 table on the whiteboard to demonstrate that population parameters are encoded with Greek letters while sample statistics use Roman letters with an accent. The symbol for the mean of quantitative data uses a variation of the letter "m" while the symbol for proportions for categorical data use variations of the letter "p." In the video-stimulated recall interview, Dr. Landry explained her rationale for attending to the structure of symbols:

> You want them to see similarities, right? And parallel constructs. So, I was trying to show them, look, these things have this in common. And they are about the same kind of variable. And we're just moving to new kind of variable that we're talking about the same kinds of ideas. Like, we know experts have organized knowledge, right? I don't want them to feel like they have to memorize a bunch of different symbols, which they do. But if they can like reason through them, then maybe they can become more fluent.

For Dr. Landry, the meaning encoded in symbols was structured and revealed patterns that students may otherwise overlook. Her directed attention to the organization of symbols was intended to support students in becoming more fluent in their thinking about statistics concepts in addition to symbols.

Instructors seem to value and encourage students to express ideas in informal language before formalizing ideas symbolically. Dr. Young spoke of the importance of having students "articulate what's happening in spoken word" and Ms. Thompson recognized that "there's a lot of technical language and complex ideas." For Dr. Scott, having students use informal language allows him an "opportunity to reassess" whether students are making sense of key concepts before attempting to encode them symbolically. It seems important for Mr. Jackson to know whether students can distinguish ideas and then apply the correct symbology to those ideas – he explained in his interview that he consistently asks students questions like, "Is it the population mean or the sample mean? And what's the symbol for sample mean?" so that students can more confidently express their ideas symbolically.

Discussion

While Mutodi and Mosimege (2021) contend that instructors are not strategic in engaging students in developing symbol sense, the instructors in this sample offered multiple examples of doing so. These instructors, who are familiar with Universal Design for Learning and attuned to challenges their students may face in the learning environment, exhibited specific strategies and approaches to attending to symbol sense development. It appears their orientation to recognizing the diversity of students' cognitive profiles helped them to empathize with their students and identify ways that symbolic expression could be a barrier to understanding and communicating in the statistics classroom.

That all instructors referred to instances in the video-recorded classroom episode where attention was paid to symbol sense indicates that they believe, as Begg and Pierce (2020) contend, "the complexities of notation and the use of symbols is a reality that cannot be avoided." Rather, they seek ways to enact UDL principles, such as ensuring that learners have "consistent and meaningful exposure to symbols to comprehend and use them effectively" (CAST, 2024). Students and instructors alike regard symbols as important tools for efficient communication. Including symbology as an instructional objective or a focus of instruction ensures accessibility for all students, recognizing that learners have "varying backgrounds, languages, and lexical knowledge" (CAST, 2024). In this study, UDL Considerations 2.1 and 2.2 functioned not only as a coding scheme but as an analytic lens that oriented the research team toward specific instructional dimensions (e.g., vocabulary clarification and decoding support) and helped surface patterns that might otherwise have been described in more generic terms. Three findings stand out as especially actionable for instructors: (1) attending explicitly to symbol morphology, including how symbols appear in different handwriting styles, addresses a source of confusion that often goes unacknowledged; (2) offering word-for-word translation between symbolic and natural language gives students a replicable strategy they can apply independently; and (3) making structural relationships among symbols visible reduces the cognitive burden of treating each symbol as an isolated fact to memorize.

Specific instructional strategies included developing lists or glossaries of key terms, consistent with Bailey (1999). This allows students to recognize the morphology of the symbol while attaching meaning and significance to it. Additionally, tools like a Frayer model graphic organizer invite students to observe how symbols interact with context as they record examples and nonexamples of uses of specific symbols. Oyoo (2022) recommends "multiple representations may be given considerable priority as its linkage to other situations of use is indispensable" (p. 127). Students find that symbol translations without context are not useful - "you can't just Google it" (Avery).

Instructors play a pivotal role in developing students' understanding of symbols. As Esuong and colleagues (2022) conclude, "mathematics teachers concentrate on demystifying the abstraction in mathematics by interpreting mathematical statements and sentences into their symbolic notation." Students appreciated this word-by-word interpretation and found the approach helpful to attempt independently. More broadly, there is a recommendation that "meaning must precede symbolization" (Mutodi & Mosimege, 2021, p. 1194). Encouraging students to express ideas in natural language before encoding them in symbols was something that students and instructors both mentioned. This is consistent with recommendations that building connections between informal and formal language helps students develop symbol sense (Mutodi & Mosimege, 2021).

Insight into how instructors can communicate the symbology ecosystem to their students was evident in Dr. Landry's classroom episode. Dr. Landry recognized that expert knowledge included appreciating the structure and organization of the assignment of symbols to ideas in statistics. Dr. Landry's attempts to communicate relationships between symbols appears to be an attempt to reduce the cognitive load required for students to distinguish between multiple symbols that express related ideas. While researchers seem to agree that learning symbology increases students' cognitive load as students hold multiple ideas simultaneously (e.g., Begg & Pierce, 2020), a strong symbol sense "accommodates new mathematics cognitive demands" (Mutodi & Mosimege, 2021, p. 1195). Dr. Landry seems to be reducing the cost of learning symbols by investigating structure and organization so that the reward of increased symbol fluency can come more naturally to students.

It is worth noting that this study documents instructional practices and students' perceptions of those practices—not learning outcomes. Students' reports of feeling less intimidated by symbols, and of developing strategies to make sense of them, represent meaningful evidence of increased access and engagement with statistical ideas. Whether those experiences translated into deeper conceptual understanding is a question for future research.

Conclusion

The diversity of student cognitive profiles and prior experiences in secondary mathematics courses increases the demands on postsecondary instructors to create responsive and effective learning environments. The Universal Design for Learning framework offers recommendations for designing environments that identify and mitigate barriers students may face. This article offers detailed accounts of student and instructor perspectives on specific practices that align with UDL Considerations 2.1 and 2.2, addressing challenges students may face in understanding symbols.

As qualitative research, this study provides thick, rich description of the lived experiences of the study participants (Creswell & Miller, 2000). Along with that depth comes the limitation of transferability to novel contexts. This study relied on student volunteers at a single institution along with their instructors. The institution's context may limit the generalizability of these findings; most institutions do not have as high a proportion of students who report learning and attention differences. Furthermore, the faculty at Curry College may possess distinct teaching philosophies and employ more student-centered instructional strategies compared to faculty elsewhere.

Future research could examine the ways instructors align their instruction to these UDL Considerations in other course subjects and at different institutions. The qualitative research methods employed in this study provide deep insight into how students and instructors experience the teaching and learning environment.

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DIMINISHED FEELINGS OF BELONGING IN AN 8TH GRADE MATH CLASS

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This classroom study describes an experienced eighth grade teacher's actions to support students' feelings of belonging in his Math 8 classroom over an academic year, based on his involvement in a professional development (PD) project to support those efforts. The teacher made progress toward the goal to support belonging with emotional decentering. He identified that validating students' contributions in class was essential to support their feelings of belonging. However, in his classroom he often did not validate cognitively—and so he often undermined this validation. Ultimately, the lack of validation contributed to frayed relationships with some students, which diminished their feelings of belonging.

Keywords: Instructional Activities and Practices; Affect, Emotions, Beliefs, and Attitudes

Feelings of belonging in school are associated with many positive outcomes for students, including engagement, motivation, and learning (Barbieri & Miller-Cotto, 2021; Osterman, 2000). Yet during middle school, students may experience a drop in feelings of belonging (Anderman, 2003). Research has found that student-teacher relationships can positively influence belonging (Booker & Lim, 2018; Maloney & Matthews, 2020). For example, teachers who supported secondary students' feelings of belonging demonstrated emotional consciousness to help students manage frustration, affirmed students as people and thinkers, practiced authentic pedagogy, and showed sociopolitical awareness of care (Booker & Lim, 2018; Maloney & Matthews, 2020). Based on this research, we designed professional development (PD) to support middle school teachers to support their students' feelings of belonging in math classrooms.

In this paper we describe an experienced eighth grade teacher's actions to support students' feelings of belonging in his Math 8 classroom, based on his involvement in our PD. The teacher, Ben Reynolds (pseudonym), identified that validating students' contributions in class was essential to support their feelings of belonging. We found he made progress toward this goal with emotional decentering. However, he was often unable to validate cognitively—and so he often undermined this basic goal. Ultimately, the lack of validation contributed to frayed relationships with some students, which diminished their feelings of belonging. Our research question is: How does a middle school teacher influence students' feelings of belonging in math class?

Literature Review: Belonging in Classrooms

Walton and Brady (2017) define belonging as a conclusion people make about "the quality of fit or potential fit between oneself and a setting" (p. 272). Belonging is experienced as "a feeling of being accepted, included, respected in, and contributing to a setting" (p. 272). Overall, in studies with middle school students, researchers have found a sense of classroom belonging to be positively correlated with motivation (Goodenow, 1993) and academic enjoyment (Sakiz et al., 2012). Feelings of belonging have not been found to directly influence achievement, but they do so indirectly by influencing motivation (Osterman, 2000). However, Barbieri and Miller-Cotto (2021) found that a sense of belonging to the academic community of mathematics was the only factor that predicted middle school students' algebra learning in a unit on quadratics. They also found that under-represented minority students reported lower belonging than other students.

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So, what influences middle school students' feelings of belonging? In a sample of 353 students at a suburban, predominantly White middle school, Goodenow (1993) found that students' perceptions of their teacher were "the most influential single component of belonging and support in terms of association with effort and achievement" (p. 37). Other researchers concur that student-teacher relationships are pivotal in supporting students' feelings of belonging (Booker & Lim, 2018; Harris-Thomas & Roseth, 2025; Maloney & Matthews, 2020; Sakiz et al., 2012). For example, Booker and Lim (2018) found that strong relationships and engaging pedagogy supported eight high achieving sixth grade African American girls to feel like they belonged in math classes taught by White female teachers. Maloney and Matthews (2020) found that secondary school teachers with more productive patterns of caring for students had emotional consciousness, a holistic view of students as mathematical thinkers and people, a collective struggle orientation, and a sociopolitical awareness of care. Students whose teachers demonstrated these characteristics were more likely to feel a sense of belonging in math class, which in turn positively influenced students' valuation of math and perceptions of the relevance of math instruction. Yet, studies have also shown a decline in perceived teacher support across middle grades students (Reddy et al., 2003). We now discuss mathematics teacher learning and decentering, a tool we use to frame teaching practice to support feelings of belonging.

Theoretical Framework: Mathematics Teacher Learning and Decentering

Like many others, we view high-quality mathematics teaching to be facilitated by high leverage practices that are implemented consistently over time (e.g., Grossman et al., 2009; McDonald et al., 2013). For example, leading a discussion is a high leverage practice that consists of many smaller practices such as anticipating students' thinking on a task, monitoring student work, and selecting and sequencing student work to support a learning goal (Jacobs & Spangler, 2017; Smith & Stein, 2018). In our view, this perspective on high-quality teaching has to include a theory of how students learn mathematics in order to guide practices such as formulating learning goals, anticipating student thinking, anticipating questions for students, and enacting that questioning in small group and whole class interactions.

Our theory of students' mathematical learning is centered on how students organize and reorganize their ways of thinking in ongoing interaction in their experiential worlds (von Glasersfeld, 1995). Social interaction in classrooms is essential in this view of learning because it provides a critical site for organizing activities that can provoke learning (Steffe, 2010). So, our theory of high-quality mathematics teaching is an interplay between high leverage practices and how students learn, where each informs the other.

Following Silverman and Thompson (2008), we frame a core of teacher learning to be teachers constructing and refining their own first-order knowledge and then constructing and refining second-order knowledge of their students. We refer to people's own ways of thinking as their first-order knowledge, which they construct to organize and understand their experiences (Steffe, 2010). Second-order knowledge refers to the knowledge a person constructs to describe and explain their observations of another person's experiences (Steffe, 2010). In turn, constructing second-order knowledge requires decentering (Bas-Ader & Carlson, 2022). Decentering involves setting one's own perspective to the side, working to adopt the perspective of another person, and using that perspective as a basis for interaction. In decentering from one's own perspective, one necessarily centers others' perspectives.

Decentering and centering can support students' feelings of belonging. For example, to demonstrate Maloney and Matthews's (2020) emotional consciousness with students, teachers need to set their own emotional experiences aside, work to understand their students' emotional

experiences, and interact with them from that basis. In this process, teachers do not assume that students' emotions are similar to their own. Instead, they aim to understand students' emotions and interact with students from that understanding, thereby centering students' experiences.

Decentering cognitively has not been addressed in studies on belonging in math classrooms, but it is an important part of mathematics teacher learning (Carlson et al., 2024; Silverman & Thompson, 2008). In cognitive decentering, teachers set their own ways of thinking aside, work to understand their students' ways of thinking, and interact with them from that basis. When a teacher interacts with students based solely on their own first-order knowledge, they may not see students' thinking, let alone center it in the classroom. In contrast, when a teacher does not assume that students think as they think, and seeks to understand how they think, the teacher is beginning to build second-order knowledge and has a chance of centering it.

Method

Participants

To support a middle school teacher to support his students to feel a stronger sense of belonging in math class, we conducted a 10-month classroom study with eighth grade teacher Ben Reynolds and one of his classes. The classroom was in a middle school located in a small town in the midwestern United States. Ben was White and was chosen because he agreed to participate. He had 14 years of experience at the time of study and taught Math 8 and algebra.

We conducted the study with Ben's Math 8 students because we thought Math 8 students might vary more in their feelings of belonging in math class compared to algebra students. Sixteen of 25 students in the class consented to participate, which meant they took surveys and assessments and participated in a focus group if asked. They also shared their work with the research team. Of the participating students, one student was Black; the other 15 were White.

Data Collection and Professional Development

In August, Ben participated in an initial interview. We sought to understand Ben's teaching and his initial ideas about belonging. Then we asked him to choose two of four PD foci for our work together: students' reasoning, math talk, social and emotional learning, or students' histories and cultures. He chose students' reasoning and math talk. We had a PD meeting approximately monthly to work on some facets of these chosen foci. In between meetings, Ben worked on an aspect of his practice related to that focus. A 4-person research team observed his class weekly and filmed monthly. Then at our next PD meeting, we debriefed, noticed and wondered, and planned for the next stretch of time (Goldsmith & Seago, 2011). At subsequent PD meetings, we reviewed artifacts, such as student work or segments of audio or video from the recently filmed class, in order to develop the current aspect of practice Ben was working on and to ground our observations in evidence.

We observed Ben's 75-minute Math 8 class every Friday from August to May and took field notes. We filmed six class periods and six PD meetings almost monthly by May (Figure 1). In the May class, students designed a brochure and did not engage in mathematical thinking, so we omitted that from our analysis. In May we conducted a final interview with Ben.

Month	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May
Meetings with Ben	Initial Interview	PD 1		PD 2		PD 3	PD 4	PD 5	PD 6	Final Interview
Math 8 Class			Class 1		Class 2	Class 3	Class 4		Class 5	Class 6

Figure 1: Project Timeline of PD Meetings and Filmed Classes

The 16 participating students completed a survey about their feelings of belonging in September, January, and May. We used the survey from Maloney and Matthews (2020) with three criteria each addressed with three items (Figure 2). Students had a choice of six responses from 1–Strongly Disagree to 6–Strongly Agree. Some students participated in focus groups in November, January, and May, each consisting of 3–7 students.

Emotional support from the teacher	My math class is like a family	My contributions are valued
• I feel that I can talk to my math teacher(s) about things that are bothering me. • My math teacher is someone I can count on to help me. • I believe that my math teacher(s) is/are interested in the things I have to say.	• Students in math class help each other, even if they are not close friends. • When I do well in math, the other students in class are usually happy for me. • My math class is like a family.	• My math teacher(s) know that I can do good work. • I want my math teacher(s) to be proud of my work. • I have a lot of opportunities to get involved in my math class.

Figure 2: Nine-Item Survey from Maloney and Matthews (2020)

Data Analysis

To analyze the data, we engaged in rounds of coding of four sets of video data: the filmed class meetings without May (5), the filmed PD meetings (6), Ben's interviews (2), and the student focus groups (3). Each video data source required a unique codebook, although there was some overlap in the codes for class and PD meetings. Many codes arose from constant comparative analysis, as there is not much research on supporting feelings of belonging in math class in middle schools. However, we also used relevant codes about decentering (Bas-Ader & Carlson, 2022). Both authors of this paper coded each piece of data, discussed coding, and ran intercoder analyses until we reached over 90% agreement. As we stabilized coding, we developed 12 categories that described key features of the filmed class meetings. Then we examined the field notes from the Friday observations (28 additional non-filmed class periods) to determine whether these categories were evident in other classes and to search for disconfirming evidence. We developed categories for the PD meeting data and summaries for each interview. Once all categories were stable, we used them to develop themes from the data.

We coded the student focus group data with constant comparative analysis, created a summary of key ideas for each group, and tracked how students' concerns changed over time.

Finally, to analyze students' belonging survey data, we computed descriptive statistics. We tested the normality of our data using the criteria that the kurtosis and skewness should have an absolute value less than 2 (Strunk & Mwavita, 2025), and they did. Plots showed most data to be approximately normal, despite the small sample size. So, we decided we were justified in running parametric tests (Harpe, 2015). We ran a repeated measures ANOVA to test for differences over three time points for each of the three criteria.

Findings

We respond to our research question—how a middle school teacher influenced his students' feeling of belonging—in three parts: Decentering and relationship-building (Fall), lack of cognitive decentering (Fall-Winter), and fraying student-teacher relationships (May). Due to space limitations, we have to summarize some parts significantly.

We begin with a brief portrait of Ben's teaching to provide context. When asked to describe himself as teacher, Ben said in the last 5 of 14 years he had moved from a competitive, computation-focused atmosphere to a relationship-building atmosphere, where he strove to get to know his students. He said he wanted students to become critical thinkers and develop a growth mindset. Ben had, over time, culled many class activities and problems that drew from a variety

of websites, including 3-Act Problems, Desmos, as well as more traditional worksheets. He sometimes used engaging, complex problems that would be classified as "doing mathematics" (Smith & Stein, 2018, p. 16). Ben did not follow a traditional text. In his class, desks were in rows, but students in the same row were in groups that consisted of 2–4 students who worked together at times. He changed these groups a couple of times during the year. During his initial interview, as well as repeatedly over the year, Ben stated how much he loved it when a student pointed out a way of thinking about an idea or problem that he had not thought of before. This often happened during the 3-Act Problems.

Fall: Decentering and Relationship-Building

In the fall, Ben primarily showed evidence of decentering and relationship-building with his students. For example, in October Ben led a number talk to start class: 12 x 25. Students were quiet, seeming reluctant to participate although they had done number talks before. Ben repeatedly and gently acknowledged that the quiet might come from students feeling like they were wrong. Eventually students shared answers of 300 and 250. A student explained that he got 300 by multiplying 25 x 10 and 25 x 2. Other students agreed. When Ben asked about another strategy, students did not respond. So, Ben ended up sharing the idea that you could count by quarters 12 times to solve this problem or count by groups of 50 cents or dollars. It turned out that the student who got 250 counted by 25s six times and got 125 instead of 150. So, this showed evidence that Ben was demonstrating ways of thinking that students had or could make sense of, an example of cognitive decentering. And, Ben decentered emotionally: He imagined the students' emotional reticence to share as distinct from his and tried to reassure them.

Indeed, the November focus group consisted of three students who were very positive about Ben. As one of them said, "he is very patient for a teacher." They said that at first Ben thought like a college student would think but then he will "do it in a way that eighth graders will understand." Their views demonstrate what seemed apparent in class to this point: Overall, the students trusted Ben and thought he was doing a good job as a teacher. These three students' comments were also positive about the class and their position in it. That said, over the year we found that these three students were among the stronger mathematical thinkers in the class. And, two of these students felt a strong sense of belonging over the year: They had the two highest averages on the survey criteria "*emotional support from the teacher*" and "*my contributions are valued*," and one had the highest average on "*my math class is like a family*." So, this focus group might not have been representative of the class as a whole.

Fall and Winter: Lack of Cognitive Decentering

In fact, in the fall Ben self-reported an interaction with a student, Kari, that pointed to a lack of cognitive decentering. We go out of chronological order here to make this point, discussing data from the September PD meeting. In that meeting, the research team presented results from the students' September belonging survey data. We noted that Kari, who had shown some struggles with ideas in class, had fairly positive scores. Kari was White. Ben said he wondered what she thought after today's class, where they were working on writing numbers in scientific notation. He had put up 107.23 and asked where to put the decimal point to express it in scientific notation. He said when he asked questions about what seemed like "simple concepts" to him, he called on students who "need a little more confidence." However, Kari could not tell him where to put the decimal. He said his strategy in these cases is "to repeat things if necessary; maybe she was zoning out; maybe she was thinking about a boy, or thinking about whatever." He restated the question multiple times, but she did not get it. He noted, "And I'm like dying, trying to get her, and she's getting frustrated and [I said] okay, we'll come back to you."

In our view, this interaction can be explained by a lack of second-order knowledge of Kari's ways of thinking about decimals. Specifically, we are not sure what Ben was relying on to simplify his question—what question or idea qualified for him as simpler. We suggest that he was trying to think from Kari's perspective socially and emotionally by wondering about her zoning out or thinking about a boy, but not from her perspective mathematically. Decimals are notoriously challenging because constructing them relies on operations involved in constructing fractions (partitioning, iterating, etc.), repeated partitioning of units into units that are ten times smaller, and relating those units (e.g., Borowski, 2020; Kosko et al., 2026; Wearne & Hiebert, 1988). It appears that Ben's simplification did not meet Kari's current thinking about decimals.

In fact, this self-reported interaction was a harbinger of things to come: Ben continued to demonstrate a lack of cognitive decentering with Kari and her partner Blair in the winter. For example, in December students worked on Turtle Time Trials, a Desmos activity where initially two turtles raced and then later a third turtle was added. Students were asked about the speed of the turtles, filled in distance values in a table with distance and time values, and then looked at a graph of each turtle's distance and time values (Figure 3). Turtle 1 traveled six feet in six seconds; Turtle 2 traveled 12 feet in four seconds; and Turtle 3 traveled 12 feet in eight seconds.

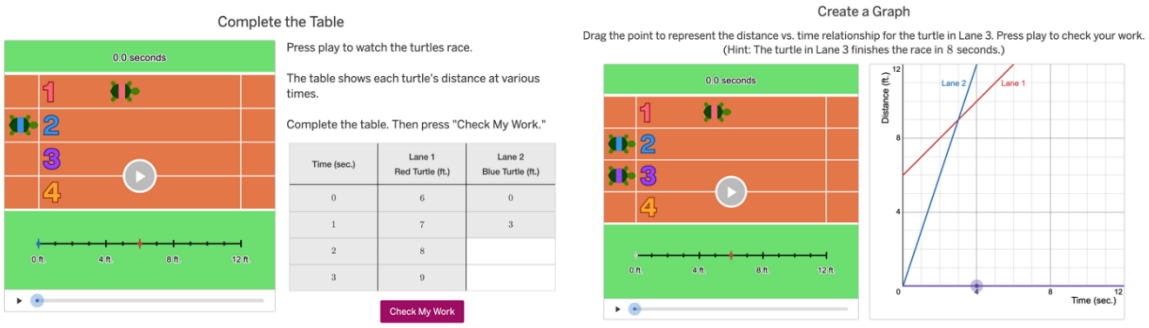


Figure 3: Turtle Time Trials

Students were asked which was faster, Turtle 1 or Turtle 3. Blair and Kari asked Ben about this question, and Ben asked what they had found. Blair said, "We said they had the same." Ben asked, "So how are you determining who is the fastest?" Blair described what they saw when they watched the race, stating, "they just look like they were going the same." Ben then repeated the question about how they were deciding how fast a turtle was, and he pointed them to the formula for speed that was on their worksheet, "speed = change in distance/change in time." Kari described what they had done for the table, where they watched the turtles race, saying, "we did 3, 6, 9" (Figure 3, left). She did not say what these numbers meant in the situation (distances in feet for Turtle 2). Ben interpreted this as them following a pattern. Then he steered them back to the formula and again asked which turtle would be going the fastest. When the students did not seem sure, Ben pointed them to a graph on a later slide (Figure 3, right) and directed them to use that. The students said they did not know how to. He then led them through using the graph.

Here, Ben appeared to assume that when the two students filled in the table, the numbers they wrote down meant distances to them. He also seemed to take for granted that they were tracking elapsed distance and elapsed time, an essential component of conceiving of speed (Hackenberg et al., 2023; Thompson & Thompson, 1994). Ben did not ask them to think about what the numbers meant and how they were related to the turtles' movements. For example, in the race and table (Figure 3, left), it is possible to see that Turtle 2 travels three feet in the first second,

and then another three feet in the next second, and then another three feet in the third second. So, the turtle is traveling three feet every second. It is not necessary to use an equation to figure that out—but it is necessary to focus on how distance in feet and time in seconds are changing together as Turtle 2 moves. Instead, Ben focused on an equation and graph that had meaning for him but not necessarily for students who are building ideas about speed. As teachers have done before him, he treated speed more as a calculation than as a quantity that needs to be constructed (Thompson & Thompson, 1994). This was one example of a phenomena that happened often in the spring.

May: Fraying Relationships

One outcome of the year was that Ben's relationships with Blair and Kari, as well as some of the other students, seemed to fray. For example, in the May focus group with six students, all said they could count on Mr. Reynolds to help them with math. However, when asked whether Mr. Reynolds understood how they thought about problems, Blair said, "probably not." Two students said that if they got something wrong and he tried to explain and they still didn't get it, he got kind of frustrated. Kari said that Mr. Reynolds "understands it how like a higher student understands it." Furthermore, Blair, Kari, and another student said they did not feel comfortable asking him questions. When asked who they went to with questions, they said their groups.

These feelings about Ben were born out in the belonging survey data. The means for *"emotional support from the teacher* and *my contributions are valued"* decreased significantly from September to May (4.35 to 3.02, with $F = 23.286$, $df = 2$, $p < .001$; and 4.62 to 3.87, with $F = 6.406$, $df = 2$, $p = .005$). Partial eta squared was 0.625 and 0.314, respectively, which indicate large effects. However, over the year there was a statistically significant increase in the means for *"my math class is like a family"* (3.0 to 4.6, $F = 17.941$, $df = 2$, $p < .001$). Partial eta squared with 0.562, also a large effect.

Taken together, we can say that students' feelings of belonging decreased on two criteria and increased on one. The teacher has influence on all three criteria, especially emotional support from the teacher. Thus, fraying relationships with teacher detracted from students' feelings of belonging. However, other students have influence on the criteria my math class is like a family. So, despite the fraying relationships, students experienced an increase in that aspect of belonging. Since students were feeling some dissatisfaction with Ben, it is possible that they bonded with each other partly in opposition to a teacher they some felt increasing distance from.

In the final interview, Ben identified that validating students' contributions in class was at the heart of supporting students' feelings of belonging. About a student contribution in class, he said: "You have to do your best, even though it's very difficult sometimes, to find a way to tie it in and give value to that and make them feel like their response was worth something." However, he acknowledged that he did not always pose problems that were sensible to students and that he needed to provide more supports. He noted that he used to teach the math support class at the school and enjoyed that because he had the time to "meet [students] on their level."

Then Ben reiterated that some of his favorite moments in teaching were when students came up with ideas he had not thought of: "you're like, mind blown, you love it. And so the student thought side of it, why are their brains thinking that this is the connection, or this isn't, or how do they not see this or whatever? It is really interesting." Here Ben spoke about how he loved understanding an idea that a student had that was new for him. In fact, his question about students' brains is a question about second-order knowledge. So, Ben seemed to see a strong need for this kind of teaching and for connecting with students' ways of thinking: He found it enjoyable; and he learned new ideas from students. However, when asked what he had learned

from his students this year, he paused. Then he said he did not know that he had learned anything new compared to prior years. Our interpretation is that despite his appreciation of students' thinking, he did not seem to have a way to regularly surface and learn it. He did learn from students, but it was rare at this point, now 15 years into his career.

One outcome of this situation was that Ben could not fully follow through on his goal to validate what students shared with him as a key piece of supporting their feelings of belonging. The reason, in our view, is that to do so requires awareness of students' first-order knowledge as distinct from one's own—as second-order knowledge to build and use as a basis for interacting. We argue that to think about students' knowledge in this way requires an orientation to do so, as well as ideas about what that second-order knowledge could be, such as understanding that students may not be constructing quantities of distance and time, let alone co-varying them, as they watch a race. This learning requires sustained professional support (Carlson et al., 2024).

Discussion

We have shown that Ben was unable to fulfill what he held most dear as a teacher, and what he identified as the heart of supporting feelings of belonging, because of a lack of cognitive decentering. We argue that this was a factor in the fraying of student-teacher relationships, which in turn diminished students' feelings of belonging in the class.

In terms of prior studies, Ben often practiced authentic pedagogy as described by Booker and Lim (2018): He posed complex problems, did not follow a textbook, and frequently used technological tools like Desmos. He also had strong relationships with some students as shown with the November focus group. Ben regularly demonstrated emotional consciousness (Maloney & Matthews, 2020) via decentering emotionally. He was seen as a trusted teacher by many, particularly in the fall semester.

And yet, these characteristics were not enough to support Ben to validate all students' responses in class, the heart of belonging for him. We found that he could sometimes validate the responses of students who had more basic ways of thinking than he would use, as shown during the number talk in October. But it was challenging for him to validate some of his students' responses, as was repeatedly evident in his interactions with Kari and Blair. So, our study suggests that decentering cognitively is a tool to support validation of students' responses in math class that in turn may support students' feelings of belonging—or at the very least, can support strong student-teacher relationships (Hackenberg, 2010). This aspect of teaching has not been unearthed in prior studies about secondary math teachers and belonging (e.g., Booker & Lim, 2018; Maloney & Matthews, 2020). So, this is one contribution of our study.

This study is limited by being a single case. The study was designed to be conducted with multiple teachers. Data collection occurred just as the country was emerging from the pandemic, and we only were able to secure one participant. From this study we cannot make claims about other teachers and classrooms, but it can provide an important foundation for further research.

Based on our study, we conjecture that cognitive decentering plays an important role in supporting students' feelings of belonging in middle school math classrooms. We propose that learning to decenter cognitively requires developing an orientation to value this type of responsiveness, along with new ways of thinking that reflect the mathematical thinking of students (Carlson et al., 2024).

A good question is: How can this work be done in classrooms, when middle school teachers meet over 100 students a day? Our response is: It cannot be done with all of those students at once. However, most middle schools have math support classes for students who struggle. We

think those classes are a potent opportunity for teachers to practice cognitive decentering with support for their professional learning. We are eager to try this in our local settings.

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A FRAMEWORK TO CONCEPTUALIZE ADAPTIVE TEACHING IN MATHEMATICS EDUCATION: THE ADAPTIVE TEACHING TETRAHEDRON

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The ability to adapt to the unexpected in the classroom is an important skill for all teachers to possess. This skill, adaptive teaching (AT), is under-theorized in mathematics education and suffers from a lack of consistency in definition. In this essay, I propose a conceptual framework called the Adaptive Teaching Tetrahedron (ATT) that is built upon themes from existing literature. The framework was developed through an iterative hermeneutic phenomenological process and conjectures a relationship between various literature bases in and around AT. Implications for the mathematics education research and teaching community are presented.

Keywords: Teacher Knowledge, Instructional Activities and Practices, Teacher Educators

The classroom environment is a complex and messy system. The preparation to teach effectively within this type of environment has been debated by mathematics education researchers for decades. Now, with the rise of scripted curriculum and new policy mandates (Vaughn et al., 2022), new standards, and diverse student situations (Parsons et al., 2018), teachers are required to *adapt* to a variety of circumstances to provide quality instruction. Further, adaptive mathematics instructors understand the complexities of student questioning and discourse, can elicit student thinking and the intricacies of various solution strategies, and use student-centered instruction to promote their students' learning (National Council of Teachers of Mathematics [NCTM], 2014). As student-centered instruction is becoming increasingly central to K-12 education, novice teachers may feel unprepared to meet the demands of being an adaptive mathematics instructor (Darling-Hammond, 2006). Adaptive teaching is a 'gold standard of instructional practice' (Darling-Hammond & Bransford, 2005; Parsons et al., 2018) that warrants the attention of the mathematics education research and teaching community. Despite the growing recognition of the importance of adaptability (Gallagher et al., 2022; Parsons et al., 2018; Remillard & Heck, 2014; Vaughn et al., 2016), there remain gaps in both the empirical study and explicit definition of adaptive teaching, especially within mathematics education. Literature has indicated various common aspects of adaptive teaching (Randi, 2022), yet has not conjectured a relationship between these aspects in the actions of adaptive teaching.

In this article, I propose a conceptual framework of the components of an adaptive mathematics teacher that builds on existing literature and incorporates mathematics education-specific components. The goal of this article is to propose this framework to the field of math education as a way to reconceptualize what it means to adaptively teach. Lastly, I call attention to the need for further study of adaptive teaching.

What is Adaptive Teaching?

This research topic is dense with entangled terms, which can make adaptability difficult to define. Loughland (2019) describes that adaptive teaching "suffers from a lack of consensus on a common term, although there is agreement that adaptive teaching, by any other name, involves a response to a range of stimuli provided by their students" (Loughland, 2019, p. 17). In addition, some researchers view adaptive teaching as only happening within the classroom in response to an unexpected stimulus (Gallagher et al., 2022) that supports student needs (Parsons et al., 2018),

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while others conceptualize it as a disposition towards teaching and a mindset both in and out of the classroom, coupled with specific practices (Corno, 2008; Loughland, 2019).

Despite this, literature has documented what an adaptive teacher looks like. Responding to students' thinking, attending to students' needs, and handling classroom complexity with flexibility are all features of adaptive teaching (Corno, 2008; Darling-Hammond & Bransford, 2005; Parsons et al., 2018). Adaptive instruction is responsive, reflective, and creative (Randi, 2022). The adaptive teacher effectively uses student thinking to progress lessons and adapt to their environment (Lin et al., 2007). Adaptive teachers assess students in the moment using content expertise, all while maintaining a flexible mindset (Randi, 2022). Likewise, effective teachers adapt to sociocultural contexts (Herbst & Chazon, 2017; Franke, 2007) and view variation among learners as an opportunity (Beltramo, 2017; Corno, 2008; Schipper et al., 2018). Adaptive Teaching has been referred to as a component of effective teaching (Vaughn et al., 2022) but has rarely been discussed or theorized explicitly in mathematics education.

Herbst and Chazan (2017) argued for increasing the subject specificity of research in mathematics teaching. Researchers such as Prediger (2022), Parsons et al. (2018), and Gallagher et al. (2022) emphasized that adaptability may not be fully understood or implemented without considering the content-specific demands of the classroom. By incorporating subject specificity into theoretical models about adaptive teaching, the mathematics education community can move toward investigating what it means to better prepare teachers to make contextually appropriate adaptive decisions, thereby building more responsive and effective teaching practices.

The definition of adaptive teaching I will use throughout this article is when a teacher adapts their instruction in such a way that it attends to content, knowledge of students and their thinking, and the pedagogical demands of the situation (Darling-Hammond & Bransford, 2005; Hoffman & Duffy, 2016). The adaptation, then, is a high-leverage move that is specific to the needs of the learners in the moment (Vaughn et al., 2022). A high leverage move is one that significantly alters the progression of the lesson to meet the needs of the students, whether that be cognitively, behaviorally, or situationally (Martin et al., 2012). In a mathematics context, the needs of the students are tied to the mathematics at hand.

Examples of adaptive teaching in math education are slim. One example in another discipline is described by Parsons and Vaughn (2013). Parsons and Vaughn (2013) observed two teachers' adaptive instruction. They describe a teacher, Ms. Bradley, who was focused on literacy and social studies education. In one instance, Ms. Bradley "adapted her instruction by initiating a discussion about how certain groups of people were left out of history: women, African Americans, Native Americans, and slaves." (Parsons & Vaughn, 2013, p. 310). Parsons and Vaughn (2013) noted that this adaptation was in response to a student contribution. The adaptation was specific to the needs of the learners in the moment, and attended to social studies content, knowledge of student thinking, and pedagogical demands. Although this example is not from mathematics education, it does demonstrate what an adaptation is in this context. The scarcity of examples from literature in mathematics education motivates the need for further study of adaptations in mathematics-specific contexts.

It is for these reasons that I call attention to the need for further study of adaptive teaching in mathematics education. This article presents the Adaptive Teaching Tetrahedron (ATT), a conceptual model built upon synthesized literature that makes visible the relationships between student thinking, subject matter expertise, and adaptive expertise, as well as conjectures some skills needed in teachers to support adaptive teaching. This framework contributes to the literature by positing a relationship between the components of adaptive teachers.

Methodology

For this article, I took a hermeneutic phenomenological approach to theory development (Annells, 2006; Boell & Cecez-Keçmanović, 2014; Creswell & Poth, 2024). A phenomenological approach requires a researcher to identify a phenomenon to study, distinguish the philosophical assumptions of the phenomenon, and then collect data, traditionally from individuals who have experienced the phenomenon. Phenomenology is focused on extracting the essence of a phenomenon across multiple instantiations (Creswell & Poth, 2024). Due to the complexity in definition and conceptualization, this approach was appropriate. Likewise, a hermeneutic approach focuses on the iterative interpretive process of the 'texts of life' (van Manen, 1990). Rather than collecting data from individuals, a literature review was conducted following the framework proposed by Boell and Cecez-Keçmanović (2014).

The framework involves two hermeneutic circles: 1) search and acquisition of literature and 2) and interpretation of the literature. The first hermeneutic cycle of search and acquisition involves acquiring, reading, identifying, refining, searching, sorting, and selecting literature. The second hermeneutic circle of analysis and interpretation includes mapping and classifying literature, critical assessment of literature, argument development, and open research questions. These cycles are intertwined and representative of the fluid nature of literature review, which allows for "the development of new linkages among concepts and theories and new synthesis" (Boell & Cecez-Keçmanović, 2014, p. 264). Boell and Cecez-Keçmanović (2014) argue that the literature review process is fundamentally iterative in nature while developing an understanding of a phenomenon.

Because there is an absence of theoretical advances and understanding of the fundamental relationships between literature bases and considerations involved with adaptive teaching, a hermeneutic phenomenological approach allowed me to synthesize data in the form of literature, generate themes from this analysis, develop textural and structural descriptions to ultimately report a composite description that represents the understanding of adaptive mathematics teaching in a written form (Creswell & Poth, 2024). By using this approach, I was able to extract the invariant essence of adaptive teaching by iteratively interpreting past literature. The result of this process was a dynamic, conceptual model, the Adaptive Teaching Tetrahedron (ATT). Here, I present this composite description along with a conceptual narrative to support the theoretical discussion of the many moving parts of adaptive teaching, specifically in a mathematics education classroom.

Results of Literature Synthesis

As I reviewed the literature on this topic, I found major themes within the literature speaking to what an adaptive teacher is and the types of actions they take. The first theme is Content Fidelity, wherein adaptive instructors adapt with respect to the content at hand. Adapting with Content Fidelity involves knowing how and when to make adjustments to instruction that are consistent with the content-specific components of the lesson. The focus on the content goals of coursework was emphasized in some adaptive teaching studies (Becker et al., 2023; Prediger et al., 2022; Prediger, 2022). Prediger and colleagues (2022) argued that adaptivity is not characterized by teachers moving in a content-independent way. This builds on the work of Parsons and colleagues (2018), who argued that more content-specific teacher moves are needed to fully conceptualize adaptability. Thus, Content Fidelity is an important aspect of adaptive teaching.

A second theme is Adaptive Expertise. Adaptive Expertise (Franyo & Dorner, 2024; Severson-Irby & Parkhouse, 2025; Suh et al., 2023; Suh et al., 2024; Vale et al., 2025; Yoon et

al., 2019) involves adaptive behavior in response to student needs that are 'novel and appropriate' (Hammerness et al., 2005; Randi, 2022). Adaptive expertise involves balancing efficiency and innovation in novel classroom situations (Anthony et al., 2015; Bransford et al., 2005; Riel & Rowell, 2017; Schwartz et al., 2005; Severson-Irby & Parkhouse, 2025). In addition, adaptive expertise draws on prior experience and knowledge (Severson-Irby & Parkhouse, 2025), which enables pedagogical practice that is flexible and able to shift to student needs (Anthony et al., 2015; Hammerness et al., 2005; Severson-Irby & Parkhouse, 2025; Hu, 2024; Spiro, 1997). This refers to a teacher's ability to perform creatively and effectively in unfamiliar situations (Severson-Irby & Parkhouse, 2025). Adaptive Teachers act in a way that is true to how and why strategies, techniques, and practices impact their students and environments (Severson-Irby & Parkhouse, 2025), which is a goal for all teachers (Barrett & Hordern, 2021).

A third theme is Student Specific Traits. The social, instructional, and linguistic needs of the student are at the forefront of instructional adaptations. Understanding student responses and in-the-moment assessment of student thinking, as well as the needs of the student, characterizes this theme (Gallagher et al., 2022; Robertson et al., 2016). In prior literature, adaptive teaching practices have generally been regarded as being in response to student need (Gallagher et al., 2022), where adaptive teachers consistently adjusting routines in the moment to meet the needs of students, especially in a content-specific manner (Riel & Rowell, 2017). These three themes summarize what adaptive teachers' actions look like in the classroom.

The Adaptive Teaching Tetrahedron

The ATT (Figure 1) is a conceptual model intended to draw together the ideas present in adaptive teaching literature to provide a basis for future efforts in adaptive teaching. The ATT is composed of the three areas previously described, where Content Fidelity, Adaptive Expertise, and Student Specific Traits form the faces of the tetrahedron. The faces of adaptive teaching represent action, while the edges are fundamental components upon which action is built. Interestingly, I found three other themes about core skills adaptive teachers possess: Formative Assessment, Subject Matter Knowledge, and Improvisation, which form the edges of the tetrahedron. The intersection of the faces and edges reciprocally supports an adaptive teacher's complex navigation of the classroom. The faces in the ATT are actions of adaptive teaching supported by the qualities presented in the edges.

Elaborating on the edges, *Improvisation* (Aadland et al., 2017; Ben-Horin et al., 2016; Holdhus et al., 2016; Thomas et al., 2023; Vinnichuk et al., 2022) refers to a teacher's ability to use premises of effective teaching and content knowledge to make spontaneous, in-the-moment changes to practice (Randi, 2022). Randi (2022) likens this to creative teaching and disciplined improvisation. Improvisation straddles general pedagogic knowledge and pedagogical content knowledge (Holdhus et al., 2016; van Driel & Berry, 2010), and is a fluid response to the constraints of the classroom structures (Holdhus et al., 2016). Subject Matter Knowledge refers to the knowledge that teachers have of the designated subject being taught. This is separate from Content Fidelity because the subject matter knowledge is part of what allows for an adaptation to be content-specific. *Subject Matter Knowledge* is also related to Student Specific Traits because it allows teachers to be faithful to the content they teach, as well as the students' content-specific thinking. *Formative Assessment* refers to the continual, informal, dynamic assessment that occurs both asynchronously and synchronously in the classroom based on elicited evidence of student thinking and understanding to inform pedagogical decisions (Allen et al., 2013; Black & Wiliam, 2009; Corno, 2008; Grigorenko, 2009; Nickl et al., 2024). In sum, each of these themes represents something a teacher would need to be adaptive: knowledge of the content itself, an

ability to improvise instruction, and assessment of student needs and understandings during classroom instruction.

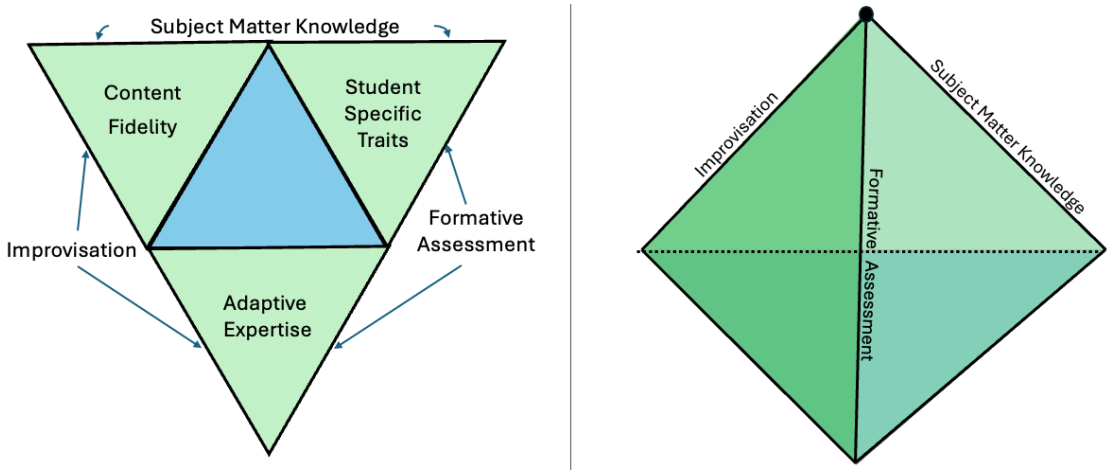


Figure 1: The Adaptive Teaching Tetrahedron

Thus, the ATT includes these three aspects as core components of adaptations. For instance, Adaptive Expertise is defined in literature by improvisation, creativity, and flexibility, as well as the continual assessment of student needs (Severson-Irby & Parkhouse, 2025). Therefore, in the ATT, Adaptive Expertise is supported through formative assessment and improvisation. An action specific to students, Student Specific Traits, is specifically related to content and is based on the students' apparent understanding of the content, which is why Student Specific Traits are supported with both subject matter knowledge and formative assessment. Content Fidelity is supported by improvisation skills and subject matter knowledge. Improvisation and subject matter knowledge support Content Fidelity because students often respond differently to the mathematics than expected, which requires the teacher to be able to improvise instruction with attention to the subject matter.

The nature of the ATT is important to consider. The ATT presents multiple aspects of adaptive teaching that consistently interact and form a complex, dynamic image of the adaptive teacher, who continually monitors and pushes the boundaries of interaction with themselves, their students, and their content (McMillan & Jess, 2021). The ATT is a dynamic entity; As teachers develop with expertise, the competencies and adaptations teachers have and do will subsequently change. For example, scholars emphasize that learning to be adaptive is possible (Granziera et al., 2019; Hardy et al., 2019), and for many pre- and in-service teachers, the edges of the ATT are still in the process of development.

Each component of the ATT supports the other when the tetrahedron is folded, so no one edge or face is independent of another. This is in support of the definition of adaptive teaching, where an adaptation attends to content, knowledge of students and their thinking, and pedagogical demands of the situation. To clarify, in this model, an adaptation is the conglomerate of all three faces. Other constructs and terms researchers use when discussing adaptations or adaptive teaching may not address all aspects of the ATT.

An adaptive teacher, however, would combine each of these aspects during instruction in a self-regulated and metacognitive manner (Randi, 2022). Further, self-regulation and metacognition are two other themes that arose from the literature search. Metacognition and self-

regulated learning (Duffy, 2005; Lin et al., 2007; Corno & Randi, 2005; Randi, 2022; Rowland, 2013) are related to adaptive metacognition (Lin et al., 2007), where teachers regulate themselves to the learning differences of a classroom (Randi, 2022). Adaptive Metacognition specifically concerns the ability for a teacher to adapt their metacognitive thinking to the events of a classroom (Randi, 2022; Lin et al., 2007). Researchers in this area are working towards developing metacognitive thought that indicates when to adapt and work towards developing an aptitude for adaptive teaching (Randi, 2022; Lin et al., 2007). In this same way, a mathematics teacher's metacognition is specific to mathematics itself and continually reflects the mathematical knowledge for teaching (Rowland, 2013) needed for any classroom situation. Furthermore, self-regulation is a vital aspect of adaptive teaching; an adaptive teacher must control their own emotions and thoughts while teaching (Corno & Randi, 2005; Randi, 2022). These components are not included in the tetrahedron because these two themes more readily describe the navigation of the complex environment; though vital to adaptive teaching, these were not included in the ATT. Future research should investigate where these components fit.

Conceptual Narrative

I employed a conceptual narrative as a reflexive heuristic, designed not for empirical analysis but to illustrate and support reflective engagement with the ATT. The narrative was constructed using evidence of adaptive practice in classrooms from literature (e.g., Parsons et al., 2018) and is intended to support the theoretical discussion of the many moving parts of adaptive teaching, specifically in a mathematics education classroom. The narrative follows Kelley, an introductory algebra instructor who uses student-centered instruction regularly. Kelley's classroom is set up in groups, and students are encouraged to participate in class.

On one particular day, Kelley is working with her students on combining like terms. This lesson is at the beginning of her content progression for this unit. She has one of her students come up to the board and solve the following expression for y: $x^2 + y = 4x^2 + 2$. After subtracting the x^2 term from both sides, the student combines the $4x^2 - x^2$ term into $3x^2$. As the student is writing this on the board, another student raises their hand and asks, "Well, they don't have the same coefficient, so how could we do that step? Why does that work?" Kelley ponders the question, recognizing that the student does not understand what it means for something to be a like-term. Kelley then reflects and realizes this is a common issue for many students. Kelley responds, "That is a great question. Does anyone know how to answer it?" The student writing on the board stumbles and says they do not know. Silence covers the class. Eventually, a student in the back of the class raises their hand and says, "Well, maybe we could figure it out with a picture or something." Kelley has a choice to make: does she follow this student's idea and sacrifice her lesson plan, or answer the question for the class? Kelley recognizes the importance of understanding what a like-term is for the rest of her students' math career, but she did not plan for this. She wants to do something that engages all the students in a discussion about math and remembers that the whiteboards would allow the class to see each other's work. Kelley responds, "That is a great challenge! Ok, everyone, in your groups, see if you can model this process using a picture, and when you have something, send a representative to the board to draw it." During this time, Kelley circles the room observing student work and passing out markers. She notes which students are on the right track and which are not. Soon, every group has a visual on the board. Most of the groups have a representation similar to one x^2 term being subtracted from 4 of an x^2 term, but not every group does. Two of the groups, in a similar variation, chose to use the exponent to combine like terms, and wrote a picture dictating $4x^2 - x^2 = x^6$. The thinking behind this was difficult to see, especially through the representation the students in this group

chose to make, and Kelley struggled to see where the students were coming from. Kelley asked two groups to explain their pictures: one group with the first representation (Group A), and the other from the second set of variations (Group B). After this, Kelley posed to the whole class, "which representation best matches what a like term is, based on the notes you took yesterday?" The students shuffled, found their notes, and began reading when someone from Group B said "wait… we didn't realize it was 4 copies of the same thing… is that what the $4x^2$ means?" Another student chimes in and says "yeah I think so… because 4 – 1 is 3 and the definition of a like term depends on the exponent because I don't think you can mess with the exponent." The rest of the students start to nod, and Kelley asks again which representation best matches the definition of a like term. The students unanimously agree that Group A is the correct representation. Kelley thanks the student who suggested using a visual representation and provides a summary of what the class just discussed, along with another example of the same idea, being sure to use pictures similar to those of her students to explain and continues with the lesson.

Discussion of Narrative

The adaptation described here is Kelley's choice to use the whiteboards in this classroom to show students' thinking and engage her students in exploring mathematical ideas. This move was not planned, and it significantly altered the course of the lesson. In addition, the adaptation was focused on content, knowledge of students and their thinking, and pedagogical demands of the situation, and was subsequently made with content fidelity, student-specific traits, and adaptive expertise. More specifically, Kelley reasoned that her students needed to know and understand what a like term was, but ultimately, her adaptation was based on where her students were in the lesson. In addition, the adaptation followed a student's contribution to mathematics. Thus, the adaptation was made with respect to student-specific traits. Secondly, the adaptation allowed student mathematical thinking to be seen in multiple representations and progressed the lesson towards collective mathematical reasoning. In this way, the adaptation possessed content fidelity. Lastly, the use of whiteboards is indicative of the application of professional knowledge and classroom constraints that ultimately allowed the adaptation to be successful, noting the mathematical benefits of using the whiteboards. Thus, Kelley demonstrated an adaptation with adaptive expertise. The edges of the ATT were also present while Kelley adapted. Kelley had to think on her feet and pivot to whiteboards to make use of the classroom structures; Kelley improvised. In addition, part of the reasoning for the adaptation was based on Kelley's content knowledge of past, current, and future mathematical needs of her students; Kelley used subject matter knowledge. Lastly, Kelley used formative assessment of her students both before and after this adaptation to create an adaptation specific to her students' needs.

Ultimately, Kelley adapted to her students' need to provide high-quality mathematics instruction that is effective and equitable because it allowed all students to engage with and make sense of mathematics (Celedón-Pattichis et al., 2018; Franke et al., 2007). Adaptive teaching is a skill that should be honed for if and when students in a classroom are in need of a change in pedagogy that would promote further mathematical understanding, such as a student in Kelley's class who may be misunderstanding like-terms based on previous instructional moves.

Discussion, Implications, and Limitations

In light of the considerable complexity in existing literature on adaptive teaching and practice of adaptive teaching, new theoretical approaches are needed to understand adaptive teaching and teacher preparation. By highlighting how each theme from the literature is related, the ATT responds to gaps in current theoretical literature to provide a tool that straddles research and

practice. In future efforts, researchers can leverage other related literature bases to fully understand how they are related to the constructs within the ATT. Examples include teacher metacognition (Lin et al., 2007), teacher noticing (König et al., 2022), ambitious teaching (Lampert et al., 2013), differentiation (Tomlinson et al., 2003), teacher flexibility (Leikin & Dinur, 2007), responsive teaching (Robertson et al., 2016), micro-adaptive teaching (Prediger et al., 2022), Building on Mathematical Opportunities in Student Thinking (MOSTs, Stockero et al., 2022), reflection (Schon, 1988), and improvisational teaching (Sawyer, 2011).

Adaptive teaching is not necessarily something that would happen in every lesson taught. In fact, in considering whether thoughtfully adaptive teaching occurs in literacy settings, Hoffman and Duffy (2016) answered *no* on the basis that adaptations occurred very rarely, attributing its lack to scripted curricula. However, provided adaptive teaching is necessary for effective teaching, and adaptive teaching should be developed in teachers, Hoffman and Duffy (2016) describe that teacher educators have the ability to promote adaptive teaching.

The construction of faces using the edges may be a useful conceptualization of task design in methods courses for learning to be adaptive. For example, a teacher educator could work with specific faces and focus on the edges that support them when engaging students in rehearsals (Kazemi et al., 2016). Future research, specifically in mathematics education, is needed to corroborate these ideas. Focusing on developing teachers' ability to make adaptations that are mathematics specific requires teachers to hold a deep understanding of the mathematics itself, the trajectories of the students in the classroom, as well as the connections between those two concepts to ultimately make an adaptation appropriate to content, students, and pedagogical demands in the class (Wang, 2026). This view requires a conceptualization of adaptability specific to the mathematical content.

In addition to a lack of research focused on mathematics education around adaptive teaching, some scholars have called attention to what affordances are in the curriculum that allow adaptive teaching to occur (Vaughn et al., 2022). Following Hoffman and Duffy (2016), future research is needed to understand a) if adaptive teaching is or is not occurring in mathematics classrooms and b) why and how it may or may not occur. Given prior research has shown policy and curricular decisions to be major points of contention for adaptive teaching, on the basis that adaptive teaching is a component of effective teaching (Vaughn et al., 2022), the mathematics education community has the right to further investigate this. Also, The ATT can be used by the practicing teacher who may be looking to improve their practice. The practicing teacher may choose to reflect on their own teaching and investigate how they view their improvisation, dynamic assessment, and content fidelity, and how these skills are reflected in their own teaching.

Limitations of this work include, firstly, that the tetrahedron is a conglomerate of separate literature bases whose connections are, at this point, theorized. The mathematics education community needs to collect data to further explore the ATT and adaptive teaching. In addition, adaptive teaching is a difficult skill to hone. Because of this, its development, while becoming increasingly important, may be difficult to ultimately develop. Secondarily, more research is needed to understand where self-regulation and adaptive metacognition fit with the ATT and adaptive teaching. For the mathematics education community, it is important to consider what support teachers need to be adaptive, what supports or constrain the development and presence of adaptive teachers at all grade levels, and where related literature bases can aid in better understanding adaptive teaching. It is my hope that communicating this framework to the field of mathematics education will spark conversation around adaptive teaching as an important aspect to develop in all teachers.

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TEACHING INFLUENCES FOR INSTRUCTORS OF GRADUATE MATHEMATICS

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Research on university mathematics teaching has largely focused on graduate student preparation and undergraduate instruction, with limited attention to faculty who teach graduate-level courses. This study examines how instructors of graduate mathematics develop teaching practices. Drawing on a sociocultural perspective (Ball & Ladson-Billings, 2020) and interview data from three faculty members, I analyze instructor-reported influences using a framework adapted from Oleson and Hora (2014). Findings indicate that experience as an instructor is a dominant influence, but its impact varies: for some, it reinforces inherited norms; for others, it prompts experimentation or identity-driven commitments. These findings suggest that teaching development is mediated by identity, affect, and institutional incentives, raising questions about how departments might more intentionally support reflective instructional growth.

Keywords: Teacher Learning, Education in Graduate Mathematics

Introduction / Problem Statement

Research on the teaching of university mathematics has documented the persistence of lecture-dominant instruction, limited pedagogical preparation, and strong inheritance of instructional norms (Bass, 2003; Schoenfeld et al., 2016; Speer et al., 2010). While scholarship has examined graduate student teaching assistants (Deshler et al., 2015; Speer et al., 2005; Musgrave & Carlson, 2016) and undergraduate mathematics instructors (Biza et al., 2014; Coppin et al., 2009; Mesa et al., 2020), considerably less attention has been given to faculty who teach graduate-level mathematics. Graduate mathematics courses play a central role in shaping future researchers, faculty, and disciplinary norms. Yet we know little about how instructors of these courses develop their teaching practices.

Teaching does not emerge in a vacuum. Prior experiences such as those as students, instructors, researchers, and community members, shape how instructors conceptualize the purpose of class time, the role of students, and the meaning of effective instruction. Making these influences visible creates space to interrogate taken-for-granted practices and to imagine alternative possibilities.

This study asks: How are the teaching practices of instructors of graduate mathematics developed?

Focusing on instructor-reported influences, this paper examines how three faculty members describe the experiences that have shaped their teaching. In doing so, it contributes to ongoing conversations about teacher learning in higher education and extends them into the largely unexamined context of graduate mathematics instruction.

Theoretical Framing

This study is grounded in a sociocultural perspective on teaching and learning (Ball & Ladson-Billings, 2020). From this view, teaching is not simply the enactment of content knowledge or technique. Rather, it is shaped by the instructor's histories, identities, relationships, institutional positioning, and the communities in which they participate. Teaching is understood as socially situated practice.

Sociocultural theory foregrounds the idea that instructors' actions are mediated by tools, norms, and prior participation in communities of practice (Lave and Wenger, 1991). Thus, when instructors teach "the way they were taught," this is not merely imitation but participation in historically situated instructional traditions. At the same time, instructors are not passive inheritors of norms; they reinterpret and reconfigure past experiences considering current contexts.

To build an analytic framework, this study draws on categories developed by Oleson and Hora (2014), who identified multiple sources of teaching knowledge in higher education. These include experiences as a student, experiences as an instructor, professional development, interactions with colleagues, disciplinary research, and non-academic influences. Using these categories deductively allows for systematic examination of the range of influences instructors report, while remaining attentive to how they interpret and prioritize those influences.

Taken together, this framing positions teaching development as cumulative, relational, and identity-laden rather than purely technical.

Methods

Design and Context

This study draws on data from a larger ethnographic comparative case study of graduate mathematics education. The broader project included interviews with 14 graduate students, five undergraduate students, ten faculty members, and two postdoctoral scholars, as well as nine graduate mathematics classroom observations and field notes. The present analysis focuses specifically on interviews with three faculty members who teach graduate mathematics courses. Each interview lasted approximately 60 minutes and was transcribed in full.

Participants

To preserve anonymity, participants are referred to by pseudonyms: Hank, Mariana, and Jason. All three are research-active mathematicians teaching graduate-level courses in a PhD-granting mathematics department.

Data Collection

Interviews were conducted either in person or over Zoom and were recorded and later transcribed. Interview questions were not framed explicitly around "influences." Rather, participants were asked questions about their teaching practices, including: "What is the purpose of class time?", "How do student responses influence your teaching?", "How do you typically plan for a class?" Responses to these prompts frequently included reflections on how participants had learned to teach and what shaped their instructional decisions.

Analysis

Full transcripts were analyzed using a deductive coding scheme based on Oleson and Hora (2014). Their coding scheme included: experience as an instructor, experience as a student, experience as a research, and non-academic experiences. Further, sub-codes of the experience as instructor code included: prior classroom experiences, professional development, reflections on feedback, and interactions with other instructors. I chose to include an "other" code to keep in mind the possibilities of other factors that may not fall neatly into one of the codes listed above. Coding of transcripts focused on identifying both the presence of these influences and the ways participants described their impact. Analytic memos were used to compare across cases and identify patterns.

Findings

Hank: Learning to Teach Through Teaching

Hank's instructional development was shaped primarily by experience in the classroom rather than by formal pedagogical preparation or structured inquiry into teaching. When describing how he learned to teach, he emphasized the near absence of formal support: "None, almost none … We just had a one-hour meeting with the equivalent of the associate chair for undergraduate, who just gave us some tips, and that was it." In the absence of sustained preparation, Hank described drawing on available local models. For example, he adopted elements of a colleague's assessment structure: "[The] format [of] my exams came from [professor], and he had a true-false question in his course, and I thought, oh, this sounds good. I just used it." These comments suggest that Hank's teaching repertoire has developed through accumulated practice and pragmatic adaptation rather than through intentional study of pedagogy. He also articulated a perception that institutional structures provide little incentive to invest deeply in teaching:

> Part of the problem is lack of incentives. So for somebody like me, there's no carrots and no sticks. There's no incentive for me to do anything really apart from my basic job. The university doesn't detectively care if I do a good job or a bad job.

Taken together, Hank's case illustrates how teaching experience itself becomes the primary site of learning. Without structured feedback or institutional emphasis, prior classroom practice—his own and his colleagues'—serves as the dominant influence on instructional decision-making.

Mariana: Identity, Community, and Teaching as Social Commitment

Mariana described a more interconnected set of influences, linking her instructional commitments to her experiences as a student and to her identity within the mathematical community. She explicitly connected her background to her pedagogical goals:

> I'm interested in supporting students who are not necessarily always thought of as the students that will be the leaders in mathematics… Because I just experienced first hand, what it is to have a class structure for other people, you know for other culture[s], for other needs.

For Mariana, prior experiences of positioning and belonging informed her commitment to designing more inclusive classroom structures. She noted directly, "Of course, now, and actually it [background] informs my practice as an instructor." Her descriptions of mathematics as collaborative further shaped her instructional orientation:

> I think one of the nicest things about math is that we work in groups to tackle hard problems, and we each bring a perspective … really in math is a place where you can kind of see it happen quite a bit.

Unlike Hank, Mariana discussed frequent student feedback, active learning structures, and intentional reflection on teaching. Her development as an instructor appears mediated not only by experience in the classroom but also by identity, values, and a vision of mathematics as communal practice. Teaching, in her account, is inseparable from questions of belonging and participation.

Jason: Experimentation Without Structured Feedback

Jason described himself as someone who actively experiments with his teaching. He reported polling students about instructional format and adjusting in response to their input: "I polled them on: do you want to continue with the active learning or do lectures? And the overwhelming response was to switch to lectures." These experiments, however, occurred largely without

formal structures for feedback or mentorship. Although Jason frequently tried new approaches, he did not describe sustained guidance on how to interpret student responses or evaluate instructional impact. Over time, this lack of structured support shaped his orientation toward pedagogy. Reflecting on his experiences, he noted:

> That experience is really pushing me towards this attitude that the details don't matter. There's not some magic way of presenting things that will suddenly make the students understand and make them engaged.

Rather than indicating disengagement from teaching, this statement reflects the challenge of experimenting in relative isolation. Jason expressed a desire to "do a good job" and repeatedly sought student input, yet the primary feedback available to him was immediate student preference rather than longer-term evidence of learning.

His case highlights how experimentation alone may not lead to sustained pedagogical development when instructors lack communities or structures to help them interpret and refine their practice. Experience as an instructor is again central—but its impact depends on the kinds of feedback and reflection that accompany it.

Cross Case Comparison

Across all three participants, experience as an instructor emerged as a central influence on teaching practice while Mariana also expressed experience as a student and personal identity as being factors. Each described classroom teaching itself as a primary site of learning. However, the meaning and impact of that experience varied, and these differences appear connected not only to identity and commitments but also to career stage.

Hank and Jason were veteran faculty members with many years of teaching behind them, whereas Mariana was in her first few years as a faculty member. This distinction matters. For Hank and Jason, accumulated experience appeared to stabilize particular orientations toward teaching. Hank described refining what already worked and drawing from established local norms. Jason described cycles of experimentation shaped primarily by immediate student feedback. In both cases, teaching practices seemed to have stagnated over time. Without sustained external feedback or structured professional development, classroom instruction functioned less as a site of transformative teaching practice and more as a mechanism of delivering content.

Mariana, by contrast, spoke from an earlier point in her faculty trajectory. Her reflections suggested a more fluid relationship to practice. She described actively experimenting with structure, seeking feedback, and intentionally aligning her teaching with her commitments around belonging and participation. While experience was still influential, it had not yet hardened into settled routines. Her case implies that early-career stages may represent a critical window in which teaching orientations are more malleable.

Identity also differentiated the cases. Mariana explicitly connected her experiences of positioning within mathematics to her instructional commitments. Hank and Jason did not foreground identity or belonging as shaping forces in their teaching. Instead, they emphasized pragmatics, experimentation, or institutional structures. Taken together, these comparisons suggest that teaching development in graduate mathematics is mediated by an interplay of accumulated experience, career stage, identity, and access to feedback. The longer one teaches without structured opportunities for reflection, the more instructional norms may become ingrained and less visible as choices.

Discussion/Implications

This study extends conversations about faculty teaching development into the context of graduate mathematics, where pedagogical preparation is often minimal and instructional norms remain largely unquestioned. The findings suggest that experience as an instructor is a powerful force in shaping teaching practice. However, experience alone does not guarantee growth or transformation. In the absence of structured feedback or meaningful incentives, experience may instead reinforce inherited norms and stabilize existing approaches.

The cases also highlight the importance of identity and affect in teacher learning. Mariana's reflections demonstrate how experiences of belonging, exclusion, and disciplinary positioning can shape instructional commitments. Teaching, in this sense, is not only a technical endeavor but also an expression of how instructors see themselves within the mathematical community. Professional development efforts that focus exclusively on technique risk overlooking these mediating factors.

Career stage appears to play a consequential role. Early-career instructors may be in a period where practices are still forming and therefore more open to intentional shaping. Over time, particularly when experimentation occurs without sustained feedback, instructional routines can become entrenched. This raises questions about when and how departments intervene to support reflective development. If teaching practices solidify through repetition, then structured mentorship and community engagement early in a faculty member's career may be especially important.

More broadly, these findings complicate the familiar claim that faculty simply teach the way they were taught. Prior experiences matter, but they are refracted through identity, institutional structures, career trajectories, and opportunities for reflection. Making these influences visible may create space for departments to reconsider how graduate mathematics instructors are supported—not only in learning new strategies, but in examining the histories and commitments that shape their practice.

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MIDDLE SCHOOL MATHEMATICS TEACHERS' ENACTMENTS OF DRAFTING AND REVISING: EXPLICIT AND IMPLICIT APPROACHES

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In this study, we examined how middle school mathematics teachers enacted Rough Draft Math [RDM]. Drawing on self-reports from 13 teachers in a larger mixed-methods project, we analyzed teachers' written reflections about how they engaged students in drafting and revising and how successful they perceived these opportunities to be. Teachers' self-reports of their enactments of RDM varied, and we categorized their reports of drafting and revising enactments into explicit and implicit enactments. We also identified differences in teachers' reports of their success of enacting RDM that aligned with whether they reported enacting RDM implicitly or explicitly. A contribution of this study is a framework to characterize RDM enactments in terms of implicit and explicit drafting and revising. Findings also illustrate the value of examining teachers' criteria for success with teaching practices.

Keywords: Instructional Activities and Practices, Middle School Education, Classroom Discourse

Mathematics teachers increasingly try to support students in sharing in-progress ideas, revising their thinking, and taking intellectual risks as they make sense of mathematical concepts. Rough Draft Math (RDM; Jansen et al., 2020) offers one such approach by positioning students' initial ideas as drafts and building classroom structures in which students are expected and supported to revise those drafts over time. Although prior work has documented a variety of RDM-aligned practices and argued that naming and reinforcing these practices can help teachers plan, enact, and reflect on them more deliberately (Jansen et al., 2026), there is a need to develop a framework of how teachers characterize their RDM enactments and how they judge the success of these enactments in their classrooms.

This study addressed that problem by analyzing middle school teachers' self-reports (diary responses) about how they enacted rough drafting and revising over a ten-day period and how successful they perceived those opportunities to be. We focused on (a) how teachers' self-reported RDM practices can be characterized as explicit or implicit enactments and (b) how teachers' success ratings and rationales for success revealed different dimensions of efficacy.

Background

Given that teachers can enact RDM in various ways, it is worthwhile to name teachers' approaches. Specifying and decomposing teaching practices "involves breaking down practices into its constituent parts for the purposes of teaching and learning" (Grossman et al., 2009, p. 2058). Once teaching practices have been named and labeled as discrete instructional moves, teachers can practice them and improve them in a more targeted manner (Grossman, 2011) due to having access to a specific technical language for teaching (Grossman et al., 2009).

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Listening to teachers' voices about their practices provides insights about what teachers can see themselves enacting in their contexts. RDM has been developed previously through listening to teachers' voices (e.g., Jansen et al., 2016) because research has a greater impact on practice when teachers play a purposeful role in the research process (Cai et al., 2017). We amplify teachers' voices to offer resonance and credibility to our investigation, which are criteria for quality in qualitative research (Tracy, 2010).

Explicit and Implicit Approaches to Rough Draft Math

Rough Draft Math takes place when teaching has been intentionally planned and enacted to provide students with opportunities to draft and revise to make sense of mathematics. Teaching with intention involves planning and making instructional decisions that are purposeful and deliberate on the part of the teacher (Fisher et al., 2016). However, intentionality behind teaching practices may be communicated to students more and less explicitly.

Some teaching practices intended to support students engaging in learning could be enacted in ways that are "useful but subtle" (Watagodakumbura, 2013, p. 303). One example is allocating time for what teachers value (e.g., time for students to engage in well-thought-out reasoning rather than recalling). Students may be aware of the literal event (e.g., more time) but they might be less aware of benefits that they could experience (e.g., greater opportunities to develop their reasoning) if teachers do not explain potential benefits to students.

In our research group's prior work (Jansen et al., 2026), we found that teachers reported that they enacted RDM with more or less explicitly communicated opportunities to revise. We interpreted explicit revising opportunities as events when students were all asked to write more than one attempt of their thinking or solution through a structured activity. Explicit revising often took place on a written document where students had specific sections to write a first draft and a revised draft. An example would be the Math Language Routine (Zwiers et al., 2017), Stronger and Clearer Each Time; students write a first attempt, discuss those first attempts with peers, and then write a second attempt. We interpreted implicit revising opportunities as instances when it was possible for a student to revise, but not every student was asked to do so. An example would be having students work in groups, but not asking students to document multiple attempts, as students could potentially revise after hearing a peer's point of view, but they were not directly asked to do so.

Teachers' Self-reports of Successful Teaching

Teachers' reports about their teaching can reveal their sense of teaching efficacy, which is beliefs about their capabilities as teachers (Tschannen-Moran, et al., 1998). According to Wheatley (2005), there are multiple dimensions for teachers to use to judge the success of their teaching: (a) teachers' perception of their ability to enact a practice (agent-means efficacy), (b) teachers' perception of whether an enactment of teaching can generate positive student outcomes (means-ends efficacy), and (c) teachers' perception of their ability to generate positive student outcomes (agent-ends efficacy). When understanding teachers' self-reports of their success with teaching, it is worth investigating which dimensions teachers report.

Data and Methods

Procedures

We collected data within a larger mixed-methods study exploring the nature of middle school mathematics teachers' rough drafting and revising practices. We instructed teachers to select a single class period from among their rotating mathematics classes to focus on. Teachers completed self-reports in the form of diary reflection activities about their typical rough drafting and revising practices in their selected class period.

Participants

Thirteen secondary mathematics teachers participated during the 2025/2026 academic year. Teachers were from two U.S. states, one in the Mid-Atlantic and one in the Pacific Northwest. The majority of teachers were Female (77%; 10 teachers), and White (77%; 10 teachers), 8% were Asian, Latino/a, and biracial, respectively (1 teacher each). Teachers ranged in years of experience as a teacher of record from 4 to 28 years. 46% taught 6th grade (6 teachers), 15% taught 7th grade (2 teachers) and 38% taught 8th grade (5 teachers).

Measures

All teachers completed diary entries during the second half of September. The goal of the teacher diary activity was to capture a general picture of the rough drafting and revising practices that are part of each teachers' daily mathematics instruction. Teachers responded to a fillable Google form which included 9 survey questions and 15 open-ended text entry questions. We asked teachers to reflect on the past two weeks (10 school days) of instruction in their focal class period and to then report the topics they taught, whether and how these topics aligned with rough drafting and revising, their enactments of rough drafting and revising practices while teaching including whether, when, how, in which contexts and to what extent they enacted these practices, and their perceptions of the success of these enactments.

For this study, we investigated teachers' open-ended text-entry responses. Open-ended questions were typically presented after teachers were asked to report on specific practices via list-selection items. For example, one list selection item asked teachers: "Over the 10 school days, during the following activities/lesson phases, did you enact revising? Select all that apply." We provided choices including "launching a lesson," "while discussing publicly displayed student work," or "at the conclusion of the lesson," among others. The open-ended item to follow was: "Look at the activities/lesson phases that you selected above. In a few sentences, tell us more about the extent to which you enact revising in each of these lesson phases."

Two members of our team assessed teachers' text entry responses according to an investigator-developed qualitative criteria for coding, using and open-ended, emergent process. We engaged in what Saldaña (2013) called "eclectic" coding, which was a back-and-forth process between identifying possible codes from the data and testing the codes on additional data points, refining as needed.

We categorized each enactment quote, for drafting or revising, as either explicit or implicit. *Explicit enactments* were described as structured activities in which all students were expected to participate. *Implicit enactments* were instances when teachers used drafting and revising language to build an environment conducive to drafting and revising or to encourage a discretionary drafting/revising activity that was not a required component of the lesson. One team member coded quotes describing drafting enactments and one coded quotes describing revising enactments. We then sub-categorized based on the function of each enactment (e.g. "activating prior knowledge" or "initial solving attempt.")

For the second research question, two members analyzed open-ended responses in which teachers reported how successful their drafting and revising opportunities were in helping students refine or reframe their thinking. We coded these responses for the criteria teachers wrote to determine success of their enactments. Teacher success ratings (very/mostly successful vs. somewhat successful) were then interpreted in relation to these coded indicators of student thinking.

Findings

Research Question 1: How can middle school mathematics teachers' self-reported enactments of drafting and revising be characterized as implicit or explicit?

Explicit Enactments of Drafting

Teachers reported multiple ways of explicitly engaging students in drafting. For instance, teachers shared that they asked students to write down initial thoughts individually before working with peers or to draft in groups before sharing at a larger level. When describing explicit drafting in their classrooms, teachers focused on different functions, such as: (a) eliciting students' first attempts, (b) activating prior knowledge, and (c) generating predictions.

Eliciting first attempts. Teachers reported using explicit drafting routines for students to start a solution process for a new problem or topic. Ms. Sponaugle explained the start of her routine: "Typically I use rough draft math the most when introducing a new topic … The first step introduces the problem and students are asked to provide a rough draft on how to solve it." For her lesson, she described: "RDM is completed with practice problems at whiteboards throughout the lesson. Students are typically given a higher-order thinking problem and asked to present a draft; whole group/small group discussion occurs before editing and finalizing their solution." Throughout their lessons, teachers used drafting as an entry point into problem-solving.

Activating prior knowledge. Teachers also used drafting to activate students' prior knowledge about new concepts. Ms. Brennan shared a specific example: "I utilize rough draft math at the beginning of lessons to see what students already know about the concept of ratios and unit rate… I had students write down what they thought a double number line was and what it could be used for." She then noted: "By having this knowledge to pull from, students were able to come up with a rough draft initial idea as to what unit rate may be." Ms. Brennan's example showed how a teacher could use an explicit drafting task, in this case, wondering about a double number line, to prompt students to share their existing knowledge of a topic to build on later.

Generating predictions. Teachers also reported using drafting to generate students' predictions about new concepts or vocabulary. Ms. Dougherty wrote: "We often set up our lesson with some predictions or ideas about a concept." Ms. Brennan described her process of drafting with whiteboards: "I have students write out on their whiteboards what they think the [vocabulary] term means and how it relates to the task." Drafting in this way required students to make predictions about concepts and connect those ideas to the lesson. This was an explicit drafting routine because students were given a specific prompt to draft with a structured goal.

Implicit Enactments of Drafting

Implicit drafting often occurred when students were working in groups or during whole-class discussion. These practices were implicit because they were not structured or planned drafting events. Implicit practices, like explicit practices, also seemed to have specific functions, such as (a) modeling drafting and (b) creating a culture of drafting in the classroom.

Modeling drafting. During a lesson, students can learn what it means to draft from peers or their teachers. Ms. Brown wrote: "In the whole group, the students who are comfortable can model what it looks like to rough draft ideas and take risks. In the small group settings, the quieter students feel more comfortable engaging in these tasks." Ms. Alderman explained that teachers can also model drafting: "During lessons in seminar space, I modeled thinking aloud through rough drafting." Teachers can use modeling, either by students or themselves, to demonstrate how to draft. These moments of modeling were implicit enactments because they

provide a space for students to understand the drafting process without being expected to do so, as drafting is not a required component of the lesson.

Creating a culture of drafting. Implicit drafting practices can foster an environment where drafting is encouraged in various stages of a lesson. Ms. Alderman shared: "Most of our time spent rough drafting happens in the heart of the lesson as students are persevering through problems." Mr. Young builds a culture of drafting in his classroom, addressing an "I don't know" by encouraging drafting: "My response of their sharing of 'I don't know' to a question has often been, 'Thank you… can we think this through as a class? We're rough drafting this…'" Although these might not have been planned drafting events, students were working in the spirit of drafting as they solved problems in class and were encouraged by their teacher to engage in drafting to problem solve, whether planned or spontaneously. Given that not all students were expected to draft in these discussions, drafting was an implicit practice.

Explicit Revising

Teachers also reported multiple ways of explicitly promoting revising, often at the end of the lesson or after a collaborative activity. If teachers had students discuss their drafts in groups (and/or in whole class discussions), some of these teachers explicitly asked their students to document revisions to their thinking that were informed by these interactions. Another explicit revising opportunity teachers reported was allowing students to revise tests or quizzes.

Revising as a reflective lesson closure. Some teachers incorporated revising by closing their lessons and asking students to write down how their thinking had changed. For instance, Mr. Young stated: "Students have also been asked to reflect at the end of a lesson with 'At first I thought ___, now I think ____.'" Similarly, Ms. Brennan described how revising occurs during and at the end of her lessons:

> As the lesson progresses, I have students go back to their definitions and add notes or diagrams to what they already had written. I also have students who were completely off describe using sentence frames, such as 'at the beginning of the lesson I thought… now I think.… My thinking changed because…' After student discussion at the whiteboards, I then allow students ample time to revise their thinking a final time during their cool down/independent work time.

Revising after collaboration. Teachers incorporated explicit revising through peer collaboration. Teachers expected students to revise their individual work after talking with partners. For example, Ms. Shisler noted: "In Sub-Unit 2, students worked on scale drawings. Once they talked with partners and also saw student work, students revised their thinking and drawings." Another teacher, Mr. Young, described revision occurring after discussion using a different color to indicate changes to students' prior drafts:

> During partner talk, partner work time, and in whole group plenary discussion I will ask students to think about whether their thinking is the same or different. When applicable, I will then encourage them to look specifically at what led to them having different answers to then decide what answer makes most sense. I encourage them to *not* erase, use a different color pencil for revisions.

If a teacher expected all students to write down how their thinking changed after collaborating, the teacher enacted revising explicitly.

Revising tests or quizzes. Lastly, teachers provided explicit revising opportunities by allowing students to revise tests or quizzes. Mr. Jenkins stated: "Students turn their quizzes/tests in and get them returned highlighted with no grade. [They can] make corrections and elaborate

on what was done incorrectly." Another teacher, Dr. Green, described: "To earn back points, they must retake the assessment, so the revisions will help them know what to review." If teachers invited every student to complete assessment revisions, this was explicit revising.

Implicit Revising

Teacher reports of students potentially revising, but having no structured activity asking them to do so, indicated implicit revising. Teachers wrote that they suggested revision opportunities to students, but these did not describe events when all students were expected to revise. Descriptions of implicit revising sounded like reminders from the teacher, perhaps in an effort to create a classroom culture of revising, in contrast to revising events or activities.

Teachers' examples illustrated this implicit positioning of revising as discretionary, rather than event-based. Ms. Brennan described: "students are reminded throughout their small group instruction at their whiteboards that they can add or change ideas to their initial thoughts at their own discretion," without identifying a specific moment when revision was expected to occur. Likewise, Dr. Green emphasized encouraging clarifying questions and invoking students' "right to revise their thinking" during small group work: "I introduced students to Torres' Rights of the Learner at the beginning of the year and I try to use the phrase 'right to revise your thinking' either by noting when students are using that right or when students can use that right." Similarly, Ms. Dougherty noted that revising could occur at any point, depending on the student: "Some students can revise their thinking as we talk in whole group, while some need more 1:1 support and instruction to really understand enough to revise." These examples demonstrate implicit revising enactments, in which teachers positioned revising as an available and valued practice but left its enactment largely unstructured.

Teachers' Perceived Success

Research Question 2: How do teachers' reported success levels and rationales differ between implicit and explicit enactments of rough drafting and revising? We observed that some teachers' success ratings for enacting drafting and revising [RDM] appeared to align with their observations of changes in students' thinking. Teachers who reported greater success with enacting RDM (very or mostly successful) and reported explicit enactments of RDM also reported criteria for success in terms of students' thinking. Teachers who reported less success or implicit enactments did not use these criteria to determine success. We interpreted teachers' reports of success through Wheatley's (2005) lenses of agent-means, means-ends.

Explicit Enactment and Very/Mostly Successful

Teachers who reported enacting RDM explicitly and rated their enactment as very or mostly successful described success in terms of shifts in students' mathematical thinking. BJ reported that rough drafting "got students thinking, they produced quality work, and it has them looking at math in a totally different way." Ms. Shisler, similarly, grounded her 'very successful' rating in how students changed their ideas: "When I move them to Mode 2, I hear a lot of agreeing and disagreeing and see students revise their original thinking. During Mode 3, when we share out as a class, students either add on to their original rough draft thoughts and answers, or they revise their thinking altogether and change their answers or drawings." For these teachers, success aligned with how RDM (means) led to changes in students' thinking (ends).

Explicit but Somewhat Successful

In contrast, teachers who reported enacting RDM explicitly but rated their enactment as only somewhat successful justified their ratings by pointing to limits in student engagement (ends) or doubts about their own enactment (agent). For example, Ms. Sponaugle wrote: "students are used to copying notes and were first confused as to what to write or how to go about each revision.

When working in small groups, not all students were actively engaged. With the introduction of a lesson, time to complete all steps was challenging." Her 'somewhat successful' rating may reflect an incomplete drafting-revising cycle and uneven participation, which did not lead to refined thinking across the class. Mr. Young also implemented explicit drafting routines; however, he wrote: "I'm not sure I am doing what I am supposed to be doing… there's so much I want to do with them that it has been hard to prioritize a sequence that helps build their capabilities (while moving along with what the curriculum requires)." Here, the teacher's rating may be tied to his uncertainty about his enactment (agent) of explicit drafting and revising routines (means). He expressed uncertainty about whether he implemented the approach correctly and described struggling to prioritize instructional steps that both support students' development and meet curricular demands.

Implicit Enactments and Success Criteria Reported

Teachers who reported implicit enactments, regardless of their reported level of success with RDM, usually described success in terms of student engagement or classroom culture. Ms. Miller, who rated her (implicit) enactment as very successful, wrote: "We are in a constant state of rough drafting and revising. 'I'm not sure but…' is used all day long. Students are encouraged to think out loud and revise whenever they need to do so," and concluded that "rough draft & revision is part of our classroom culture. We are 2 months in, and students have taken ownership of the process." Her evidence for success centered on students' willingness to participate and take ownership, not on descriptions of changes in students' thinking. This was a means-ends reflection on teaching effectiveness. Her ends outcome (e.g., engagement) differed from the means-ends reflections among teachers reported explicit, strongly successful enactments. Mr. Pryzbylewski rated his implicit enactment as somewhat successful, and similarly highlighted: "I have not succeeded much with formal revising… we don't have a routine or system where we intentionally plan to revise something." This was an agent-means reflection because he reflected on his own success with enacting RDM. He added: "I think the spirit of rough draft thinking is in the classroom." This part of his rationale for success emphasized using RDM to guide his instruction (means) and cultivating the "spirit" of rough draft thinking (ends of classroom culture) rather than changes in students' thinking (ends of students' thinking).

Discussion

We determined that teachers reported enacting RDM with explicit and implicit drafting and revising practices. We also determined that teachers reported a range of success levels, but the teachers who reported strong success and explicit RDM enactments also reported changes in their students' thinking. Our findings indicate that teachers can enact RDM in their classrooms in different ways with a range of functions, and that there were different ways that RDM added value to their teaching. These benefits included changes in students' thinking over time and changes in students' engagement or the culture of the classroom.

Teachers' Explicit and Implicit Enactments of Drafting and Revising

Teachers shared that they enacted drafting and revising in a variety of ways, as found previously (Jansen et al., 2026). Some reported explicit RDM activities and others reported more implicit RDM enactments. Teachers explained that drafting can create a basis for later revisions, either by revising individually or in groups. Implicit RDM practices can also be valuable for sustaining a classroom culture of ongoing learning. When the language and spirit of RDM are present, students may already be accustomed to the ideas of drafting and revising, even without structured activities. This study extends this research team's previous work (Jansen et al., 2026)

by adding more clarity to definitions for explicit and implicit RDM enactments and providing additional examples of explicit and implicit RDM enactments.

Teachers' Perceptions of Successful Enactments

Our analysis revealed differences in teachers' criteria for success, and these criteria varied both in terms of teachers' perceptions of success and whether they reported enacting RDM implicitly or explicitly. We interpreted that teachers drew on different perspectives on teaching efficacy (Wheatley, 2005): (a) agent-means efficacy [their (agent) perceived ability to enact RDM (means)], (b) means-ends efficacy [whether RDM (means) generated positive student outcomes (ends), such as revisions to students' thinking or improvements to students' engagement]. Teachers who reported implicit enactments of RDM or teachers who reported less success with RDM did not refer to students' thinking as evidence of success with RDM. Our analysis extends Wheatley's (2005) research by showing how RDM enactment type and perceived levels of success with RDM may align with different efficacy judgments.

Limitations and Future Research

We acknowledge that this study is limited due to a relatively small sample size. Additionally, with written reflections, we cannot probe for future elaborations as we could during an interview. However, the value of this analysis is that we were able to identify explicit and implicit RDM practices, which could be studied in other classrooms. These practices are also worth naming so teachers could adopt them into their own practice (Grossman et al., 2009).

Future research could investigate how teachers' self-reports of explicit versus implicit practices align with researchers' observations of their teaching. Teachers might not report components of their teaching practice that a researcher could see as productive in a video-recorded observation. Finally, we studied only middle school teachers at a single point in time. Further research could explore self-reports of teachers at other grade bands and over time.

Implications for Teaching Practice and Teacher Education

If teachers develop an awareness of a variety of explicit and implicit drafting and revising practices, they could reflect upon enactments to try in their own classrooms. Drafting and revising can serve multiple functions in a classroom. Some teachers might choose to enter into enacting RDM implicitly by using drafting and revising language to create a culture of drafting and revising in their classrooms. Alternatively, teachers may first try explicit enactments to help students understand the value and purpose of RDM. Professional development for teachers could introduce teachers to various RDM enactment approaches and support teachers with recognizing criteria that they use to determine success. Teachers could consider how to improve their practice and see strengths in their teaching that they can build upon through identifying their RDM enactments and considering a variety of dimensions for interpreting success of their teaching.

Conclusions

Our study of self-reports of teaching showed that teachers enacted RDM practices in their classrooms in multiple ways and with multiple functions. Teachers enacted drafting and revising explicitly (through specific activities) and fostered a culture of drafting and revising (through implicit practices). Secondly, teachers reported varying levels of success with enacting RDM, but those who reported more success and explicit RDM enactments also reported changes in their students' thinking. We continue to stress that teachers' voices are a valuable resource for understanding their practices in the classroom, and we also advocate that researchers benefit from trusting teachers' self-reports as evidence of their classroom practices.

Acknowledgments

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ECOLOGICAL FACTORS INFLUENCING MATHEMATICS TEACHERS' INSTRUCTIONAL DECISION-MAKING: A CROSS-CASE SYNTHESIS

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In this qualitative multiple-case study framed by Bronfenbrenner's ecological systems theory, the author conducted semi-structured interviews with three elementary teachers in districts across northern Utah. These interviews utilized the novel Ecological Factors Map to explore how teachers described the interplay of factors within their ecological systems and the influence of these factors on their implementation of mathematical student discourse. After developing within-case themes through a thematic analysis, the author performed a cross-case synthesis to identify themes spanning the three cases. These cross-case findings highlight the complexity of teacher decision-making and show that various factors within each participant's environment interact to affect their decision-making. This paper incorporates extant literature into a discussion of the cross-case findings and provides recommendations for educational leaders.

Keywords: Instructional Activities and Practices, Classroom Discourse, Elementary School Education

Introduction

As educational leaders work to promote stronger mathematics academic gains for students, the instruction that takes place in the classroom does not always align with what is considered best practice (Remillard, 2005). Though mathematics education research is robust, there remains a gap between research and the practices that take place in real classrooms (Silver & Lunsford, 2017). Because the instructional decision-making of mathematics educators is complex and affected by various factors (e.g., Bartell, 2011; Conner & Singletary, 2021; Herbel-Eisenmann, 2006; Ruef, 2021; Schifter, 1995), there is a need for research that examines the interplay of factors that influence teachers' instructional decisions in the mathematics classroom. This exploratory study used ecological systems theory (Bronfenbrenner, 1974, 1977) to explore how teachers described their decision-making related to one extensively researched teaching strategy—mathematical student discourse (MSD). Here, MSD refers to the opportunities for students to engage in mathematical discussions in whole-group, small-group, or peer-to-peer settings (Chapin et al., 2009). In this study, participating teachers used MSD to refer to many different forms of mathematical discourse, including whole-group discussions, brief opportunities for students to share an idea with another student, choral or individual responses to questions posed by the teacher, and partner work. The purpose of this study was to explore factors influencing teacher decision-making for instruction involving MSD (as well as mathematics instruction more generally) in order to better understand the support teachers need both individually and collectively to provide rigorous mathematics instruction to all students.

Theoretical Perspective

Ecological Systems Theory

Ecological systems theory, developed by Urie Bronfenbrenner (1974, 1977), suggests that humans' development is impacted by the interactions of aspects of their complex environments, or ecosystems. Bronfenbrenner (1977) conceptualized four systems that make up an individual's environment and influence the individual's development: (1) the microsystem—the

environmental settings and elements with which the developing individual directly interacts, (2) the mesosystem—the interrelations between the microsystem elements, (3) the exosystem—interrelations between social structures that influence the individual's setting, and (4) the macrosystem—the overarching culture and patterns encompassing the developing individual's world.

Though ecological systems theory is a theory of human development most often used in social work, educational studies that have used this theory as a framework (Buchanan, 2020; Falloon et al., 2022; Smith et al., 2017) demonstrate that the highly complex and social field of education can also benefit from such a holistic examination. In this study, the theory provided a meaningful lens through which to view the experiences of teachers and the interplay of ecological factors that affect their decision-making for mathematics instruction, particularly related to the facilitation of MSD.

Figure 1 illustrates the conceptual framework for this study, framed with an ecological systems perspective. At the center of the framework are the teacher and components of the teacher (Remillard, 2005). An arrow connects the teacher to the teacher's decisions for MSD, representing the effect that the teacher and the components of the teacher have on their implementation of MSD. Factors shown to influence teacher decision-making sit within the teacher's ecological systems. Bidirectional arrows show the influence that these factors may have on each other, and on the teacher and the teacher's decision-making.

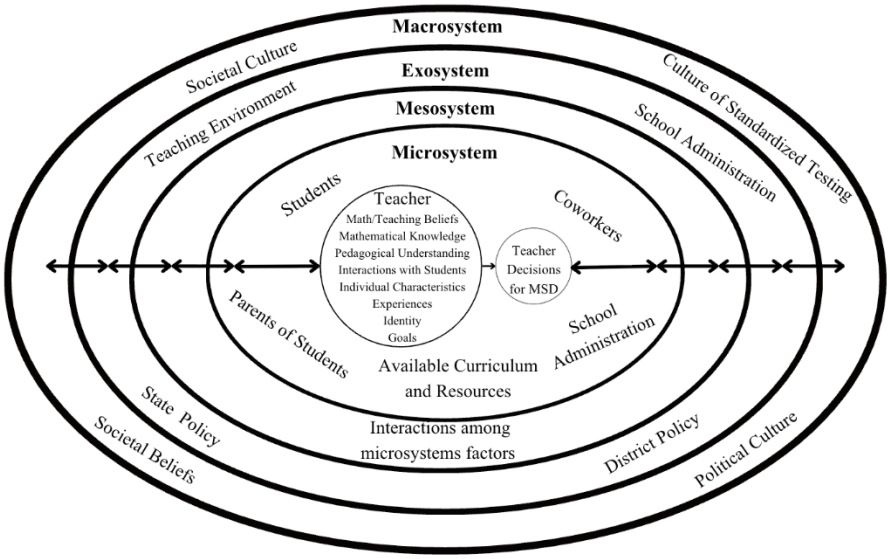


Figure 1: Conceptual Framework

Method

I conducted a multiple-case study (Yin, 2014) of three elementary teachers who teach mathematics in separate districts across northern Utah. I conducted teacher interviews, which were audio recorded and transcribed, with each participant utilizing the novel Ecological Factors Map (EFM, see Figure 2). I created the EFM based on the ecological systems theory (Bronfenbrenner, 1974, 1977) and the research literature describing factors that influence teacher decision-making in mathematics classrooms (Ayalon & Even, 2016; Bronfenbrenner, 1974, 1977; Davis et al., 2020; Herbel-Eisenmann et al., 2006; Smith et al., 2017; Remillard, 2005).

The EFM was displayed on a single 8-inch by 11-inch document and included several factors from each ecological system shown in the conceptual framework (see Figure 1).

The participants were three generalist elementary teachers referred to by the pseudonyms Taylor, Sarah, and Carmen. All participants had taught elementary mathematics in a general education setting for at least three years. Taylor was a white male fifth-grade teacher; Sarah and Carmen were both white, female first-grade teachers. Carmen taught in a dual-language immersion (DLI) classroom where she and her students spoke in Spanish.

In the teacher interviews, I asked the participants to describe the MSD that takes place in their classroom. I then asked the teachers to take a careful look at the concepts listed on the EFM and asked them to create a visual on the EFM that illustrated how certain factors influence how they implement student talk in their classrooms, either positively or negatively. I asked the participants to talk about their actions as they performed them. This use of the EFM allowed the participants to discuss complex influences of various interconnected factors on their decision-making and teaching experiences. The completed EFMs from each interview are displayed in Figure 2.

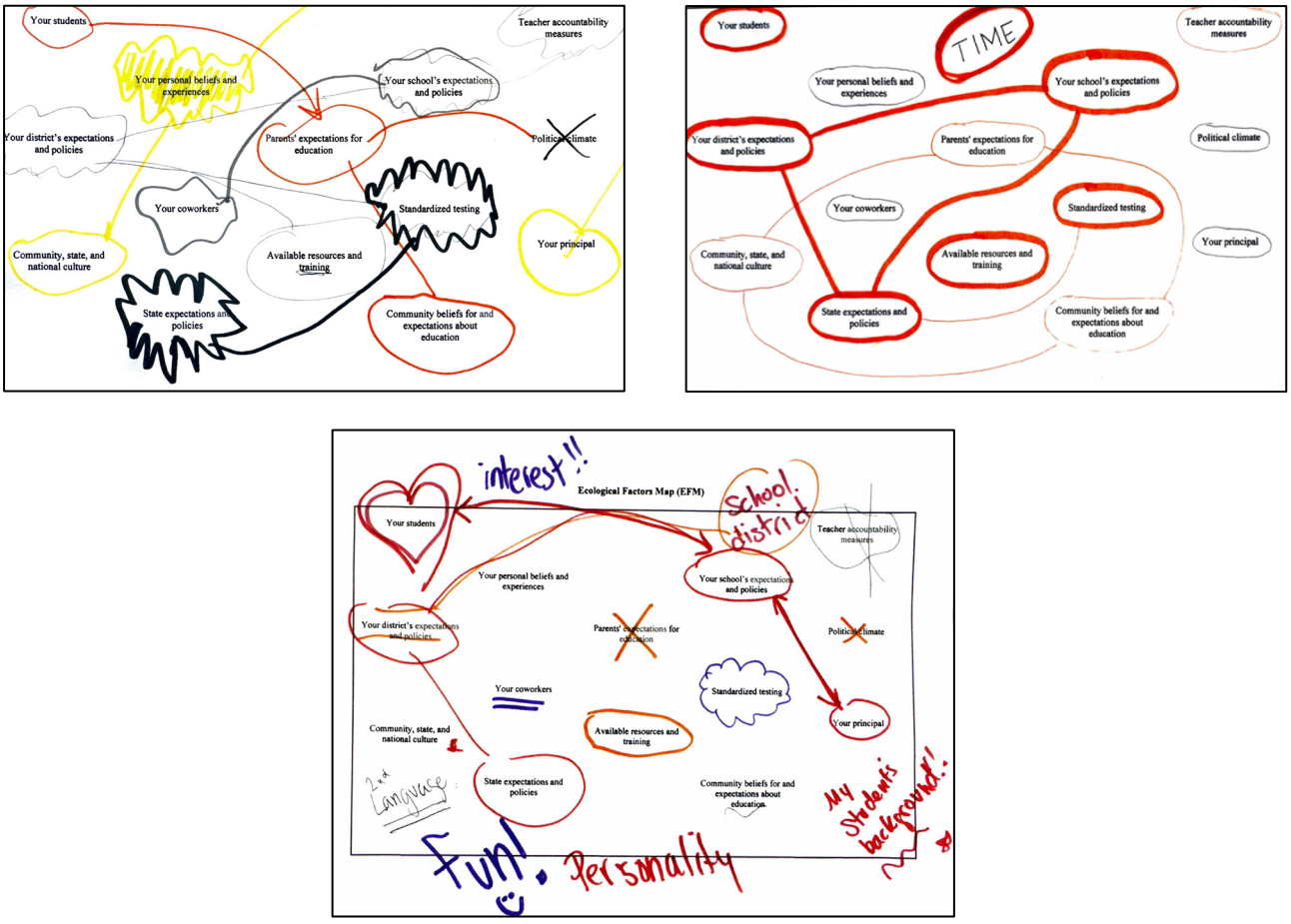


Figure 2: EFM Visuals Created by Taylor (top left), Sarah (top right) and Carmen (bottom)

While decision-making for MSD served as a focal point and the primary interest of this study, it was anticipated that the teachers may discuss decision-making for mathematics

instruction generally. This proved to be the case, with the teachers occasionally discussing ideas and experiences related to mathematics instruction broadly. Such ideas were seen as important elements of the teachers' complex experiences as educators and were included in the analysis.

Data Analysis

I conducted a thematic analysis (Nowell et al., 2017; Riger & Sigurvinsdottir, 2016) of the data sources described in the previous section. This process involved becoming familiar with the data in the form of audio recordings, transcripts, and EFM visuals; taking analytic notes; generating initial codes in the transcripts using MAXQDA; searching for themes; iteratively reviewing and altering themes that were eventually defined and reported; and performing member checks with each participant (Nowell et al., 2017). The unit of analysis was a concept or idea shared by the participant, whether it be a sentence or a paragraph from the transcripts.

I first conducted three within-case analyses of the data from each teacher's interview and EFM. I triangulated the data for these analyses with administrator interviews and classroom observations, which are not directly discussed in this report. After completing the within-case analyses, I conducted a cross-case synthesis (Yin, 2014) by searching for patterns across the within-case themes (see Figure 3). The development of cross-case themes emerged from iterative reviews and alterations as I returned to raw data and took detailed analytic notes. This paper includes a discussion of the cross-case synthesis connections to relevant extant literature.

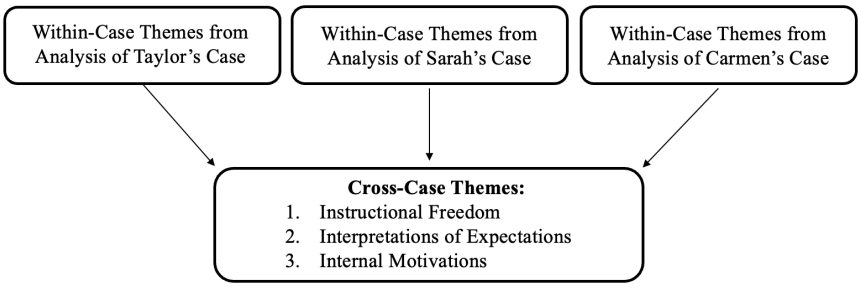


Figure 3: Analysis Process for the Development of Cross-Case Themes

Findings and Connections to Literature

Each teacher expressed unique ideas throughout the interview as they created their visuals on the EFM. While full reports of each case are reported elsewhere (Lund, 2026), this research report focuses on the cross-case themes synthesized from the findings of all three cases. To provide context for these cross-case themes, a brief overview of the within-case findings is reported here.

Taylor discussed conflicts between his personal views about mathematics instruction—which centered around problem-solving and real-world application with natural connections to MSD—and expectations from his students' parents, his district, and the state. He shared a perception of "foggy" expectations from his district—especially related to standardized testing—and his negative feelings toward state policy and standardized testing as "enemies to education" that, in his mind, are both unhelpful and discouraging. Sarah described her focus on student needs through instructional strategies such as MSD and a sense of instructional freedom that was largely based on a personal sense of confidence and the ability to view expectations from educational bodies as "guidelines," creating a flexible framework for her instructional decisions. Carmen also described a focus on her students, including their needs, interests, backgrounds, and

sense of fun. She shared ways that she chooses to use MSD to meet student needs and described expectations from her school and district that aligned with this teaching style.

Cross-Case Themes

This section includes the three cross-case themes discovered from the cross-case analysis. Each theme is discussed in connection to extant literature relevant to the findings.

Cross-Case Theme 1: Instructional freedom. The three teachers shared experiences connected to instructional freedom that influenced their instructional decision-making. Sarah and Carmen experienced strong instructional freedom within their schools and districts. They discussed ways that they use this freedom to prioritize MSD, flexible problem-solving, and conceptual understanding to meet their students' academic and personal needs and to align instruction with their own personalities and values. Sarah and Carmen seemed to enjoy this sense of freedom and did not express a desire for more guidance. Carmen noted how working in a DLI classroom afforded her more instructional freedom than her English-teaching counterparts:

> We have the same curriculum, but English teachers don't have that freedom like [what] we Spanish or Portuguese teachers, we have. And English teachers, they have, like, every single time measure, and they have a routine to do it, and they give them the routine, and even the words that they have to say. But [if] we want to do routines, or we want to do something, we have to create our own.

Though Taylor did not specifically discuss the degree of instructional freedom allowed him, he shared pressures to follow strict, testing-focused guidelines promoted by his district and school. While he expressed a desire to teach more in accordance with his personal beliefs and experiences through application-based pedagogy and real-world problem-solving, he expressed frustration at the misalignment between this personal compass and the realities of standardized testing and state expectations. However, rather than express a desire for more instructional freedom, Taylor shared his search for clarity among foggy expectations. Feeling that he does not naturally know what his students need, he described his desire for a clear outline of what to teach and how. "I'm not one of those teachers that just knows my kids and knows what they need," he shared, "… so I want to get [clear guidance], and it's not there."

These nuanced experiences with instructional freedom are supported by literature on teacher autonomy (Cribb & Gerwirtz, 2007; Jerrim et al., 2023; Narayanan et al., 2024). Teacher autonomy is complex and not experienced the same by all teachers. In this study, Sarah and Carmen seem to benefit from feelings of instructional freedom and autonomy. Taylor does not experience the same sense of autonomy, but it is unclear if he wants it. Though he wishes he could make different instructional decisions aligned with his personal teaching values, he desires clearer expectations to guide his instruction.

Cross-case Theme 2: Interpretations of expectations. Each teacher shared a unique view of expectations from their school, district, and state. Sarah referred to state expectations as "guidelines" and explained how her sense of confidence allows her to implement MSD without excessive supervision. She stated that she would implement MSD "no matter what" even in the face of conflicting expectations because of how important MSD is for young students growing socially and academically. She said:

> I take the district expectations, the state expectations, the school's expectations, the standardized testing, the resources … I take all of that, but … I don't let it like affect my decision-making process. I trust myself enough to know that the things that I'm doing are

> within [these] expectations. But it is essentially what's best for my students, and at the end of the day for me, that's what's most important.

Carmen expressed similar sentiments, describing the freedom and support she is allowed by her principal and district. Carmen described a lack of tension between her desired teaching style—which involves lots of movement and student talk—and expectations from her school and district. She feels encouraged to incorporate MSD into her classroom to support her students who are growing socially, academically, and linguistically, and she adheres to standards while modifying her curriculum as she and her team teacher see fit.

Taylor described expectations from his school, district, and state as more strict than the other teachers did. As mentioned earlier, he described feelings of pressure to adhere to testing expectations and a desire for more clarity about these expectations. Speaking of standardized testing and state expectations and policies, he said:

> [They] are there to make life more difficult, teaching less effective, and to basically rub our noses in us not doing good enough … Very negative connotations in my mind from those two. I feel like state expectations and policies and standardized testing is the enemy of education.

The different interpretations of expectations among the teachers may be connected to standardized testing expectations. Taylor, as a fifth-grade teacher, must prepare his students for end-of-year state-mandated standardized tests in mathematics, writing, reading, and science (Utah State Board of Education, 2025). As first-grade teachers, Sarah and Carmen do not administer these high-stakes standardized tests. Research supports the idea that testing pressures have a strong influence on teacher experiences and instructional decision-making (Moon et al., 2007; Ramsay-Jordan, 2020; Stevens, 2023; Youn, 2018). Standardized testing can cause feelings of limitation in instructional practices, a narrowing of the curriculum, and a dependency on district-provided resources (Stevens, 2023). High testing expectations can also lead to teacher frustration, a lack of instructional flexibility, limited demand for rigorous and cognitively demanding instruction, difficulty implementing culturally responsive pedagogy, and a decrease in teachers' feeling of empowerment (Moon et al., 2007; Ramsay-Jordan, 2020; Youn, 2018).

While Sarah and Carmen seemed confident in their understanding of expectations overall, Taylor repeatedly described expectations and policies as "foggy." His district provides many resources for mathematics instruction, but he is often unsure of how to use them to meet strict expectations. This interpretation aligns with Allen's (2023) findings from a study on teachers' sensemaking of expectations regarding a newly mandated curriculum, where the author differentiated between *ambiguity* and *uncertainty*. (i.e. "ambiguity" as confusion in the face of too many possible interpretations and "uncertainty" as confusion in the face of too little information) to understand the teachers' experiences. Based on the interview data, Taylor may be experiencing uncertainty because of too little information provided about how to utilize the testing-focused resources he is expected to use. Or perhaps his experience is better described as ambiguity: he has access to a lot of resources and curricular materials with too many interpretations for their use. Unfortunately, Taylor has not found ways to uphold the spirit of his school's, district's, and state's expectations without being weighed down by specifics, as it seems Sarah and Carmen have. As a result, it seems that Taylor feels less confident implementing MSD through real-world application and problem-solving tasks like he wishes he could, instead focusing on improving his students' test scores and grades and meeting other school and district expectations.

Cross-case Theme 3: Internal motivations. The teachers' internal motivations, sitting within themselves at the center of their ecological systems, strongly influenced their decision-making. Sarah and Carmen stated that their students most strongly influenced their decision-making, even considering school and district expectations, state policy, and parent expectations. Taylor, in contrast, described his personal beliefs and experiences with mathematics—focused on real-world application—as the factor that allows him to do his best teaching. He expressed a desire to better align his teaching with this internal motivation as a result of our interview and mused that focusing more on understanding his students as individuals would help him look past oppressive testing policies and focus on his internal motivations.

These findings reflect a description of the microsystem as including the individual's personality, temperament, and beliefs (Crawford, 2020). From an ecological perspective, these findings prompt questions about how certain internal motivations situated within an individual's microsystem may interact with other factors within the individual's mesosystem, exosystem, and macrosystem. For example, Sarah and Carmen engage in daily interactions—or reciprocal proximal processes (Bronfenbrenner & Morris, 1998)—with their students, encouraging them to modify their instruction and curriculum to meet their students' needs even if it means pushing back on expectations from their students' parents, their school, their district, or their state. It is possible that, because these teachers are oriented toward meeting their students' needs and because they work in a low-stakes testing environment with administrators who do not discourage them from these modifications, this kind of reconciliation between internal motivation and external expectation is easier achieved than it is for Taylor. Taylor, who is more oriented toward his personal beliefs and experiences with mathematics, works within a high-pressure testing environment and lacks clarity in testing-focused expectations. In this context, he may find it difficult to push back on expectations and focus his instruction on real-world application and other instructional activities more aligned with his internal motivations.

Implications for Educational Leaders

A goal for exploring teacher decision-making through ecological systems theory was to inform educational leaders how to support teachers both individually and collectively so as to contribute to the closing of the research-practice gap for rigorous mathematics instruction. The following recommendations based on the findings of this study offer a clearer vision of what educational leaders can do—in northern Utah and beyond—to support this cause.

Recommendations for District Leaders and Principals

The following are three recommendations for district leaders and principals based on the findings of this study: (1) provide clarity, (2), balance instructional freedom and professional learning, and (3) recognize differing internal motivations among teachers.

1) Providing clarity. District leaders and principals should strive to provide clarity to their teachers about their expectations for mathematics instruction. This includes clarifying logistical expectations for standardized testing preparation. They should articulate the purpose and limitations of these assessments for accountability and instructional improvement. District leaders in particular should clarify the importance of high-impact instructional strategies like MSD, even considering testing pressures. They should ensure that teachers and principals understand how standardized testing expectations size up against high-impact instructional strategies that may or may not improve testing outcomes. Providing this clarity to teachers, of course, requires district leaders to first formally determine their position on these matters.

2) Balance instructional freedom and professional learning. To support high-quality mathematics instruction, district leaders and principals should reasonably support instructional

freedom for classroom educators. Instructional freedom should exist alongside quality professional learning that provides educators with the skills and knowledge they need to provide rigorous and equitable instruction (Cribb and Gewirtz, 2007). For example, professional learning on ambitions number talks (Cavanna et al., 2021; Kazemi & Hintz, 2023; Woods & Salem, 2024), open-ended math talks, (Hufferd-Ackles et al., 2004), classroom discussions (Smith & Stein, 2018), and quality questioning, (Hennessy et al., 2023; LaRusso et al., 2024; O'Connor et al., 2017) could support teachers to make their enactment of MSD more rigorous. Furthermore, teacher beliefs about mathematics, about how mathematics should be taught, and about their students influence what MSD looks like in the classroom (Bartell, 2011; Conner & Singletary, 2021; Davis et al., 2020; Kooloos et al., 2022; National Council of Teachers of Mathematics, 2014; Nilsson, 2019; Ruef, 2021; Stockero et al., 2020). Districts and schools should prioritize professional learning that helps teachers develop beliefs that support high levels of student learning, extend noticing skills, encourage self-reflection, and disrupt biased beliefs (Hand, 2012; Mendoza et al., 2023; Ruef, 2021; van Es et al., 2022) so MSD and other high-impact instructional strategies are meaningful and equitable for all students.

3) Recognize differing internal motivations. District leaders and principals should be careful not to assume they know what matters most to teachers or what they find valuable in the teaching profession. Some teachers may feel in tune to their students' needs and may be primarily guided by what they believe is best for their students (like Sarah and Carmen). However, some teachers may be motivated by their personal experiences with mathematics (like Taylor) or something else entirely. These teachers may need more guidance to determine what their students need, to navigate conflicting expectations, or to implement high-impact instructional strategies like MSD.

Recommendations for State Leaders and Policy Makers

Like district leaders and principals, state leaders and policy makers should work to provide clarity in expectations, facilitate instructional freedom and professional learning, and recognize the diverse needs and motivations of teachers. Additionally, state leaders should develop metrics (likely qualitative) to determine how MSD and other rigorous strategies are being utilized in their state. Utah has expectations for the implementation of rigorous instructional strategies (Utah State Board of Education, 2019; Utah State Board of Education, 2022b), which focus in large part on the Standards for Mathematical Practice (Common Core State Standards Initiative [CCSSI], 2010). However, it is not clear if and how state leaders are encouraging districts and schools to use them. Districts may face barriers in promoting these strategies. Maintaining expectations for rigorous instruction alongside expectations for standardized testing results would help close the research-practice gap for strong mathematics instruction and promote student achievement resulting from strong instruction rather than teaching to the test.

Conclusion

This study explored the experiences and instructional decision-making of three elementary teachers through an ecological systems lens. The findings demonstrate the unique experiences of individual teachers within their personal educational environments and provides actions educational leaders can make to better support teacher decision-making for high-impact instructional strategies like MSD. Future research similarly exploring teacher decision-making will help researchers and educational leaders identify and address the complex needs of teachers both individually and collectively as they work together to provide rigorous mathematics instruction to all students.

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INTERNAL CONFLICTS IN DESIGNING A MEASURES OF CENTER TASK: A SELF-STUDY

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This paper explores how five categories of conflict in collaborative task design can be applied in a self-study focused on the design of a measures of center task. The arenas of conflict theory is combined with Hermans' dialogical self theory to extend the construct of conflict from group planning settings to the intrapersonal experience of a teacher-researcher planning independently. Through this extension, the paper demonstrates how the arenas of conflict illuminate key decisions in the design process and can be used to examine how these choices functioned in implementation. Further, the study illustrates the potential for this framework to inform task revision.

Keywords: Preservice Teacher Education, Design Experiments, Data Analysis and Statistics, Instructional Activities and Practices

Selecting and adapting appropriate classroom tasks are key aspects of planning and implementing meaningful mathematics instruction. Sometimes, however, an adequate task that matches students' needs, explores the intended mathematical content, and leverages an engaging context cannot be found, and teachers must take on the role of task designer. In my recent reading about students' initial development of measures of center concepts, I found myself questioning whether the suggested tasks would function as intended, often due to mismatches I perceived between students and the context or between mathematics and the context. In the initial phase of a design-based research project reported here, I designed a measures of center task, implemented it with a small group of pre-service teachers (PSTs), and analyzed the design process through the lens of Barbosa and de Oliveira's (2013) arenas of conflict in task design. Using a self-study approach, the purpose of this project is to illuminate key decisions in the design process, the outcomes of those decisions, and paths for design revision. Specifically, I explore the research question: *What insights about task design are developed through an initial cycle of designing, implementing, analyzing, and revising a cognitively demanding math task focused on measures of center?*

Theoretical Framework

Implementing reasoning and problem-solving tasks is identified as one of the eight mathematics teaching practices in *Principles to Actions* (National Council of Teachers of Mathematics, 2014). A mathematical task can be defined as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea," and tasks are often categorized by the cognitive demand required to complete them (Stein et al., 1996, p. 460). High cognitive demand tasks have been associated with student learning gains (Hiebert & Wearne, 1993; Stein & Lane, 1996), as well as rich classroom discourse (Tekkumru-Kisa et al., 2020), increased student engagement (Hourigan & Leavy, 2023), and positive mathematics dispositions (Sullivan & Mornane, 2014). However, alongside these benefits for students come significant challenges for teachers, including selecting or designing tasks that appropriately align mathematical ideas, student needs, and instructional goals (Crespo & Sinclair, 2008).

Researchers who study children's development of statistical learning offer several recommendations for designing tasks that support students' conceptions of measures of center. First, students benefit from exploring data sets embedded in meaningful contexts and from comparing two or more sets (Konold & Pollatsek, 1999). Comparing data sets provides a legitimate need for students to determine a value that is representative, average, typical, or usual for a set (Mokros & Russell, 1995). Second, researchers suggest delaying the introduction of the arithmetic mean in favor of building an understanding of mode and median first (Mokros & Russell, 1995). When mean, median, and mode are taught simultaneously and algorithmically, students can become convinced that the mean is the only acceptable measure of center (Jacobbe, 2012), losing sight of the usefulness of the other measures (Friel, 1998). Finally, Konold and Pollatsek (1999) hypothesized that repeated measures could be effective contexts for helping students think about measures of center, but they acknowledged that in elementary instruction "repeated measures are rarely used, perhaps because they often lack pizzazz" (p. 18). Designing tasks that attend to these recommendations is a concrete way to connect research and practice.

When designing tasks for classroom use, teachers must weigh mathematical needs against the practical needs of their students and the instructional context (Crespo & Sinclair, 2008). To further illuminate this friction, Barbosa and de Oliveira (2013) identified five arenas of conflict or tension that surface during mathematical task design: the mathematical *context* of the task, the use of *language* to increase or decrease rigor, the *structure* or degree of openness of a task, the *distribution* or the content focus of the task, and the positioning of teachers and students as participating *subjects*. Although the researchers originally observed these conflicts within a collaborative group of academics and teachers (Barbosa & de Oliveira, 2013), I argue that the same tensions operate within myself as I design a mathematical task. Hermans' (1996) dialogical self theory conceptualizes the self as a multiplicity of I-positions that fluctuate across time and space and can be in conversation with each other. Drawing on this theory, I view myself as occupying multiple I-positions at once, each with a unique experience, perspective, and voice. In the design process, *I as a teacher* and *I as a researcher* are in dialogue and negotiating compromises, much like the middle school teachers and academics of Barbosa and de Oliveira's collaborative group (2013). These theoretical lenses provide a structure for analyzing my planning process and the resulting task implementation, and they help surface possible ways in which the measures of center task can be improved in future iterations.

Methods and Data Analysis

Here I report on the first iteration of a design-based research study, an iterative approach that connects theory and practice through the design, implementation, and refinement of educational materials (Bakker & Van Eerde, 2015). The following phases formed the first iteration of the study: designing a high cognitive demand task based on measures of center literature recommendations, implementing the task, soliciting participant feedback, analyzing data, and revising the task. During the design phase, I used generative artificial intelligence (AI) as a thought partner to brainstorm potential contexts and refine task instructions and questioning (Beavers, 2025; Fleischmann, 2026). A thought partner is defined as "another entity (human or AI) that works with an agent to push forward some aspect(s) of thinking" (Collins et al., 2024, p. 1852). The transcript of the AI conversation and the initial draft of the task were collected as data sources. I then facilitated the task with three undergraduate volunteers from an elementary math methods course in which I serve as a graduate teaching assistant. We met for two hours outside of class, and participants received a $10 gift card for their participation. They completed the task collaboratively and provided both written and verbal feedback on the experience. Data sources

from this phase include video and audio recordings of the session, the transcript generated by Otter.ai, participants' work on the task sheets and whiteboards, and verbal reflections recorded after completion of the task.

The collected data were uploaded into ATLAS.ti, a web-based tool for qualitative coding, and I completed an initial cycle of coding to cull episodes that were mathematically or pedagogically relevant to the research question. For example, I removed off-topic conversations that took place while students were stacking and kept instances of comparisons, predictions, and mathematical connections. Within the remaining relevant episodes, I identified a recurring tension of control present within myself: How can I design a task that will be as open and student-led as possible, while still ensuring that all intended learning objectives will be met? Barbosa and de Oliveira's (2013) five arenas of conflict in task design were used as codes for my second cycle of coding. Applying these new codes allowed me to identify an illustrative example of each conflict and further explore the impact of these tensions on the task implementation.

Measures of Center Task

The task I designed prompted students to stack five different classroom manipulatives (see Figure 1) and determine which type of tower can hold the most items before it topples. This was a repeated measures task because students were instructed to work together to do ten trials for each object and then determine one representative, usual, or typical number for each manipulative. Follow-up prompts and questions were constructed with the intention of discussing several different measures of center and the advantages and disadvantages of each.



Figure 1: Classroom Manipulatives Used for Stacking

Results

For each of the task design conflicts described by Barbosa and de Oliveira (2013), I first illustrate ways in which the conflict manifested in the planning process. Second, I address how my related planning decisions impacted the task implementation. Finally, when the analyses point to a need for related task revisions, the changes are shared as well.

Context

Konold and Pollatsek (1999) hypothesized that data representing repeated measures may be "conceptually the easiest to think about as having central tendencies" (p. 18). As a researcher, I felt drawn to test their hypothesis, but as a teacher, I struggled to think of a manageable and engaging context that would authentically involve repeated measures. This internal conflict is illustrated in the prompt that I gave ChatGPT during planning:

> I want to have students produce a series of repeated measures on something.
>
> Two examples come to mind: (1) How fast can you run around the basketball court? But then we have to deal with students comparing themselves, which will get sticky.
>
> (2) How far does your paper airplane fly? This just gets kids all wild and crazy.

> But what are other ideas like this where it won't take much time to do repeated measures, it's not completely random (not like rolling a die) but there is variation, it won't create problems if the kids compare themselves, and it won't cause too much chaos.

I as researcher articulated the need for an appropriate context to explore with repeated measures that would naturally have some variation, and I offered two contexts that would undoubtedly work mathematically. At the same time, I as teacher drew on classroom experience to know that running laps or throwing paper airplanes would cause unintended management consequences that would distract from the true purpose of collecting a meaningful dataset to analyze. After several rounds of ChatGPT brainstorming followed by clarifying prompts from me, I settled on the stacking context. ChatGPT affirmed this choice by appealing to both of my I-positions:

> The stacking idea is excellent—it's tactile, low-mess, quiet, naturally variable, and totally avoids comparison about physical ability. It also generates rich data: you can count the number of items stacked (a clear, discrete measure), explore patterns across materials, and discuss averages, variation, and even stability factors.

During implementation, the stacking context produced within-object and between-object variation as intended (see Figure 2), providing ample opportunities for rich conversations. As students compared the datasets, they discussed that bears scored decisively low, cubes scored decisively high, and the other three were somewhat harder to order. Additionally, the dice data were bimodal, symmetrical, and had a very small range, while the rhombi had nine different values among just ten trials. As planned, the data collection was easy and accurate, but the participants and I both expressed concern about the large amount of time it took to collect the data. They anticipated that some students would lose interest, while I worried that teachers would not be willing to devote so much class time to data collection. To address this in the next iteration, I plan to increase the group size from three to five students, decreasing the trials each student must conduct and the total time required for this phase of the task.

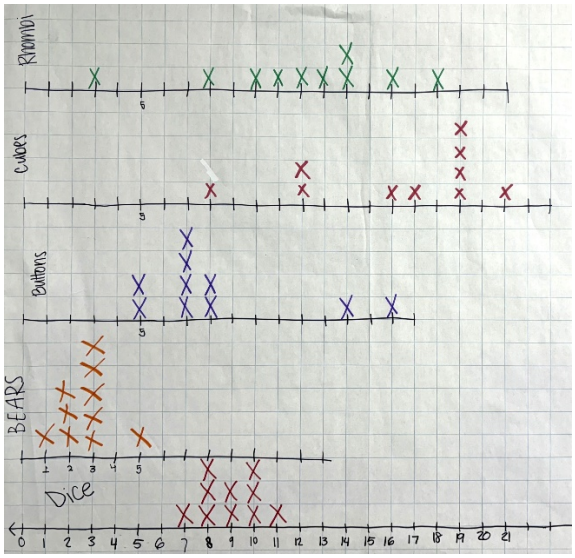


Figure 2: Student-Created Line Plots of Tower Counts for Ten Trials Per Manipulative

Language

Barbosa and de Oliveira (2013) describe the conflict of language as related to "the level of rigor tied to the task" (p. 546). While designing the task, I as researcher was set on using the words *representative*, *usual*, and *typical* to prompt students towards thinking about measures of center (Mokros & Russell, 1995), but I as teacher wanted to confirm that my directions would be understandable. When I asked ChatGPT to help me revise them for clarity, it suggested giving this instruction to students: "Brainstorm several ways to find a number that could represent your data (for example, the most common number, the middle number, or an average)." While these revised directions might be easier for a student to answer, including the more specific language of *most common* and *middle number* decreases the rigor of the task by identifying ways to numerically represent a dataset. This exchange caused me to recognize that I as teacher value openness in tasks and want to give students as much space to construct their own understanding as possible. I chose not to use ChatGPT's revised, less rigorous version, which reflects my belief in students' abilities to make meaning. My past experiences as a teacher have convinced me that student learning is worth the productive struggle that accompanies rigorous tasks.

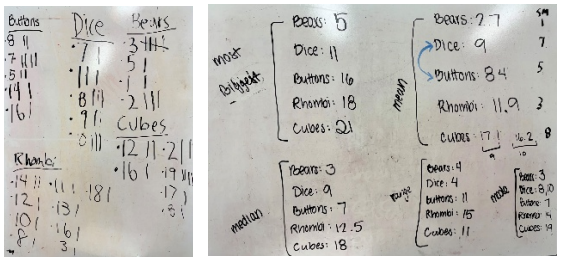
When I began planning this task, I imagined implementing it with fifth graders, based on the timeline of our state standards. However, because of easy access to college students in my current role, I chose to first test the task with PSTs. The participants completed data collection and sharing what they initially noticed and wondered, and I prompted them as planned: "It could help to have a single number for each one of those that might represent the data, or be usual or typical for each one of the objects." Immediately, one student answered, "Guys, I think we're finding the average, the mean." I felt slightly perturbed because they went straight to the algorithm, rather than even considering any other possibilities. I had tried to create a more open task with my intentional choice of language, but they approached this moment as if there was one right answer. In revisiting this moment, I recognize another I-position that I occupy: math teacher educator (MTE). I as MTE want my PSTs to experience authentic, elementary, cognitively demanding tasks from the learner perspective to cast a vision for the type of teaching I hope they will do in the future. But I as a new MTE am also still learning the types of modifications necessary for adult students who are experiencing tasks meant to build concepts they have already constructed. Leaning into the idea of multiple I-positions, it still feels acceptable and reasonable to design a task with both fifth graders and future teachers in mind. In the next iteration, which I still plan to do with PSTs, I want to make language adjustments that take their background knowledge into account, but also encourage them to consider how the same lesson could be done in an elementary classroom.

Structure

Structure refers to the "degree of openness" in a task (Barbosa & de Oliveira, 2013), and I experienced a conflict related to structure when deciding how students would record their data. I as teacher and MTE typically advocate for the highest level of openness possible in a task, which in this case would mean allowing students to record their trials and analysis in whatever format made sense to them. On the other hand, I as researcher knew that a purposefully organized table, especially for the measures of center, would be beneficial when trying to facilitate conversations comparing the different objects. I ultimately decided to leave the recording format completely up to students in order to demonstrate the open structure that I as MTE advocate for when helping PSTs plan their own tasks.

As shown in Figure 3 (left and middle), the students chose unexpected structures in both phases of their recording. As they built towers, they recorded the count and added a tally under

the tool name, hence the random order of the values in each list. Each time the tower count was repeated for a tool, an additional tally was added (i.e., 8 | | represents two towers of eight). On the second board, the students recorded measures of center and spread in multiple lists, and they kept the order of the tools constant in each list. Interestingly, when asked to offer ways to improve the overall task, one student suggested they should be given "a way to write our original data" because the tally marks caused some confusion. A different student countered that they appreciated the opportunity to choose their recording structure, and explained that in a full classroom setting, students could benefit from seeing and comparing a variety of student-created methods. In this instance, I argue that not requiring a specific recording format helped me meet the MTE goal of demonstrating the value of an open structure to PSTs. At the same time, I recognize that the students' comparison of the calculated measures on the second board was not as rich as it could have been if a more organized recording method was used. In the next iteration, I plan to discretely fill out a premade table like the one shown on the right in Figure 3 as students calculate their measures of center and spread. When the comparison conversation stalls, I will introduce the additional table to help spark new insights. This revision should maintain the benefits of an open structure while also supporting meaningful discourse.



Item	Mean	Median	Mode	Min	Max	Range
Bears	2.7	3	3	1	5	4
Buttons	8.4	7	7	5	16	11
Dice	9	9	8, 10	7	11	4
Rhombi	11.9	12.5	14	3	18	15
Cubes	16.2	18	19	8	21	11

Figure 3: Student-Created Data Collection (left), Student-Created Data Analysis (middle), and Suggested Format of an Analysis Table (right)

Distribution

For Barbosa and de Oliveira (2013), the distribution conflict refers to tension over the "content to be focused on" in a task (p. 546). I as former elementary teacher expected that the stacking context would be an effective match for the content goal of measures of center. I felt fairly confident that upper elementary students could "reinvent" the concepts of mode and median when prompted to develop a representative number for each stacking object to compare them. Mean could also surface in discussion if students have prior background from completing science experiments or averaging grades, but mean was not intended to be the focus. Distribution was not a conflict for me during the planning of this task because I as elementary teacher and I as researcher both believed in the benefit of delaying the introduction of mean in favor of building a conceptual understanding of mode and median first (Mokros & Russell, 1995).

Unsurprisingly, the task prompted the PSTs to discuss a much wider array of content than just mode and median. During the data collection phase, the students made spontaneous comments that explicitly or implicitly referenced range, variation, and maximum—all of which are related to spread. In addition to discussion about measures of center, their conversations during the analysis phase also covered topics such as consistency or clusters in the data, and how close together or spread out the data were. After creating line plots for each object, one student commented, "When I was learning range, oh my gosh, I had no idea what they were talking

about. But as soon as we started doing this, I could see, okay yeah, the farther apart they are, the bigger the range." This statement and others indicated to me that the PSTs were deepening their statistical content knowledge through this task. Neither the researcher nor the teacher in me is disappointed at the breadth of the content this task evoked. However, I as MTE missed an opportunity to deepen PSTs' mathematical content knowledge because I had not adequately anticipated or planned for the wide range of topics they would discuss. One element I will add to the next iteration of this task for PSTs is an open sort of the measures they calculated in the analysis phase (e.g., mean, median, mode, range, maximum, minimum). I ultimately want PSTs to be able to explain that measures of center and measures of spread are different but interrelated.

Subjects

The final area of conflict, subjects, refers to the "way the task positions students and teachers" (Barbosa & de Oliveira, 2013, p. 546). In other words, task designers must consider the type of interactions the "subjects" will have. Will the task be structured enough for students to progress through it on their own, or will it require perturbations or scaffolding from the teacher at preplanned moments? Overall, I planned the task to fall somewhere in the middle. I wanted students to read and follow the instructions I had prepared without my interference, attempting to confirm that the task could stand on its own and be implemented by other instructors. At the same time, I knew I wanted the freedom to interject in a way that matched our natural classroom interactions—both to help me learn more about the students' understanding and to push their thinking forward. A significant internal conflict surfaced over how I would handle the line plot portion of this lesson (Figure 2). I as researcher envisioned the five line plots that students could create from the data, aligned one over the other. I anticipated that this representation could enrich students' discourse because they would be able to see all distributions at once. However, I could not think of a way to get students to this point, other than, "Now please make a line plot for each object, and stack them." I as teacher felt hesitant to directly prescribe such a specific representation because it did not match the way I positioned myself throughout the rest of the task. I had not resolved this internal subjects conflict by implementation time, so I decided to mostly just wait and see what would happen.

After the students collected the data and shared initial reflections, I showed them a single line plot I had discretely created while they were collecting data. I did not give it a title, and they were tasked with determining which object it matched. This was not much of a challenge for the PSTs, but I am curious if fifth graders without prior knowledge of line plots could do the same. I set the chart aside and told them they could return to it later, still attempting to position students as the main decision-makers. About 20 minutes passed, and I asked them to put on their "teacher hats" and imagine what fifth graders might do when asked to find representative numbers for each object. Immediately a PST said, "I think they would make a chart," and motioned to the single line plot I had created. Though I did not expect this question to necessarily lead us back to the line plot, it turned out to spark a natural, student-led segue into creating the other four graphs. In future iterations of this task, I will purposefully listen for openings like this so that I can still position students as the task drivers as much as possible.

Discussion

In this design-based research self-study, I investigated the following question: *What insights about task design are developed through an initial cycle of designing, implementing, analyzing, and revising a cognitively demanding math task focused on measures of center?* Here I share three main insights.

First, the five arenas of conflict (Barbosa & de Oliveira, 2013) served as an effective framework for analyzing my own task design process when coupled with Hermans' (1996) concept of multiple I-positions. The five conflicts allowed me to focus my analysis of the process and surface the dilemmas and decisions that take place almost subconsciously during task design. The tensions between I as researcher and I as teacher were especially apparent when reviewing my ChatGPT prompts, and this "collaboration" helped me clarify the perspective and values of each I-position. Additionally, the combination of these frameworks revealed another I-position at play, I as MTE, and helped me articulate some of the challenges inherent in designing a task for two very different audiences simultaneously.

Second, this pair of frameworks (Barbosa & de Oliveira, 2013; Hermans, 1996) is also helpful for identifying potential areas of revision within the task design. Barbosa and de Oliveira (2013), when describing the design process, wrote that "the resulting task holds the markers of the conflicts" (p. 547). Extending this idea, I envision revisiting those "markers" during analysis of the task implementation to assess the effectiveness of the decisions I made amidst conflict. In this case, I identified my rationale for using the stacking context, and considered whether or not that context met my expectations during implementation. All mathematical and pedagogical expectations were met except one—time—and this became my focus for task revision.

Finally, intentionally considering multiple I-positions and five conflicts during planning may enhance future task design endeavors. For example, consciously acknowledging my MTE and elementary teacher positions during planning should prompt me to adjust appropriately and purposefully for my audience. Additionally, I can imagine myself considering the five conflicts to self-check that I have been intentional about aspects of the task design and its delivery.

Conclusion

In this study, I report on the first iteration of a design-based research self-study on the design of a measures of center task. I combined the theories of Barbosa and de Oliveira (2013) and Hermans (1996) to analyze my design process and identify areas of improvement in the task. This novel combination of theories contributes to the field of math education and task design by extending the conflicts framework from a collaborative setting to an individual's internal, dialogical tensions. Additionally, I argue that the conflicts framework may be helpful in organizing one's thoughts and intentions during the planning process and assessing the effectiveness of design decisions after implementation. Teachers and other task designers may find that clarifying or justifying their choices related to context, language, structure, distribution, and subjects prompts them to address overlooked aspects of the task during planning or refine task elements based on the values of their various I-positions.

I as researcher am eager to conduct the next round of this design-based research project with PSTs, implementing the revisions that my framework and analysis helped me develop. In this round, I used Barbosa and de Oliveira's (2013) five conflicts without critique, looking for evidence of each conflict within my planning process. In future rounds, I plan to consider the possibility of conflict arenas that they have not named, as well as other I-positions that may be at work during my task design process. Additionally, the findings reported here are limited by a small sample size of three students. Implementing the task with multiple groups of students in a classroom setting will provide further opportunities for task refinement.

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THE CONNECTION BETWEEN PRODUCTIVE STRUGGLE AND A SENSE OF BELONGING

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This theoretical brief explores the connection between mathematical identity components (i.e., mathematical authority and agency), productive struggle, and a sense of belonging. Specifically, the role of a sense of belonging and productive struggle in students' comfort with struggling in whole groups or small collaborative groups. We present a theoretical model for not only creating mathematical tasks to foster a sense of belonging, but also an environment conducive to thickening a sense of belonging over time, allowing students to become comfortable with the vulnerability needed to routinely engage in ambitious mathematics.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Culture and Asset-Based Pedagogies; Instructional Activities and Practices

Both a sense of belonging and opportunities for productive struggle have been identified as critical to students' engagement in mathematics. While recent works have addressed the cultivation of a sense of belonging (National Council of Teachers of Mathematics [NCTM], 2025) and productive struggle (Jarry-Shore & Richardson, 2024) separately, it is imperative to discuss a more purposeful approach to these constructs beyond lessons of interest or names in word problems (Gutiérrez, 2012). Productive struggle is crucial for students' mathematical learning (Hiebert & Grouws, 2007; Jarry-Shore & Richardson, 2024; Warshauer, 2015). Since ambitious mathematics promotes student collaboration, students need to form relationships and develop a comfort with productive struggle (Jarry-Shore & Richardson, 2024; Langer-Osuna, 2017). Thus, fostering students' sense of belonging in mathematics is necessary for students to become comfortable with the vulnerability required for productive struggle in collaborative settings, especially for students from culturally and linguistically diverse communities, who have often been positioned as mathematically deficient (Buli, 2025; Civil, 2007; Gutiérrez, 2012).

In this brief, we propose a theoretical model for attending to students' mathematical learning while humanizing the need to productively struggle with problems in collaborative classroom interactions. Furthermore, we discuss a sense of belonging as both an outcome of productive struggle and a condition for it, and the complexity of the components needed for this to occur. While a sense of belonging can be cultivated through caring relationships with students or leveraging Funds of Knowledge (Moll, 2019) in isolated mathematics tasks, we argue there is an astounding need to strengthen belonging through routine practices and over time, to not only cultivate a sense of belonging, but to repair possible endured bias or traditional approaches to mathematics instruction (Aguirre et al., 2017; The Collaborative for Academic, Social, and Emotional Learning [CASEL], 2022; Cohen, 2022; Thanheiser et al., 2025).

Productive Struggle

Student learning necessitates productive struggle (Hiebert & Grouws, 2007; NCTM, 2014). When students wrestle with and work through cognitively demanding tasks, they develop a stronger understanding of mathematical concepts (Hiebert & Grouws, 2007). Therefore, students should be given ample opportunities to engage in cognitively demanding mathematics tasks that promote productive struggle (NCTM, 2014; Sengupta-Irving & Agarwal, 2017; Warshauer,

2015). In this paper, we are particularly interested in Jarry-Shore & Richardson's (2024) typologies of struggle, including mathematical and social aspects of struggle during collaborative tasks.

A Sense of Belonging, Authority, and Agency

A student's sense of belonging refers to their feelings of inclusion and value in both the classroom and the larger mathematics community (NTCM, 2025). Students of all genders, languages, abilities, races, and ethnicities have the right to feel a sense of belonging in school and in mathematics (CASEL, 2022; Cohen, 2022). Watson (2023) identifies five factors of belonging: (1) You see me, (2) I see myself, (3) Others see me, (4) I am valued, and (5) I can grow. Thus, students' sense of belonging hinges on being seen by both the teacher and their peers as a valued member of the community. For students to establish themselves as participants in the community, educators must share authority with them. Mathematical *authority* resides with whoever is regarded as mathematically legitimate (Dunleavy, 2015). Teachers delegate authority to students when students decide which solutions and ideas are mathematically valid and why (Gresalfi & Cobb, 2006). Thus, supporting students in developing a sense of mathematical legitimacy within themselves and amongst their peers (Langer-Osuna et al., 2017), and ultimately, affording the students the space and ability to use their *agency* to solve problems as they see fit (Dyer et al., 2023). When students have the opportunity to engage in instruction that provides regular opportunities for student agency, they are more likely to share their mathematical thinking and ideas, and to advocate for representations that make sense to them in public and group settings (Aguirre et al., 2013; Ramos-Duke, 2024; Turner, 2003).

The Connection Between Productive Struggle and a Sense of Belonging

We propose that a sense of belonging is not just an outcome of productive struggle, but a condition for it in the mathematics classroom. Furthermore, we recognize Funds of Knowledge (FOK; Moll, 2019), Rights of the Learner (ROTL; Kalinec-Craig, 2017), and Social-Emotional Learning (SEL; CASEL, 2022) as supportive constructs in the development of productive struggle and a sense of belonging amongst mathematics students in the classroom. Funds of knowledge, the premise that students walk into the classroom with knowledge (e.g., in different languages, contexts, interests); thus, leading to planning and constructing mathematics classroom environments with asset-based views. Additionally, Kalinec-Craig's (2017) first right, *You Have the Right to be Confused*, establishes that productive struggle is necessary for students to feel supported without "judgment or ridicule" (p. 5). Socio-emotional learning helps students develop self-management, self-awareness, responsible decision-making, relationship skills, and, importantly for group work, social awareness (CASEL, 2022; Joswick & Taylor, 2022). Thus, supporting a sense of comfort and belonging while working together and productively struggling through mathematical problems.

The Thickening of Sense of Belonging and Productive Struggle Over Time

The model in Figure 1 depicts a thickening of concepts over time through the implementation of asset-based practices and frameworks. Turner (2003) describes students' mathematical agency as a fragile construct that necessitates routine day-to-day attention. Therefore, because students need to use their mathematical agency to engage in productive struggle in collaborative settings to solve cognitively demanding tasks, we extend this notion to the whole model. Thus, mathematical authority, agency, productive struggle, and a sense of belonging are fragile. This fragility is lessened by the asset-based practices and frameworks used to build a positive relationship and a sense of belonging with(in) mathematics (Aguirre et al., 2013; Cohen, 2022).

Therefore, if mathematical practices routinely support students' mathematical authority, agency, ability to productively struggle, and a student's sense of belonging, and do so purposefully in tandem, these student-centered concepts will thicken and strengthen over time.

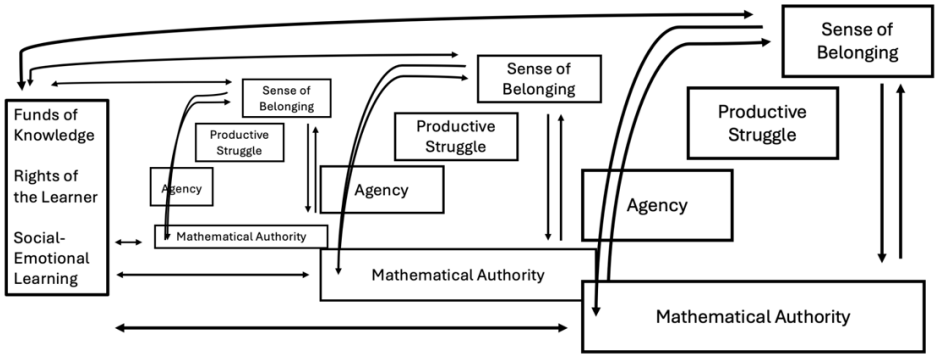


Figure 1: Building a Sense of Belonging (CASEL, 2022; Cohen, 2022; Dunleavy, 2015; Hiebert, J., & Grouws, 2007; Jarry-Shore & Richardson, 2024; Kalinec-Craig, 2017; Langer-Osuna et al., 2017; Moll, 2019; Turner, 2003)

Productive Struggle in Groups Necessitates Positive Relationships

Working together in mathematical groups requires positive relationships (Langer-Osuna et al., 2021). Furthermore, productive struggle in collaborative settings requires that each student be heard and included (Jarry-Shore & Richardson, 2024). To support productive struggle within groups, we link CASEL (2022)'s notion of a sense of belonging, a student's sense of being part of the community or a group, with seven elements identified by the Annie Casey Foundation (2021). These elements aim to support equity and inclusion and address the fundamental human need to belong: (1) an extravagant welcome (You Belong!), (2) being seen, (3) being affirmed, (4) culture, (5) being heard, (6) self-determination, and (7) a focus on well-being. This link brings to the surface elements necessary for students to engage in collective productive struggle in a group setting, rather than hang back or abandon a problem.

When students even feel slightly as if they do not belong, there can be missed opportunities for learning (Annie E. Casey Foundation [AECF], 2021). According to Walton and Cohen (2007), "One of the most important questions that people ask themselves in deciding to enter, continue, or abandon a pursuit is, 'Do I belong?'" (p. 94). Moreover. The Annie E. Casey Foundation (2021) states that cultivating belonging requires addressing culture and identity. Therefore, to support positive relationships and address elements of belonging in mathematics groups (AECF, 2021; Langer-Osuna et al., 2021), we intertwine Funds of Knowledge, the Rights of the Learner, and Social-Emotional Learning (CASEL, 2022; Kalinec-Craig, 2017; Moll, 2019). In our theoretical model, a sense of belonging is purposefully cultivated in a mathematics environment that is steeped in respect for the knowledge (FOK) and culture students bring to the group; students are routinely afforded the ROTL in the group; and students are supported in developing the emotions (SEL) conducive to building relationships, in order to activate the authority and agency needed to productively struggle together as a group when solving mathematics problems (CASEL, 2022; Cohen, 2022; Jarry-Shore & Richardson, 2024; Kalinec-Craig, 2017; Langer-Osuna, 2021; Moll, 2019; NCTM, 2014; Ramos-Duke, 2024; Saba & Johnson, 2025). Thus, the social aspects of struggle in groups are fostered, including "the need to be heard and included" (Jarry-Shore & Richardson, 2024).

Discussion

We acknowledge that the theoretical model discussed in this brief is only a portion of the complex ecosystem required to support the development of a sense of belonging. However, we hope to emphasize the connection between productive struggle and a sense of belonging, as both concepts are crucial for mathematical progress and growth. As K-16 educators, we see this model as especially salient for multilingual learners and others who have traditionally been harmed in mathematics classrooms, and who might not be comfortable with the vulnerability required to engage in productive struggle in a group setting (Aguirre et al., 2013; Ramos-Duke, 2024; Jackson et al., 2017). Therefore, routine, persistence, and tending to the complex nature of developing a sense of belonging are imperative. Although a sense of belonging can be supported through math tasks with relatable contexts or connections to students' lives and interests, teachers need to do the heavy lifting of creating a culture of belonging. This culture must develop and thicken over time.

Implication and Future Use of Our Model

This emergent theoretical framework is intended to guide research, teacher preparation, and the longitudinal implementation of productive struggle in K-16 mathematics courses. As a crucial aspect of learning, productive struggle, and a potent part of collaborative culture, a sense of belonging seems necessary for students to succeed and engage in ambitious mathematics and cognitively demanding tasks.

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HOW DO ELEMENTARY TEACHERS INCORPORATE NON-DIGITAL MATH GAMES INTO INSTRUCTION? A MIXED METHODS STUDY

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This convergent mixed-methods study examined how K-2 teachers in Maine use non-digital math games to support equitable mathematics instruction, grounded in Universal Design for Learning (UDL) and embodied cognition. Survey data from 127 teachers revealed that 87% use non-digital games at least weekly, primarily to foster engagement, fluency, conceptual understanding, and differentiation. Interviews with three teachers aligned with survey findings and showed that games are adapted to support diverse learners, with a preference for non-digital over digital activities. Despite barriers such as time and resources, findings suggest non-digital math games are often embedded within core instruction and are adapted with practices aligned with UDL. Implications point to the need for educators' professional learning around non-digital math games and further research to deepen their instructional impact.

Keywords: Elementary School Education, Instructional Activities and Practices

Introduction

Mathematical games have been used in classrooms for decades, tracing back to Dienes' theory of learning mathematics (Dienes, 1963). According to Dienes (1963), games provide playful and embodied experiences, making mathematical concepts more tangible and accessible to learners. Ernest (1986) further advocates for the use of games, highlighting their effectiveness in teaching mathematics, particularly for practice, acquiring concepts, developing problem solving skills, and motivation. Mathematical games include both digital (use technology) and non-digital formats and are defined as activities that teach math skills through skill, strategy, rules, and interactive play (Russo et al., 2024). A classic non-digital math game is *Shut the Box*, in which players use dice and arithmetic operations to close numbered tiles, with the goal of closing all the tiles on the box (Stebler et al., 2013). This simple design affords opportunities for players to strengthen their fact fluency through haptic-tactile interaction and strategic thinking. Design-based researchers, cognitive scientists, and teachers, especially those working in alternative or experimental settings (e.g., Montessori Schools), have long heralded the educational value of math games. In a systematic review of 43 empirical studies focusing on K-12 math games, Pan et al. (2022) found that learning games in K-12 mathematics education are generally effective in enhancing math learning, however, they highlight the need for more research examining game design and use to support meaningful mathematical understanding. Similarly, Russo et al. (2024) analyzed 32 studies and found that math games are potentially effective for developing math proficiencies and improving motivation towards math. Math games also have a multifaceted role in mathematics teaching practices (Applebaum, 2025). Buchheister et al. (2017) suggest math games have the potential to meet the needs of individual students, "bridg[ing] the type of instruction alongside the needs of the individual student to positively impact students' mathematical development" (p. 12).

While research supports the effectiveness of math games broadly, an important distinction emerges between digital and non-digital formats, particularly in how teachers use them instructionally and how they support students' development of mathematical understanding. In a

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recent study that surveyed Australian primary school educators, teachers reported a clear preference for non-digital games, emphasizing their affordance of collaboration, communication, use of manipulatives, and flexible differentiation (Russo et al., 2023). Even as schools in the United States of America work to close the digital divide and more schools have access to technology, non-digital games remain a powerful, accessible, and pedagogically valuable tool for fostering mathematical thinking and engagement (Hall et al., 2024). Physical tools allow students to touch, feel, grip, spatially interact, and engage socially (Shin et al, 2017). This is further supported by theories of embodied cognition emphasizing that mathematical understanding is grounded in physical interaction and manipulation with the environment (de Freitas & Sinclair, 2014). Consequently, non-digital math games may provide unique opportunities for children to discover mathematical relationships, develop strategic reasoning, and construct understanding through embodied, social, and multimodal interactions (Tran et al., 2017).

However, despite growing recognition of these affordances, they remain on the periphery of many teachers' instructional practices and relatively little is known about how teachers incorporate non-digital games into everyday mathematics instruction or adapt them to support the diverse needs of learners. Orim and Ekwueme (2011) found that teachers feel games are a waste of time because they are too time consuming and unmanageable. Teachers also face constraints due to lack of resources and limited training in effectively implementing games in their classrooms (Applebaum, 2025). Contrary to this, the Maine Education Policy Research Institute found that elementary educators "suggested more hands-on materials and games for concrete representations and engagement in learning and less reliance on paper worksheets" (Lech et al., 2024, p. 52). Therefore, the current state of the research suggests a disconnect between the growing body of scholarship highlighting the instructional potential of math games and their integration into everyday classroom practice. Therefore, the purpose of this study was to investigate how elementary teachers incorporate non-digital mathematics games into their teaching and adapt them to meet diverse student needs. This study's focus on non-digital math games is important because it highlights hands-on, technology-free, socially interactive, and differentiated learning experiences without the use of technology.

Purpose of the Study and Research Questions

This study explored how primary grade teachers (K-2) use non-digital math games to support student learning. Specifically, the researchers investigated teachers' instructional goals, adaptations, and perceptions of the benefits and challenges of using math games. The findings contribute to a deeper understanding of both the pedagogical role of non-digital math games and the ways teachers tailor them to meet diverse student needs. The following research questions guided this study:

1. How frequently do teachers use non-digital math games, and what pedagogical aims guide their use?
2. How do teachers implement and modify non-digital games to support all students, and what factors support or constrain their use?

Theory: UDL and Embodiment

We frame our inquiry into teachers' use of non-digital mathematics games from the perspective of Universal Design for Learning (UDL; Meyer et al., 2014). UDL came from the CAST Center in Boston in the 1980s and is an educational framework for instruction that is rooted in the learning sciences and neuroscience (Meyer et al., 2014). UDL encourages

"mathematics teachers to move from deficit-focused pedagogies that seek to "fix" children toward asset-based pedagogies" (Lambert, 2021, p. 662). Since all students vary in how they see, hear, move, remember math concepts, focus attention, and respond emotionally to mathematics, teachers must account for this learning variability. Only by doing so can they effectively support mathematical learning of all students (Meyer et al., 2014; Rose, 2017). The goal of UDL is to develop expert, strategic, and lifelong learners, rather than focusing on content.

Non-digital math games align closely with the UDL framework by offering flexible, multimodal entry points to mathematical ideas. Through physical manipulatives, visual cues, oral language, movement, and choice of strategies, games provide multiple means of representation and action, allowing students to access mathematical ideas in ways that align with their strengths. Importantly, games make student thinking visible through moves, talk, and decisions, creating rich opportunities for equitable feedback and formative assessment. Math games also position students as capable problem solvers, support productive struggle, and affirm diverse ways of mathematical understanding. Therefore, through a UDL framework, non-digital math games have the potential for helping all learners, particularly students with disabilities, develop positive mathematical identities and sustained engagement with mathematics.

This study is grounded in the theoretical perspective advanced by de Freitas and Sinclair (2014) conceptualizing mathematical learning as an embodied, material, and relational process rather than a purely cognitive one. From this perspective, mathematical thinking emerges through the dynamic entanglement of learners' bodies, gestures, spatial movement, and material artifacts within learning environments (de Freitas & Sinclair, 2014). This framing is particularly salient for early elementary learners, whose mathematical understanding is closely tied to sensorimotor experience and physical engagement with their environment.

Non-digital mathematics games in K-2 classrooms provide a powerful context for examining these material entanglements. Such games often involve manipulatives, spatial organization, turn-taking, bodily movement, and peer interaction. For example, when students count by moving along a number line, build structures with blocks to compare quantities, or use hand gestures to represent numerical relationships, mathematical ideas are enacted through embodied practice. In these moments, bodies, materials, and mathematical concepts co-produce understanding. By using de Freitas and Sinclair's (2014) framework, this study examines how K-2 teachers leverage non-digital math games to create embodied mathematical experiences and how these practices reflect underlying beliefs about how children learn mathematics. This lens allows for an analysis of teaching practices that attends not only to instructional strategies, but also to how classroom materials, spatial arrangements, and bodily actions shape opportunities for mathematical thinking.

Methods

This study employs a convergent mixed-methods design (Tashakkori & Teddlie, 2016) to investigate K-2 teachers' use of non-digital mathematics games, with a particular focus on how teachers design, adapt, and facilitate games to support equitable mathematical learning, including for students with disabilities. Through the convergent mixed methods design, we simultaneously collected and analyzed both quantitative data from surveys and qualitative data from interviews, then merged the results to compare findings, identify alignment or divergence, and gain a more comprehensive understanding of our research questions (Tashakkori & Teddlie, 2016).

Survey Participants

A total of 127 K-2 teachers in Maine completed a survey that focused on how they use non-digital math games in their classrooms. Although Maine represents a convenience sample, its

highly localized school systems and the instructional autonomy afforded to teachers provide a valuable context for this study. Maine educators' perspectives offer an opportunity to surface a diverse range of perspectives on game-based learning practices in a rural state. Participants included classroom teachers (n = 71), special education teachers (n = 37), math interventionist (n = 10), math coaches (n = 3), gifted and talented teachers (n = 3), behavior interventionists (n = 2), and an education technician (i.e., paraeducator; n = 1). These K-2 teachers indicated they worked in rural (n = 78), suburban (n = 37), or urban (n = 12) settings. Overall, participants were relatively experienced teachers with 70% (n = 89) of participants having over 10 years of experience.

Survey Design

The survey was distributed using Qualtrics, an online survey platform to reach all K-2 teachers in the state of Maine. The survey defined non-digital mathematics games as an activity that: "has specific mathematical cognitive objectives; requires students to use mathematical knowledge to win the game; is enjoyable and with potential to engage students; is governed by a definite set of rules and has a clear underlying structure; involves a challenge against either a task or an opponent(s) and interactivity between opponents; includes elements of knowledge, skills, strategy, and/or luck, but not luck alone; has a distinct finishing point;" and "does not employ a digital device or tool to support the playing of the game" (Russo et al., 2024, p. 3). The survey questions were adapted from Russo et al. (2021) and focused on frequency and pedagogical aims for playing non-digital math games.

Survey Analysis

Survey responses were analyzed descriptively and are presented in the following section. Frequencies and percentages were calculated for each survey item and used to identify common patterns and trends across participants' responses. No statistical analyses were conducted, as the survey data were used primarily to contextualize subsequent qualitative analyses.

Interview Participants

Fifteen K-2 Maine teachers agreed to participate in follow-up interviews after completing the survey. This paper focuses on three of these teachers, whose backgrounds are presented in Table 1. These participants were purposefully selected to illustrate a range of perspectives on the use of math games across diverse teaching roles, levels of experience, and educational settings.

Table 1: Background of Three Participants: Teacher A, Teacher B, and Teacher C

	Teacher A	Teacher B	Teacher C
Position	Grades K-2, Math Interventionist	Grade 2, General Education	Grades 1-2, Special Education
Teaching Experience	35th year teaching	19th year teaching	2nd year teaching
Type of School	K-2 School	K-8 School	PreK-2 School
Student Population	300 Students	360 Students	600 Students
School Setting	Suburban	Rural	Suburban
Frequency of playing non-digital math games	"I do multiple times a week… at least two times a week"	"I shoot for five times a week… at least three days"	"I use games almost everyday"

Interview Design

Semi-structured, one-on-one interviews were conducted virtually over Zoom. To have a more in-depth understanding of teachers' experiences, interview questions were designed by transforming selected survey items into more open-ended prompts, allowing teachers to elaborate on their instructional practices related to implementing non-digital math games. Prompts were aligned with the Universal Design for Learning (UDL) framework to explore how teachers provide multiple means of engagement, adaptations, and differentiation. For example, the interview protocol included questions such as, "Describe how you use non-digital math games in your classroom," and "Thinking about students who struggle in math, how do you adapt math games to support their learning?"

Interview Analysis

Interviews were thematically analyzed through iterative coding, theme development, and ongoing interpretation of patterns across participants' responses (Braun & Clarke, 2021).

Survey Results

Survey responses from 127 K-2 teachers in Maine were analyzed to describe how teachers use non-digital math games to support their mathematics instruction. Teachers were asked to report how frequently they played non-digital math games, ranging from "all the time" to "never." Table 2, below, summarizes these results. Overall, 87% of participants (n = 110) reported playing non-digital math games at least once per week in their classrooms, while only one participant reported never playing non-digital math games.

Table 2: Frequency Teacher Reported Playing Mathematical Games

Frequency of math game use	Frequency (n=127)	Percent
All the time (e.g., about 4–5 times per week)	34	27%
Often (e.g., about 2–3 times per week)	48	38%
Sometimes (e.g., about once per week)	28	22%
Rarely (e.g., less than once per week)	16	13%
Never	1	1%

Teachers were also asked how frequently they incorporated non-digital math games into different components of their classroom routines and math lessons. The same frequency scales were used, ranging from "all the time" to "never." Table 3, below, summarizes these results. 57% of participants indicated that they use non-digital math games to support fluency practice either all the time or often. Furthermore, 48% of participants indicated that they use non-digital math games to support students who are low achieving in math either all the time or often. Using math games outside of math lessons, as a warm-up activity in a math lesson, and/or at the closure of a math lesson were less frequently reported by teachers.

Table 3: Frequency of Ways Teachers Use Mathematical Games

Component of classroom routines or math lessons	All the time (4–5 times per week)	Often (2–3 times per week)	Sometimes (about once per week)	Rarely (less than once per week)	Never
Outside of my math lessons	12%	22%	31%	24%	12%
As a warm-up activity in a math lesson	8%	22%	30%	30%	10%
As the main learning activity in a math lesson	7%	15%	45%	21%	13%
As the closure activity in a math lesson	11%	21%	30%	20%	17%
To support fluency practice	20%	37%	28%	12%	3%
To support students who are low achieving in math	16%	32%	30%	16%	6%

Teachers were asked to evaluate the effectiveness of non-digital mathematics games for 11 pedagogical aims on a 5-point Likert scale, ranging from strongly disagree to strongly agree. Table 4, below, summarizes the results. In general, teachers indicated that non-digital math games were effective for achieving the pedagogical aims outlined in the survey. The vast majority (97%) of participants indicated that they either strongly agreed or agreed that non-digital games were effective for engaging students in mathematics lessons. Furthermore, the majority of participants also indicated that they either strongly agreed or agreed that math games were effective for building procedural fluency (88%), conceptual understanding (91%), mathematical reasoning (88%), and problem-solving skills (91%), and for differentiating (90%). In contrast, only 54% of participants indicated that they either strongly agreed or agreed that math games were effective for supporting connections between home and school.

Table 4: Teacher Agreement on Effectiveness Across 11 Pedagogical Aims

Pedagogical Aim	Combined Agree or Strongly Agree	Only Strongly Agree	Only Agree
Engaging students in math lessons	97%	49%	48%
Building procedural fluency	88%	31%	57%
Maximizing on-task behavior	81%	28%	53%
Building conceptual understanding	91%	31%	60%
Generating rich mathematical discussions	73%	17%	56%
Building mathematical reasoning	88%	33%	55%
Differentiating for different performance levels	90%	38%	52%
Building problem-solving skills	91%	40%	51%
Focusing students on important mathematical ideas	85%	30%	55%
Assessing student learning	76%	24%	53%
Supporting connections between home and school	54%	16%	38%

Interview Results

Pedagogical Aims for Using Non-Digital Math Games

Teachers' responses regarding their use of non-digital math games aligned with trends identified in the survey data. Their responses included using games to support fact fluency, serve as warm-ups, increase student engagement, provide hands-on learning opportunities, and support students who struggle in mathematics through differentiation. Teachers described using games flexibly within math lessons, across small-group and independent contexts, and as part of broader instructional routines. For example, Teacher A explained, "I will embed them in the lesson, typically at the beginning, just to warm up our brains, to have something fun to get us started, and then it's really nice when I can tie it into the lesson later on." Consistent with prior research, teachers in this study viewed games as effective tools for engaging students in meaningful mathematics (Hall et al., 2024; Russo et al., 2023; Tran et al., 2017). Extending this literature, the findings illustrate how non-digital games are embedded within and integrated across daily classroom instructional routines.

Teaching Practices

Across all three teachers, there is strong alignment with posing purposeful questions, eliciting and using evidence of student thinking, and facilitating discourse. For example, Teacher B explained, "I try to go around and help facilitate with each group and ask prompting questions." Together, their statements illustrate how non-digital math games can serve as rich contexts for student-centered, discussion-driven, and conceptually focused instruction. These practices are also consistent with the National Council of Teachers of Mathematics' (NCTM) vision of effective mathematics teaching, which emphasizes that high-quality mathematics instruction should foster conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NCTM, 2014; NCTM, 2020).

Adaptations to Support Students

Across all three teachers, adapting math games reflects strong UDL-aligned practices by reducing unnecessary barriers to participation, adjusting challenges to support productive engagement, offering flexible representations and supports, and prioritizing student success as a pathway to deeper learning. For example, Teacher C described, "If there is a student that learns better with a number line, there's an adaption for that game." Together, these adaptations illustrate how non-digital math games can be flexibly designed and implemented to support inclusive, learner-centered mathematics instruction consistent with UDL principles. Additionally, all three teachers' adaptations support embodied mathematical learning by engaging students in physical and social interactions with game materials (e.g., moving pieces, using number lines, and negotiating turns). Consistent with de Freitas and Sinclair's (2014) perspective, these teachers' adaptations create conditions for mathematical understanding to emerge through the entanglement of students' actions, social relationships, and material artifacts.

Preference for Non-Digital Games Over Digital Activities

All three teachers' perspectives further illuminate why non-digital math games remain central in their mathematics instruction, especially when comparing it with technology or digital activities. As Teacher A explained, "I never use digital math games with my students…the math games that we play need to be interactive … I need to be watching as we play games" (Teacher A), emphasizing the importance of interacting with students and observing students' thinking in real time. Similarly, Teacher C described how "they look at screens all the time," and explained that digital activities are "really hard to monitor" and their "just doing it solo, online, zero interaction," contrasting this with hands-on experiences where students work with partners,

"explain[ing] why they know what they're doing" (Teacher C). Together, all three teachers' emphasize the importance of and preference for non-digital math games.

Barriers to Implementation

Across interviews, teachers described consistent barriers to implement non-digital math games in their classrooms including time, challenges with student behaviors, access to ready-made resources or shared materials, collegial collaboration, funds, and support from administration. Teachers valued curated game kits and shared repositories that reduced preparation demands and increased instructional quality. Teacher A even noted the value of "the plethora of games that we now have with the Building Fact Fluency [Kit], instead of me hunting, trying to find games. I really like the Building Fact Fluency games, because it's not just math facts, there's also strategy involved." Guidance varied by context and experience, ranging from school-supported initiatives and coaching cycles to informal mentoring and RTI-based supports. Behavioral challenges required ongoing adaptation, including modifying rules, scaffolding emotional regulation, or pausing gameplay altogether. For example, one teacher explained, "For probably at least a third of the students in here, it was too hard, so I ended up changing the rules" (Teacher B). Therefore, teachers' wishes centered on increased administrative support, dedicated time to design and observe game-based instruction, shared spaces for resource creation, and funding for physical games rather than digital platforms, highlighting a desire for systemic alignment to sustain equitable, effective implementation.

Discussion and Implications

This study contributes to the growing body of research on the pedagogical value of non-digital math games in early elementary classrooms. Findings from surveys and interviews suggest that non-digital math games are frequently embedded within core instructional routines in K-2 classrooms and have important implications for equitable mathematics instruction. The findings from this study provide evidence that teachers' practices using non-digital math games often align with the UDL framework by offering multimodal access to math content, while also supporting embodied learning by connecting mathematical understanding to sensorimotor experiences and physical engagement with the environment. Together, these affordances create more accessible and inclusive opportunities for students to participate in mathematical learning.

Despite growing pressure to adopt digital technologies, there remains a persistent preference for non-digital approaches. Survey and interview data suggest educators intentionally avoid digital activities and prefer activities that requires interactive, hands-on play where teachers can observe thinking and facilitate social interactions. Research supports these findings, indicating that educators not only use non-digital math games more often than digital games, but they also express a clear preference for them to support mathematics teaching and learning (Russo et al., 2024). Furthermore, while digital tools are increasingly embedded in early mathematics instruction, excessive screen exposure may limit opportunities for hands-on, social, and movement-based learning that are developmentally important in the early grades (Munzer et al., 2026). Non-digital math games offer an appropriate alternative by providing interactive mathematical experiences without reliance on screens. Moreover, as artificial intelligence (AI) becomes more deeply integrated into educational tools, opportunities for embodied and sensory-motor engagement may become further limited. This shift underscores the continued importance of tactile, hands-on learning experiences, particularly in elementary classrooms, where embodied interaction plays a critical role in mathematical learning. Accordingly, the intentional design of non-digital learning activities is increasingly essential to ensure that they meaningfully support mathematical development within AI-driven educational contexts.

However, while teachers reported frequent use of games, particularly to support low-achieving students, they were not always leveraging them to their fullest, such as by extending them into rich mathematical investigations that promote deeper reasoning (Russo & Russo, 2020; Russo et al., 2021). Therefore, further research is needed to identify how non-digital math games can best function across diverse classroom contexts and student populations. This paper also highlights a need for educator professional learning that supports teachers in critically examining how to leverage non-digital math game interventions as a component of teaching mathematical problem solving, "in which the main goal is for students to develop a deep understanding of mathematical concepts and methods" (Gray et al., 2025, p. viii).

The findings from this study also provides a strong foundation for future design-based research (DBR) grounded in Math UDL (Lambert, 2021). The next phase of this work will involve collaborative partnerships with K-2 teachers to iteratively co-design, implement, and refine non-digital math game interventions that align with UDL Math principles and classroom realities. Through DBR, educators and researchers can work together to create engaging, accessible, and meaningful mathematical experiences that position all learners as capable participants in rich mathematics learning.

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POSITIONING STUDENTS AS MATHEMATICS DOERS: TEACHER DISCOURSE AND LEARNING OPPORTUNITIES IN GHANAIAN ELEMENTARY SCHOOLS

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This study investigates how elementary school teachers from Ghana position their students as mathematics doers and how such positioning shapes instructional practices. Using positioning theory and sociocultural learning theory, the study analyzes semi-structured interviews and classroom observation data from five teachers. The analyses trace connections between teachers' beliefs about students' ability and their communicative acts and resulting teaching moves. The findings reveal three positioning practices: inclusive, mixed, and restrictive, that mediate students' access to mathematical reasoning and participation. These are not fixed traits but interactional stances that emerge during instruction. The study highlights the role of teacher discourse in shaping learning opportunities and suggests making positioning an explicit focus of professional development.

Keywords: Positioning Theory, Teacher Discourse, Inclusive Positioning, Restrictive Positioning

Introduction

Teachers play a major role in how learners respond to mathematical learning. They help students develop problem-solving skills, connect math to real-world contexts, and build confidence through meaningful feedback (Mathematics, 2020). How teachers communicate, deliver instruction, and facilitate classroom discussions significantly influences how students see themselves and engage with mathematics (Boaler, 2013; Davis, 2018; Kofi & Chaiklin, 2015). Research shows that when teachers encourage open dialogue, pose thoughtful questions, and validate student contributions, students' participation, confidence, and mathematical abilities improve (Davis, 2018). This research positions classroom interactions as essential in mathematics teaching and learning and meaning-making.

At the core of mathematics education is the relationship between how math is discussed and how students are positioned (Bishop, 2012; Langer-Osuna, 2011). Bishop shows that students' mathematical identities are formed and reinforced through classroom interactions. Mathematical discourse includes how teachers and students represent, think, talk, agree, and disagree when working on math problems (Moschkovich, 2007). Classroom discourse not only conveys math content but also positions students as either competent, valued thinkers or as struggling, passive learners, depending on teacher expectations and interactions (Moschkovich, 2002; Turner et al., 2013). Thus, discourse extends beyond learning terminology and involves participating in mathematical practices and ways of thinking.

Teachers are instrumental in positioning students in the classroom. Teacher positioning refers to how teachers perceive and interact with students, assign roles, and shape identities that affect students' participation and learning (Turner et al., 2013; Yamakawa et al., 2009). This includes how teachers frame expectations, provide feedback, and manage the classroom dynamics. These interactions can lead to either positive positioning,, for example, when a teacher acknowledges a student's effort and potential, positioning them as a capable and valued, or negative positioning, when a teacher repeatedly points out errors without constructive feedback, positioning the student as a struggling or passive learner (Davies & Hunt, 1994; Suh et al., 2013). Positioning

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

students as competent involves using specific, explicit affirmations to recognize that they have contributed something meaningful to the classroom community (Davies & Hunt, 1994; Gallagher et al., 2024).

In Ghana, recent curriculum reforms promote student-centered, discourse-rich math instruction (National Council for Curriculum Assessment [NaCCA], 2019), but little is known about how teachers position students in practice. This study addresses that gap by qualitatively examining how Ghanaian elementary teachers understand and enact positioning of themselves and their students as "math doers and thinkers," and the instructional strategies and classroom interactions they operate.

Conceptual Framework

This study draws on positioning theory and sociocultural perspectives on mathematics education (Lerman, 2000) within Ghana's educational system. Harré and van Langenhove's (1999) positioning theory is the primary lens (Wagner & Herbel-Eisenmann, 2009) for examining how Ghanaian elementary teachers position their students in mathematics classroom discourse. McVee (2018) acknowledges that positioning theory (PT) encompasses diverse communicative practices that can be observed in actions and gestures. Therefore, we as educators must recognize that although many theorists concentrate on spoken interactions, Harré and others acknowledge a range of symbolic representations and actions as positioning methods. Consequently, as this study employed Positioning theory, it considered various symbols, not just verbal communication, that could position individuals within or in relation to specific narratives or storylines. Positioning theory highlights how rights and responsibilities are negotiated in interaction and how these negotiations shape participants' roles in shaping individual roles within a given social context. It emphasizes the rights and responsibilities that individuals assume during these interactions. Sociocultural learning theory complements this view by emphasizing learning as mediated through social interaction (Vygotsky, 1978).

Guided by this framework, the study addresses an overarching research question, with two sub-questions:

1. How do Ghanaian elementary teachers position their students as mathematical doers?
    a) What perception do teachers hold about their students' abilities?
    b) How do teachers position students in classroom interactions?

Methods

This qualitative study took place in Winneba, a municipality in the Central Region of Ghana, and involved five (Teachers A, B, C, D, and E) Grade 3 teachers from five different schools, allowing for varied positioning practices. Data were collected through classroom observations, video recordings of lessons, and semi-structured interviews with teachers about their positioning practices and perceived influence of positioning on their instructional moves.

Each teacher was observed across three consecutive mathematics lessons to identify patterns in how they positioned students and the associated participation structures and teaching strategies (Drageset & Ell, 2024). The analytical framework integrated the MIELL Observation Rubric with Wilson et al.'s Seven Forms of Practice. Using positioning theory's constructs—communication acts, positionings, and storylines—I analyzed teachers' interactions with the students (Herbel-Eisenmann et al., 2015). Video data were segmented into 7.5-minute chapters in line with the MIELL Observation Rubric. Within each chapter, instances (Wilson et al., 2019) were documented to determine the frequency of each practice.

Description of the codes of analysis. Wilson et al. (2019) identified the seven forms of practice. They conducted a comparative analysis of teaching practices in two sets of middle-grade mathematics classrooms: Phase 1, where African American students outperformed expectations on standardized assessments, and Phase 2, where they did not. The findings suggest that while all Phase 1 classrooms demonstrated conceptually oriented instruction, several additional teaching practices were more prevalent in these settings compared to the Phase 2 classrooms. The authors argue that these additional practices are important for providing equitable access to high-quality mathematics education. The seven identified teaching practices that correlate with improved mathematics outcomes for marginalized student groups are attending to the classroom community, attending to language, attributing mathematical authority to students, attending to students' local context, coaching students, making expectations explicit, and positioning students as competent. These practices served as the codes used in this study.

The analysis focused on teachers' strategies to foster students' participation and engagement, such as revoicing students' contributions, asking probing questions, providing feedback, clarifying concepts, explicitly validating students' reasoning, elevating students' responses, and opening spaces for students to share their ideas.

Findings

Cross-Case Synthesis of Teachers' Form of Practices Positioning Students as Math Doers in the Observations

This cross-case synthesis of fifteen lessons examines how teachers' enacted positioning practices reflected their perceptions of students' abilities and beliefs about learning. The summary highlights key consistencies, contrasts, and tensions across cases.

Attending to community. All teachers aimed to cultivate a sense of classroom community. Teachers A, C, and D created robust participatory environments through chants, clapping, songs, and shared rituals, positioning students as members of a learning community and fostering belonging. Teacher B combined similar rituals with strict behavioral norms, positioning students as community members within a disciplined environment. Teacher E also used rituals and occasionally encouraged students to validate one another's answers, but discipline tended to override positive community-building. Teacher D went further in constructing group identities and explicitly balancing participation between boys and girls, naming and recognizing groups to support equitable involvement.

Attending to language. All teachers emphasized correct mathematical vocabulary. For Teachers A, B, C, and D, language functioned as a bridge between informal student talk and formal mathematical discourse. Teachers A and C consistently revoiced students' informal expressions into formal terms and provided frequent bilingual scaffolding. Teacher D systematically built vocabulary over time (e.g., from perimeter to time routines and clock-related terms). Teacher B also used bilingual support and revoicing to transition students from familiar to formal language. In contrast, Teacher E treated language as a gatekeeper mechanism, equating precision with mathematical competence and insisting on exact terminology (e.g., "If you don't write the unit, you are wrong").

Attributing mathematical authority. Teachers C and D most clearly decentralized mathematical authority, distributing it across the classroom and positioning students as legitimate validators and correctors. They invited students to validate peers' work, identify and correct errors, and self-correct at the board. Teachers A and B allowed some shared authority, but Teacher B retained strong teacher-centered authority with limited peer validation. Teacher A encouraged students to test, estimate, and propose answers, yet reserved final judgment. Teacher

E most strongly centralized authority: she relied on teacher validation, and peers only affirmed correct answers rather than engaging in shared reasoning.

Attending to local context. Teachers A, C, and D actively positioned students as everyday mathematicians by drawing on local contexts. Teacher A richly incorporated familiar objects (e.g., bowls, bottles, and polytanks). Teacher C strongly anchored lessons in local practices (e.g., farming, routines, local measurements), while Teacher D consistently used everyday situations (e.g., church, TV, meals, wristwatches, phones). Teacher B referenced local contexts such as trading and measurement, but less frequently and less extensively. Teacher E rarely used local context, favoring abstract, decontextualized tasks with limited connections to students' lives.

Coaching and making expectations clear. All teachers made expectations for participation explicit. Teachers A and C framed expectations through coaching and encouragement, treating mistakes as part of learning. Teachers B and E focused on rules and compliance, pairing procedural guidance with discipline. Teacher D integrated expectations into group work, emphasizing neatness, balanced turn-taking, and group norms to promote fair participation.

Positioning students as competent. Teachers A, C, and D defined competence broadly, encompassing effort, reasoning, and improvement, not just correctness. They framed errors as opportunities for learning, avoiding shaming or negative labels, thus positioning students as contributors and sense makers. Teacher B frequently revoiced students' contributions but, like Teacher E, linked competence more narrowly to correctness, neatness, and adherence to rules, with only occasional recognition for improvement.

Overarching storylines. Each teacher's interactional patterns produced a particular storyline about what mathematics is and who counts as a "math doer." Teachers A, C, and D organized their classroom around participation, reasoning, and improvement. For Teacher A, Mathematics was practical and something to reason through together; for Teacher C, mathematics was reasoning in a safe community where errors are opportunities; for Teacher D, mathematics was participatory, shared by everyone, and connected to real life. Teachers B and E reinforced storylines of mathematics as rule-bound, precise, and teacher-controlled, making students' status as "doers" conditional on compliance. For Teacher B, mathematics was fun, but success was ultimately defined by correctness and careful procedural work. For Teacher E, math was procedural and precise, with competence equated almost restrictively with correct answers.

Integrated insight. Taken together, the five cases form a continuum of positioning practices, from inclusive to mixed to restrictive, that shapes participation. In classrooms where community, local relevance, and productive use of errors were (A, C, and D), students were positioned as capable contributors even when their responses were partial or incorrect, or regardless of the correctness. In classrooms where the focus is on discipline, rules, and precision (E), or those that combine engagement and correction with limited reasoning (B), students are seen as competent only when they are correct and compliant.

The cross-case and integrated syntheses reveal a clear pattern in teachers' practices, distributed along a continuum of positioning students as "Math doers." Viewed together, these patterns form a spectrum of positioning practices, from inclusive to mixed to restrictive, which shape classroom participation, engagement, and learning.

Teachers A, C, and D anchor the inclusive end of the spectrum, consistently treating students as competent learners whose contributions, even when approximate, warrant attention and refinement. Teachers B and E lean toward more restrictive positioning, tying competence by correctness and compliance. Teacher B, however, occupies a middle position: his emphasis on discipline and correctness coexists with inclusive practices such as revoicing, building

community through songs, providing enjoyable activities, drawing on local contexts, and using bilingual scaffolding. This mix generates tension; even as his practices invite participation, his recognition of competence remains largely confined to correct responses. Figure 1 illustrates the spectrum of positioning observed in the findings and cross-case analysis.

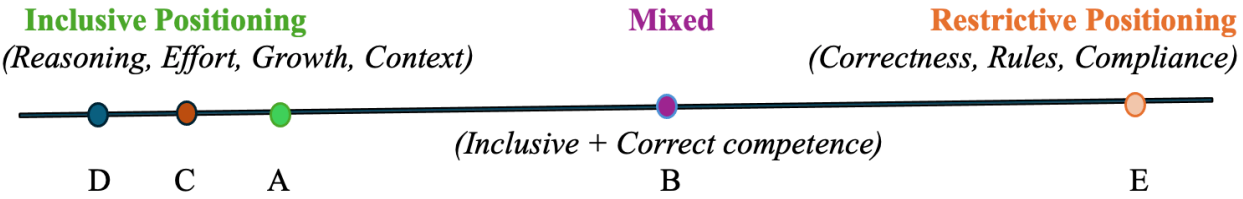


Figure 1: Spectrum of Teacher Positioning

Discussions

Teachers' perceptions about students' abilities. Interview findings showed that teachers generally held positive views of their students' mathematical abilities and expressed confidence in their capacity to learn mathematics, though with notable differences in depth and nuance. Most rated their students as "good" or "very good." For example, Teacher D believed her students could succeed and frequently work independently with minimal explanation, signaling strong trust in their capabilities. In contrast, some teachers framed students' struggles as evidence of limited potential. For instance, Teacher E viewed certain students as unlikely to improve substantially and treated ability as unstable and heavily dependent on teacher-led instruction. This stance reflects Dweck's (2006) concept of a fixed mindset, in which teachers see students' abilities as constrained and any progress as fragile. Such orientations may yield gains in procedural fluency but are unlikely to promote lasting conceptual understanding. Students may perform accurately in familiar contexts yet struggle to adapt to novel problems (Skemp, 1976). Other teachers emphasized emotional and relational aspects. Teacher C highlighted students' emotional sensitivity, demonstrating care for their well-being, while Teacher B described a range of abilities, noting that students were more cooperative and higher achieving. These patterns align with Myers et al. (2013), who found that teachers tend to group students by perceived ability or prior achievement, particularly during professional development on learning trajectories (LTs). Position theory (Harré & van Langenhove, 1999; Herbel-Eisenmann et al., 2015) suggests that perceptions and the roles teachers assign to students shape motivation, self-efficacy, and achievement. Accordingly, teachers' perceptions influence students' skills and participation in mathematics (Wood et al. 2019). Overall, these findings support prior research indicating that teachers' views and practices strongly influence students' engagement and learning (Yamakawa et al., 2009; Turner et al., 2013). The teachers in this study enacted positioning practices that reflected their beliefs about students' abilities and learning, supporting Mosvold and Bjuland's (2019) argument that managing students' positioning is central to effective mathematics instruction.

How teachers position students as math doers. Positioning students as active participants in mathematics is more than praising them or inviting participation. It is a sociocultural process through which teachers signal who is seen as mathematically competent, whose ideas are valued, and who holds mathematical authority (Yamakawa et al., 2009; van Langenhove & Harré, 1994). For the five Ghanaian elementary teachers in this study, positioning practices were shaped by their beliefs, language, instructional choices, storylines, and interpretations of mathematical

learning, in line with Anderson's (2019) mediational approach to positioning theory. Drawing on combined interviews and observation data, the study offers a coherent and nuanced account of how these teachers shape students' identities as mathematical participants. Across cases, positioning was frequently articulated in affective terms, with relational and affective positioning emerging as central to all five teachers' practices.

Across the five cases, Ghanaian elementary teachers positioned students as "math doers" through (a) their discourse patterns (affirming or evaluative), (b) the extent to which they share mathematical authority, (c) the forms of practice they employ (collaborative or procedural), (d) the storylines they construct about the nature of mathematics is, and their beliefs about student capabilities. In summary, Teachers A, C, and D tended to position students as competent sense-makers; Teacher B positioned students as enthusiastic yet dependent learners, and Teacher E positioned students as passive followers of procedures.

Implications of the spectrum. The spectrum illustrates that teacher positioning is not merely rhetorical but is embodied in routine classroom practices, such as daily rituals, scaffolding, error framing, language use, affirming, revoicing, and contextualization. As shown in Figure 1, it helps illuminate how subtle variations in teacher practices shape student participation, demonstrating that positioning is not binary but fluid and context-dependent (Davies & Hunt, 1994). This perspective suggests that teachers may move along the spectrum in response to lesson goals, classroom dynamics, subject matter, or individual student needs. Nonetheless, a teacher's dominant orientation, whether inclusive or restrictive, ultimately shapes students' mathematical identities and opportunities.

Implications for Future Professional Development and Recommendations

Teacher education programs should therefore move beyond a narrow focus on content knowledge and pedagogy to include:

- developing awareness of how teachers' perceptions and beliefs about student ability influence positioning.
- Provide intentional training in inclusive discourse practices.

Making positioning an explicit focus of professional development could support a shift away from primarily corrective practices toward a more expansive recognition and use of student competence.

Additionally, the findings suggest that teachers' everyday instructional decisions, particularly their communication acts, play a significant role in positioning students as either competent or restricted learners. To promote mathematical engagement, teachers should:

- Pose probing, open-ended questions that invite student reasoning.
- Attribute mathematical authority to students by validating and extending their ideas.
- Create space for mistakes and multiple solution strategies.
- Consistently affirm students' responses.

Future research should examine positioning practices among a larger and more diverse sample of teachers and across multiple contexts to clarify how these practices vary and how they might be reshaped to promote more equitable participation and learning in mathematics.

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TEACHERS' QUESTIONING DURING INTERACTIVE MATHEMATICS READ-ALOUDS

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Reading picture books to students can promote joy and curiosity while also supporting them in making meaningful connections to mathematical ideas. In particular, the types of questions asked during interactive mathematics read-alouds can target different levels of mathematical thinking. However, we need a clearer understanding of what happens in practice and what research suggests as effective questioning practices during interactive read-alouds that engage students in making sense of mathematical ideas. For this paper, we conducted a systematic literature review, focusing on detailing how mathematical read-alouds are used. In particular, we focus on a subset of the papers that describe the types of questions asked during interactive mathematics read-alouds. We analyzed 52 papers and found that, beyond detailed questioning categories, teachers also included questions related to mathematical affect.

Keywords: Elementary School Education, Instructional Activities and Practices

One of the National Council of Teachers of Mathematics' key recommendations for catalyzing change includes having students "experience the wonder, joy, and beauty of mathematics" (Huinker et al., 2020, p. 11). Key to this recommendation and other recommendations is to both provide the opportunity for students to wonder and ask questions, as well as for teachers to ask purposeful questions. One type of motivating mathematical exploration that can promote connection to and joy of mathematics is through picture books (e.g., Columba et al., 2009; Hintz & Smith, 2020; van den Heuvel-Panhuizen et al., 2016; van den Heuvel-Panhuizen & van den Boogaard, 2008), which builds on students' joy of listening to stories. The development and design of picture books that explicitly introduce mathematical concepts is a practice that has drawn increasing attention with the creation of the Mathical book awards (MSRI/Simons Laufer Mathematical Sciences Institute, 2026). However, researchers and educators also identify the benefits of mathematizing picture books that are not explicitly mathematical (e.g., *Yard Sale,* Bunting, 2015; see Columba et al., 2009) during interactive read-alouds. Interactive read-alouds are when the teacher stops at key points to invite discussion. Recently, Hintz and Smith (2022) outlined three types of instructional practices that teachers can use to help students notice and wonder, and think more deeply about mathematical ideas embedded in the story's content or illustrations as they listen to picture books read aloud. While important, their focus was on eliciting students' thinking through general questions, such as, "What do you wonder?" As they demonstrate, not only is questioning an important teacher practice in the mathematics classroom and in elementary contexts, but incorporating questioning within picture books during read-alouds is also a common and important practice. However, there are many additional types of questions, beyond asking about notices and wonderings, which may elicit both mathematical and non-mathematical ideas, that teachers might ask during picture book readings to engage students with the story's mathematical ideas.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Torbeyns et al. (2024) identify mathematical questions as those that encourage students to provide responses that relate to different mathematical domains (e.g., number and operations, measurement, geometry, space and location, patterns, etc.). Further, they delineated four levels of abstraction that could occur with mathematical questions posed during picture book reading. At level 1, questions focus on features that are directly perceptible, such as a numeral printed on the page or pointing to where the *inside* of the box is. At level 2, questions go a step further by having children integrate multiple components of the visuals, perhaps including some recall, such as finding the total number of apples on the tree. At level 3, questions require students to use information that is not immediately present, such as combining information with that on a previous page or making judgments based on the story's content or illustrations (e.g., determining the total number of apples across multiple pages). Finally, level 4 questions involve predictions. While there are several accounts of researchers and teachers using questions when reading picture books within mathematics lessons, the field lacks a clear and cohesive understanding of the scope of these questioning practices across the elementary grades. In this brief research report, we take a portion of our systematic literature from an educational database to answer the following research question: What types of questions do teachers and researchers use during interactive mathematics read-alouds with the picture books?

Methods

By using specific search terms in the educational database ERIC (EBSCO), applying inclusion and exclusion criteria on abstracts and titles, and later reviewing the entire papers, we landed on 52 sources that involved interactive mathematics read-alouds (see Figure 1).

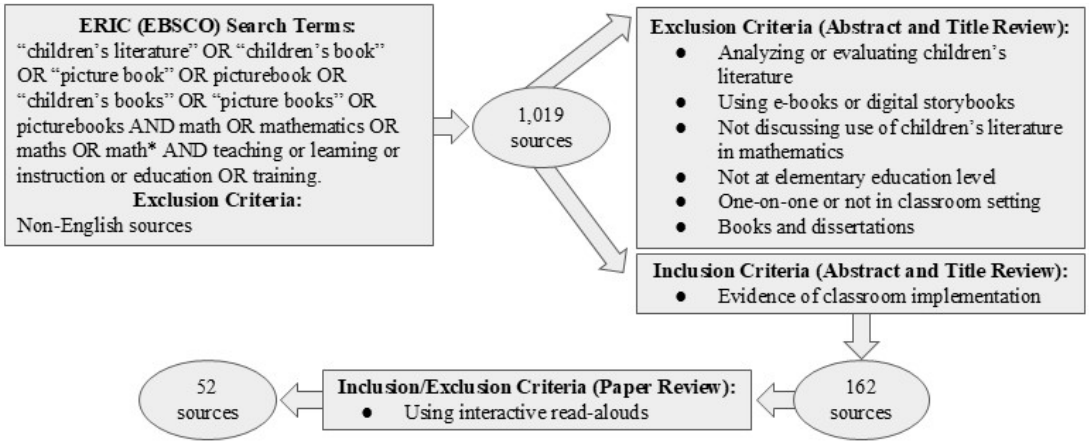


Figure 1: Inclusion and Exclusion Criteria and Results

We focus our analysis on the questions asked during the interactive mathematics read-alouds of these 52 sources. First, we identified whether the questions were non-mathematical or mathematical. Second, we focused on classifying the mathematical questions by content area (e.g., number and operations, measurement, geometry, etc.) and the four levels of abstraction that reflected mathematical questions (Torbeyns et al., 2024). Table 1 provides descriptions and examples from these 52 sources for the mathematical and non-mathematical questions, as well as the four levels of abstraction.

Table 1: Descriptions and Examples of Non-Mathematical Questions and Four Levels of Abstractions for Mathematical Questions

Question Types		Description (Torbeyns et al., 2024)	Example
Non-Mathematical Questions		Questions not related to mathematical ideas in the story.	"What do you notice?" "What do you think the party is for?"
Mathematical Questions: Levels of Abstraction	Level 1	Questions related to salient perceptual information in the text or visuals of a page.	"What number is this?"
	Level 2	Questions related to specific aspects and integration of separate components of salient perceptual information in the text or visuals of a page.	"How many ducklings do you think there are altogether?"
	Level 3	Questions not related to perceptual information in the text or visuals of a page. Instead, they require children to reorder or restructure information from the page or previous pages.	"How many cats are there now? How do you know?"
	Level 4	Questions related to predicting what would happen in the story.	"How many ways could we spend a nickel buying candy marked one cent, two cents, and five cents?"

In the analysis, we found additional question types that did not align with any of the levels of abstraction and mathematical questions that emerged from non-mathematical questions. We discuss these question types in the results below.

Results and Discussion

Of 52 sources, 36 were studies conducted in the United States, and 16 were outside of the United States, with target grade levels ranging from pre-kindergarten to sixth grade. Among these sources, three did not include examples of any questions despite clearly indicating the use of mathematics interactive read-alouds and asking questions in their studies. By far, questions related to number and operations were the most popular across the papers, with at least 22 of them specifying questions asked on these topics. Examples include, "Did the caterpillar eat more strawberries or more oranges?" (Green et al., 2018, p. 10) and "How many ducklings do you think there are all together?" (Burns, 2005, p. 28). Nine others focused primarily on geometry and space related questions, three focused on patterning, and two integrated geometry and measurement topics within perimeter and area questions while one integrated number and measurement topics through time questions. Other papers discussed multiple books, and therefore, the questions spanned multiple subject areas.

Twelve sources shared non-mathematical questions. Some examples of these questions are "Why does Noodle start to sneak away?" (Petronzi et al., 2024, p. 149) and "How did the

caterpillar feel after eating all the food items on Saturday?" (Green et al., 2018, p. 10). Although five of these sources focused exclusively on these non-mathematical questions, in other sources, these questions led to sharing and discussing mathematical questions at different levels of abstraction. For example, in Whitin and Whitin (2006), the teacher started by asking, "What do you notice?" for each picture in the *More, Fewer, Less* book and later guided children by asking, "How many cheeses have orange edges? How many have red edges? Are there more cheeses with orange edges or more with red edges?" (p. 197). These questions reflect levels 1 and 2 of abstraction. Likewise, some of the non-mathematical questions taken out-of-context of a particular classroom may not sound mathematical but within the particular context are interpreted as mathematical. For example, "What kind of questions could we explore about this story?" (Wickett, 1997, p. 476) could be used in a non-mathematical way and is open to that interpretation. However, in the context of the classroom where this question was used, students clearly understood it as referring to mathematics. These results point to the importance of understanding the context in which the questions were asked.

Another interesting result related to non-mathematical questions was sources focusing on other aspects of mathematics rather than content areas. For instance, one paper focused on questions around mathematical anxiety using a book that was designed for a study on supporting mathematical identities. The questions accompanying this book focused on how the character feels about math and why, while also posing questions to the listeners, "Have you ever felt scared about math?" (Petronzi et al. 2024, p. 149). Another paper (Crawford-Mckinney & Özgűn-Koca, 2022) hinted at a similar theme around mathematical anxiety and identity, where the teacher used *When Sophie Can't* and asked, "Why do you know who is 'smart?,'" "What makes them smart?," "Why is the word yet important?" (p. 627). Although these questions do not appear mathematical, they are meant to probe students' thoughts about who is smart at mathematics. Therefore, our results from this systematic literature review provide another category of question type related to mathematical affect.

Regarding the four levels of mathematical questions, we found 12 sources with level 1, 17 with level 2, 21 with level 3, and 13 with level 4 questions. In one researcher-design book aimed at teaching children patterning, the questions align directly with aspects of patterns and patterning across the four levels (Wijns et al., 2023). For example, a level 2 question included "What color streamer do they need to put up next?" (p. 737), which could be determined by looking at the accompanying picture. However, a level 3 question required children to make connections between two similar patterns and how an auditory pattern describing them both would be the same: "Why does the 'beep boop' sound the same when the cake has different colors?" (Wijns et al., 2023, p. 737). Teacher-led book readings also included questions that required students go beyond what was on the page. For example, in one story, a teacher asked a rhetorical question (focusing on numbers and operations) about apples on the page, "This basket has more apples than this basket, doesn't it?" but then later asked, "If we add one more, how many would we have?" (Hojnoski et al., 2016, p. 681). Beyond these levels, questions focused on eliciting students' justifications and reasoning. Overall, the results point to integrating and extending mathematical thinking as important practices in interactive mathematics read-alouds.

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AI-GENERATED FEEDBACK AND PRESERVICE TEACHERS' UPTAKE IN MATHEMATICS TEACHING SIMULATION

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This study examines AI-generated evaluative feedback as a learning tool for supporting preservice teachers' (PSTs) self-assessment and subsequent mathematics teaching practice. Using data from 10 PSTs with complete paired simulations, the analysis compared AI evaluations, PST reflections, and simulation transcripts to identify alignment, unnoticed areas of practice, and evidence of feedback uptake. Findings suggest that AI feedback confirmed PSTs' perceptions of encouragement while drawing attention to questioning, mathematical precision, and checks for understanding. Uptake was uneven: PSTs who explicitly engaged with AI suggestions in reflection were more likely to enact related moves later. The study treats AI evaluation as a useful but incomplete feedback resource that requires human mediation and structured opportunities for uptake.

Keywords: Preservice Teacher Education, Instructional Feedback, Artificial Intelligence

Introduction

Learning to teach mathematics involves more than learning pedagogical strategies. Preservice teachers (PSTs) also need to learn how to interpret their own instructional decisions and decide what to revise. This is difficult work. PSTs may notice whether students seem engaged but miss whether their questions elicit reasoning, whether their language is mathematically precise, or whether their feedback keeps opportunities for sense making open (Santagata & Yeh, 2016; Sleep & Boerst, 2012). Teacher preparation programs therefore need repeated, low-stakes approximations of practice in which PSTs can rehearse instruction and receive feedback specific enough to support revision (Grossman et al., 2009; Zeichner, 2012).

Generative AI simulations offer new possibilities for this kind of rehearsal (Zhang et al., 2026; Zhuang & Zhang, 2025). PSTs can respond to AI-student errors, ask for an evaluation of their instructional feedback, and then compare that evaluation with their own reflection. Yet feedback does not support learning simply because it is immediate. Learners have to interpret it, decide whether it is useful, and use it to guide later action (Boud & Molloy, 2013; Winstone et al., 2017). In teacher education, this distinction matters because AI evaluations may attend closely to visible discourse moves while missing relational, contextual, or equity-oriented dimensions of teaching.

This study examines AI-generated evaluative feedback as a resource for developing PST self-assessment and uptake in mathematics teaching simulations. Two questions guided this study: (1) How does AI feedback align with PSTs' self-assessments, and what aspects of practice does it make visible? (2) To what extent do PSTs incorporate AI feedback into a later simulation, and what patterns distinguish stronger from weaker uptake?

Theoretical Perspective

The study draws on three perspectives. First, research on feedback emphasizes that useful feedback clarifies goals, provides information about current performance, and offers guidance for closing the gap between current and desired practice (Hattie & Timperley, 2007). Boud and Molloy (2013) argue that feedback should be designed as a learner-centered process rather than

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

information transmitted to a passive recipient. This point is important for this study: AI-feedback may be immediate and detailed, but its value depends on how PSTs use it.

Second, self-assessment and reflection are treated as developing forms of professional vision, not simply as introspection. Beginning teachers often focus on surface features of lessons and may not yet recognize the instructional significance of eliciting student reasoning, pressing for explanation, or checking understanding (Lee, 2005; Schön, 1983; Santagata & Yeh, 2016). External feedback can help by making differences visible between how PSTs perceived their teaching practice and what they actually did (enacted practice).

Third, the study draws on work on responsive mathematics teaching. Productive feedback to students involves selecting follow-up moves that advance mathematical thinking rather than simply correcting answers (Jacobs & Empson, 2016; Robertson et al., 2016). For that reason, the analysis focused not only on whether AI evaluations were positive or negative, but also on whether they pointed PSTs toward teachable moves such as asking open questions, using representations, maintaining mathematical precision, and eliciting student explanations.

Methods

Nineteen PSTs enrolled in a mathematics methods course participated in AI-student simulation activities. The analytic sample included 10 PSTs who consented and submitted complete data across two simulations conducted six weeks apart. In each simulation, PSTs interacted with an AI-simulated student around a mathematics task, requested follow-up evaluative feedback, and wrote reflections on their learning and on the AI evaluation experience. The data sources were simulation transcripts, AI-generated evaluations, and written reflections.

For each simulation, the analysis compared three records of practice: what the PST did in the transcript, what the AI evaluation emphasized, and what the PST recognized in reflection. This comparison identified convergent awareness, AI-revealed gaps, PST-identified concerns, and contradictions. Analysis proceeded in two cycles. First, AI evaluations were coded for strengths, improvement suggestions, specificity, and actionability. Second, a cross-case matrix was used to examine alignment and uptake. A Simulation 1 suggestion was coded as taken up when Simulation 2 transcript evidence showed a corresponding instructional move. Given the small sample, percentages are descriptive rather than inferential. Uptake coding was anchored in transcript evidence; for example, a recommendation to ask for reasoning counted only when the second transcript showed the PST inviting explanation or justification. Analytic memos were used to compare discrepant cases, including instances in which PSTs raised affective concerns that AI did not address.

Findings

AI feedback: Confirmed and Extended PST Self-assessment

Across AI-evaluation and PST-reflection pairs, PST reflections and AI evaluations often converged around broad strengths, especially encouragement, patience, and growth-oriented feedback. Nine of ten PSTs recognized that they were supportive, and AI evaluations usually affirmed this perception. PSTs also sometimes recognized when they over-explained rather than elicited student thinking. However, as shown in Table 1, 31% of instances involved AI-revealed gaps, most often related to questioning, precision, and checks for understanding.

AI evaluations were most useful when they connected a general judgment to a specific alternative teaching move. Broad praise rarely added much because PSTs already recognized those strengths. More useful feedback named transcript feature and translated it into a next move: ask for reasoning, pause before introducing a procedure, use a representation, or restate a

rule with place-value language. In these instances, AI feedback worked less like a score and more like a source of possible moves to try.

Table 1: Alignment Between AI Evaluations and PST Self-Assessments

Type	%	Description	Example
Convergent Awareness	62%	PST and AI identify same strengths/weaknesses	PST: "I told too much" AI: "Balance telling with eliciting"
AI-Revealed Gaps	31%	AI identifies issues PST missed	AI: "Use more open questions" PST reflection: no mention of questioning
PST-Identified Concerns	18%	PST recognizes issues AI didn't mention	PST: "worried about making students feel bad" AI: no mention
Contradictions	7%	PST and AI disagree about quality	PST: "explained clearly" AI: "explanations too complex"

Note. Percentages are descriptive and based on 10 PSTs with complete data. Percentages exceed 100 because cases could contain more than one alignment type.

The most frequent AI-revealed gaps were pedagogically consequential. AI recommended more open-ended questions in 45% of evaluations, whereas only 20% of PST reflections named questioning as a growth area. Mathematical precision appeared in 40% of AI evaluations but only 15% of reflections; comprehension checks appeared in 35% of evaluations but 10% of reflections. These patterns suggest that AI feedback can draw attention to features of instructional discourse that novices may not yet notice on their own.

PST self-assessment was not simply inaccurate; it varied by dimension of teaching. PSTs were more accurate when judging tone, encouragement, and whether they had explained a procedure. They were less accurate when judging how their discourse positioned the student as a mathematical thinker. This matters because supportive feedback can still limit learning if it closes down student reasoning.

Uptake: Depended on Explicit Engagement with Feedback

Across PSTs, average uptake was 58%, with variation by individual and suggestion type. Uptake was highest when suggestions were concrete and easy to translate into discourse moves, such as asking students to explain reasoning or using a visual representation. Uptake was lower for mathematical precision, which may require stronger content-specific knowledge and more deliberate rehearsal.

Table 2: Uptake of AI Suggestions by Type in Simulation 2

AI Suggestion Type	Uptake Rate	Evidence of Subsequent Uptake
Open-ended questioning	70%	"Can you tell me why we need the denominators to match?"
Visual representations	65%	"Let's try and draw a picture to see if that helps us"
Checks for understanding	60%	"Does this clear up your questions?"
Balance telling/eliciting	55%	Shifted from direct explanation to guided questions
Mathematical precision	45%	Limited evidence of sustained implementation

Uptake did not always mean better (or improved) mathematical interaction. Some PSTs adopted the surface form of a suggestion, such as asking whether the student understood, without consistently eliciting mathematical reasoning. Stronger uptake involved more diagnostic prompts, such as asking why denominators must match or using a representation to support explanation. Teacher educators therefore need to help PSTs move from simply following a suggestion to understanding instructional purpose behind it.

PSTs who explicitly referenced AI feedback in reflection (n = 6) showed higher descriptive uptake (M = 72%) than those who did not (n = 4, M = 38%). For example, PST RK named two AI suggestion - asking the student to explain reasoning and using visual or concrete models—and then enacted both moves in the second simulation. In contrast, PSTs who wrote general reflections without naming specific suggestions tended to show little change. Thus, AI evaluation alone appeared insufficient; structured reflection helped turn feedback into instructional action.

PSTs Valued Immediacy and Specificity but Recognized Limits

Eight of ten PSTs described the AI evaluation as helpful because it gave immediate, specific comments connected to their own conversation. Several were surprised when AI noticed missed comprehension checks or limited questioning, which supports the interpretation that AI feedback revealed real blind spots. At the same time, two PSTs questioned whether AI could judge teaching because teaching involves relationships, classroom context, and emotional responsiveness. AI feedback was most convincing when it addressed textual evidence of instructional moves and least convincing when judged relational dimensions of teaching.

PST trust in AI feedback was conditional. PSTs were more willing to accept evaluations that matched course language about eliciting student thinking, using representations, and checking understanding. They were more skeptical when AI judged relational quality without broader context. This kind of conditional trust is useful: instructors can treat AI evaluations as texts to examine alongside transcripts and course frameworks rather than as final judgements.

Discussion and Implications

The findings contribute to practice-based teacher education by showing how AI-generated evaluation may support self-assessment within repeated approximations of mathematics teaching. First, AI feedback drew attention to discourse moves that PSTs often overlooked, especially questioning, precision, and comprehension checks. These moves matter because they shape whether feedback only corrects an answer or also advances student reasoning (Jacobs & Empson, 2016; Sleep & Boerst, 2012).

Second, the study underscores the importance of uptake. PSTs who explicitly processed AI feedback were more likely to enact related moves later, aligning with Winstone et al.'s (2017) argument that learners need opportunities to engage agentically with feedback. In methods courses, AI evaluation should be paired with routines in which PSTs select one high-leverage suggestion, locate transcript evidence, plan a revised move, and rehearse it. Such routines preserve AI feedback's immediacy while keeping teacher educators responsible for quality, equity, and mathematical substance.

Third, AI should be a supplementary feedback resource rather than an authoritative judge. Teacher educators remain necessary for interpreting AI suggestions, linking them to course goals, attending to relationships and equity, and helping PSTs recognize incomplete or misaligned evaluations. Although limited by a small sample, single context, and descriptive design, the study suggests that AI-generated evaluation can be productive when embedded in routines that emphasize reflection, evidence, and deliberate uptake.

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EXAMINING RELATIONSHIPS BETWEEN THE FIVE PRACTICES AND CLASSROOM DISCOURSE LEVELS IN ELEMENTARY MATH CLASSROOMS

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Elementary teachers focused on mathematics instruction in a sequence of three eight-week courses, with data from the first two courses included in this preliminary analysis. During each course, part of their assignments consisted of creating a task to implement, using the Five Practices (Smith & Stein, 2018) as a guide. Following task implementation, teachers completed a self-assessment of their classroom discourse and a reflection on the task implementation. These submissions were analyzed qualitatively to determine the teachers' assessment of their expertise in using the Five Practices and the relationship to classroom discourse. Findings showed that as teachers increase their fidelity in the Five Practices, particularly the transition from monitoring work to connecting student ideas, the discourse levels in the classroom increased accordingly.

Keywords: Classroom Discourse, Elementary School Education, Instructional Activities and Practices

The Five Practices, introduced by Smith and Stein (2018), consist of the preliminary practice of setting a goal and selecting a task, followed by anticipating student responses, monitoring student work, selecting, sequencing, and connecting student solutions. While each of these activities is important on its own, the intentional linking of all Five Practices has been seen to improve classroom discourse and student learning. The Five Practices provide a guide for teachers to use to improve their enactment of student-centered classroom discourse (Stein et al., 2008). However, implementing a task using Five Practices with fidelity is challenging for many teachers. This research examined the extent to which the intentional implementation of Five Practices was related to student-centered classroom discourse as measured by Hufford-Ackles et al.'s (2014) mathematical discourse rubric.

Literature Review

Stein et al (2008) presented the Five Practices as a pedagogical model intended to support student discussion while making the process manageable for teachers. Although classrooms at the time exhibited presentations of student solutions, researchers found that simply making ideas public does not engender the same results as eliciting student explanations with probing questions (e.g., Mueller et al., 2014). Stein et al. therefore recommended the Five Practices as "teaching practices that take students' ideas as the launching point, but shape class discussions so that over time important mathematical ideas are surfaced, contradictions exposed, and understandings developed or consolidated" (p. 327). The Five Practices support equity by "revoicing student work in the whole class environment, juxtaposing standard approaches with nonstandard strategies, positioning mistakes as areas to grow, elevating the status of nonparticipators, and allowing students to amplify another student's work in light of new information." (Nabb, 2022, p. 300). Rupe and Borowski (2022) recommended the Five Practices as essential for pre-service teachers, in order to gain experience teaching student-centered mathematics, especially focusing on making connections between representations.

The Five Practices depend on setting goals and selecting tasks. According to the National Council of Teachers of Mathematics (NCTM) (2014), goals must clearly state the mathematics

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students are to understand, as they guide decision making during lessons. Goals should focus on what students will learn, rather than on the actions students will perform (Smith et al., 2020). Tasks that require high-level cognitive demand are recommended, as research has shown that high cognitive demand tasks lead to improvement in solving complex problems as well as increased student motivation (Ni et al., 2018). Anticipating student responses includes not only anticipating different solution pathways students may take, but also planning assessing and advancing questions, as well as thinking ahead about what aspects of work should be noticed while students are working (Smith et al., 2020). When goals are procedure-based rather than learning-based, anticipated solutions are more likely to focus on procedural errors without considering the underlying gaps in conceptual understanding (Kartal et al., 2020; Morrissey et al., 2019), as goals drive the problem representation, launch, and anticipated solutions (Vale et al., 2019).

The quality of questions planned is important, as these questions will be asked during the practice of monitoring student work. Developing strong questioning skills is an important component of pre-service teacher education, considering that Wimer et al. (2001) and Boaler and Brodie (2004) found many teachers ask mostly fact-gathering questions. Teachers' reasoning for selecting and sequencing solutions varies. Teachers may select the solution of a student who is comfortable sharing, or select a correct solution of a student who needs to build confidence (Dunning, 2023; Kersaint, 2017). Alternatively, teachers may show different representations or procedures in order to build up to an efficient procedure or to expose misconceptions (Dunning, 2023; Kersaint, 2017; Meikle, 2016). Meikle (2016) further noted that regardless of the method used for selecting and sequencing solutions, it is vitally important to connect solutions.

According to NCTM (2014), "Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (p. 29). Classroom discourse around different solutions and their connections is intended to advance the conceptual understanding of the whole class. To achieve this, the Five Practices (Stein et al., 2008) provide a roadmap for teachers to orchestrate discussions where students share ideas, justify their reasoning, and make connections between solutions (NCTM, 2014). The successful implementation of these practices serves as a vehicle for moving a classroom along a developmental framework defined by Hufford-Ackles et al. (2014). This framework describes levels of discourse across five domains: teacher role, questioning, explaining mathematical thinking, mathematical representations, and responsibility for learning. As a classroom moves from Level 0 to Level 3 on those domains, the teacher shifts from being the sole authority to a facilitator.

Given that the ultimate goal of the Five Practices is to create effective classroom discourse, this study investigated "how does teachers' implementation of the Five Practices relate to the level of classroom discourse?"

Methods

Teachers at a Southeastern University enrolled in a three-course sequence of eight-week advanced mathematics methods of teaching were assigned to teach a task using the Five Practices during each of the eight-week courses, with the goal of improving their fidelity of implementation and improving their classroom discourse over the three courses. Seven teachers participated in the study, one Latino male and six females (two White and four Black), with a range of teaching experience from one to eight years. Data collected in this study were from the first two of the three-course sequence.

In the first course, prior to the Five Practices assignment, teachers read and discussed Smith and Stein's (2018) book on the Five Practices, along with an NCTM publication (NCTM, 2014), an article about transforming mathematics education (Green, 2014), and the *Launch, Explore, Discuss Lesson Framework* (North Carolina Collaborative for Mathematics Learning, 2017). Teachers then engaged in the Five Practices assignment along with reading and discussing selections from 2 textbooks (Van de Walle et al., 2018a; Van de Walle et al., 2018b), an article related to the Five Practices (Meikle, 2016), and an article on equity (Thomas et al., 2024). In the second course, teachers read and discussed additional articles related to equity (Gallivan, 2020; Roberts et al., 2022; Schmidt et al., 2022; Wilson & Smith, 2022; Yeh et al., 2020), as well as pedagogy (Boaler et al., 2021; Candela et al., 2020).

For the Five Practices assignment, teachers designed or selected a high-cognitive demand task that allowed multiple entry points, required students to provide justification or explanation, connect diverse representations, engage in student-student discourse and identify mathematical patterns to make conjectures and/or form generalizations. These tasks were discussed with classmates to gather feedback and ideas for improvement. Teachers then created anticipated solutions and planned assessing and advancing questions for each solution. The course instructor (one of the authors) also provided feedback on tasks, anticipated solutions, and planned questions. Teachers used a monitoring chart to monitor which students provided each of the anticipated solutions (as well as other, unanticipated solutions), and decided in what order to present the solutions to the class. Finally, teachers implemented the task, video-recorded the lesson, wrote a reflection on the implementation, and self-assessed the classroom discourse while reviewing the video. Teacher reflections and discourse self-assessments from each of the first two courses were collected and analyzed.

Instruments

In each course, after implementation of the task, teachers completed a reflection, detailing how the experience of task implementation was different from their previous instruction, for both themselves and their students. Teachers then used the *Levels of Discourse Rubric* (Hufford-Ackles et al., 2014) to assess classroom discourse, and commented on shifts in their practice specifically related to the Rubric. Finally, teachers reflected on how to refine their practice moving forward. The *Levels of Discourse Rubric* moves from Level 0 to Level 3 as teaching moves towards more student-centered discourse in the areas of Teacher Role, Questioning, Explaining Mathematical Thinking, Mathematical Representations, and Building Student Responsibility within the community.

Data Analysis

To analyze the relationship between the Five Practices and Classroom Discourse levels, first, we used a qualitative deductive content analysis on the teachers' written reflections from the two courses. We coded each reflection for the Five Practices (Anticipating, Monitoring, Selecting, Sequencing, and Connecting) using three levels as Weak (minimal or no evidence), Moderate (theoretical mention or partial enactment), and Strong (clear, evidence-based execution). For example, a teacher wrote "I carried around my question cards and my progress monitoring sheet to help support my students and my own teaching practice" which is evidence that monitoring was purposeful and tied to instructional decisions, not just observation, and hence coded as "strong." Another teacher wrote "I selected my sequence based on the most concrete examples to most abstract" and "I selected her to go first based on my sequencing plan" which was coded as "strong" for Sequencing practice. This coding focused on the depth of teacher reporting rather

than just the presence of a practice. Second, we aligned the implementation strength of the Five Practices with teachers' self-assessed discourse levels (0–4).

Findings

Given the small sample size ($N = 7$), statistical tests were not appropriate as they lack the necessary power to generalize findings; instead, we described patterns. Teacher reflections revealed a distinct hierarchy in the implementation of the Five Practices. Teachers generally excelled in the active, "in-the-moment" phase of monitoring ($M = 2.79$) but struggled with the high-level orchestration required for sequencing ($M = 1.36$) and selecting ($M = 1.79$) across both iterations. Reflections often showed that while teachers noticed interesting student work (Monitoring), they lacked a strategic plan for which students should share (Selecting) and in what order (Sequencing) to build a mathematical argument, and many defaulted to having everyone share on the carpet. Teachers were at the "moderate" level on anticipating ($M = 2.00$) and Connecting ($M = 2.14$).

The discourse rubric data showed a cohort transitioning from Level 1 (Teacher-Led) toward Level 2 (Shared Responsibility). Teacher Role, Questioning, and Explaining Math Thinking all shared a mean of 1.71. This indicates that teachers have successfully moved away from traditional lecturing and are consistently asking "Why?" and "How?" to prompt student justification. Math Representations ($M = 1.43$) and Student Responsibility ($M = 1.50$) were the lowest-scored domains. Teachers noted in their reflections that while they are asking better questions, students are not yet independently connecting different representations or taking full ownership of the mathematical authority in the room.

When looking at the numerical relationship between specific practices and the overall discourse level, distinct patterns emerge: There is a strong association between a teacher's ability to "Connect" and the quality of classroom discourse. Teachers who excelled at asking students to build on each other's ideas achieved the highest discourse levels. This suggests that "Connecting" is the bridge that moves a classroom from teacher-led to student-driven. Effective monitoring showed a moderate-to-strong association with discourse levels. This confirms that a teacher's ability to "hear" student thinking in real-time is a prerequisite for facilitating a high-level discussion. Sequencing and Anticipating practices were less directly associated with the real-time discourse scores. This implies that a teacher may have a strong plan (Anticipating), but if they cannot execute the "Connecting" moves during the lesson, the discourse level remains capped.

Overall, the findings suggest that as teachers increase their fidelity in these instructional moves, particularly the transition from monitoring work to connecting student ideas, the discourse levels in the classroom increase accordingly.

Discussion

Important findings to note include the connection between teachers' struggles with selecting and sequencing, and students' inability to independently connect different representations. As teachers improve their fidelity in the "later" practices (Selecting, Sequencing, Connecting), the responsibility for learning shifts from the teacher to the students. This finding aligns with Meikle's (2016) belief that it is vitally important to connect solutions. Because connecting representations is vital to a complete understanding of mathematical reasoning, the selecting and sequencing skills need to be further developed in teachers. Teachers need to understand that sequencing should build on previous solutions and focus on the meanings and relationships between the solutions (Smith & Stein, 2018). It was also noted that, when a teacher is able to

successfully guide their students to connect their solutions to the solutions of others, classroom discourse is improved, resulting in a move towards student-driven classrooms. This finding lends further weight to the importance of connecting representations.

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PATTERNS IN ALGEBRA INSTRUCTION: SUMMARIZING TWO STUDIES ACROSS 14 STATES IN THE UNITED STATES

PATRONES EN LA ENSEÑANZA DEL ÁLGEBRA: RESUMEN DE DOS ESTUDIOS EN 14 ESTADOS DE EE. UU.

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Mathematics education scholars have been promoting instructional reforms for decades, but it is important to consider the extent to which these efforts have yielded changes in mathematics classrooms. Having completed three lesson observations per year of 36 algebra teachers in the Midwestern United States (Study 1) and of 54 algebra teachers across multiple regions of the United States (Study 2), this brief report summarizes general features of the observed instruction. Findings include a consistent instructional pattern characterized by teacher-led and procedurally-focused whole-class instruction and student tasks that are 93% low cognitive demand. Implications are discussed with regard to how these patterns in algebra instruction might form a basis for incremental rather than transformational instructional change.

Keywords: Instructional Activities and Practices, Classroom Discourse, Algebra and Algebraic Thinking

For decades, mathematics education scholars have promoted a "reform-oriented" vision for mathematics classrooms (e.g., Lott et al., 2010; National Council of Teachers of Mathematics, 1980, 1989, 2000, 2014). The accompanying professional development (PD) efforts with teachers have sought to transform instruction away from conventional or "traditional" patterns of classroom teaching (e.g., teacher-led lecture and an emphasis on procedures) and toward a complex but relatively unified alternative form of teaching. Sztajn et al. (2017) pointed to this unified aim in their broad review of PD literature, stating "the majority of [PD studies] embraced a similar vision of [reform-oriented] K–12 mathematics instruction … designed to support teachers in implementing instruction that builds on rich mathematical tasks, attends to student thinking, values interactions as a learning mechanism, and is considered 'ambitious' (Lampert et al., 2010) in its goal of advancing mathematical understanding for all students" (p. 796).

To what extent has mathematics education, as a field, had success in these efforts? If, say, one were to visit mathematics classrooms across the United States, would one observe rich tasks, visible student thinking, learning through robust interactions, and an emphasis on understanding over rote procedure? There have been "pockets of success" (Arbaugh, 2025) regarding the implementation of reform instruction, yet our decades of work in mathematics PD led us to wonder about the prevalence of those pockets. This wondering led us to examine baseline data from two of our large-scale studies to look broadly at the type of mathematics instruction that was occurring when we "first walked in," so to speak. The summaries presented in this brief report depict general aspects of instruction in algebra from 36 teachers in the Midwest United States (Study 1) and 54 teachers from across the entire United States (Study 2). Although not

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indicative of instruction in other grades or courses, Algebra 1 is an important culmination of middle school mathematics and the launchpad for high school mathematics, and, as is shown below, conventional instruction seems firmly entrenched within this linchpin course.

Background Literature and Theoretical Perspective

Space restrictions do not allow a full treatment of instructional history in North America, but it suffices to say that conventional mathematics instruction, characterized by a teacher defining terms and demonstrating procedures so students may memorize those terms and execute those procedures, has been dominant for more than a century (Karp & Schubring, 2014; Stanic & Kilpatrick, 2003; Stigler & Hiebert, 1999). Various efforts to transform instruction were met with significant resistance (Davis, 2003; Schoenfeld, 2007; Wilson, 2003), and none of the reforms had achieved widespread adoption even before the backlash. More recent instructional change efforts have involved a New Mexican School curriculum reform (Nava-Guzmán et al., 2025), with similarities to NCTM's standards-based reform, as well as many PD efforts focused on specific elements of the reform vision, such as cognitively-demanding tasks (e.g., Boston & Smith, 2009), noticing and building on students' thinking (e.g., Jacobs et al., 2024), student collaboration (e.g., Tabach & Schwarz, 2018) and mathematical discourse (e.g., Herbel-Eisenmann et al., 2013; Staples & Colonis, 2007), or teachers' content knowledge and the development of conceptual understanding (e.g., Garet et al., 2016). But PD success remains difficult to achieve in these areas (Ahl et al., 2023), and even if a pocket of success is achieved, the question remains of how likely it is for the instructional reforms to spread (Otten et al., 2025).

In this brief report we sought to view instruction holistically with an eye toward the extent to which some key features of reform instruction were present. Both studies reported herein viewed students' mathematical learning as a sociocultural process whereby they come to participate actively in a mathematical community (Vygotsky, 1978) and thus learning occurs through and is defined by students' participation in forms of mathematical discourse (Lemke, 1990). For these reasons, the lesson observation protocol (Otten et al., 2021) identifies various activity structures such as whole-class discourse, independent work time, group work time, and non-instructional time. The protocol also attends to holistic aspects within those structures, as described below, as well as the cognitive demand of tasks posed to students (Stein et al., 1996).

The following research questions about the broad features of the algebra instruction observed within two large-scale studies guided this study:

1. How is class time allotted for the activity structures of whole-class discourse, independent work, group work, and non-instructional time?
2. Within whole-class instruction, to what extent are key features of reform instruction present?
3. Within independent or group work, what were the levels of cognitive demand of the tasks assigned to students?

Methods

Classroom data for Study 1 comes from the 2018–19 and 2019–20 school years in two Midwestern U.S. states (36 teachers; 108 lessons). Classroom data for Study 2 comes from the 2022–23 and 2023–24 school years across 14 states in various regions of the United States (54 teachers; 162 lessons). Details can be found in de Araujo et al. (2017) and Otten et al. (2025), respectively. Both studies involved Algebra 1 (or equivalent) classes in Grade 8 and Grade 9, with some Grade 7 and Grade 10 students (ages approx. 11–14). The participating teachers were not necessarily representative of the general population (i.e., they were willing to join a

university-based project and so may be more amenable to university-promoted forms of instruction than are teachers who declined to participate) but the school districts were purposefully chosen in both studies to vary by locale (rural, suburban, urban) and student demographics (race or ethnicity, socioeconomic status) in ways that roughly follow state or national characteristics.

In Study 1, no intervention was performed with the teachers; they were asked to teach in their normal manner for all observations. It should be noted, however, that half of the teachers in Study 1 were implementing flipped instruction (i.e., videos assigned as homework) (de Araujo et al., 2017) and so their use of class time is likely not representative of other algebra teachers. The other half of teachers in Study 1 were in school contexts that matched the first half but they were not implementing flipped instruction. In Study 2, a PD intervention occurred in subsequent years but data reported here were from each teacher's baseline year, prior to any intervention. Teachers in Study 2 were recruited because they were teaching Algebra 1, regardless of instructional style.

In both studies, our research teams conducted at least 3 lesson observations per school year. When more than 3 observations occurred, we randomly selected 3 for inclusion in the present summary analysis. Observations were carried out in person for Study 1 and via video recordings for Study 2, with a process check that the two modalities remain comparable (Wonsavage et al., 2024). All class materials were also gathered from the teachers. Each observation was done by at least two trained research team members, filling out the protocol (Otten et al., 2021) separately and then meeting to reconcile discrepancies. Lesson segments were parsed to the nearest minute and instructional features, described below, were rated on a 3-point scale based on categories, not levels (and 0 was included for features that could be absent).

Findings

Use of Class Time in Algebra Lessons

The most common lesson segments we observed were whole-class discourse (all students expected to attend to shared public discourse) and independent work time (students expected to be working either individually or in voluntary groups but collaboration is not required). Whole-class discourse (WCD) was observed in 92.1% percentage of lessons, averaging 27 minutes out of a typical lesson duration of 57 minutes (47%) across the entire data sets. Independent work time (IND) was observed in 93.2% of lessons, averaging 21 minutes out of 57 (37%). In contrast, group work (GRP), when students were expected to work collaboratively, was observed in just 31.6% of lessons, with an average duration of only 5 minutes (9%) across the entire data sets, or an average of 17 minutes (30% of class time) in those few lessons where GRP occurred.

Table 1: Mean Durations in Minutes of Various Activity Structures in Algebra Lessons (With Median Durations Indicated in Parentheses)

	Class Period Duration	Whole-Class Discourse	Independent Work	Group Work	Non-Instructional
Study 1 (n = 108)	55 (50)	22 (23)	22 (19)	4 (0)	7 (5)
Study 2 (n = 162)	60 (54)	30 (30)	19 (17)	6 (0)	4 (3)
Combined (n = 270)	58 (51)	27 (27)	21 (18)	5 (0)	5 (4)

With regard to class duration, 102 minutes was the maximum observed and 27 minutes was the minimum, with a standard deviation of 16 minutes. The class periods tended to be broken into nine separate segments, with a standard deviation of four segments.

Instruction During Whole-Class Discourse

The observation protocol distinguishes between mathematical authority (choosing the problems to work on, validating correctness, etc.) that lies predominantly with the teacher or textbook (coded as 1) or that is shared among the classroom community (coded as 3). As depicted in Figure 1, the average algebra lesson we observed was 1.1 (SD = 0.3), indicating that nearly all lessons were teacher/textbook authority. The nature of the discourse (Staples & Colonis, 2007), either sharing (univocal with speakers & listeners attempting to receive meaning, coded as 1) or collaborative (dialogic with interlocutors building meaning together, coded as 3) was even more consistent, averaging 1.0 on the scale (SD = 0.2). Although the discourse was univocal, a majority of students tended to contribute at some point to the WCD (mean = 2.3, SD = 0.7), though lessons varied substantially and students' contributions tended to be short (e.g., one-word responses). The development of mathematical ideas was much more often procedural than conceptual (Hiebert & Lefevre, 1986), averaging 1.3 toward the procedural end (SD = 0.6). Only 4% of observed lessons had a predominantly conceptual emphasis. Mathematical representations (equations, tables, graphs, words, etc.) could be solo or only presented in isolation (coded 1) or they could be strongly integrated (coded 3); the average was 1.3 but there was more variation for this characteristic than the others (SD = 1.1).

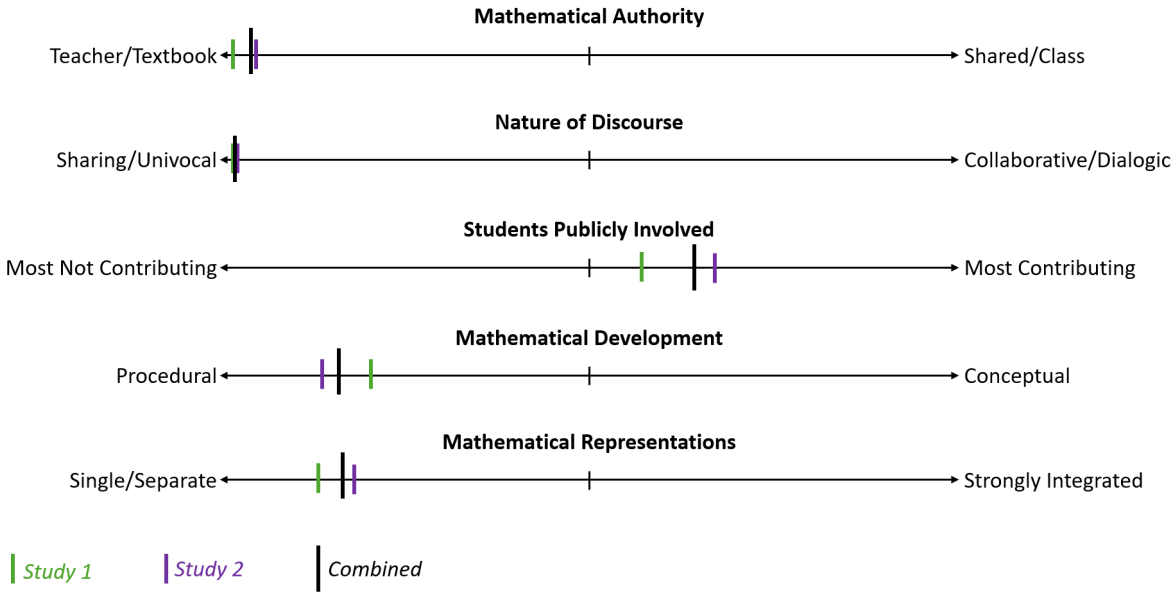


Figure 1: Average (Mean) Characteristics of Whole-Class Instruction in Algebra 1 Lessons

Tasks Implemented During Student Work Time

We previously reported on Study 1 results in Otten et al. (2024), categorizing tasks by level of cognitive demand (Stein et al., 1996). Of those 2,378 mathematical tasks (16.8 per lesson), 2,208 (92.9%) were low cognitive demand involving either *memorization* or the application of *procedures without connection to meaning*. Approximately 6% of tasks were *procedures with connections to meaning* and 0% were *doing mathematics*. Study 2 corroborates those findings: out of 1,695 tasks (10.5 per lesson), 1,577 (93%) were low cognitive demand and only 118 (7%) were cognitively demanding. Only 10 tasks (0.6%) were doing mathematics. Note these tasks were coded as written; they may (and often did) decline in cognitive demand during enactment.

Discussion

Overall, we found many aspects of conventional instruction to be prevalent in algebra classrooms, particularly teacher-led instruction with an overwhelming focus on procedures and low-cognitive-demand tasks. Collaboration was allowed but not typically expected. The key features of reform instruction as described in the introduction were extremely rare. We do not report this as an indictment of mathematics teachers. They are working hard and supporting students within conventional contexts. We share this as a potential reality check for those in mathematics education who, like us, are curious about the extent to which visions of reform instruction have become a reality. At least based on what we have seen, it is not a reality in algebra classrooms in the United States. These findings might provoke reflection upon transformational approaches to PD and instructional change or whether incremental approaches that build directly upon patterns of conventional instruction might be worthwhile (Otten et al., 2025).

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PROMOTING EXPECTANCY VALUE IN AN EXPERIMENTAL BUSINESS CALCULUS CLASS

PROMOTION DE L'ATTENTE-VALEUR DANS UN COURS EXPERIMENTAL DE CALCULUS POUR LES SCIENCES DE GESTION

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In this brief report we demonstrate how a redesigned Business Calculus course sought to promote students' expectancy and utility value. Grounded in Expectancy–Value Theory, we analyze students' responses to an assignment in which students interviewed professionals in their intended careers about their use of mathematics. Data include 100 written reports collected across two semesters. Using two-cycle qualitative coding, we identified themes related to expectancy (e.g., increased confidence, mentoring) and utility (e.g., use of spreadsheets, basic quantitative reasoning). While few professionals reported direct use of calculus, most described frequent use of spreadsheets and financial calculations, some of which were rooted in calculus. These findings suggest that structured relevance interventions in the context of an ongoing redesign effort can shape students' perceptions of the usefulness of calculus for business majors.

Keywords: Calculus, Undergraduate Education, Design Experiments, Expectancy-Value Theory

Business Calculus is often taught as a terminal course with few prerequisites (May & Bart, 2024; May et al., 2020). Multiple researchers have identified a lack of alignment between the content and pedagogy of Business Calculus and the mathematics found in business and economics (Dahma et al., 2022; May et al., 2020; Spencer-Tyree et al., 2024). This misalignment is an issue in other client-discipline calculus courses, and these issues have inspired interventions in introductory math courses.

At our institution, Business Calculus is usually taught as a large lecture. The DFW rate from 2022-2024 was 28%. Leveraging an institutional grant focused on increasing student success, we created an experimental section of Business Calculus designed to address (a) students' varied preparation, (b) low student engagement during lecture, and (c) the perceived lack of relevance of course content. In prior works we have presented data on student preparation and outcomes and student engagement (Zahner & Sekone, 2025, 2026). In this Brief Report, we delve into the issue of relevance by analyzing student responses to a utility value assignment. Our research question is How do students in an innovative Business Calculus class describe the meaning and use of calculus in their subsequent business courses and envisioned careers?

Theoretical Perspective and Prior Research

Our course redesign drew on the pillars of inquiry-based mathematics education (Laursen & Rasmussen, 2019). For this analysis we focus on an intervention that was rooted in Expectancy Value Theory (EVT). EVT incorporates self-efficacy (expectancy) and perceived utility value. Motivation theorists have documented a positive relationship between expectancy and student success within a domain (Wigfield & Eccles, 2000), as well as a positive relationship between students' perceptions of utility value and success (Hulleman et al., 2010). In other words, theorists in this tradition supported that an individual's beliefs about how well they will do on an activity (expectancy) and the extent to which they value the activity influence their choice,

persistence, and performance. In relation to mathematics, expectancy for success refers to a person's beliefs about their ability to perform well on upcoming mathematics related tasks, either in the immediate or longer-term future (Wigfield & Eccles, 2000). The achievement values portion of EVT include components such as intrinsic value and utility value (Eccles et al., 1983). Intrinsic value is about the enjoyment one gains from doing the task, and utility value or usefulness of the task refers to how a task fits into an individual's future plans, for instance, taking a math class to fulfill a STEM degree requirement and/or to learn skills that will be useful later in one's career. Interventions targeting expectancy value have shown positive results in undergraduate STEM (e.g., Asher et al., 2023; Hulleman et al., 2010; Rozek et al., 2017). For instance, Hulleman et al. (2010) demonstrated that students in the relevance condition, who completed a written task on the utility value of a mathematics activity to their life, were more interested in the mathematics activity at the end of the session than those in the control condition.

As described in more detail below, for this work we asked business students to identify a working professional in their envisioned future career and then interview that person to learn how that professional uses mathematics in their work. This assignment is a twist on the more typical "mathematical autobiography" assignment that is often found in, for example, courses for prospective teachers (Towers et al., 2017). While we anticipated this assignment might increase EVT, we also noted there was some risk: recent research has called into question the utility of (school) mathematics in the workplace (Walkington et al., 2025). Thus, we were not sure whether the EVT intervention would have the desired effect.

Research Methods

Context and Data Sources

We taught the redesigned Business Calculus in a classroom amenable to active learning. In place of the standard textbook, we adopted an open educational resource textbook that used business applications and Excel spreadsheets to introduce each topic (May & Bart, 2024). The course reform was structured as a design experiment, with iterative cycles of implementation and revision (see Brown, 1992; Cobb et al., 2003). The enrollment in the innovative sections was 68 in fall 2024 and 72 in Fall 2025. By way of contrast, each section of the traditional lecture class had a capacity of 300. Given the number of students and repeaters, and the fact that Business Calculus was a required pre-major class, all sections were usually filled each semester.

To document the impact of the intervention, we collected multiple sources of data including student surveys, interviews, and responses to course assignments focused on expectancy value. We also tracked student outcome data such as DFW rates. For this Brief Report we focus on one of the expectancy value interventions, called the "Math in your Major" assignment.

Table 1: Overview of Interviewees' Career Categories

Career Categories	Fall 2024	Fall 2025
Finance, Accounting, and Banking	10	12
Marketing and Sales	9	7
Business Owner / Entrepreneur	9	7
Retail and Operations Management	7	9
Technical and Specialized (e.g. professor of economics)	6	9

Data Collection

Here we analyze 100 submissions representing 85 unique individuals who were interviewed. Table 1 shows the broad categories of careers of interviewees. The student-written reports provided another perspective on the usefulness of calculus for business careers. In accordance with our institution's guidelines for research on student work produced as part of a class, all analysis for the purpose of this research was done after student grades were submitted.

Analysis

We used a two-cycle coding process drawing on the qualitative methods in Miles et al. (Miles et al., 2014). In the first cycle, we re-read the papers students submitted and we used open, descriptive coding focused on statements in the student write-ups related to the expectancy and utility value of mathematics for business. For example, one student wrote that a winery owner "uses spreadsheets to calculate how much the [gift] baskets cost in order to calculate the price to get the best profit margin for the shipment." We coded this excerpt with a descriptive code "Budgeting and Financial Calculations" since the use of mathematics was for budgeting. In the second cycle of coding, we organized the open codes into categories. The first author (W. Z.) did the majority of the first pass coding, and both authors reviewed the coded segments and categories. In a final step we organized the clustered codes to identify six major themes from this assignment.

Results

Table 2 provides a summary of the results of the coding process. We divided the themes according to the two superordinate categories of Expectancy and Utility. The numbers in Table 2 correspond to the count of interview write ups that included at least one code from each theme area. Some documents addressed multiple themes so the sum is greater than 100.

In the category of expectancy, we identified two major themes: Motivation/Self Esteem and Networking/Mentoring. One illustration of the Motivation/Self Esteem theme was found in the account of a student who interviewed an accountant at a poultry farm in his hometown in rural California. Talking to the accountant raised the student's sense of efficacy after the interview: "I find accounting/finance even more interesting which is deeply reassuring since often I questioned whether I really wanted to go into this field. I found it surprising that the work can be so simple and that I don't need to be a genius to prosper in my field." The expectancy for success construct captures student's belief in their ability to succeed in the course or perform specific mathematical tasks.

In the category of utility, the use of basic mathematics (but not calculus) was mentioned in the majority of the written reports. The second most common response was that spreadsheets were useful. Some students were surprised to learn that business professionals used the same spreadsheet formulas that they were learning in class. For example, one student wrote "What did surprise me a lot, though, is how much he actually relies on Excel. I thought most people who work in real estate use it every now and then, but [interviewee] said he uses it in almost any project."

Finally, in the category of utility value, we did find some students who learned that school mathematics is not likely to be useful in their future career. For example, one student wrote, "I learned that even though I like math itself, like solving equations as we do in class, it won't be a part of the real world."

Table 2: Summary of Coded Interviews

Category & Theme	Example	Number of Reports
Expectancy		
Motivation / Self Esteem	Students identified that they could be successful and they "don't need to be a genius" to go into the field.	17
Networking / Mentoring	Interviewee identified the importance of networking and mentors for career success	14
Lack of Expectancy	<Did not appear in data set>	0
Utility		
Usefulness of Calculus	An options trader uses first and second derivatives to make trades	10
Usefulness of Excel/ Spreadsheets	Real estate professional uses Excel to do most calculations	33
Usefulness of Basic Mathematics (not calculus)	Small business owner uses math for budgeting and financial decision daily	76
Lack of Utility	Interviewee explicitly mentions calculus not used	11

Discussion and Conclusions

In the foregoing we have described student statements related to expectancy value on the basis of an interview assignment in an experimental Business Calculus class. We note that the primary source of data for this analysis is limited by the potential bias of students reporting what they thought we would like to hear. That is, the students may have written their responses to the interview assignment in a way that aligned with our hopes as course designers. Nonetheless, the ubiquity of responses that described the importance of spreadsheets and business mathematical calculations was encouraging. Also, many students explicitly said that they did not see the application of calculus in their interviews, which indicates that likely the students were honest in their responses.

The results of this work can inform future reforms to Business Calculus and the design of other introductory mathematics and statistics courses designed to serve client disciplines. In particular, aligning course content with the tools and practices of working professionals may be one way to increase students' expectancy and utility value. Our findings suggest that greater emphasis on spreadsheet-based modeling and authentic business applications may strengthen these connections. Additionally, this assignment—interviewing a professional—has significant potential for helping students (a) see the relevance of what they are learning and (b) develop networks with professionals in their intended career field. Future implementations could expand this effort through more career-relevance activities such as industry-based projects.

Acknowledgments

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IMPACT OF IMPLEMENTING THE BUILDING THINKING CLASSROOMS MODEL IN UNDERGRADUATE MATHEMATICS

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Keywords: Undergraduate Mathematics Education, Teacher Education, Classroom Discourse, Active Learning

Purpose and Theoretical Framing

Building Thinking Classrooms (BTC) (Liljedahl, 2021) has been widely adopted in K–12 mathematics, where research documents increased student discourse, engagement, and problem solving through visibly random grouping, vertical non-permanent surfaces (VNPS), and rich tasks. In contrast, empirical work examining BTC in undergraduate mathematics remains limited. Emerging higher education studies report promising shifts in participation and collaboration (Reiser, 2023; Parrott, 2023; Douglas, 2024), yet systematic investigation of how BTC structures mediate discourse, knowledge mobility, and classroom community in tertiary mathematics contexts is still developing.

Related scholarship on collaborative problem-based learning and public workspace design in university mathematics similarly suggests that spatial and social structures shape mathematical interaction and visible idea refinement. Although Liljedahl's earlier work includes undergraduate participants, the full BTC model has not been extensively examined across diverse postsecondary mathematics settings. This gap is particularly notable given increasing calls for active learning and discourse-oriented instruction in undergraduate STEM education.

This study investigates how BTC structures function in several undergraduate mathematics courses—including 100-level general education courses, a 400-level geometry course for prospective teachers, and traditional calculus courses—at a regional comprehensive university. The guiding research questions are: (1) How does implementation of BTC structures influence student engagement and mathematical discourse? and (2) In what ways do students describe changes in their mathematical understanding and sense of classroom community? While the study has implications for teacher education, it is positioned primarily as a contribution to research on undergraduate mathematics learning in general.

Context and Method

BTC structures were implemented during the 25-26 school year. Core practices included daily visibly random grouping, problem solving at vertical whiteboard spaces, non-routine tasks (e.g., combinatorial generalizations; geometric investigations of Cevians and Menelaus), and structured teacher micro-moves such as rotating writers and group accountability for collective understanding.

Data sources include classroom observations, photographic documentation of vertical work, student written artifacts, end-of-course reflections, and instructor field notes. Analysis examines patterns of discourse, evidence of knowledge mobility, and student-reported perceptions of learning, engagement and community.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Preliminary Findings and Contribution

Preliminary analysis suggests increased student-to-student discourse, visible iteration of ideas, and greater willingness to publicly revise thinking compared to prior lecture-based iterations. Students describe a shift from individual performance toward collective problem solving and report deeper conceptual understanding and greater engagement.

By providing systematic documentation of BTC implementation across undergraduate mathematics contexts, this study extends emerging postsecondary BTC research and contributes to PME-NA conversations about instructional design, knowledge mobility, and active learning in tertiary mathematics classrooms.

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TEACHING MATHEMATICS AS STORYTELLING: BRIDGING MICRO- AND MACRO-NARRATIVES IN ELEMENTARY CURRICULAR DECISION MAKING

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Keywords: Curriculum, Elementary School Education, Instructional Activities and Practices

We use storytelling to interpret mathematics teaching. Prior work treats stories literally, as word-problem contexts (Chapman, 2006; Depaepe et al., 2010), or metaphorically, as a property of curriculum (Dietiker, 2015; Dietiker et al., 2023). We instead adopt a narrative lens (Fisher, 1985): Teachers propose settings and events, and students co-narrate, reproducing and adapting narratives as they learn (Clark & Rossiter, 2008; Erickson, 2004; Ochs et al., 1992).

Micro-narratives are stories specific to a group's experience. Macro-narratives are broadly understood accounts of concepts (Devine et al., 2014), reproducible across settings and codified in standards. Teaching is then a loop rather than a transfer. Micro-to-macro bridging generalizes a classroom story into concepts students carry elsewhere; macro-to-micro bridging situates a concept within experiences that make sense to particular students. Closing the loop needs both rote reproductions, by which students internalize a story as taught and teachers assess what was learned, and creative reproduction, by which students apply a concept in a new setting.

We interviewed nine teachers at two elementary schools about their resources and coherence across lessons. To demonstrate the possibilities of a storytelling lens, we limit our findings to data from two teachers (Audrey and Sarah), both teaching second grade at the same midwestern school. These two teachers share curricular resources and institutional contexts and work toward the same macro-narrative. Yet they propose very different micro-narratives.

Audrey presented an Oreo taste test and built a whole-class line plot, holding that students must be "completely engaged before you can start teaching." Her whole-class lesson focused on generating one story, co-narrated by the group and reproduced as taught. Sarah's Survey Says had pairs poll classmates on questions they chose, allowing students to retell a known graphing story. This creative reproduction represents a micro-to-macro move. Sarah nonetheless feared hearing "you're not preparing them well enough for the test." implying a tension between standards-driven macro-narratives and locally meaningful micro-narratives.

A storytelling lens resists two binaries. Micro- and macro-narratives function as ends of a bridge rather than as a dichotomy: Audrey and Sarah work in both registers, differing in the direction and means of crossing. Attending to bridging surfaces the tension teachers manage between standards and particular students, and shows them meeting it through different strategies, advocating an empowered and diverse account of teachers. Seeing students as co-narrators credits what they contribute to the lesson and argues for enlarging that role. Rote and creative reproduction are similarly developmental rather than opposed: rote reproduction internalizes a story and lets teachers assess it, preparing the retelling in new settings that constitutes the macro-narrative.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

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TECHNOLOGY AND LEARNING ENVIRONMENT DESIGN

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

PRESERVICE TEACHERS' FEEDBACK ON DIGITAL GAMES BASED TOOLS ON FRACTIONS AS FUTURE DESIGN IMPLICATION

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Digital game-based learning has gained increasing attention in mathematics education and research shows positive effects of them for supporting fraction understanding; however, research highlights the need for further examination of effective design features and instructional integration. This study investigates two purposefully designed digital fraction tools, one game and an interactive app, and examines how Preservice Elementary Teachers' (PSTs) feedback offers insights for the future improvement of these tools. The study uses the ADDIE model (Hidayat et al., 2023) to guide the design of the tools and to analyze PSTs' feedback on the effectiveness of game elements and suggested modifications. The findings provide insights for educational game developers and educators on designing digital games that can be effective in supporting students' learning and knowledge of development of fractions.

Keywords: Digital Games, Preservice Teachers' Perspective, Fraction, Game Design Elements

Understanding fractions is a challenging concept for both students and preservice teachers (Gabriel et al., 2013; Satriyo et al., 2025). Digital game-based learning has gained increasing attention in mathematics education for its potential to foster engagement and support students' mathematical understanding (Beşaltı, & Kul, 2021; Fadda et al., 2021; Hussein et al., 2021). Research suggests that educational digital games can be particularly effective for teaching complex topics such as fractions (Beşaltı, & Kul, 2021; Hunt et al., 2025; Tangkui & Keong, 2021; Wardat et al., 2022; Zhang et al., 2020). While many studies have explored the effectiveness of educational digital games, further research is needed to examine the design features and key elements that make learning games effective for mathematics instruction (Gui et al., 2023; Pan et al., 2022), specifically in fraction learning (Thoma et al., 2023).

Research indicates that mathematics teachers perceive digital games as valuable instructional tools (Da Cruz et al., 2025; Mazuru & Popa, 2025); however, limited knowledge about how to effectively integrate digital games into instruction can discourage their use as a primary teaching approach (Matic et al., 2023).

This study explores the use of two purposefully designed digital tools for fraction instruction, a mobile video game and an interactive app. In this study, we integrated these digital tools into a mathematics education program for elementary preservice teachers and examined their feedback regarding the challenges they encountered, areas for improvement, and recommendations for redesigning the tools. Thus, the research question is:

How do preservice elementary teachers evaluate the effectiveness of digital game-based tools on fractions, and what implications does their feedback offer for the future design of the tools?

Literature Review and Theoretical Framework

There is substantial empirical evidence indicating that students' understanding of fractions predicts later achievement in advanced mathematics (e.g. Bailey et al. (2012), Hansen et al., (2017), and Siegler et al. (2012)). However, many students still experience difficulties with fraction concepts (Di Lonardo Burr et al., 2022; Hariyani et al., 2023). Research also shows that fraction as measurement (Wilkins & Norton, 2018), fraction as part–whole (Gabriel et al., 2013:

Pramudiani et al., 2022; Rosli et al. (2020), and fraction equivalence (Deringöl, 2019; Girgin & Altay, 2023; Hariyani et al., 2023) are challenging topics for both PSTs and elementary students. Furthermore, a review of 70 research studies by Perry (2023) showed that using measurement and number lines helps PSTs improve their general fraction knowledge.

This study draws on the ADDIE model by Hidayat et al. (2023) which has been developed based on constructivism learning theory (Zhang et al., 2023) and supports the alignment of instructional materials with learners' needs and educational objectives. This model has widely been used to develop digital teaching materials and educational games (Nuntawisuttiwong & Dejdumrong, 2024; Widyastuti, 2019; Zhang et al., 2023) and suggests five steps in game development: Analysis, Design, Development, Implementation, and Evaluation (See Figure 1).

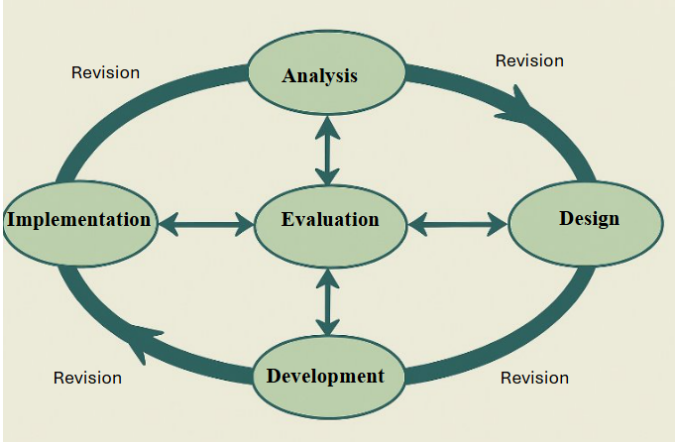


Figure 1: ADDIE Model by Hidayat et al. (2023)

In this section, we elaborate on the stages of this model based on the descriptions provided in Zhang et al. (2023). The *analysis* stage phase includes problem identification, needs analysis, and identifying the target audience. The *design* stage includes creating a learning environment outline, mapping evaluation methods, developing a communication strategy, aligning learning objectives with specific game elements, and creating concept art. The development stage is the production of game products: creating game strategies, designing media and instructional methods, and developing and evaluating assessment tools. The *implementation* stage involved launching the game, training participants, and carrying out the game's use in the study. The *evaluation* stage involved assessing the game overall, examining its effectiveness, evaluating participants' interactions with the game, and reviewing each stage of the design and implementation process. Specifically, in this study, we zoomed in on the *evaluation* stage with preservice teachers on two different digital tools developed around fractions. More details about the two digital tools and the context of the data are described in the Methods section. The following result section addresses preservice teachers' feedback on the two tools drawn on the lens of the evaluations stage of the ADDIE model.

Methods

The study was implemented in two mathematics education content courses for elementary preservice teachers in Spring 2025, with the goal of exploring their perspectives on using a mobile game and a game-based app, both of which were developed by the second author, for teaching fraction concepts. Data were collected at the end of the semester, after students had

learned the mathematics concept of fractions. These courses are designed to deepen preservice teachers' conceptual understanding of elementary mathematics topics, including fractions. During class time, students engaged individually with the mobile game and the digital app, followed by survey questions prompting reflection on their experiences. Participants included 66 volunteer PSTs who registered for the courses and agreed to complete the study.

Designing and Developing the Game and the Interactive App

In the game-based app, players need to figure out the length of a green stick based on the blue stick length being 1. Players can change the number in the blue box by manipulating the two arrows. Based on the selected blue number, blue divider sticks are created on the number line (Figure 2-left). If the selected blue number is a proper value as a denominator, the green box gets filled with a corresponding value. The design intends for players to coordinate the green number and the blue numbers to construct the meaning of numerator and denominator in a number line context. The digital game incorporates more graphics and interactive elements. Players are required to create a cloud by manipulating blue and green numbers in the boxes to block a stone falling from the sky (Figure 2-right). To create a cloud at a position players want, they need to manipulate the blue and green numbers, where blue numbers represent denominator and green numbers represent numerator. If players block at least one stone in each stage, they can move to the next stage (check the link for the app and the game for more details). The app was designed more as an interactive tool to be used by the teacher during instruction, so the teacher was expected to monitor students' gameplay as they engaged with the app. In contrast, the game was designed more as a structured fraction-learning game which players can figure out how to play the game (e.g., the tutorial stage) without supports from others. Within the game, even though the teacher could provide students with instruction, it was designed more as a structured fraction-learning game which players can figure out how to play the game (e.g., the tutorial stage) without supports from others; the design intends to support players to develop the understanding on fractions while proceeding the game stages.



https://rambo429.itch.io/playwithfractions-v2

App



https://rambo429.itch.io/bc-v2

Game

Figure 2: Developed App and Game in Development Stage

Data Collection

Participants engaged with both the game and the app for approximately 20 minutes for each tool at the end of two class sections, after having learned about the fraction concept during the semester. Each participant worked individually, allowing them to explore the app and game at their own pace. During this time, the teacher observed and provided minimal guidance, ensuring

that participants could navigate the tools independently while applying their understanding of fractions. From the students enrolled in both classes, 36 responded to the app survey question, and 30 completed the Digital Game survey. Both surveys included three scaled questions, effectiveness, clarity, engagement and two open-ended questions around improvement and challenges in common. We report on the participants' responses to the 5 questions in this paper. In the scaled questions, PSTs rated their perspectives on the effectiveness, clarity and the engagement of game and app on a scale of 1 to 10. Open-ended questions were "What challenges did you face?" and "What improvements would you suggest?" In the scaled questions, PSTs rated their perspectives on the effectiveness and the engagement of game and app on a scale of 1 to 10. Open-ended questions were "What challenges did you face?" and "What improvements would you suggest?" Qualitative responses to the open-ended questions were analyzed using open coding (Williams & Moser, 2019); responses were initially coded by two authors independently and then reviewed collaboratively to identify emerging themes.

Results

In this section, we describe the results based on the *Evaluation* stage of the ADDIE Model. PSTs' responses to the question "How effective did you find the app/game in enhancing understanding of fractions?" indicated that overall, both the app and the game received positive user feedback. The game's ratings are slightly more concentrated in the higher range (7–10), while the app's ratings are more spread out across the mid-level range (3–6) (see Figure 3).

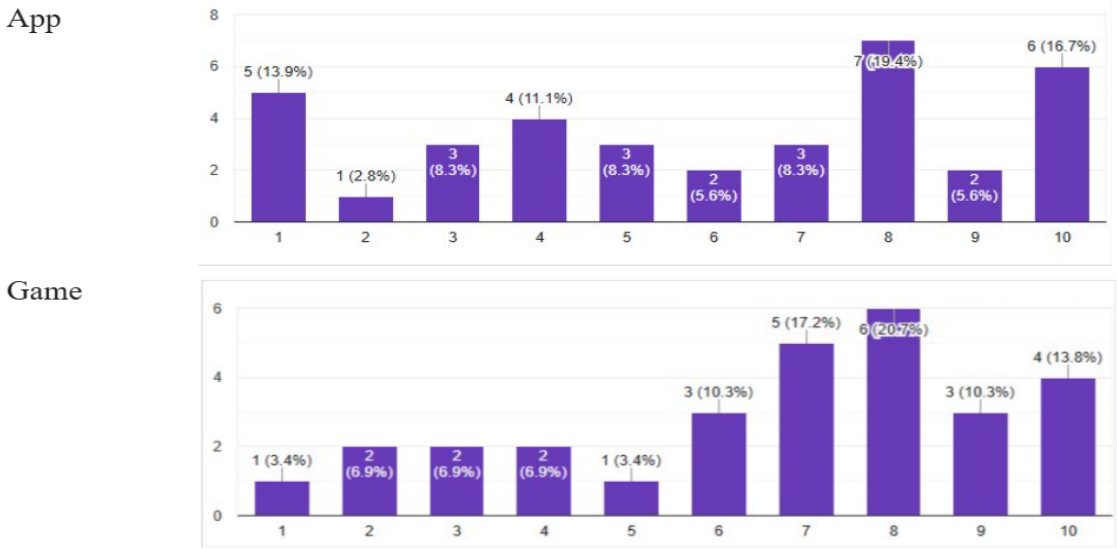


Figure 3: PSTs Responses to the Effectiveness of the App and Game in Understanding Fractions Concepts

Similarly, in PSTs' response to the question "Were the objectives and instructions clear and easy to follow?," the app rating shows a more evenly distributed pattern, whereas the game is more heavily concentrated in the higher range (7–10) (Figure 4). The game's strong peak at 8 suggests higher overall satisfaction compared to the app. This implies that PSTs perceived that the game has clearer instruction than the app does.

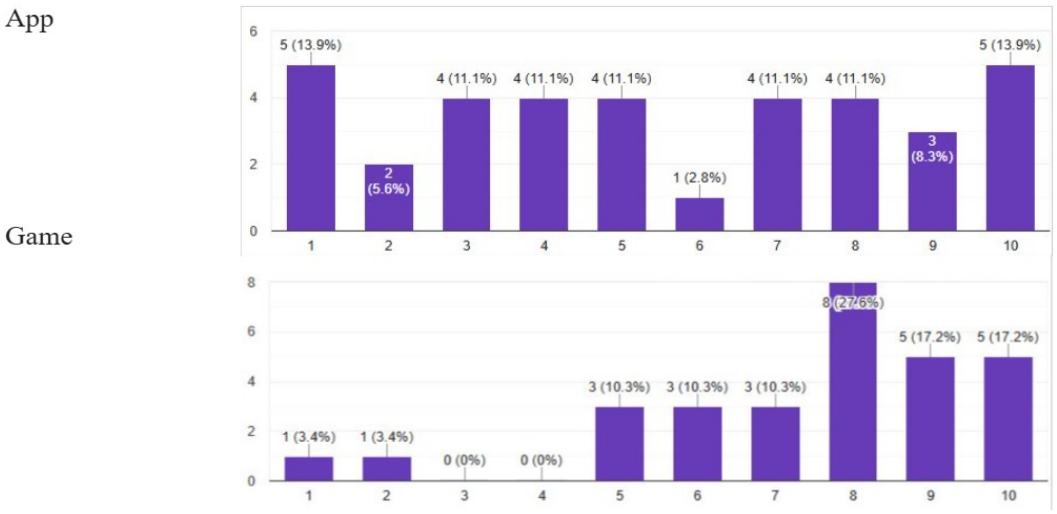


Figure 4: PSTs Responses to the Clarity of the Instructions

In response to the question "How engaging did you find the App/game for teaching the fraction concept?," PSTs' responses revealed that the app received a mix of ratings centered around five and seven, while the game showed a much stronger skew toward high engagement and satisfaction.



Figure 5: PSTs Responses to the Engagement of the App and Game

In analyzing PSTs' responses to two open-ended questions about the challenges they faced and their suggestions for improvement, several themes emerged, including instruction, immediate feedback, game loading time, time limits, repetitiveness, and math concepts (see Figure 6).



Note. For the Digital App, the number of responses was 36 and for the Digital Game, it was 30.

Figure 6: Challenges PSTs Faced and Their Suggestions for Improvement

Out of the responses collected about the game, 15 participants (50%) stated that they did not encounter any challenges, suggesting that the game was generally easy to use and accessible for PSTs. For the app, nine participants (25%) reported no challenges. From the participants that reported challenges, unclear instruction on how to play either the game or app (coded as Instruction) was the most frequently mentioned challenge for both the app and the game (50% of the responses for the app and 20% for the game). Consistent with this challenge, 56% of PSTs recommended adding clearer instructions, a tutorial, or a trial feature to the app as a possible improvement. For the game, this percentage was 17%. Although the game already included a tutorial, some participants still commented on the need for clearer instructions on how to play. This suggests that clarity of instructions played an important role in PSTs' experiences.

Lack of immediate feedback was mentioned by three PSTs as a challenge for the app (coded as Immediate feedback), while no comments about feedback were identified as a challenge for the game. However, in their improvement suggestions for both tools, PSTs highlighted that providing immediate feedback for correct or incorrect solutions is an important factor for enhancing learning and engagement in digital tools.

Four PSTs mentioned challenges with the *math content* covered in the game, while only one PST referred to this issue in the app. In their suggestions for improvement, participants recommended adding a tutorial to review the concepts, explaining the concepts within the questions, or including a video or voice guidance to provide clearer instruction on the math content.

Discussion and Implications

In this study, we explored PSTs' perspectives on an app and a game designed for fraction concepts. Specifically, we examined their feedback to see how their feedback can be used as implications for further design refinement. As the cyclic features of the ADDIE model suggest, the results from the survey questions will lead us to redesign the app and the game, making them more effective tools for instructional use.

From these findings that showed the current design of instruction was the most frequently mentioned challenge for both tools, and that over half of the PSTs recommended adding or improving instructional support for the app, we conclude that clear and accessible instructions for digital tools are what PSTs perceive as an important factor when considering using the tools for math learning. This will guide us in the next phase of redesigning the app to add more clear instructional support. The redesign will include (a) Task-specific micro-instructions: place a 1–2 line explanation about "What to do" panel at the top of each activity, (b) Try/Practice mode: A friction-free "trial round" with no scoring, just exploration to be used before the main activity page shows up.

Lack of immediate feedback was mentioned by some PSTs as a challenge. In their improvement suggestions, PSTs highlighted that providing immediate feedback is an important factor for enhancing learning and engagement in digital tools. This finding is supported by previous research. For example, a meta-analysis by Van der Kleij et al. (2013) showed that elaborated feedback produced larger effect than feedback that only indicated the correctness of an answer or provided the correct answer. Also, it showed that the effects of the feedback were affected by delayed feedback timing. Furthermore, Barana et al., (2021) defines interactive feedback as "a step-by-step interactive solving process that guides learners in completing a task after one or more independent attempts" (p. 6) in technology environments, based on theories of automatic and formative assessment. They found that interactive feedback, not correct or incorrect or rewards, is effective at helping learners improve their performance. In the redesign of the app and the game, we plan to incorporate both interactive feedback for content-related feedback (e.g., guidance on math concepts and problem-solving strategies) and technology-based feedback (e.g., rewards, stars, and scores) to further motivate and engage learners.

Furthermore, based on the finding that we received more comments about the math content in the game compared to the app, a plan for the next step for redesigning the game is as follows: the game requires additional scaffolding for math content to maximize both its effectiveness and student engagement. Because the design intention of the interactive app and game is to be used as instructional tools to be integrated into learning, rather than being used as a review tool only after students have learned the concept, we plan to include a brief conceptual review within the tool following completion of the game and app. This addition is intended to support students who may not fully grasp the concept after interacting with either tool alone. Also, even though we received only one comment noting that the game and app were repetitive, this prompted us to add more variety to the questions in both tools.

This study primarily focused on collecting feedback and perspectives of PSTs, rather than measuring actual learning gains or changes in teaching practice. In our larger study, we administered a pre-test and post-test knowledge check, which we plan to report in future work. Additionally, given the tools were designed specifically for teaching fractions and were limited to the particular app and game used in this study, the findings may not generalize directly to other math topics or digital tools. Nevertheless, the study provided valuable insights into the design and instructional use of educational technology from the PST's perspective; the results provide insights for educational game developers and educators on how to design digital games that are both engaging and effective in supporting teachers and students in teaching and learning fractions.

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THE PROMISE OF PERSONALIZATION: OPERATIONALIZING MATHEMATICAL KNOWLEDGE MODELS IN THE AGE OF AI

LA PROMESA DE LA PERSONALIZACIÓN: MODELOS OPERATIVOS DE CONOCIMIENTO MATEMÁTICO EN LA ERA DE LA IA

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Mathematics learning is cumulative, yet many representations of development remain descriptive rather than operational. Persistent difficulty in geometry and data analysis is often attributed to topic-level complexity but may reflect unresolved gaps in earlier knowledge. This conceptual paper examines the gap between descriptive progressions and models capable of inferential personalization. Drawing on Mastery Learning, the Zone of Proximal Development, and Knowledge Space Theory, we articulate three requirements: sufficient granularity, explicit precursor–successor encoding, and inferentially coherent knowledge states. We bring two forms of evidence to bear: error patterns in generative AI systems lacking structured knowledge models, and the architecture and pilot results of a system that satisfies them. Formalizing these dependencies is necessary for diagnosing cumulative difficulty.

Keywords: Learning Theory, Learning Trajectories and Progressions, Assessment, Technology

Introduction

Large-scale assessment data continue to reveal persistent and troubling patterns in middle and secondary mathematics performance. National and international assessments indicate that students struggle with domains such as geometry, data analysis, statistics, and probability (National Center for Education Statistics [NCES], 2022; Organisation for Economic Co-operation and Development [OECD], 2023). These findings are often interpreted as evidence that particular mathematical domains are inherently difficult. However, growing evidence suggests that many of these difficulties reflect unresolved gaps in earlier foundational knowledge rather than domain-specific complexity (Burleigh & Durkin, 2026).

Mathematics is structurally cumulative. Later competencies depend on earlier conceptual structures, including whole number understanding, multiplicative reasoning, spatial structuring, and relational thinking (Clements & Sarama, 2020; Siegler et al., 2012). When this foundational understanding is incomplete, students may develop what is sometimes described as "Swiss cheese" foundations: knowledge that appears intact on the surface but obscures hidden structural gaps. These gaps may remain undetected for years before emerging as difficulty in algebra, geometry, or data analysis. In such cases, the visible struggle is not located in the advanced topic itself, but in earlier, underdeveloped precursor understandings.

Research on early number development illustrates this cumulative structure. As an example, Baroody and colleagues (2009) argue that fluency with basic number combinations does not arise from rote memorization alone, but from the gradual construction of interconnected numerical relations and reasoning strategies. When these relational structures are insecure or incomplete, later learning becomes difficult and unstable. In some cases, incomplete foundational development may be misinterpreted as mathematics disability rather than as a constellation of unresolved precursor gaps that could be identified and addressed (Baroody et al., 2009).

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Foundational learning theories rest on similar structural assumptions. Bloom's Mastery Learning presumes that prerequisite understandings can be identified and secured before advancing (Bloom, 1968). Vygotsky's (ZPD) of Proximal Development presumes that for each learner, adjacent and reachable knowledge can be determined within a structured space of understanding (Vygotsky, 1978). Likewise, research on mathematics learning trajectories articulates increasingly sophisticated forms of reasoning and developmental ordering (Clements & Sarama, 2020; Fonger et al., 2018). Across these perspectives, learning is treated as structured and cumulative. Yet there remains a representational gap: while these frameworks describe how understanding develops, they do not always specify how prerequisite relations should be formally encoded so that learner readiness can be systematically inferred and automated, such as in adaptive learning systems.

This paper argues that if AI-driven personalization is to address cumulative mathematical difficulty rather than merely respond to surface performance, it must be grounded in formally structured knowledge models that decompose mathematical knowledge into sufficiently granular components with explicitly represented dependency relations. This is a conceptual paper, and its argument proceeds analytically rather than through the test of a hypothesis. The sections that follow articulate why descriptive progressions are insufficient for this purpose and specify three representational conditions that content-structural personalization requires. We then bring two forms of evidence to bear on those conditions: a documented set of recurring failures in generative AI systems that lack structured knowledge models, which shows what their absence produces, and the architecture and pilot results of an adaptive system built to satisfy them, which shows that they can be implemented. We close with implications for how the field interprets and addresses mathematical difficulty.

Progressions Are Not Enough for AI-Driven Personalization

The mathematics education community has invested substantial effort in articulating learning trajectories and progressions that describe how students' mathematical understanding develops over time (Clements & Sarama, 2020; Confrey et al., 2014). These frameworks have advanced the field by identifying typical developmental pathways, highlighting common misconceptions, and emphasizing the cumulative nature of mathematical knowledge.

However, learning progressions are primarily descriptive. They specify common sequences of development but do not always formalize the structural dependencies required for systematic inference about individual learners. A progression may indicate that Concept B tends to follow Concept A but does not necessarily specify whether mastery of A is logically required for B, whether partial mastery suffices, or whether certain knowledge states are structurally impossible. Developmental order is not the same as structural constraint.

This distinction becomes especially consequential in early childhood mathematics. Foundational domains such as whole number understanding are often described at a relatively broad level, for example, through constructs such as counting, cardinality, or part–whole reasoning. Yet each of these contains multiple interdependent components. Stable count sequence, one-to-one correspondence, cardinality, relational comparison, additive composition, and unitizing are distinct but structurally related forms of understanding. Treating these as a single construct obscures the internal dependency relations that determine whether later learning is accessible.

When knowledge is represented at a coarse grain size, important precursor relations remain implicit. A learner may demonstrate correct performance on a task associated with "cardinality" while still lacking flexible part–whole reasoning. The absence of specific precursors constrains

what successor understandings are reachable, but without sufficient granularity, such constraints remain invisible. Adaptive systems may interpret surface accuracy as evidence of readiness for advancement without distinguishing which successor states are accessible and which remain out of reach. In such cases, personalization may recommend tasks that appear aligned with recent performance but reinforce fragmented foundations rather than strengthening them.

Structural Requirements for Content-Structural Personalization

If mathematical knowledge is cumulative and structurally organized, then personalization requires representational commitments that make prerequisite relations explicit and inferentially usable. Adaptive systems must reason about structure, not merely about performance. This imposes three representational conditions on any knowledge model intended to support automated and principled personalization: sufficient granularity, explicit encoding of precursor–successor relations, and inferential coherence of knowledge states.

(1) Granularity as a Representational Threshold

Personalization presupposes that knowledge can be represented at a level of resolution fine enough to identify missing components without collapsing distinct forms of understanding into broad constructs. Macro-level categories such as "number sense," "place value," or "fractions" conceal the internal structures that determine whether later learning is accessible.

As noted above, whole number competence comprises many interdependent components (Clements & Sarama, 2020; Siegler et al., 2012). Baroody and colleagues' (2009) work further emphasizes that fact fluency emerges from structured numerical relations rather than isolated memorization. Each of these components supports distinct successor structures. When knowledge is represented too broadly, diagnostic precision suffers. It becomes impossible to determine which specific precursors are missing. Faithful modeling of early mathematics may therefore require representing hundreds of distinct but interconnected knowledge units (KUs) even within a limited developmental span. Such a level of resolution better reflects the genuine complexity of foundational learning rather than the convenience of assessment design.

(2) Explicit Precursor–Successor Encoding

Granularity alone is insufficient. Representing discrete KUs does not guarantee that their relations are formally specified. Operationalizing personalization requires explicit encoding of which relations are logically or developmentally constraining.

Learning progressions describe typical developmental orderings. However, operational knowledge models must go further by specifying which successor understandings depend on which precursors (often representing multiple such relationships). Knowledge Space Theory provides one formal articulation of this requirement by representing knowledge as a set of states governed by prerequisite relations (Doignon & Falmagne, 1985; 1999). Within such a framework, mastery of specific concepts constrains which successor states are reachable. If a learner lacks a particular precursor, certain successor understandings remain inaccessible regardless of exposure or practice.

This formalization clarifies the cumulative nature of mathematical difficulty. Consider multiplicative reasoning, which underlies fraction equivalence, proportional reasoning, and aspects of algebraic thinking (Siegler et al., 2012). When multiplicative structures are incomplete, later struggles may appear in domains that are distal from early number development. Spatial structuring in early geometry likewise predicts later facility with area, volume, and formal proof (Clements & Sarama, 2020). Difficulties observed in middle school domains such as geometry or data analysis may therefore originate in earlier, unresolved structural gaps. Without explicit encoding of precursor–successor relations, an adaptive system

can only respond to the present visible performance; it cannot recognize that advancement is constrained by an absence it has no means of representing or locating within the adaptive system. Accurate diagnosis requires a model of dependency, not merely a record of outcomes.

(3) Inferential Coherence and Knowledge State Closure

A third requirement concerns inferential coherence. Personalization requires reasoning not about isolated skills, but about structured knowledge states. If adaptive systems are to approximate principles such as mastery learning or leverage an individual learner's ZPD, they must identify what forms of understanding are adjacent, reachable, and possible given a learner's current knowledge state (Betts et al., 2024).

Within formal knowledge space frameworks, a learner's knowledge state is defined as a set of mastered KUs that satisfies closure under prerequisite relations (Doignon & Falmagne, 1999). States that violate dependency constraints are excluded as impossible. This constraint-based representation allows systematic reasoning about a learner's space of possible next steps and makes visible which understandings remain out of reach.

Without this third condition, personalization is driven by patterns in performance rather than by the architecture of knowledge itself. A learner who answers several items correctly may be advanced to new content with no representation of what that advancement presupposes. This is distinct from advancing a learner past content that has not been directly observed. Closure under prerequisite relations makes that inference possible: when a learner demonstrates a successor understanding, the precursors it depends on may be inferred as mastered rather than treated as skipped, which is what allows structured systems to assess efficiently and to probe the upper reaches of a learner's ZPD. The difference is diagnostic. In a structured model, a failed probe localizes the dependency on which the inference rested; in a performance-driven system, the same failure is only an incorrect answer. Undiagnosed gaps tend to re-emerge as the demands of later content increase, and a system without dependency structure cannot distinguish a genuine gap from an unobserved mastery.

These three conditions, (1) granularity, (2) explicit dependency encoding, and (3) inferential coherence, form a mutually reinforcing representational threshold: granularity without dependency encoding produces isolated fragments; dependency encoding without coherent state reasoning cannot determine which next steps are reachable; state reasoning without sufficient granularity operates on constructs too coarse to support diagnosis. Any system that omits one condition undermines the others.

The Representational Gap in Practice: AI Errors in the Absence of Structural Models

Systems that attempt to support early mathematics learning without the representational requirements specified in the previous section fail in observable and predictable ways. To examine those failures, we prompted several largen language models (LLMs), including ChatGPT, Claude, Perplexity, and Gemini, to generate early mathematics activities targeting specific KUs drawn from a granular developmental progression spanning numeracy, geometry, and measurement (Betts, 2025). In each case the model was supplied with the full KU learning progression as text, including the position of the target KU within the sequence. The patterns reported in this section are those that recurred reliably across prompts, models, and content domains.

LLMs are increasingly used by parents and educators to generate early mathematics activities, often by those with the least domain-specific training and therefore the greatest need for reliable guidance (Betts, 2025). Yet across platforms, these systems exhibit persistent patterns of developmental misalignment, concept conflation, and boundary violation when

generating content for early mathematics. These errors are the structurally predictable consequences of operating without the representational infrastructure that content-structural personalization requires.

Let us consider a concrete case. When prompted to generate activities for the foundational measurement KU "*Shows awareness of size, and uses appropriate language to describe such as big, small, giant, tiny*," one LLM immediately proposed a sorting activity requiring children to classify objects into size categories (Betts, 2025). Sorting presupposes comparison and classification, which are developmentally subsequent to recognition and labeling. The system had itself articulated where this KU fell within the sequence. Yet it continued to consistently generate activities targeting skills several steps ahead of the intended level. When corrected, it acknowledged the error accurately, and then produced another activity requiring children to order objects from big to small, targeting a later KU in the same progression.

This pattern (i.e., surface-level acknowledgment followed by repeated structural violation) is diagnostic. The progression was available as descriptive text, but the precursor–successor relations were not encoded in a form that could govern the AI's inference. Without explicit structural constraints, the system treated the progression as a list of related topics rather than a set of ordered dependencies. It could recognize a mismatch when told about one, but could not prevent the next occurrence, because it had no mechanism for reasoning about which knowledge states were reachable from a given developmental position.

A second class of error illustrates the consequences of missing granularity. The system conflated size vocabulary (big, small) with length vocabulary (tall, short), describing a flower as "short and small" within a single activity (Betts, 2025). In early mathematics, size and length refer to distinct measurable attributes that require different mental models, tools, and units. This distinction underlies later differentiation of area from perimeter, volume from surface area, and one-dimensional from multi-dimensional measurement. When a system collapses these attributes into interchangeable descriptors, it models a conceptual conflation that, if implemented by a non-expert adult, may become embedded in the child's developing understanding as a misconception.

These failures were not isolated. Across numeracy, geometry, and measurement, generative AI systems routinely confused verbal count sequence with one-to-one correspondence, proposed backward counting when forward skip counting was specified, conflated part–whole reasoning with conservation of number, and generated contextually implausible activities disconnected from children's lived experiences (Betts, 2025). Each category of error maps onto the absence of a specific representational condition: boundary violations reflect missing dependency encoding, attribute conflation reflects insufficient granularity, and contextual implausibility reflects the absence of inferentially coherent knowledge states.

Most AI-embedded educational products for early mathematics are, in practice, general-purpose language models with an educational interface (Betts, 2025). Very few have been trained on rigorously vetted, research-based knowledge models, largely because such models barely exist at the level of granularity and formal structure required for truly adaptive AI-driven "smart" learning systems. This reflects a broader "triple expertise gap:" (1) practitioners lack deep domain-specific knowledge of early mathematics, (2) society perceives early math as simple, and (3) the field has too few specialists to build the knowledge infrastructure that both educators and AI systems require (Betts, 2025; Ban et al., 2024). AI systems trained on this fragmented knowledge base do not invent new misconceptions; they reflect and amplify existing ones with a confidence that makes detection by non-experts unlikely.

Content-Structural Personalization in Practice

If the preceding section illustrates what happens when the representational conditions for content-structural personalization are absent, PAL (Personal Assistant for Learning), an adaptive learning system for early mathematics, stands as evidence that those conditions can be operationalized within an adaptive system, producing measurably different design decisions and outcomes (Betts et al., 2026).

PAL is an adaptive learning system grounded in sociocultural theory, Knowledge Space Theory, and learning sciences principles that supports early mathematics learning by guiding parents and teachers to facilitate hands-on, socially mediated interactions aligned with each child's ZPD (Betts et al., 2026). Its architecture instantiates each of the three representational conditions identified in this paper. Firstly, sufficient granularity is realized through hundreds of developmentally sequenced Knowledge Units (KUs), each representing a discrete concept or skill at a resolution fine enough to distinguish between components routinely collapsed in curricular standards (e.g., between labeling objects using size vocabulary and sorting objects by size, etc.). Secondly, explicit precursor–successor encoding is realized through a directed conceptual graph in which prerequisite relations among KUs are formally specified, capturing the branching, converging nature of early mathematical development and preventing the system from recommending activities that target unreachable knowledge states (Figure 1). And lastly, inferential coherence is realized through an explore–exploit adaptive engine that probes emerging readiness on adjacent KUs while consolidating understanding in areas of strength, informed by lightweight adult feedback (e.g., "did independently," "needed help," "struggled") and brief KU-aligned digital assessments.

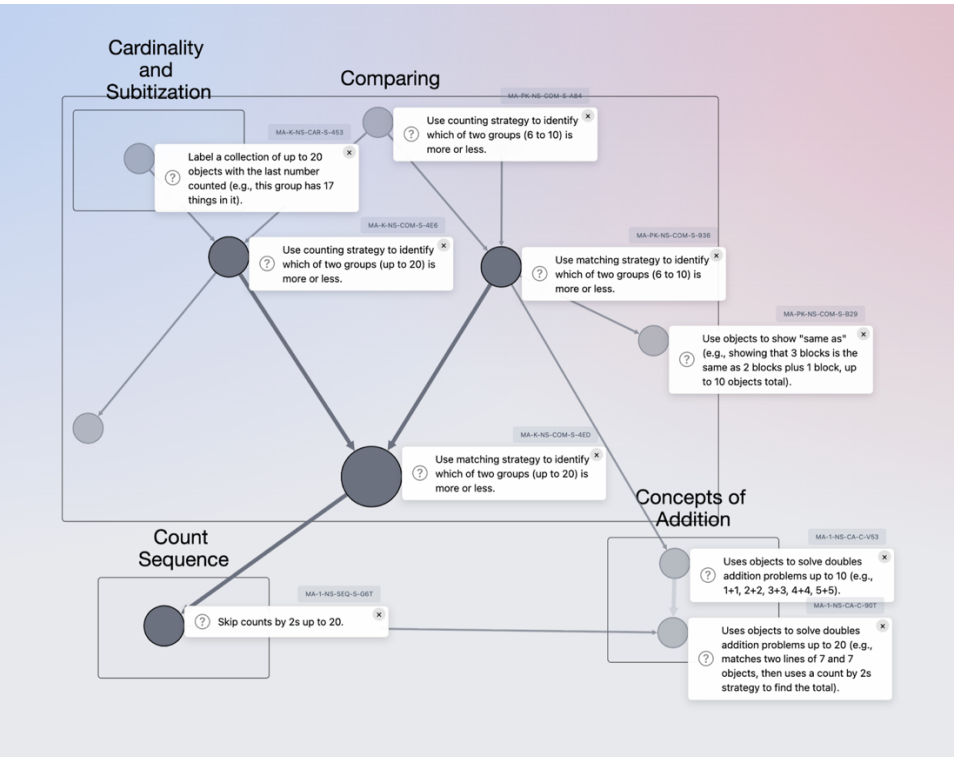


Figure 1: Model of Precursor–Successor Relations Among Knowledge Units in Early Number Sense, Comparisons Topic

A notable feature of PAL is that it does not require children to interact directly with a digital device. PAL's personalization operates on structural knowledge states inferred from adult-reported evidence and periodic assessments rather than on real-time screen-based performance, and learning takes place through hands-on, adult-mediated activities embedded in everyday routines, making them more developmentally appropriate for young learners. While PAL does feature periodic digital assessments, these are not required and can also be done in an offline manner. This approach follows from the representational architecture itself. If the knowledge model is granular enough, the dependencies explicit enough, and the state inference coherent enough, then the system can guide learning through any interaction modality that produces evidence at the appropriate grain size. The adult becomes both the mediator of learning and a source of structured evidence, operationalizing sociocultural theory's concept of the "more knowledgeable other" (Vygotsky, 1978) within a computationally grounded adaptive framework.

Early evidence from two exploratory implementations is consistent with these predictions. In a randomized pilot with prekindergarten families from under-resourced communities ($n = 18$), children whose parents received PAL showed greater growth on the TEMA-3 than a business-as-usual comparison group ($p < .01$; Cohen's $d = 1.71$), and parent engagement was positively associated with gains (Spearman $r = .62$, $p = .003$; Betts et al., 2026a). A second study varying nudge frequency over seven months ($n = 39$) found that the number of nudges a family received predicted children's growth ($r = .51$, $p < .001$), a relationship that held after adjusting for baseline achievement (partial $r = .54$, $p < .001$; Betts et al., 2026b). Both samples are small and neither was designed to isolate the knowledge model's contribution from that of adult mediation; we report them as directional. What they establish here is that these representational commitments are implementable at the granularity this paper argues for, and a system so structured can generate next steps that children are ready to take.

Implications for Mathematics Education Research

The analysis presented here carries several implications for how the field interprets mathematical difficulty, designs adaptive systems, and directs research effort.

First, content-structural personalization reframes the interpretation of difficulty itself. When dependency relations are not explicitly modeled, observed performance deficits tend to be attributed to domain difficulty, insufficient effort, or learner disposition. If knowledge is structurally cumulative, however, many instances of apparent difficulty may instead reflect unresolved precursor gaps. The cascading patterns observed in large-scale assessment data (e.g., student struggle with geometry, data analysis, algebra, etc.), are consistent with this structural interpretation. A student struggling with proportional reasoning may receive remediation in algebraic procedures without identification of incomplete multiplicative foundations. Similarly, a student labeled as having a mathematics disability may exhibit a constellation of missing precursors that were never systematically diagnosed. A structurally explicit knowledge model does not eliminate the possibility of genuine learning disabilities; it reduces the risk of conflating missing learning opportunities with inherent incapacity, shifting the focus from labeling to structural diagnosis.

Second, this perspective clarifies what constitutes theoretically grounded personalization in the context of adaptive technologies. Without explicit structural modeling, adaptive systems reproduce the same descriptive limitations found in traditional progressions, in other words, responding to visible performance rather than diagnosing the structural conditions that produce

it. The evidence from generative AI systems confirms that access to descriptive progressions alone is insufficient; the errors these systems produce are precisely those predicted by the absence of formal dependency encoding. Content-structural personalization requires that knowledge models make cumulative structure visible and diagnostically actionable.

Third, the challenge for mathematics education research is not merely to describe how understanding develops, but to formalize the structural relations that make development inferentially tractable. This is a different kind of research enterprise than articulating learning trajectories; it requires specifying which relations are constraining, validating prerequisite structures empirically, and building the knowledge infrastructure that both human educators and AI systems need. If mathematical learning is cumulative, then its representation must be structurally cumulative as well.

Conclusion

This paper has argued that AI-driven personalization in mathematics education requires explicit representational commitments that most current systems do not make: sufficient granularity to differentiate component knowledge structures, explicit encoding of precursor–successor relations, and inferentially coherent knowledge states. These conditions are not optional enhancements to existing adaptive approaches. They constitute the minimal infrastructure for personalization that can diagnose structural gaps rather than merely react to surface performance. The observable failures of generative AI systems in early mathematics confirm what the theoretical argument predicts, and the design of systems like PAL demonstrates that these conditions are achievable. What remains is the harder work: building, validating, and sharing the structured knowledge models that this form of personalization demands—and reckoning with the cost, to children and to the field, of continuing without them.

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GENERATIVE AI AS A MATHEMATICAL LEARNING TOOL: AN ANALYTICAL FRAMEWORK PROPOSAL

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Generative artificial intelligence (GenAI) gained unprecedented public support upon its launch in November 2022. Initially dismissed by many educational stakeholders, the educational establishment is gradually welcoming it. GenAI resembles earlier mathematical action and communication technologies but also has unique capabilities, such as conversing with a student primarily in writing. Consequently, this theoretical research report reviews frameworks for examining students' use of technology to do and learn mathematics, which I describe as a mathematical learning tool. This report proposes an analytical framework comprising nine dimensions that builds on prior frameworks, addresses their gaps, and accounts for the unique capabilities of GenAI. Some of these dimensions include equity and access, GenAI function and instrumentalization, epistemic control, discourse, and technology mediation.

Keywords: Technology, Artificial Intelligence, Analytical Framework

Introduction

ChatGPT, the first large language model (LLM) and generative AI system (hereafter referred to as GenAI), became publicly available in November 2022 and reached 100 million users within two months, making it the fastest-downloaded application in history (Hu, 2023). While GenAI initially struggled with mathematical tasks (e.g., Davis, 2025b), it has since become highly effective across a range of mathematical tasks (Metz, 2024). Consequently, they join a long list of mathematical technological tools that were developed for other purposes but found a home in schools (e.g., spreadsheets), as well as tools specifically designed for mathematics education, such as virtual manipulatives. They certainly share features with prior mathematical tools, such as calculators, but they are also used in different ways (e.g., via natural language) and have different capabilities (e.g., in developing mathematical arguments). Moreover, although research on GenAI has used previously developed frameworks (e.g., Davis, 2025a), these frameworks do not capture the unique capabilities of GenAI or the primarily text-based ways users interact with the technology. This paper argues that, given GenAI's unique capabilities, a new analytical framework is required to investigate its role as a *mathematical learning tool* for students. While frameworks unique to GenAI are needed, it is important to note that such a framework may need to be in a constant state of flux as the capabilities of these systems change rapidly.

Background

Categorizing Educational Technologies

Dick and Hollebrands (2011) used the terms "conveyance technology" and "mathematical action technology" to distinguish two types of technologies that can be leveraged for educational purposes. The first refers to technologies that convey information in an educational setting, such as a document camera. Mathematical action technologies operate on different representations of mathematical objects, such as performing calculations, plotting graphs, constructing geometric objects (e.g., a line perpendicular to another), and manipulating algebraic symbols, as in a computer algebra system (CAS). A common example of a mathematical action technology that incorporates all these functions is a TI-Nspire CX CAS handheld calculator.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Drijvers, Stacey, and Trouche (2024) used the terms *communication infrastructure* and *computational infrastructure* to denote distinct types of technology. The former aligns with Dick and Hollebrand's (2011) conception of conveyance technology, but the latter is somewhat broader. It includes technologies that operate on or produce mathematical representations, as well as technologies that execute computer code and employ earlier forms of AI, such as those used in facial recognition. Recently, GenAI has demonstrated the ability to construct mathematical arguments (Liu et al., 2025). Thus, the term computational infrastructure should be further broadened to encompass these new mathematical capabilities. Furthermore, GenAI more closely integrates communication and computational infrastructure within a response, suggesting that these components should be more closely connected.

Not only does GenAI expand the capabilities of computational infrastructure and suggest a more closely intertwined computational and communicational infrastructure, but it also enables interactive instruction for students through tutoring. Indeed, the U.S. Department of Education (2023) has argued that one of the educational strengths of GenAI is its ability to provide personalized feedback to students as they learn a mathematical idea. While students can interact with prior mathematical technologies to deepen their understanding, GenAI enables them to pose their own questions to further their understanding.

Students' Use of Technology to Offload or Learn Mathematics

When students use mathematical action technologies or computational infrastructure to perform a task, they can do so in two ways. First, it can be used to perform, execute, or offload mathematical work, which Drijvers (2012) refers to as *doing mathematics*. Second, it can be used to help students learn mathematics by helping them understand why the output is reasonable or by exploring a mathematical idea through interconnected representations, which Drijvers refers to as *learning mathematics*. Buchberger (1990) argued that if students understand how to perform the mathematics being offloaded to technology, it is a use of technology as a *white box*. When that technology performs the mathematics faster and more accurately, Pea (1985) described this use of technology as an *amplification* of human capabilities. If students do not understand the mathematics performed by the technology, Buchberger described this as using the technology as a *black box*. When technology helps students understand a mathematical idea in a new way or grasp an idea they previously did not understand, it is used to *reorganize* the user's understanding (Pea). Cedillo and Kieran (2003) described this use of technology as a *grey box*, illustrating a hybrid of black and white box metaphors.

Instrumental genesis is the progressive development through which a digital artifact becomes a functional *instrument* for mathematical activity, as users construct and stabilize schemes for acting with it (*instrumentation*) while simultaneously shaping the artifact-in-use through selection, configuration, and repurposing (*instrumentalization*) (Vérillon & Rabardel, 1995). In the instrumental approach, an artifact (e.g., a CAS, dynamic geometry environment, or graphing technology) is not automatically an instrument; rather, an instrument is conceived as a coupled system of **(**a**)** the artifact *as used* and **(**b**)** the user's schemes of utilization, recurrent, goal-directed organizations of action that include operational invariants (implicit concepts-in-action and theorems-in-action) and control structures for monitoring and validating results (Rabardel, 1995).

Frameworks for Thinking about Users' Interactions with GenAI

Mollick (2023) conceptualized individuals' relationships with GenAI in the workplace as *centaurs* and *cyborgs*. Working with GenAI as a centaur keeps a sharp boundary between what the human does and what the technology does. Much like the distinct human and horse halves in

the centaur myth. In this approach, responsibilities are deliberately divided so each partner focuses on its comparative strengths, with work handed back and forth as needed. In contrast, cyborg work dissolves the boundary between person and machine by integrating GenAI into the flow of thinking and producing. Rather than cleanly handing off entire tasks, the human and GenAI collaborate in small, iterative exchanges, repeatedly crossing the jagged frontier as one partner begins drafts, phrases, or ideas and the other completes or refines them.

The OECD GenAI Literacy framework (2025) describes four ways students can engage with generative AI: *engaging with AI* (using AI tools while recognizing their presence and evaluating output quality), *creating with AI* (co-producing ideas or products through iterative prompting and refinement), *managing AI* (delegating tasks strategically while retaining human judgment and oversight), and *designing AI* (examining how data and design choices shape model behavior and impacts, oriented toward shaping AI for human good). The United Nations Educational, Scientific, and Cultural Organization (UNESCO) GenAI competency framework (2024) for students treats learning about the technology as preparation for responsible participation in societies, framing students as informed users and, where appropriate, co-creators. It describes four components: human-centered mindset**,** ethics of GenAI**,** GenAI techniques and applications, and GenAI system design. The U.S. Department of Education (2023) frames AI as most valuable when it strengthens human judgment rather than replaces it. For learning, it highlights AI's potential to support more adaptive learning experiences through timely, targeted feedback and guidance (e.g., tutoring systems that respond to students' solution steps), while cautioning that AI models are always partial and must avoid narrow deficit-only conceptions of learning

Equity Issues Surrounding GenAI

Equity concerns about AI in education span the full life cycle of these systems, from the material infrastructure required to build and operate them to the content they generate to how schools and families are positioned to use them effectively. At the infrastructure level, expanding data-center capacity for GenAI can concentrate environmental burdens (e.g., electricity demand and water use) in particular regions and communities, raising questions of resource and environmental justice (Mozur, Satariano, & Rodriguez Mega, 2025).

Metz (2019) highlights the largely hidden, low-paid annotation labor raises fairness, privacy, and embedded-bias concerns. Moreover, generative models can reproduce and normalize historical patterns of bias because they are trained on large-scale internet text and other data that reflect existing social inequalities (Gómez Marchant & Hardison, 2024).

Equity issues also arise from uneven institutional capacity. Recent district survey data suggest that, although many districts anticipate providing professional learning on GenAI, such training is not distributed uniformly, with urban districts less likely to deliver such support during the study period (Diliberti et al., 2024). Finally, student opportunities to benefit from AI are stratified by household resources and dispositions. For example, national survey data indicate that students whose parents have higher educational attainment report more frequent use of GenAI for school and hold more favorable views than peers whose parents have lower educational attainment (Adair et al., 2025).

Framework Gaps

Like other technologies, such as Amplify, GenAI combines both mathematical action/computation infrastructure and conveyance/communication infrastructure. Additionally, in Amplify, content can be communicated between students and the teacher, but in GenAI, content is communicated between a student and the GenAI and from the GenAI to the student. An additional difference between GenAI and previous educational technology is the nature of the

interactions between users and the technology. The lingua franca of prior mathematical action technologies has been mathematical representations, whereas the currency in student-to-GenAI communication is primarily text-based. Additionally, these prior mathematical action technologies could be used to learn mathematics, as Drijvers (2012) has pointed out, but this typically requires working through a predesigned activity, often involving dynamic representations or student freedom to engage with the technology in mathematical ways. GenAI enables users to ask specifically where they are struggling to understand an idea.

Furthermore, non-GenAI forms of technology *constrain* student actions to the functions of the technology (e.g., graphing calculator) or how technology was embedded within an activity structure involving imperatives and questions (e.g., dynamic sketch with constrained student actions); however, GenAI provides users (students within an educational setting) with much more *freedom*. Specifically, GenAI enables users to perform multiple query-response-evaluate cycles, and the query or prompt given to the technology can vary considerably. Thus, the user must examine responses for mathematical fidelity (Dick, 2008) to assess their validity, given the possibility of hallucinations. Furthermore, GenAI users need to reflect metacognitively (e.g., Schoenfeld, 1987) on how well the answer aligns with their understanding and whether it moves them closer to the mathematical learning goal. In general, much has been written about prompt engineering (Qian, 2025), i.e., crafting a query to submit to a generative AI system. Although Mollick (2025) has argued that prompt engineering is less important than it once was, users must still evaluate the GenAI response against their own knowledge to determine whether a further prompt is needed and, if so, what its nature should be. GenAI shares similarities with other Internet-based technologies, such as Google, in that it requires a query, but it differs in that users conducting a Google search may not submit additional queries as they might when using GenAI.

Several frameworks exist for designing and critiquing dynamic manipulatives involving mathematical action technology (e.g., Sherman & Cayton, 2015). Yet at the time of this writing, GenAI cannot create dynamic, manipulative sketches. Hence, while GenAI expands the capabilities of mathematical action and conveyance technologies, it is limited in other ways. Consequently, frameworks associated with the dynamic capabilities of mathematical action technology were excluded from the framework presented here.

In their examination of frameworks for learning and teaching mathematics with technology, McCulloch et al. (2021) identified three gaps in existing frameworks: equity, noticing student thinking, and discourse. Equity and access should be treated as an explicit framing rather than an afterthought: the authors note that equity concerns are often not directly addressed across the sampled frameworks, even though technology-access criteria (e.g., what devices, accounts, connectivity, and policies are available) routinely determine what can be used in practice. In a GenAI-oriented framework, this means positioning access constraints, such as account availability, platform restrictions, bandwidth reliability, and data/privacy policies, within the technological strand itself and treating them as design constraints that shape pedagogical intent.

McCulloch et al. (2021) emphasize that noticing changes in student thinking during technology-mediated activity is challenging: when students use mathematical action technologies, evidence of their thinking is often embedded in tool interactions, creating a need for explicit guidance on what counts as evidence and how it should inform instruction. For GenAI, the conversational trace, prompts, follow-up questions, revisions, and corrections can become a rich resource for noticing, but only if teachers and students have a shared interpretive language (for example, distinguishing verification from delegation, or identifying moments of conceptual repair).

Finally, McCulloch et al. (2021) position classroom discourse as a central object rather than a peripheral concern, highlighting that discourse is another underdeveloped intersection and that technology can reshape how students justify, argue, and coordinate meanings. A GenAI-specific implication is that discourse now includes a non-human conversational partner; therefore, a framework should specify when and how this new discourse supports justification and argumentation, and how the classroom establishes shared criteria for what counts as a good justification.

GenAI as a Mathematical Learning Tool: Proposed Analytical Framework

Although the nine dimensions described below arise from distinct sources, they are unified by an organizing principle, the distribution of epistemic agency across the student-GenAI-classroom environment. Unlike previous mathematical action technologies, whose mathematical fidelity was a design property, GenAI is a fallible conversational partner, in which the location of mathematical authority is negotiated during each interaction. Each dimension examines a facet of this negotiation. The conditions under which students are positioned to exercise agency (Dimension A), the enactment of agency in positioning the tool, verifying its output, justifying claims publicly, monitoring one's own understanding, and using the tool in ethically informed ways (Dimensions B-G), and the visibility and consequences of that agency in the conversational trace and the final mathematical product (Dimensions H-I). In this way, the upper levels of each dimension describe a student who retains intellectual autonomy (Yackel & Cobb, 1996) in partnership with the technology, while the lower levels describe authority, delegated by default or design, to GenAI.

Unit of Analysis

I define an *episode* as a single problem or task from initiation through completion and reflection, comprising both student work and the GenAI conversational trace, including the student's prompts to the system and the technology's responses. A *turn/move* consists of a student prompt and a GenAI response. Space constraints prevent me from providing all the details of the analytical framework. Instead, I present the framework's dimensions and subdimensions. Dimensions A, B, F, and G are descriptive in nature with no increase in sophistication among the subdimensions, while Dimensions C, D, E, H, and I are developmental, showing increased sophistication from lower to upper levels.

Dimension A: Equity and Access as Design Constraints

Following McCulloch et al. (2021), the first construct encompasses equity and access as design constraints. This construct assesses whether access and policy realities are treated as explicit constraints that shape the learning pathway. It consists of three subdimensions. The first is material/platform access, which includes the type of GenAI used (free or paid), school bandwidth features, and school-level blocked features. The second is privacy constraints, which involve what can and cannot be entered into the AI and the norms governing anonymization. The third is institutional support capacity, which involves the availability of guidance, norms, prompt templates, and training. Each of these subdimensions is coded across four levels: unarticulated, noted, planned around, and strategically leveraged and documented. Some of these subdimensions measure components at the school and episode levels, specifically what students have been taught to develop prompts.

Dimension B: GenAI Function and Instrumentalization in Mathematical Activity

This dimension captures how students shape GenAI as an artifact-in-use. It codes the role the student assigns to GenAI and the ways the student configures that role through prompts, constraints, and iterative reconfiguration. It does not code learning outcomes or quality of

mathematical reasoning. It codes the intended tool role and configuration moves, regardless of whether they succeed. The first category involves an answer-retrieval role, in which the GenAI is positioned as a source of answers. In the second category, the student positions the GenAI in a procedure generation role. In the third category, GenAI is positioned in an explanation/representation-provisioning role, regardless of whether the student learns from their interactions. The last category involves an inquiry-orchestration role in which the student iteratively reconfigures GenAI, for example, by examining the effects of a parameter on a graphical representation. This last category aligns with Mollick's (2023) metaphor of work with the GenAI as a cyborg.

Dimension C: Epistemic Control and Verification

This category refers to correctness that is established through mathematical validation rather than GenAI authority. This category aligns with the OECD (2025), which asks students to engage with the outputs of GenAI; UNESCO (2024), which positions students as important arbiters of the validity of GenAI results; and the U.S. Department of Education (2023), which emphasizes strengthening human engagement rather than replacing it. There are four subdimensions within this dimension. In the lowest level, students do not verify the GenAI response, and there is no control structure with authority delegated to the technology. The next subdimension is weak/fragile control, in which students may ask the GenAI system questions in different ways but use its responses as their own without critically examining them. The third subdimension involves the student performing a form of mathematical work (e.g., estimation) to assess the validity of the GenAI answer, thereby demonstrating some degree of control. Within the last subdimension, students may evaluate the response in multiple ways and correct the GenAI response when it is incorrect, thereby demonstrating robust control.

Dimension D: Classroom Discourse and Justification with a Non-human Partner

This dimension addresses the gap in the previous technological frameworks identified by McCulloch et al. (2021) and the fact that in a GenAI environment, students engage in conversation with a non-human partner. The dimension comprises two subdimensions: the role of GenAI in discourse and the norms governing what counts as valid justification. There are four components of GenAI's role in discourse. 1) Proxy discourse where the GenAI provides reasoning. 2) Assisted discourse consists of paraphrasing the GenAI response and limited warrants. 3) Dialogic support consists of the student using the GenAI to test and refine while the student author warrants. 4) Argumentation environment consisting of warrants, rebuttals, and definitions, with GenAI playing a role, strengthens reasoning under classroom norms.

There are four subdimensions under norms. 1) Unspecified, where the student states that the GenAI agrees. 2) There are partial or inconsistent norms. For instance, a student may ask why but receive only a procedural explanation or ask a peer for an explanation but accept a GenAI response without making the same request. 3) Explicit criteria are used, consisting of definitions, warrants, validity checks, and representation coordination. 4) Under enforced or shared criteria, students critique GenAI and peers using agreed standards. The key is not just that criteria exist, but that they are collectively enacted. Students actively police their own arguments and revise them, including those generated by GenAI, until they meet the standards.

Dimension E: Noticing and Metacognitive Reflection

This dimension relates to the importance of metacognition (Schoenfeld, 1987) in interacting with GenAI while students work on a mathematical task. An important precursor to reflecting is noticing. Additionally, it highlights the importance of students reflecting on their work involving GenAI (OECD, 2025). There are four subdimensions within this dimension. First, students may

exhibit no reflection on an episode involving the GenAI. Second, students may provide only vague reflections, such as "I get it now." Third, the student identifies the misconception, insight, or shift in understanding, and describes how working with GenAI led to that change. Fourth, the student identifies a general strategy and applies it to a case related to the problem initially addressed using the technology.

Dimension F: Life-cycle Equity Awareness and Enactment

This dimension captures the equity issues across the GenAI lifecycle, as outlined above. The first subdimension concerns infrastructure/environmental burdens associated with GenAI. A student who recognizes this does not merely acknowledge the risks; they adjust their use of the technology to limit their interactions. The second subdimension involves understanding hidden labor, accountability, and privacy risks. For example, the student does not upload a photo of their quiz with their name on it; instead, they write out the steps associated with the quiz to receive feedback on those steps without jeopardizing their privacy. The third subdimension recognizes the bias associated with the use of GenAI in providing context for a mathematical idea (Gómez Marchant & Hardison, 2025) or in overgeneralizing a mathematical procedure. The last subdimension involves an opportunity to benefit from stratification, where the student understands and explicitly states that they have access to the paid version of a GenAI system, which provides them with more capabilities than their classmates, who only have access to a free version.

Dimension G: Instrumentation Effects and Technology Mediation

This dimension captures the mediated effects of GenAI on the student's mathematical activity and thinking. The first subdimension captures whether GenAI's involvement orients the episode toward doing mathematics (offloading/producing answers) versus learning mathematics (sense-making, explaining, connecting representations), consistent with the doing/learning distinction (Drijvers, 2012). The second subdimension examines how GenAI-mediated work affects the transparency of mathematical reasoning, drawing on black box, white box, and grey box perspectives (Buchberger, 1990; Cedillo & Kieran, 2003). The coding here is based on the student's own mathematical account and monitoring, not on the level of detail in GenAI's explanation. The last subdimension captures whether GenAI primarily amplifies performance (e.g., speed, formatting, reduced cognitive load) or reorganizes understanding (e.g., new representation, new schema, new connections), consistent with amplification/reorganization (Pea, 1985).

Dimension H: Trace-based Noticing of Student Thinking

This dimension examines the extent to which the GenAI conversational trace is interpretable as evidence of student thinking (subdimension H1) and whether teachers and students have a shared language to examine conversations between the student and the GenAI (subdimension H2). It addresses a gap identified by McCulloch et al. (2021) while aligning with the OECD (2025) by engaging with GenAI to evaluate output quality and to create with GenAI. Subdimension H1 contains four levels. The first level corresponds to thin or one-shot interactions, such as when a student asks for an answer, and the GenAI responds with it. At the second level, GenAI provides procedures for the student to verify. At the third level, the student shares steps with the GenAI, requests verification of these steps, and, if there is a mistake, works with the technology to determine the source of the error. In the final level, the GenAI and the student each propose steps and solutions to solve a problem, with the student ensuring they understand the technology's responses.

Subdimension H2 contains four levels: none, basic labels, shared distinctions, and instructional actionable. At the lowest level, the teacher notes that the student used ChatGPT, but there is no further parsing of how the student used GenAI. At a basic label level, the student states that they used GenAI for help, and the teacher responds with, "So, you got the steps from AI." The terminology used by the teacher and the student describes how GenAI was employed to solve a problem. For instance, the student may state that this part of the work was *delegated* (where the GenAI completed the algebraic manipulation for the student). In the final level, common terminology is used to determine the next set of interactions between the student and the GenAI, which are teacher-directed and interpreted by the student.

Dimension I: Mathematical Integrity of the Final Product

The last dimension examines the quality of the final mathematical product (solution and explanation) independent of GenAI authorship. This dimension differs from dimension C, which examines how correctness is established. Dimension I differs from dimension E, which examines how the student articulates changes in their understanding. Finally, this dimension differs from dimension G, which examines how GenAI mediates mathematical work. There are four levels to dimension I: (1) incorrect/unsupported, (2) partially correct, (3) correct with adequate justification, and (4) correct, justified, and communicatively strong (coherent, clear, and anticipates conditions/objections).

Illustrating the Framework

Maya, an Algebra 1 student with school-provided access to a free GenAI system (Dimension A), types a quadratic equation into the system rather than photographing the work and submitting that to the system, consistent with classroom privacy norms (Dimension F). She prompts, "Show me how to solve $x^2 - 6x + 2 = 0$ by completing the square and explain each step (Dimension B, explanation-provisioning role). The GenAI response contains a sign error. Maya substitutes the proposed root into the original equation, finds it fails, and prompts, "Check step 3. I think $(x - 3)^2$ gives +9, not –9." The technology provides a corrected response, which she verifies (Dimension C, robust control; H1, level 3, an interpretable trace of error-locating work). Asked to reflect, Maya writes, "It makes sense now" (Dimension E, vague reflection). The next day she tells her group, "the AI showed me," without stating the warrant herself (Dimension D, assisted discourse). Maya's solution is correct with adequate justification (Dimension I, level 3). In the episode, GenAI mediates learning rather than offloading (Dimension G, doing mathematics, white box). Maya exercises strong epistemic control privately while ceding authority publicly, illustrating the power of combining verification and discourse in the framework.

Conclusion

This analytical framework is an initial attempt to capture the capabilities of GenAI as a mathematical learning tool for students. It builds on prior frameworks, addresses gaps identified in them, such as a lack of equity, incorporates components from international competency frameworks around GenAI (e.g., OECD, 2025), and includes capabilities specific to GenAI. While this framework is GenAI-specific, it does not cover all GenAI capabilities, such as agentic AI, which refers to systems that can perform their own tasks with minimal supervision. GenAI is transforming rapidly, and as Mollick (2024) has written, "…whatever AI you are using right now is going to be the worst AI you will ever use" (p. 61). Consequently, any GenAI framework is likely to be revised; however, it provides a foundation on which others can build or renovate as needed. The unique natural-language interaction currently required of users suggests that mathematics educators may need to work with literacy specialists to analyze how students use GenAI to deepen their understanding and where they may struggle with this math learning tool.

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INTEGRATING COMPUTATIONAL THINKING IN ELEMENTARY MATHEMATICS: COMPARING TEACHER INTENTIONS AND RESEARCHER INTERPRETATIONS

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This study examines how elementary teachers conceptualize the depth of computational thinking (CT) integration in mathematics lessons they design. Eleven teachers enrolled in a graduate STEM education program course designed standards-aligned mathematics lessons incorporating coding. They categorized the level of CT integration with an exist-enhance-extend integration (Waterman et al., 2020). Using deductive content analysis, we analyzed the lesson designs and compared our categorizations with teachers' self-categorizations. Findings suggest that teachers often associate integration depth with the presence of code rather than opportunities for mathematical reasoning. Implications for professional development include explicitly supporting teachers in leveraging iteration, parameter variation, and modeling to deepen conceptual understanding of mathematics through computing.

Keywords: Computational Thinking, Computing and Coding, Elementary School Education, Professional Development

With the growing recognition that computational thinking (CT) is a cross-cutting skill that is best developed in STEM learning contexts (Grover, 2021), the mathematics education community has a timely opportunity to change perceptions of mathematics as procedures, skills, and facts that have been known for decades or even centuries (Li & Schoenfeld, 2019). There is a natural overlap between CT and mathematics in relation to abstraction, generalization, and modeling. (Silva et al., 2020; Strickland et al., 2021; Weintrop et al., 2016). This "mutually supportive potential of computing and mathematics" (Weintrop et al., 2022, p. 39) has only begun to be explored in elementary education (Wu & Yang, 2022).

CT in the form of coding can be an especially powerful medium for developing conceptual understanding of mathematics (Boote & Galanti, 2023; Ye et al., 2023). Yet efforts to integrate CT into school curricula remain largely compartmentalized, and teachers have limited opportunities to build their own confidence in learning and teaching coding (Mills et al., 2025). Most coding activities in elementary grades are not aligned with content standards or are taught as a standalone subject (Kakavas & Ugolini, 2019). When elementary teachers learn to facilitate students' use of computing as both a process and product of dynamic mathematical thinking (Cui et al., 2023; Dahshan & Galanti, 2024; Galanti, 2022, Weintrop et al., 2016), they can leverage the natural overlap between CT and mathematics for deeper learning.

Recent research on the preparation of elementary educators to leverage CT as a vehicle to enrich students' exploration of mathematical patterns and structure has largely focused on preservice teacher education (e.g., Dobgenski et al., 2025; Gleasman & Kim, 2020). There is also a need to understand how experienced elementary teachers perceive the integration of CT and mathematics learning in their own lesson designs. These insights can in turn inform more relevant professional development experiences that build teachers' abilities to integrate CT for the purpose of deepening mathematics learning (Kotsopoulos et al., 2019).

This vision for computing-integrated mathematics teaching and learning will not be realized without the development of teacher capacity to both design and facilitate these experiences

(Kong et al., 2020). This study focuses on elementary teachers' categorization of the level of CT integration in mathematics activities. The following research questions guided our study:

1. How do elementary teachers enrolled in a graduate-level CT course categorize the depth of CT integration in mathematics lesson design assignment with a coding component?
2. What similarities and differences emerge in how teachers and researchers interpret and apply an exist-enhance-extend framework (Waterman et al., 2020) when describing the depth of CT integration in mathematics lesson designs?

Theoretical Perspectives

While there is still no scholarly consensus on the definition of CT (Cansu & Cansu, 2019), there is general agreement on its close connection to the process of problem solving. CT has specific characteristics related to problem formulation; organization and analysis of data; use of abstractions such as models and simulations; efficiency of algorithmic solutions; and generalizability of solutions (International Society for Technology in Education & Computer Science Teachers Association, 2011). CT integration can emphasize both computing skills (e.g., sequencing, loops, conditionals, debugging) and learning processes (e.g., communication, creativity, exploration/engagement) (Israel & Lash, 2020). Students often learn mathematical concepts by memorizing procedural approaches with limited opportunities for logical thinking. Well-designed CT activities encourage students to make sense of mathematical concepts beyond procedural memorization and apply computational strategies to representing and deepening their mathematical sensemaking (Wang et al., 2022; Weintrop et al., 2016).

CT integration has potential to increase the depth of mathematical learning in PK-12 classrooms (Hjalmarson et al., 2020; Rich et al., 2020), but there is a need for evidence-based professional development experiences to build this CT integration capacity (Bull et al., 2020; English, 2018). Yeni and colleagues (2024) offered several recommendations for teacher professional development on CT integration, including an emphasis on lesson design aligning subject matter and computational principles. They also highlighted the value of collaboration between teachers and researchers to scaffold CT knowledge and to select an appropriate level of CT integration.

Most of the research on levels of CT integration in K-12 has taken a technology-centric approach by adapting Puentedura's (2011) Substitution, Augmentation, Modification, and Redefinition model. In this conceptualization, depth progresses from subject-specific and low-tech substitution and augmentation to more transformative, interdisciplinary transformation (Yeni et al., 2024). Grover's (2021) CTIntegration framework, informed by Kohler and Mishra's (2007) TPACK principles, focuses on teacher knowledge for design of K-12 CT integrated learning. This framework includes the subject domain and computing as distinct but overlapping fields informed by content, practices, and pedagogies in each of these fields. The application of this framework to examine the development of pedagogical knowledge for CT integration in teacher professional learning has elicited the tensions between teachers computing knowledge and their readiness to integrate CT into practice (Lane et al., 2023).

Waterman and colleagues (2020) conceptualized three levels of CT integration in the specific context of elementary mathematics and science education. This *exist-enhance-extend* framework has been applied to adapt existing district curriculum to develop CT skills and practices (Waterman et al., 2020) and to analyze teacher lesson design and implementation (Coenraad et al., 2022; Nolte et al., 2024). While prior research on this framework has focused on the assessment of existing curriculum or teacher-generated lesson plans, we see the important

potential to use these ideas in the design of professional development. By using the *exist-enhance-extend* framework as both an instructional resource and a reflective tool in professional development, teachers can build their capacity to integrate CT with intentionality in mathematics instruction.

In the paragraphs that follow, we elaborate on our conceptualization of the Waterman et al. (2020) framework in the specific context of elementary mathematics education.

Adaptation of Waterman's (2020) Framework for Levels of CT Integration in Mathematics

At the *exist* level, teachers may draw attention to CT concepts, skills, and practices that are embedded within mathematics instruction. Common mathematical practices such as decomposing a problem into steps, following and evaluating algorithms, or recognizing patterns, can be intentionally described as CT. For example, when students follow a standard procedure for solving an equation, the teacher may highlight this process as an algorithm and discuss how similar step-by-step instructions are required for a computer to perform calculations. No new tasks or technologies are required at this level; rather, the teacher highlights how CT is already present within a mathematical activity.

At the *enhance* level, teachers may add tasks or tools that support existing mathematics instruction without changing the mathematical questions students are asked to answer. Computing may be used to make problem solving or investigations more efficient, but tools do not necessarily create new opportunities for mathematical exploration. For example, students might program a computer to recreate a graph they have already made by hand or to create representations they will interpret using previously learned strategies. In these cases, engaging in CT practices such as abstraction, pattern recognition, and debugging may deepen students' mathematical understanding.

At the *extend* level, opportunities for mathematical reasoning are fundamentally changed because of the integration of computing. Teachers may design programming activities in which students can explore and apply mathematical concepts in ways that would not be possible without a computer. Students learn mathematics and CT more deeply in a mutually beneficial way (Weintrop et al., 2022) by creating, testing, and refining computational models of mathematical ideas. For instance, students might write programs to model multiplication as repeated addition or simulate probabilistic events. They might translate mathematical relationships into algorithms and debug their reasoning when results do not match expectations. These experiences extend mathematics instruction beyond traditional representations, positioning students to engage with mathematics as a dynamic, computationally mediated discipline.

Coenraad and colleagues (2022) suggested "exhibit" as a fourth level of CT integration based on their analysis of practicing teachers' CT-integrated science lessons. This level is characterized by the use of block-based programming to animate a science concept. Since the integration of programming only served as a way of presenting knowledge and not as a way of deepening science learning, it did not fit within any of the three established levels.

Although the exist-enhance-extend framework provides shared terminology for describing the depth of CT integration, exposure to this framework in professional learning does not necessarily ensure shared conceptualizations of what constitutes meaningful integration. Teachers and researchers may bring different assumptions informed by prior beliefs and experiences with mathematics, computing, and curriculum. For example, teachers may emphasize the presence of code, while researchers may focus on how computing deepens opportunities for mathematical reasoning. Examining how teachers interpret and apply CT

integration frameworks in comparison with researcher interpretations can inform professional learning and better support teachers in leveraging CT for mathematical sensemaking.

In this study, we chose to employ the "*exist-enhance-extend*" framework as a structure for lesson design, teacher reflection, and research analysis. With these multiple perspectives, we sought to understand the ways in which teachers might perceive CT integration differently than mathematics education researchers.

Methods

In a Summer 2025 synchronous online CT course for elementary teachers who were enrolled in a STEM+C graduate certificate program, teachers (n = 11) collaboratively explored the relationship between CT and mathematics learning in a series of instructor provided mathematics tasks integrating both unplugged and plugged CT approaches (e.g., BeeBots and Scratch block-based programming). As part of their reflection on their experience with these tasks, teachers analyzed CT concepts and practices (Grover & Pea, 2018) and the Standards for Mathematical Practice (NGA & CCSSO, 2010) for conceptual overlap.

The unit of analysis for this study was the CT-integrated mathematics lesson design created by teachers during Week 7 of the 12-week course. Prior to creating their lessons, teachers read two chapters from *Computational Thinking in Pre-K: Empirical Evidence for Integration and Future Directions* to learn about levels of CT integration in Grade 5 science lessons (Coenraad et al., 2022) and CT integration in Grade 4 mathematics (Weintrop et al., 2022). Teachers also researched an integrated CT lesson from the freely available Digital Promise library. They categorized the level of CT integration in their chosen lesson using descriptions of *exist*, *enhance*, and *extend* levels of CT integration. They further reflected on how they would teach the relevant subject area differently because of the CT integration.

For the integrated CT and mathematics lesson design assignment, teachers were prompted to design a mathematics activity (30–45 minutes) that emphasized the Standards of Mathematical Practice, CT, and coding. In the activity planning template, they were provided with a set of design scenarios to illustrate varied levels of CT integration. In addition to identifying relevant CT components (pattern recognition, abstraction, decomposition, and algorithms) and programming skills (iterative development, debugging) in their activities, they categorized the level of CT integration using the *exist-enhance-extend* framework.

The lesson design template also included a Launch-Explore-Summarize model structure that supports inquiry and mathematical sensemaking. Teachers had the choice of what coding tools to integrate. Two of the teachers integrated Bee-Bots in Grade K-3 mathematics lessons, while the other nine teachers integrated Scratch in Grades K-5 lessons. All teachers who integrated Scratch used starter code provided by the instructor or from CT education websites.

Using a deductive content analysis approach (Elo & Kyngäs, 2008), we collaboratively classified the lesson design artifacts using the levels of CT integration as apriori codes. The first author, the course instructor, initially assigned a CT integration level to each of the lesson designs, using both the template content and the formative feedback that she provided when she graded the assignments to support her inferences. She assigned her level based on lesson text coupled with identifiable entry points to deepen either CT or mathematical reasoning within the lesson design. The second author independently assigned integration levels. Finally, we collaboratively discussed our categorizations, comparing them to the teachers' categorizations to identify patterns and emergent themes. Trustworthiness was supported through collaborative coding, consensus-building discussions, and documentation of our interpretations.

Results

We identified five distinct patterns in teacher and researcher categorizations of CT integration (see Table 1). Our categorizations of the levels of CT integration agreed with teachers' categorizations in five of the 11 lesson designs, with two cases of agreement at the *enhance* level and three cases of agreement at the *extend* level.

Table 1: Patterns of Alignment Between Teacher and Researcher Categorizations of CT Integration

Teacher	Researcher	Number of Lessons
Enhance	Enhance	2
Extend	Extend	3
Enhance	Exhibit	2
Exist	Potential to Enhance	1
Enhance	Potential to Extend	3

In the other six lesson designs, three additional patterns emerged. We found two cases of teachers planning to use code for students to display mathematical knowledge and four cases of teachers understating the integration potential of coding for mathematics learning. Across these four cases, we found differences in teachers' descriptions of their mathematics learning goals and the potential for deepening conceptual understanding of mathematics through coding.

In the sections that follow, we highlight cases from three different mathematics domains to surface how teachers designed lessons at different levels of integration and to highlight how teachers might understate the mathematical power of computational integration.

Grade 3 Number and Operations Lesson (Enhance → Potential to Extend)

In this mathematics lesson, Caryn (Grade 3 teacher) described her learning goals as reinforcing number patterns and fact fluency through coding. She planned to prompt students to create a "Multiplication Machine" in Scratch that would model multiplication by taking two numbers as input and displaying the product. She classified the level of CT integration as *enhancing* their mathematics learning by "showing their understanding of multiplication as a process as they combine programming to compute how to multiply." She emphasized her motivation to use the "Multiplication Machine" to increase students' multiplication fact fluency.

Caryn also provided a link to an instructor-provided Scratch project (see Figure 1) automating the generation of multiplication facts with meaningful variable names, understanding of counting by one, and skip counting by three in expressing sequences of multiplication Students could also apply their understanding of multiplication tables to generate sequences of products. However, this code was not explicitly integrated in her planned lesson enactment.

Caryn's expectations for her students did not leverage opportunities to demonstrate conceptual understandings of multiplication through coding. In the Launch phase of her lesson design, she planned to model the coding for the "Multiplication Machine" in a different way than in the coding project she referenced. She would show her students how to use four blocks of code to input two numbers and multiply two numbers to display a product. She would then release her students to work in pairs to create their own projects.

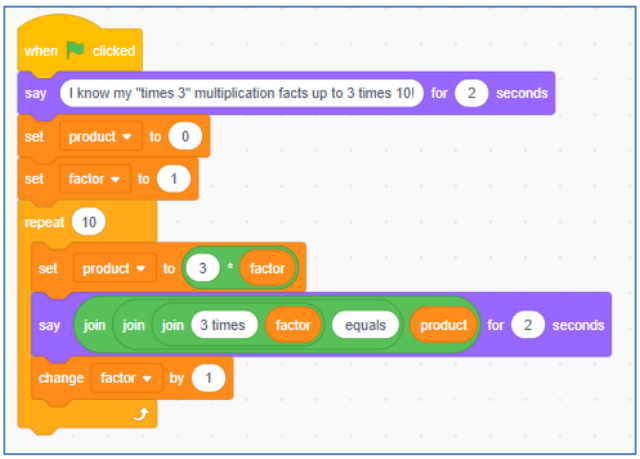


Figure 1: Instructor-generated Scratch Project Modeling Multiplication Patterns

We described the CT integration potential within this lesson as *extending* because of the opportunities to explore multiplicative patterns and structure through repetition. In lieu of a demonstration of code as a calculator for isolated multiplication facts, Caryn could provide students with a subset of the blocks in the instructor's example project. With purposeful scaffolding, students could then connect coding to reasoning about patterns in multiplication. They could also create equal groups with linking cubes, number lines, or rectangular arrays to connect the functionality of their code to multiplicative structure and patterns. This emphasis on iteration as a CT skill and connections to mathematics representations within Caryn's lesson design would raise the integration potential from *enhancing* to *extending*.

Grade 4 Geometry Lesson (Enhance → Potential to Extend)

Paula's Grade 4 "Drawing Polygons with Code" lesson supported her learning goal of understanding the relationship between angle measurements and polygons. She classified the level of CT integration as *enhancing* mathematics learning because students were measuring angles and connecting angles to turn in their code. During the Launch phase of her lesson, she planned to introduce her students to the concept of exterior angles and model how knowledge of exterior angles could be used to draw a square in Scratch coding. During the Explore phase of the lesson, students would use protractors to measure exterior angles on other polygons and then modify the square drawing Scratch code (See Figure 2) to create regular triangles, pentagons, hexagons and octagons. Paula also planned to lead a whole class conversation about the shapes that the students made and how they debugged their code to draw their shapes.

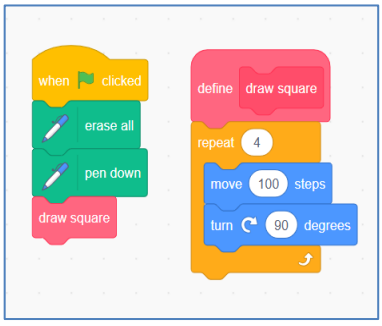


Figure 2: Instructor-generated Scratch Project Modeling Shape Patterns

Because students are modifying the repeat parameters and the amount of turn in the starter Scratch project, they can understand angle not only as a static quantity to be measured with a protractor but also as units of rotation. Because the actions in the code can deepen students' concept of angle and make more advanced geometry concepts (e.g., interior and exterior angles, transformations) more accessible, we classified the CT integration potential in this lesson as *extending*. We noted that in Paula's lesson design, some students might simply replace the amount of turn in the coding block with the exterior angle they measured on the worksheet without understanding the concept of rotations. Paula could make use of paper plate spinners, pattern blocks, or physical motion to scaffold these connections.

Grade 3 Patterns and Data Lesson (Extend → Extend)

Laine integrated both Scratch coding and Excel spreadsheet formulas in her "Pattern Detectives" lesson. Her goal was for students to learn to model arithmetic patterns through code, test pattern rules, and interpret real-world data trends in tables and graphs. Her opening question was "Can a computer recognize patterns like you can?" She provided a starter Scratch project in the Launch phase of her lesson and prompted pairs of students to modify the parameters in the Scratch project to create repeating number patterns. She would later contrast these experiences with predictable, automated patterns (constant differences) with the natural patterns and trends that they might find in real-world weather patterns (varying differences). The teacher would model the use of subtraction formulas in the Excel spreadsheet to evaluate varying differences. Her description of the depth of CT integration connected these two computing opportunities.

Laine categorized her lesson as *extending* the mathematics concepts by integrating programming and spreadsheet tools to explore patterns in data over time. Rather than just identifying patterns on paper, students would use Scratch to simulate numerical sequences and spreadsheets and analyze real-world data, applying CT practices like abstraction, debugging, and automation.

We also described the purposeful pairing of generating patterns in Scratch programming and looking for patterns in temperature data as *extending* the mathematics learning opportunities. Laine's sequencing of pattern explorations created opportunities for rich conversations about constant differences and varying differences from one data point to the next. We also noted that Laine's students could also use the Excel spreadsheet to plot the data from the Scratch project alongside the weather data, creating rich conversations about patterns and sequencing.

Discussion and Implications

Our analysis of teachers' categorizations of the interplay between CT and mathematical sensemaking revealed variation in the ways that teachers saw the purpose of code and how it related to mathematics learning. Several teachers seemed to categorize their CT integration as *enhancing* mathematics learning because students could see mathematics concepts through code. In Caryn's case, we wondered if she perceived coding as a means of making mathematics more engaging as opposed to deepening conceptual understanding of multiplication. She did not seem to see code as a tool for reasoning about mathematical relationships beyond efficient calculation of answers. In contrast, our categorizations focused on how coding might deepen opportunities to model mathematical structure, vary mathematical parameters, and reason about patterns in new ways. These categorizations are consistent with prior research on how CT practices can serve as productive entry points into mathematical reasoning about structure and generalization (Silva et al., 2020; Yilmaz et al., 2024; Wang et al., 2022).

We also noted how expressed learning goals for the lesson related to teachers' categorizations. Caryn and Paula were focused on coding to achieve mathematical products of

calculating multiplication expressions and drawing shapes correctly. We as researchers saw the potential for exploring patterns and rotations through the process of coding. Laine used her students' experiences with coding numerical patterns to build a conceptual foundation for her learning goal related to investigating patterns in real world data. In her case, the code opened the door to more meaningful mathematical conversations. Across all three cases, we envisioned ways in which physical or graphical representations coding output could create further opportunities for reasoning about mathematics through coding.

Prior application of the *exist-enhance-extend* framework for CT integration has focused primarily on researcher analysis of lesson design and enactment (Coenraad et al., 2022; Nolte et al., 2024). In our study, the framework functioned simultaneously as an instructional tool, a reflective scaffold for teachers, and an analytic tool for researchers. This positioning surfaced differences in how teachers and researchers conceptualized the same lesson designs. Teachers often emphasized coding and engagement, while researchers more often focused on opportunities for mathematical modeling, parameter variation, and conceptual reasoning. These differences are best understood as entry points for professional learning design. Making these differing interpretations explicit may help mathematics teacher educators better scaffold teachers' noticing of how coding can deepen mathematical understanding. At the same time, researchers can learn from teachers' practical perspectives on feasibility and classroom enactment when designing CT-integrated learning experiences.

As mathematics teacher educators who are excited about the potential realizing the mutual benefits of CT and mathematical thinking in elementary classrooms, we have new questions about how to better scaffold teachers' facilitation of deep mathematical learning through code. We recognize that we need to do more than offer coding experiences and a library of starter Scratch projects tied to elementary mathematics content standards. Consistent with Grover's (2021) CT Integration framework, teachers' realization of CT integration potential may be as much of a reflection of the depth of their mathematics knowledge for teaching as it is of the depth of their knowledge of coding. Distinguishing between using code as a more engaging way of doing mathematics and using computing to model mathematical ideas should be an explicit focus in teacher professional learning. Articulating a learning goal that emphasizes how CT skills and practices can deepen explorations of mathematics concepts is also critical to realizing the integration potential.

The juxtaposition of teachers' and researchers' interpretations of CT integration frameworks in this study advances knowledge about the design of professional development for elementary teachers. Future research on teachers' use of the *exist-enhance-extend* framework for CT integration should focus not only on lesson design artifacts but also on enacted classroom instruction. By examining the differences between intended and enacted CT integration, mathematics teacher educators can more productively support teachers in leveraging coding not merely as an engaging activity but as a vehicle for deepening students' mathematical sensemaking.

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UNDERGRADUATE MATHEMATICS FACULTY NARRATIVES ABOUT GENAI INTEGRATION

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As generative artificial intelligence (GenAI) reshapes higher education, undergraduate mathematics faculty face new questions about integrating it into instruction. This mixed-methods study examines how 41 faculty describe their GenAI experiences using a framework integrating the Technology Acceptance Model and the Will, Skill, Tool model. We collected concurrent Likert-scale and open-ended survey data. Results indicate that faculty are generally open to implementing GenAI into teaching but report low confidence and institutional support needs. Even among those who have already redesigned aspects of their teaching, hesitancy remains due to concerns at the individual, student, and institutional levels.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs; Teacher Noticing; Technology

Introduction and Literature

With a rapid emergence of generative artificial intelligence (GenAI) since 2022, teaching and learning have been impacted across disciplines and educational levels in various ways. While GenAI offers personalized, adaptive feedback that can benefit students and teachers (Pardos & Bhandari, 2024; Wardat et al., 2023), concerns persist around inaccurate outputs, overreliance, and academic integrity (Lim et al., 2023; Michel-Villarreal et al., 2023; Yoon et al., 2024).

In secondary mathematics, it has become increasingly common for students to use GenAI (Kasneci et al., 2023). Some studies reported that GenAI benefits them. For example, middle school students treat GenAI as a learning companion and facilitator, rather than merely an answer machine, resulting in larger improvements in their mathematical knowledge (Song et al., 2024; Xing et al., 2025). Wu et al. (2024), through their mathematics classroom experiment, found that high school students who used ChatGPT outperformed those who did not use it. However, other studies reported contrasting findings. Bastani et al. (2025) found that when high school students used ChatGPT, they often relied on complete solutions, which hindered skill development. Moreover, researchers have pointed out that K-12 students do not often recognize or understand the ethical issues related to using AI (Gouseti et al., 2025).

In undergraduate mathematics, GenAI presents both opportunities and unique challenges compared with secondary education given the more abstract and rigorous content. Regarding opportunities, Pardos and Bhandari (2024) found that ChatGPT-generated hints produced greater learning improvements than no assistance or human tutoring, while Gouia-Zarrad and Gunn (2024) reported improved coding skills and engagement when students used ChatGPT in a differential equations course. Liu et al. (2026), in a meta-analysis of 22 empirical studies, reported a moderately positive impact on learning outcomes. However, Wang et al. (2024) found that extensive AI use widens the gap between AI-supported performance and actual mastery, as students are less likely to internalize concepts. Similarly, Yoon et al. (2024) and Klawa et al.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

(2025) noted that students often accept AI responses without critical evaluation and struggle to balance AI assistance with the cognitive demands of proof construction.

Based on our review, research on undergraduate mathematics faculty use of GenAI remains limited compared to the studies focused on students. To address this gap, this study explores undergraduate mathematics faculty experiences and perspectives on GenAI use, guided by the research question: How do undergraduate mathematics faculty perceive GenAI integration?

Theoretical Framing

This study is guided by two complementary theoretical frameworks: the Technology Acceptance Model (TAM; Davis, 1989) and the Will-Skill-Tool model (WST; Petko, 2012). The TAM emphasizes two key beliefs shaping technology integration: perceived usefulness (PU), the degree to which users believe technology improves performance, and perceived ease of use (PEU), the degree to which users believe technology requires minimal effort (Venkatesh & Bala, 2008). WST outlines three conditions for successful technology integration: Will (teachers' motivation, attitudes, and beliefs), Skill (teachers' technical and pedagogical competence), and Tool (access to technology, infrastructure, and institutional support), all of which must be present simultaneously.

Eight Attitude Dimensions (Table 1)

Based on TAM and WST, we identified eight attitude dimensions (Joung et al., 2026): (a) Openness/Curiosity (TAM's PU), reflecting faculty interest in GenAI's teaching potential; (b) Student Impact Concerns (TAM's PU), capturing faculty beliefs about GenAI's effect on student learning; (c) Integration Confidence (TAM's PEU), reflecting faculty belief that GenAI integration will be manageable; (d) Traditional Methods Preference and (e) Curriculum Modification Willingness (WST's Will), capturing faculty motivation toward or resistance against changing established practices; (f) Equity/Access Concerns (WST's Will and Tool), reflecting faculty values about fairness and equitable resource access; (g) Institutional Support Needs (WST's Tool), encompassing required training, policies, and infrastructure; and (h) Professional Practice Influence (TAM's actual system use and WST's integration success), capturing behavioral changes associated with faculty attitudes. This framework guided survey development and served as an analytic lens for our study.

Table 1: Eight Attitude Dimensions Aligned with TAM and WST Frameworks

Study Dimension	TAM Alignment	WST Alignment
1. Openness/Curiosity	Perceived Usefulness	WILL
2. Student Impact Concerns	Perceived Usefulness & Attitude Formation	WILL
3. Integration Confidence	Perceived Ease of Use	SKILL
4. Traditional Methods Preference	Attitude Formation	WILL
5. Curriculum Modification Willingness	Behavioral Intention	WILL
6. Equity/Access Concerns	Attitude Formation	WILL & TOOL
7. Institutional Support Needs	External Variables	TOOL
8. Professional Practice Influence	Actual System Use	Integration Success

Methods

This study employed a convergent mixed-methods design (Creswell & Plano Clark, 2018), in which quantitative (Likert-scale) and qualitative data (open-ended) were collected concurrently

through a single survey instrument. Quantitative data were analyzed by examining statistically significant patterns in participants' responses. Qualitative data were analyzed using structured content analysis (Schreier, 2012), involving iterative coding and category development.

Description of Survey

Quantitative items were developed across eight dimensions: (a) openness/curiosity, (b) student impact concerns, (c) confidence to integrate, (d) preference for traditional methods, (e) curriculum modification willingness, (f) equity/accessibility concerns, (g) institutional support needs, and (h) influence on professional practice, with 3–7 items each (total = 37 items). Each item used a 5-point Likert scale (1 = *strongly disagree*, 2 = *disagree*, 3 = *neutral*, 4 = *agree*, 5 = *strongly agree*). Example items included "I am interested in using GenAI in my mathematics courses" (openness/curiosity), "I worry that GenAI might hurt students' ability to solve math problems on their own" (student impact concerns), and "My attitudes toward GenAI have influenced how I teach" (influence on professional practice).

Qualitative items consisted of five open-ended questions that asked faculty to (a) describe overall thoughts on GenAI in mathematics classes, (b) explain comfort level and readiness factors, (c) discuss potential curricular changes and concerns, (d) identify needed institutional support, and (e) provide examples of how their attitudes have influenced teaching practice. Content validity was established through review by three faculty outside the research team, and several items were revised based on their feedback.

Data Collection

Data were collected over a three-week period in Fall 2025 at a large public university in the western United States. The institution offers a wide range of mathematics courses, from developmental to upper-division levels from two mathematics departments (the Mathematics Department and the Mathematical and Quantitative Reasoning Department). Forty-one ($N = 41$) faculty members from these two mathematics departments voluntarily participated in an online survey via Qualtrics. Participants reported teaching a variety of courses, including developmental math and quantitative reasoning, college algebra, statistics, calculus, differential equations, and teacher education courses. They also indicated experience teaching in multiple instructional formats of in-person, online, and hybrid. They completed the two parts of a single survey in 15-20 minutes.

Data Analysis

All 41 participants completed the Likert-scale items. Data were screened for outliers and distributional assumptions prior to statistical analysis. Descriptive statistics (means and standard deviations), effect sizes (Cohen's d), and Pearson correlation coefficients were calculated. Negatively worded items were reverse-coded using the "six minus original response score" method to ensure directional consistency within each dimension, and all dimensions met acceptable reliability standards (Nunnally & Bernstein, 1994). A multiple regression analysis examined relationships among dimensions and predictors of professional practice influence.

Qualitative response rates ranged from $N = 21$ to $N = 34$ across the five open-ended items. Using an inductive-deductive content analysis approach (Schreier, 2012), each author independently coded responses line by line, then grouped similar codes into categories. Authors met to discuss and resolve discrepancies across multiple meetings. Inter-rater reliability was assessed using Fleiss' kappa (Fleiss, 1971), ranging from .424 to .904 (mean $\kappa = .743$).

Results

This section presents summary measures of faculty experiences and views about GenAI integration and faculty narratives reflected in the open-ended responses.

Faculty Views about GenAI Integration Reflected on Likert-scale

Table 2 shows descriptive statistics and internal consistency for the eight dimensions. Dimensions 1 and 7 have mean values (*M*) around 3.5 or higher with effect sizes (*d*) ranging from 0.4 to 0.9, suggesting that faculty members are highly open and curious about integrating GenAI into their teaching and express a strong need for institutional support. Dimensions 3, 5, and 8 indicate that faculty members have moderate confidence in integrating GenAI, are neutrally willing to modify their curriculum, and perceive that their attitudes have had little impact on their teaching practices. For the reverse-coded dimensions of 2, 4, and 6, faculty members reported a relatively strong preference for traditional teaching methods, moderate to high concern about the impact of GenAI on students, and relatively low concern regarding student equity or accessibility.

Table 2: Descriptive Statistics and Internal Reliability for Eight Dimensions

Dimensions	*M*	*SD*	*d*	No. of Items
1. Openness/Curiosity	3.46	1.16	0.40	4
2. Student Impact Concerns*	2.29	1.16	0.61	5
3. Confidence to Integrate	2.85	1.11	0.14	6
4. Preference for Traditional Methods*	2.27	0.92	0.79	3
5. Curriculum Modification Willingness	3.10	1.15	0.09	4
6. Equity/Access Concerns*	2.69	1.11	0.28	4
7. Institutional Support Needs	3.86	0.96	0.90	4
8. Influence on Professional Practice	2.95	1.13	0.04	7

Note. $N = 41$ participants with complete data. *d* = Cohen's *d*. *Reverse-coded dimensions.

Pearson correlation coefficients (*r*) indicated that faculty high in openness/curiosity (D1) were less likely to prefer traditional methods (D4), more willing to modify their curriculum (D5), and more likely to perceive GenAI as influencing their teaching (D8). Faculty preferring traditional methods (D4) were also less likely to modify their curriculum (D5).

A multiple regression analysis examined how the seven other attitude dimensions predict faculty perceived influence of GenAI integration (D8). Although the overall model was statistically significant, only openness/curiosity (D1) emerged as a significant predictor, suggesting that faculty openness/curiosity toward GenAI is the primary factor shaping their perception of its impact on teaching practices, while the remaining dimensions did not contribute significantly. For the significant dimension, openness and curiosity, the standardized regression coefficient was $\beta = 0.5$ ($t = 0.73$, $p = 0.013 < 0.05$), indicating a moderate to large effect. This suggests that the more open and curious faculty members are about GenAI integration, the greater the perceived influence on their professional practice.

Faculty Views about GenAI Integration Reflected on Narratives

In the open-ended section, faculty participants described their perspectives in response to five questions. These included faculty members' (a) overall views on integrating GenAI into their teaching, (b) their comfort levels with using it, (c) the types of changes they would make in the future, (d) the kinds of institutional support they hope to receive, and (e) the ways in which their experience with GenAI so far have impacted or are likely to impact their teaching.

Efficient tool for faculty, efficient but potentially harmful for students. Table 3 summarizes faculty participants' responses to Question 1. The largest portion of participants

expressed *concerns* about integrating GenAI into their teaching. Some pointed out that GenAI-generated outputs can be inaccurate: "The danger for students using AI in math is that it can frequently get things wrong." Others noted that students may over rely on it: "I do worry some about them becoming too dependent on AI. If students use too much AI as a crutch it will hurt them…" Additional responses raised academic integrity issues: "My mean [main] concern is not knowing if a student used it to cheat on an exam and that it make[s] cheating easier and harder to detect." These concerns center on potential impact of GenAI on student critical thinking development.

Table 3: Faculty Attitudes Toward GenAI Use in Math Instruction

Q1: Tell me about your thoughts on using AI in your math classes. Are you excited about it, worried about it, or somewhere in between? Why do you feel this way?		
Concerned	38.3%	Potential inaccurate outputs; Student overreliance; Academic integrity; Reduced development of student critical thinking
Mixed feelings	26.5%	Recognized instructional benefits for teaching while concerned about negative learning impacts
Excited	17.6%	More efficient teaching; More efficient learning
Undecided	17.6%	Uncertain how GenAI could be integrated into instruction

Other faculty members expressed *excitement* about GenAI integration. They viewed GenAI as an efficient teaching and learning tool: "we had a virtual assistant named Sage. I think that it was valuable to the students to be able to get feedback when they needed it." Similarly, one participant noted, "AI chatbots in education is that they create a low-pressure space for learning." Some faculty also described using GenAI for their own professional learning: "Excited … I've used it personally to delve into subjects." Responses categorized as *Mixed Feelings* included both concerns about potential hindrances to student learning and recognition of GenAI efficiency as a tool.

Comfort for teaching preparation but discomfort in class due to unreliability. Table 4 summarizes faculty participants' responses to Question 2. The largest portion of participants expressed *discomfort* with integrating GenAI into their teaching due to concerns about its unreliability. Some pointed out that GenAI-generated outputs can be unpredictable: "GenAI can take you in directions you don't plan on going." Others noted concerns about inaccuracy: "proven techniques can help students develop conceptual understanding of material better than interacting with something I don't trust to be accurate." Similarly, one participant stated, "GenAI is too unreliable in explaining or working through math problems to confidently use them for those purposes."

A close proportion of faculty participants expressed *comfort* with using GenAI, primarily for class preparation and exploration of deeper contents. Example quotes include: "I am comfortable when I use GenAI to help me explore new topics, new ideas, and new perspectives." Another noted, "Using AI in the classroom is a powerful way to generate examples and clarify concepts." However, many of those who expressed comfort also emphasized the need for additional training: "I am comfortable, but we need to be trained first." Others expressed concerns about students' ability to use it effectively: "[I am] very comfortable. I don't trust the students to use AI effectively."

Table 4: Faculty Comfort Levels in Using GenAI for Teaching

Q2: How comfortable do you feel actually using GenAI when you teach? What makes you feel ready or not ready to try them?		
Uncomfortable	41.4%	Concerns about inaccurate AI-generated outputs; Unpredictability of system responses; Potential negative impact on student critical thinking development
Comfortable	37.9%	Effective tool; Comfortable but seeking additional training
Developing comfort	6.9%	Increasing familiarity but limited integration experience
Neutral	6.9%	Neither positive nor negative stance toward AI integration
No use (N/A)	10.3%	No prior experience using GenAI in teaching

Note. Percentages do not sum to 100% as a few participants reported feeling both comfortable and uncomfortable.

Wanting to change teaching materials and assessments once confident to do so. Table 5 summarizes faculty participants' responses to Question 3. Two response groups accounted for the largest proportions; one expressed interest in changing *teaching materials*, while the other reported *lacking confidence* in making changes.

Table 5: Faculty Proposed Future Changes in GenAI Integration

Q3: If you had to integrate GenAI to one of your math courses, what would you change?		
Teaching materials	29.6%	In-class activities; Tasks
No confidence	29.6%	Training needed; Not convinced to change with AI yet
Assessment methods	22.2%	No more take-home exams; Essay type assessment
Enhancing AI literacy	14.8%	Evaluating AI generated outputs; Correcting results from AI
No change needed	3.7%	GenAI as just a tool

Those who indicated a willingness to modify their teaching materials mentioned revising in-class activities, tasks, and assignment types. For example, one participant noted, "AI use would be best in mathematical discussions or even anything that involves writing about mathematical processes/explanations." Another intended adjusting homework structures: "we could adjust homework so that students do one problem of each type, and work with an AI to give them new problems until they get one right by themselves."

In contrast, an equally large group expressed a lack of confidence, pointing to the need for training and concrete examples before making changes or being convinced that GenAI would benefit students. The following quotes reflect this perspective: "I don't know enough to know what I would change." "I need to see some options and ideas." "Other than my using it to prepare materials, I wouldn't know what is possible."

Additionally, some faculty participants stated that they would change *assessment* type by replacing take-home portion with in-person and assigning more essay-type exams. Another group noted that they would leverage GenAI imperfections to help students *develop AI literacy*. Example quotes include: "I might use it as something where students have to evaluate its correctness." "They need to look at the results of AI and critically assess it for errors."

Need guidance to help us know the direction. Table 6 summarizes faculty participants' responses to Question 4. The two largest groups of faculty expressed differing degrees of *need for institutional support* in integrating GenAI into their teaching: one group emphasized the need

for clear rigid rules, while the other preferred flexible guidelines. Both groups noted that current institutional guidelines are either lacking or confusing with comments such as, "The institution's guidelines regarding AI are currently unclear. It appears that the decision to use AI in coursework is left entirely to the instructors' discretion" and "Official statements from the university/department are helpful to include in the expectations set forth in my syllabus." Faculty in these groups also highlighted the need for guidance on handling students' unethical use of GenAI, for example: "I would like clear, university policy clarifying ethical and unethical uses of GenAI."

Table 6: Institutional Support Needed for GenAI Integration

Q4: What kind of institutional support (policies, guidelines, protocols) would you need from your institution to start using GenAI in your teaching?		
Clear policy & guidelines	52.4%	To handle ethical issue
Flexible policy & guidelines	14.3%	General guidelines needed applicable across different courses;
No need of guidelines	9.5%	Individual instructor in control; No need of institutional guides
Training	9.5%	Training to implement GenAI
N/A	14.3%	No responses

A smaller portion of responses expressed *the need for training focused specifically on teaching practices*. These participants voiced interest in learning ways to implement GenAI effectively in their instruction and in receiving teaching support from the department or institution. Example quotes include: "Advice on what types of assignments to include." And "More of training how to implement the use of GenAI in the classroom [,] then policies, guidelines, and protocols can follow."

Have redesigned assignment, assessment, and responsible GenAI use. Table 7 summarizes faculty participants' responses to Question 5. The response groups were roughly equal in size. One group reported modifying *assignments and/or assessments* so that GenAI can benefit students or, GenAI cannot hinder student learning. These changes include revising testing policies, requiring assignments on paper, eliminating take-home exams, and incorporating more challenging and creative tasks. The following quotes illustrate this perspective: "It has helped me reshape my assignments as well as give me more ideas on how to assess students using more creative projects," and "One thing I've done is make sure that homework assignments cannot be completed through the use of AI, since it is so easy for students to type in the prompt and receive an answer."

Reflecting concerns about students' overreliance on GenAI or the potential inaccuracies that may reduce student critical thinking, some faculty reported spending time discussing what makes *responsible GenAI use* and how to use it responsibly by emphasizing the importance of developing students' mathematical reasoning in class. For example, one noted, "Teaching students how to use AI effectively to learn and warning them off the copy-paste situation," and another said, "occasionally recommending students talk to GenAI about topics as a tutoring resource."

Table 7: Changes (Past and Future) in Teaching Practices Due To GenAI

Q5: Give me an example of how your feelings about GenAI *have already changed* something you do in teaching, or how they *might change* something in the future.		
Assignments and assessment	31.8%	Modifying assignments; Modifying assessments
Guide Students' AI Use	22.7%	Spending time to discuss how to use AI responsibly; Restricting GenAI access during tests
AI as a prep/grading tool	22.7%	Checking own work; Grading student work
No changes	27.3%	No changes to my teaching

Another group of faculty mentioned that GenAI has changed *the way they prepare for class*, such as generating examples or identifying potential errors. Comments included: "I mostly just use it to check things I've made to identify errors more efficiently," and "I use GenAI to check my work and to show students mistakes it makes, which helps them learn how to reason through problems."

Discussion

Based on the quantitative and qualitative analyses, we found that undergraduate mathematics faculty are generally open/curious to integrating GenAI into their teaching; however, they do not appear fully convinced of its effectiveness. Factors shaping the views can be interpreted at three levels: Individual, student, and institutional levels.

Individual Level

Faculty expressed discomfort stemming from GenAI's unpredictability in classroom settings and concerns about potential unethical student use. Those with limited experience hesitated to implement it and indicated a need for training to better understand available tools and how AI might function in their courses. These findings align with TAM's Perceived Ease of Use (PEU) regarding AI's unpredictability, and with the WST model's Will component, reflecting hesitancy in addressing ethical issues, and Skill component, reflecting the need for training to effectively implement AI.

Student Level

Faculty expressed strong concerns that GenAI may reduce student critical thinking due to overreliance and GenAI-generated inaccuracies. They are also worried about unethical student use on assignments and assessments. As GenAI emerges as a new authority figure in mathematics learning- replacing or competing with faculty, problems arise when students accept AI output without critical evaluation, raising issues of academic integrity. This level heavily relates to the TAM's Perceived Usefulness (PU), as faculty question whether GenAI supports student learning. It also connects to the WST model's Will component, as uncertainty about its pedagogical value reduces faculty willingness to integrate it.

Institutional Level

Across both Likert scale and open-ended responses, faculty indicated a need for institutional support, including policies and guidelines addressing ethical and pedagogical issues identified at the individual and student levels. For example, faculty reported lacking departmental or institutional protocols for handling unethical student use, as well as adequate resources and systemic support for course redesign. These findings align with TAM's Perceived Ease of Use (PEU), as unclear or absent guidelines complicate implementation and limit practical options, and with all three WST model components, as insufficient institutional support reduces

willingness to integrate AI (Will), limits skill development (Skill), and constrains access to appropriate tools and resources (Tool).

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PROBLEM-SOLVING APPROACHES THROUGH THE COORDINATED USE OF GEN-AI AND GEOGEBRA

ESTRATEGIAS DE RESOLUCIÓN DE PROBLEMAS MEDIANTE EL USO COORDINADO DE IAG Y GEOGEBRA

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This exploratory study analyzes the coordinated use of Generative Artificial Intelligence (GenAI) and GeoGebra in solving geometric problems. Through an a priori analysis of the problems, it examines how the articulation of the affordances of both tools facilitates the transition from visual and empirical exploration to formal geometric argumentation. The findings suggest that articulating the use of both tools reduces GenAI inaccuracies, facilitates the formulation of conjectures, and consolidates mathematical justifications. This coordinated use transcends an operational approach, delineating cognitive routes that strengthen geometric reasoning and support the design of didactic strategies in problem-solving.

Keywords: Problem-Solving, Geometry and Spatial Reasoning, Technology

Introduction

The emergence of Generative Artificial Intelligence (GenAI), catalyzed by the public release of ChatGPT, has served as a catalytic force in accelerating its integration within educational contexts. Initial studies on its use in mathematics education highlighted a marked propensity for "hallucinations" and the generation of erroneous responses (McGalliard & Otten, 2025; Wardat et al., 2023). Nevertheless, the relevance of these findings is diminished in light of the rapid advancement in the capabilities of contemporary GenAI models. This shifts the focus from mere model evaluation toward an effective incorporation into teaching scenarios. An area of particular interest is the coordination of GenAI with other consolidated digital tools, such as GeoGebra.

To date, the literature has addressed this combination predominantly from the perspective of computational thinking in the works of Yunianto and colleagues (Yunianto, Galic et al., 2024; Yunianto, Lavicza et al., 2024; and Yunianto et al., 2025), who have explored how construction in GeoGebra mediated by GenAI commands fosters the development of programming skills. Meanwhile, in other works (Heo, 2025; Segal & Klemer, 2025), GeoGebra is only used to verify the AI's responses, without delving into the coordinated use of both tools.

Unlike these uses, Medina-Alarcón (2025) proposes addressing this coordination specifically in mathematical problem-solving. In a first approach, the specific contributions of each tool are distinguished: GenAI tends to favor approaches based on algebraic concepts and procedures through verbal and symbolic explanations, as opposed to the visual and dynamic modeling offered by GeoGebra. However, documentation remains limited concerning: (a) How do GenAI and GeoGebra coordinate during problem-solving? (b) Which strategies can be used? and (c) How does this coordination impact the problem-solving process?

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The present work forms part of an ongoing doctoral research project that investigates, with a view toward the design of didactic tasks, various mathematical problems through the coordinated use of two tools: GenAI and GeoGebra. This report documents the successive approaches employed in addressing a selected problem using these tools and presents a preliminary analysis of the problem-solving experiences thus undertaken. This analysis is essential for identifying the possibilities—including both advantages and obstacles—inherent in the user's interaction with the two tools prior to their large-scale implementation.

Conceptual Framework

The conceptual framework is based on problem-solving as a process and is structured around four axes: the stages of problem-solving (Polya, 1945); resources, heuristics, beliefs, and metacognition (Schoenfeld, 1985); the transformations that digital technologies introduce into the problem-solving process (Santos-Trigo et al., 2016); and the way in which their coordinated use enhances it (Santos-Trigo & Reyes-Martínez, 2019).

Polya (1945) identified a four-phase approach: understanding the problem, devising a plan, carrying out the plan, and looking back. Alongside these phases, he systematized a set of heuristics—working with specific cases, establishing analogies, reformulating the problem, etc.—that guide the solver when there is no obvious path. Schoenfeld (1985) expanded this model with three additional domains: resources, which encompass the available mathematical knowledge, facts, procedures, and concepts; metacognitive control, referring to decision-making and monitoring; and beliefs, which condition the student's actions.

The incorporation of digital tools requires extending these frameworks. Santos-Trigo et al. (2016) position the use of digital technologies as a part of the problem-solving process rather than as an auxiliary resource. A central concept is affordances, understood as the possibilities for action that each tool offers the solver: in GeoGebra, the construction of dynamic models and the dragging of objects; in computer algebra systems, symbolic manipulation; in GenAI, natural language queries and the critical evaluation of responses. It is the affordances that determine the forms of reasoning in problem-solving.

The full utilization of these possibilities is not achieved with a single tool, but through their coordinated use. This principle implies that the affordances of each tool complement each other during resolution: formulation, comprehension, representation, exploration, generalization, and communication (Santos-Trigo & Reyes-Martínez, 2019). Empirical evidence reveals that students who engage with multiple tools in mutually reinforcing manner evidence a richer array of problem-solving strategies and navigate more fluidly between visual representations and formal mathematical arguments (Santos-Trigo & Reyes-Martínez, 2019). Santos-Trigo (2024) points out that coordinated use opens new areas of research aimed at analyzing the cognitive effects of coordinating multiple tools.

Methodology

This work is part of a research project currently in an exploratory phase of design and a priori analysis. In this stage, a selection and resolution of geometry problems have been carried out, simulating interactions with GenAI and GeoGebra.

By analyzing the possible problem-solving paths, this study aims to anticipate the cognitive routes and obstacles students might encounter. Specifically, it analyzes how the verbal and algebraic solutions generated by the GenAI are complemented by exploration in GeoGebra; and how this coordination allows for extending, generalizing, or particularizing the original problem.

The data and files presented derive from this internal validation phase and constitute a hypothetical coordination model that will serve as the basis for field work.

The problem addressed was:

a) Consider quadrilateral ABCD with A(0,2), B(1,0), C(4,0), D(4,3). Find the point P in the plane that minimizes PA + PB + PC + PD. (Convex case)
b) Given the quadrilateral ABCD with A(0,2), B(2,2), C(4,0), D(4,3). Find the point P that minimizes PA + PB + PC + PD. (Non convex case)

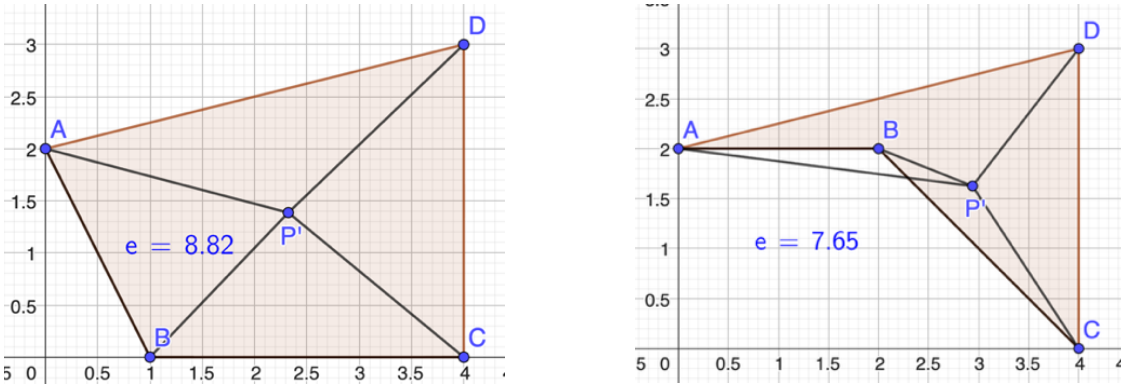
The exploration was conducted using the ChatGPT 5.5 and Gemini 3.5 Flash models. Queries were conducted in the browser interface, without logging in, to simulate student use and prevent the memory of these models from affecting the interactions. Furthermore, given the inherent variability of GenAI, each instruction was entered into each model at least five times in a new chat, with the aim of avoiding reporting anomalous responses. The interactions are available at the following link: https://bit.ly/3PDVgmM.

Results

The problem was approached through three routes: (1) construction of a dynamic model in GeoGebra, to explore different positions of point P and pose conjectures to the GenAI; (2) entering the problem without additional instructions and performing empirical verification with GeoGebra; and (3) restating both subsections into specific cases—square, rectangle, and parallelogram—to arrive at the general case using both tools.

Dynamic Model

For the first interaction, a dynamic model of Part A of the problem was built in GeoGebra (Figure 1). This construction was used to dynamically explore different positions of point P and conjecture that the solution is obtained with the point P(2,1). In addition to constantly verifying the responses of the GenAIs.



(a) (b)

Figure 1: Dynamic Models for Subsections A and B

The conjecture for Part A of the problem was posed alongside the problem to both GenAIs. Interactions with Gemini are recorded in Table 1, and interactions with ChatGPT are recorded in Table 2.

Table 1: Interactions with Gemini Regarding the Conjecture for Part A

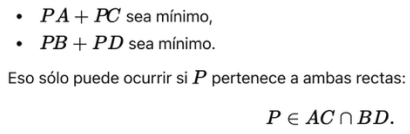
Summary of the Five Interactions	Responses
The model initially rejected the proposed conjecture; however, it correctly identified the procedure to obtain the solution (Figure 2).	¡Casi lo tienes! Tu intuición de que el punto debe estar en esa zona es muy buena, pero el resultado exacto es ligeramente diferente. El problema que estás intentando resolver es un clásico de la geometría y se conoce como el **Problema de Fagnano** o una variante del **Punto de Fermat-Torricelli** extendido a cuadriláteros convexos. Aquí te explico cómo encontrar la respuesta exacta de forma muy visual y sencilla. **Figure 2: Rejection of Conjecture by Gemini**
In four of the interactions, Gemini validated the result after solving the system of equations and generated a final message justifying its initial rejection (Figure 3).	**Conclusión** ¡Tu respuesta $(2, 1)$ es absolutamente **correcta**! Disculpa el amago de duda al inicio, ¡quería asegurarme de que viéramos todo el desarrollo matemático juntos! El punto que minimiza la suma de las distancias es, exactamente, $P(2, 1)$. ¡Buen trabajo de intuición geométrica! **Figure 1: Correct Response and Correction**
In the fifth trial, an algebraic error prevented reaching the solution, yielding incorrect coordinates that Gemini presented as definitive until the flaw was pointed out (Figure 4).	$$x = 2 \cdot \frac{3}{3} \implies x = 1$$ **Corrección / Nota importante:** Al resolver la intersección de las diagonales con los puntos dados, vemos que si $x = 1$, entonces $y = 0$. ¡El punto de intersección es exactamente el vértice $B(1, 0)$! **Figure 2: Arithmetic Error by Gemini**

Gemini begins by rejecting the initial conjecture in all five interactions. In four of them, after identifying the correct method, it arrives at the correct solution that matches the conjecture. In the fifth interaction, it makes an arithmetic error identified with the information provided with GeoGebra when verifying each element: slopes and equations of the line.

Table 1: Interactions with ChatGPT Regarding the Conjecture for Part A

Summary of the Five Interactions	Responses
ChatGPT validated the conjecture through calculus methods—partial derivatives—in two interactions (Figure 5).	$f(x,y) = \sqrt{x^2 + (y-2)^2} + \sqrt{(x-1)^2 + y^2} + \sqrt{(x-4)^2 + y^2} + \sqrt{(x-4)^2 + (y-3)^2}$ Las condiciones críticas son $$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0.$$ **Figure 3: Use of Partial Derivatives**
It applied the Fermat-Weber point—general solution for *n* points—in three interactions (Figure 6).	$$\sum \frac{\vec{P_i P}}{|P_i P|} = 0,$$ que es exactamente la condición de mínimo. **Figure 4: Use of the Fermat-Weber Point**

When explicitly asked for a geometric solution in all cases, ChatGPT responded similarly to Gemini, based on the intersection of the diagonals (Figure 7).


- $PA + PC$ sea mínimo,
- $PB + PD$ sea mínimo.

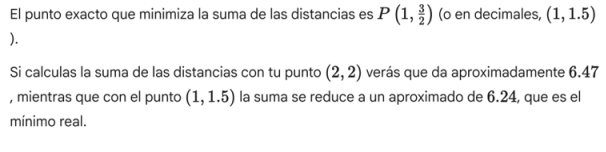
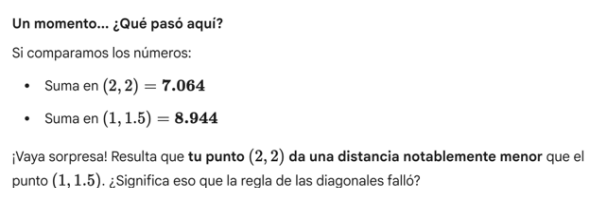
Eso sólo puede ocurrir si P pertenece a ambas rectas:

$$P \in AC \cap BD.$$

Figure 5: Geometric Solution

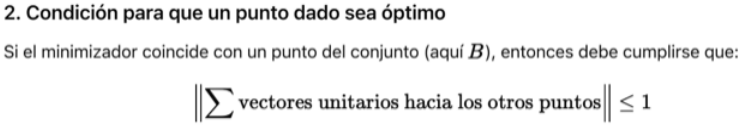
ChatGPT starts by verifying the conjecture with calculus methods; however, when asked for a geometric justification, it was able to generate it.

Table 2: Interactions with Gemini Regarding the Conjecture for Part B

Summary of the Five Interactions	Responses
Gemini tried to use the intersection of the diagonals again (a method that does not work when the quadrilateral is concave), so it discarded the conjecture and gave the intersection as the solution (Figure 8).	 El punto exacto que minimiza la suma de las distancias es $P\left(1, \frac{3}{2}\right)$ (o en decimales, $(1, 1.5)$). Si calculas la suma de las distancias con tu punto $(2, 2)$ verás que da aproximadamente 6.47 , mientras que con el punto $(1, 1.5)$ la suma se reduce a un aproximado de 6.24, que es el mínimo real.
	Figure 6: Erroneous Solution
Instead of directly pointing out the error, it was asked to compare the sums of the distances (Figure 9). Although the model correctly obtained the individual distances, in three interactions it performed the sum incorrectly. Upon pointing out this contradiction, it corrected its error.	 **Un momento... ¿Qué pasó aquí?** Si comparamos los números: • Suma en $(2, 2) = \mathbf{7.064}$ • Suma en $(1, 1.5) = \mathbf{8.944}$ ¡Vaya sorpresa! Resulta que **tu punto $(2, 2)$ da una distancia notablemente menor** que el punto $(1, 1.5)$. ¿Significa eso que la regla de las diagonales falló?
	Figure 7: Comparison of Sums

Gemini did not identify the type of quadrilateral, which led it not to validate the correct answer. When asked to compare distances, Gemini made arithmetic errors that led to confirming that its original answer was correct; upon noticing its error, Gemini validates the initial conjecture without justifying it.

Table 3: Interactions with ChatGPT Regarding the Conjecture for Part B

Summary of the Five Interactions	Responses
In four interactions, ChatGPT validated the conjecture: one without justification and three applying the condition for a given point to be a solution (Figure 10).	 **2. Condición para que un punto dado sea óptimo** Si el minimizador coincide con un punto del conjunto (aquí B), entonces debe cumplirse que: $$\left\| \sum \text{vectores unitarios hacia los otros puntos} \right\| \leq 1$$
	Figure 8: Validation with Optimization Condition

In the fifth interaction (Figure 11), it rejected the conjecture and resorted to numerical methods to provide an incorrect solution.

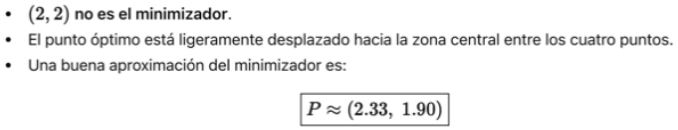


Figure 9: Erroneous Validation

In both cases for Part A of the problem, a geometric solution to the problem was obtained. For Part B of the problem, both GenAIs validate the conjecture—ChatGPT with the first prompt and Gemini after providing feedback—however, neither provides a geometric justification. Nevertheless, by exploring the dynamic model in GeoGebra (Figure 1b), a justification can be constructed.

Direct Problem

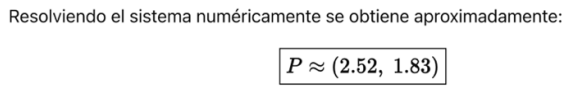
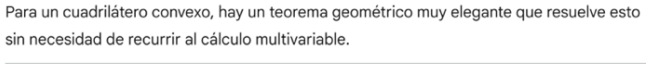
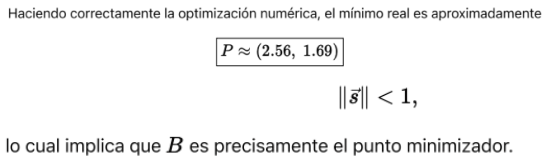
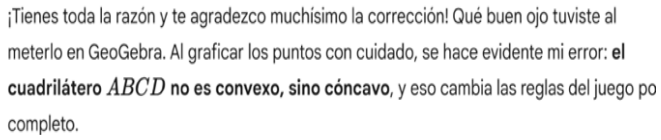
In this approach, the problem was entered into each GenAI without additional instructions. For subsection a) (Table 5), no additional prompt was provided after the first response. In contrast, for Subsection B (Table 6), after receiving an initial erroneous response, a different prompt was entered into each GenAI to assess whether they could correct their result. In each interaction, the GenAI's response was verified with the dynamic model in GeoGebra (Figure 1a).

Table 5: Interactions for Part A — Direct Problem

ChatGPT	Gemini
Aplicando el método de Weiszfeld (iteración estándar para el punto de mínima suma de distancias), se obtiene convergencia rápida. Una aproximación numérica estable es: $P \approx (2.12,\ 1.05)$	**1. El Principio Geométrico** Para cualquier conjunto de cuatro puntos que forman un cuadrilátero convexo, el punto P que minimiza la suma $PA + PB + PC + PD$ es exactamente el **punto de intersección de sus diagonales**, AC y BD.
Figure 10: ChatGPT's Solution	**Figure 11: Gemini's Solution**
ChatGPT used numerical methods in all of its interactions and generated a different point P on each occasion, none of them correct (Figure 12).	Gemini offered the same point P(2,1) in each interaction and provided the geometric argument that supports it: that point is the intersection of the diagonals (Figure 13).

When entering the direct problem to both GenAIs, only Gemini provides the correct solution. ChatGPT generated only approximations to the solution.

Table 6: Interactions for Part B — Direct Problem

ChatGPT	Gemini
	
Figure 12: Approximation by ChatGPT	**Figure 13: Erroneous Solution by Gemini**
ChatGPT once again used numerical methods and generated a different point P in each interaction, none of them correct (Figure 14).	Gemini did not generate the correct response: it identified the quadrilateral as convex instead of concave, which changes the solution method (Figure 15).
	
Figure 14: Change in ChatGPT Response	**Figure 17: Correction by Gemini**
Upon pointing out the existence of points, for example B, with a lower sum than its response, ChatGPT affirms in two interactions that the solution is a different approximation, and in three that the solution is vertex B, without explaining how it reached that conclusion (Figure 16).	When Gemini was informed that the quadrilateral was concave, Gemini corrected its response (Figure 17).

Initially, the GenAIs generated an incorrect solution, but by providing them with prompts indicating the error, the majority of interactions obtained the correct solution.

Particular Cases

An approach more aligned with problem-solving consists of considering specific cases (square, rectangle, parallelogram) of the two initial problems to later obtain a more general problem: in any quadrilateral with vertices A, B, C, and D, determine the location of point P whose sum of distances to the four vertices is minimal, and then examine a series of specific cases. First, an exploration was carried out in GeoGebra for each specific case (Figure 18); in each of them, the solution was found at the intersection of the diagonals.

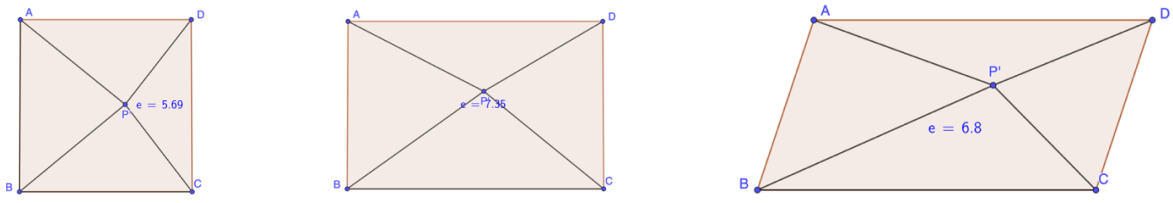
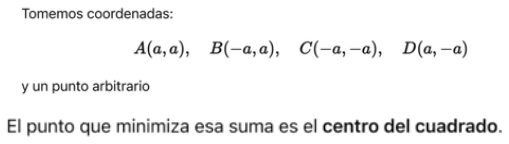
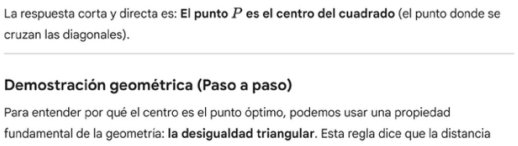


Figure 15: Particular Cases in GeoGebra

Subsequently, each specific case was presented separately to each GenAI. Table 7 presents the summary of the interactions regarding the square; however, the interactions for the rectangle and parallelogram are similar.

Table 7: Specific Case—Square—with ChatGPT and Gemini

ChatGPT	Gemini
	
Figure 16: Case of the Square with ChatGPT	**Figure 17: Case of the Square with Gemini**
ChatGPT used the geometric argument in three conversations and, in the remaining two, placed the square on the plane and generated a function to minimize (Figure 19).	Gemini employs the triangle inequality argument in all five interactions (Figure 20).

In all interactions, the GenAIs arrive at the correct solution, the intersection of the diagonals, allowing a pattern to be identified for exploration in the general case.

General Case

The general case was explored with GeoGebra (Figure 21), identifying two types of quadrilaterals: convex and concave. For the convex case, the solution from the specific cases can be generalized; however, for the concave case, with the evidence from GeoGebra, the solution from the specific cases is discarded and a new conjecture is formulated for the concave case.

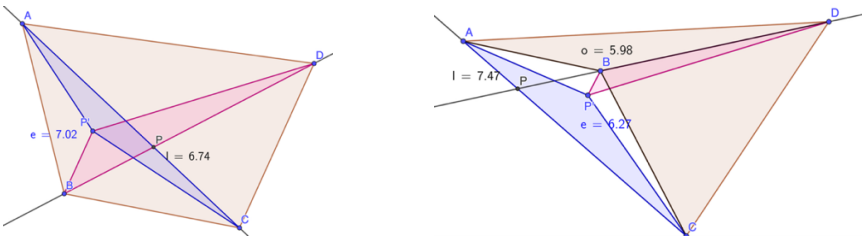
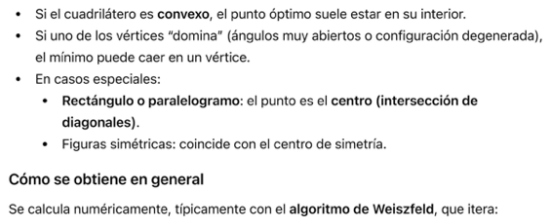
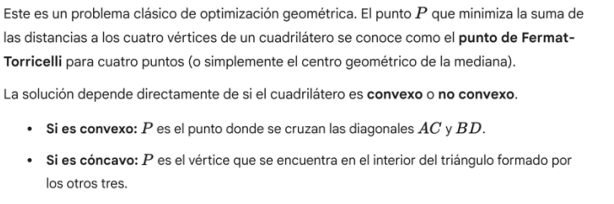


Figure 18: General Cases – Convex and Concave Quadrilateral

Table 8 presents the results of posing the general case to both GenAIs.

Table 4: General Case in ChatGPT and Gemini

ChatGPT	Gemini
	
Figure 19: General Case by ChatGPT	**Figure 20: General Case by Gemini**
ChatGPT highlights the solution for special cases, but does not arrive at the general result (Figure 22).	Gemini properly identifies the convex and concave cases. Justifying only the convex case (Figure 23).

Gemini managed to provide the answer by separating it into the two possible cases—convex and concave; however, it only justifies the convex case. On the other hand, ChatGPT only identifies special cases where the pattern of the specific cases is generalized. In this scenario, GeoGebra plays a crucial role in exploring and determining the two cases; furthermore, with the exploration of the dynamic model, it is possible to determine the justification for the concave case, which neither GenAI provided.

Conclusions and Final Considerations

The analysis of the explored problem is discussed below, with a necessary methodological caveat: as this is a single exploratory case prior to its implementation, the conclusions illustrate the potentialities and emergent dynamics of the coordinated use of these technologies, rather than generalizable results.

Regarding the GenAI, its responses to the same prompt are not deterministic. This variability is due to the stochastic nature of each model and the differences in architecture and data between them. Given this, a recommended heuristic strategy is to enter the same prompt into two different models. The goal is not for the models to compete, but to contrast approaches, compare the validity of the arguments, and discern what elements to extract to synthesize a robust solution.

Regarding the coordination of tools, two approaches are contrasted. During the direct approach, the GenAIs showed dependence on numerical methods and an inability to discern key topological properties, such as convexity. In this scenario, the information obtained through GeoGebra was crucial for the models to correct their responses. In contrast, when employing a guided conjecture approach, where GeoGebra functions as an experimentation environment, the landscape changes. Providing the GenAI with data derived from dynamic exploration mitigates its mathematical hallucinations and allows natural language processing to be redirected toward the construction and correction of valid geometric arguments.

On the other hand, transforming the initial task into a progression of specific cases and their subsequent generalization transcends the search for a solution. This reformulation turns the problem into a mathematical exploration activity, fostering a space to formulate conjectures and identify their limitations.

Finally, this study acknowledges two main limitations: a) it presents hypothetical problem-

solving trajectories, requiring the collection of empirical evidence through classroom implementation; and b) the analysis was limited to three specific approaches, leaving out explorations with custom GenAI assistants, prompt engineering, or multimodal interactions with images from the dynamic environment, among other approaches. These areas constitute open lines for the development of ongoing research work.

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FIVE CONSIDERATIONS FOR THE POTENTIAL VALUE OF STUDENT-CREATED MATHEMATICS LEARNING OBJECTS

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Renewable assignments invite students to create Learning Objects (LOs) that are shared publicly, openly licensed, and of educational value to others (Wiley & Hilton, 2018). Although the creation of such LOs has proven beneficial to the student creators, there is limited evidence of the value these LOs provide to others. We examined a collection of student-created mathematics LOs to understand the nature of the value they could bring to other learners. We examined five considerations related to the potential value of such LOs: content quality, learning goal alignment, presentation design, interaction usability, and visibility. The results show that some intended users of these LOs could benefit from using them. However, the intended users may not come across the LOs even if they searched specifically for them. Thus, LOs created for use in future iterations of the same class likely have a higher potential value to others.

Keywords: Instructional Activities and Practices, Technology, Undergraduate Education

Homework assignments are a regular part of most mathematics classes. Typically, however, these assignments are disposable—they serve no purpose beyond local learning goals within the class. To address this limitation, Wiley (2013) introduced the idea of renewable assignments. Renewable assignments ask students to create, share publicly, and openly license an artifact, or Learning Object (LO), that has educational value to others (Wiley & Hilton, 2018). Some examples include video tutorials and study guides. Whether an LO is student-created, shared publicly, and openly licensed is relatively straightforward to ascertain. Whether it has value to others, however, is quite subjective. One might wonder how student-created LOs could "provide a lasting benefit to the broader community of learners" (Wiley & Hilton, 2018, p. 137) when their creators are often still learning the very mathematics their LOs are designed to address. So, just what is the value of student-created mathematics LOs? This is a question that should concern anyone who considers implementing renewable assignments. In this paper, we report the results of a study that explored the value that student-created mathematics LOs bring to the broader community.

Literature Review

Researchers have found that renewable assignments benefit student creators in various ways. For example, student creators developed a deeper understanding of the topics covered by the assignments (Croft et al., 2013; Reyna & Meier, 2018; Schreiber & Klose, 2017b) and performed better than their peers when they were given renewable assignments (Crow, 2006). Students also found the work they were asked to do more meaningful (Allan et al., 2017; Andone et al., 2020; Ashman, 2023; Pitt, 2017). And as a result, they were more motivated (Clinton-Lisell & Gwozdz, 2023; Schreiber & Klose, 2017a) and engaged (Lansdown, 2010) in the process. Other benefits include opportunities for the student creators to represent their personal identities (Clinton-Lisell & Gwozdz, 2023), and improve their creative thinking (Ashman, 2023; Rosedale et al., 2021), research (Lower, 2015), and digital literacy (Hoban et al., 2013) skills. Despite these many benefits to the student creators of the LOs, the limited amount of literature shows a mixed opinion from the teachers. On one hand, some teachers may be hesitant to replace

traditional learning resources with student-created LOs (Lansdown, 2010). On the other hand, some teachers see a distinctive characteristic of student-created LOs as beneficial, which is a student perspective on the content (Croft et al., 2013).

To date, there is little evidence on the value student-created LOs bring to the broader community. A few researchers have investigated the value of student-created LOs to others. One study found that student-created LOs were more effective in preparing students for labs of a chemistry course when compared to TA instructions (Jordan et al., 2016). Croft et al. (2013) showed that student-created LOs may bring a special perspective to its users that traditional education resources cannot. It is unclear, however, what attributes of these LOs contribute to the added value. In order to further research related to student-created LOs for renewable assignments, we first seek to better understand the dimensions of these student-created LOs that could potentially contribute to the educational value of their users—the potential value of student-created LOs.

Theoretical Framework

As mentioned previously, three of the four criteria for renewable assignments are easy to ascertain. The criterion of educational value to others, however, is subjective. In this study, we use *potential value* to capture the learning "benefit or impact on the larger community" (Seraphin et al., 2019, p. 91) or "value beyond supporting its creator's learning" (Wiley & Hilton, 2018, p. 137) of student-created LOs from renewable assignments. In doing so, we acknowledge that the potential value of an LO likely differs when looked at through different perspectives. We focus on understanding the potential value of student-created LOs to its intended users in the context of mathematics learning and from the perspective of mathematics educators. Furthermore, for a student-created LO to bring an added value, its quality should be at least similar to what is already available. Thus, we chose to view the LOs in our study through an adaptation of the Learning Object Review Instrument (LORI) (Leacock & Nesbit, 2007)—a nine-dimension instrument for users to numerically rate professionally-created LOs. We adapted the LORI in two ways. First, we used the only four dimensions of quality relevant to capturing the potential value of student-created mathematics LOs: content quality, learning goal alignment, presentation design, and interaction usability. (An additional consideration for potential value—visibility—emerged during analysis.) Second, instead of simply assigning a numerical rating, we sought to characterize the LOs qualitatively, and in a way that is specific to student-created mathematics LOs. This study aimed to answer these two research questions: (a) How may the potential value of student-created mathematics LOs be captured? (b) What is the potential value of student-created mathematics LOs?

Methods

To explore the potential value of student-created mathematics LOs, we sought out a collection of real-life examples. We looked for a collection of LOs done under the direction of the same teacher. Such a controlled sample would thus meet certain common criteria such as the topics of mathematics, and the time and/or effort expected in the creation of the LOs. We located the website of a mathematics professor who had shown interest in Open Educational Resources (Misseldine, 2018) and had incorporated renewable assignments into his courses since 2017. Furthermore, he had documented his descriptions of the assignments and publicly shared the openly licensed LOs created by his students (Misseldine, n.d.). When we accessed the student-created LOs in 2021, there were a total of 66 LOs in the online collection. We analyzed the 52 LOs from Calculus III and Modern Geometries. The most prominent types are lesson plan, visual

aid, and video tutorial. Some descriptions of the LOs are included throughout the results section. (For a more detailed description of the collection of LOs, see Wong & Leatham, 2026.)

For our analysis, we first considered questions that could give us insights about the qualities of the LOs. We generated these questions by thinking about what we, as mathematics educators, would care about if we were to recommend these LOs to our students. The questions considered include the following: What, if any, are the mathematical inaccuracies in the LO? Does the LO accomplish its perceived purpose? How might a teacher/expert create this LO differently? Is it likely for someone to come across this LO through a Google search? These questions were answered and the answers were coded and grouped based on their similarity using a peer review process. The first author would apply the code, and then the two authors would talk about how the codes applied. The two authors then would discuss how the codes fit together under a common theme. These groups gave us insights about the qualities of the LOs in our dataset. We then sorted these qualities using the four dimensions of the LORI framework, with the exception of visibility which is not captured by LORI because LORI was designed for users to rate an LO that they have already used.

Results

We describe five considerations related to the potential value of the LOs in our dataset: content quality, learning goal alignment, presentation design, interaction usability, and visibility. We elaborate on each of these considerations that emerged from our analysis. Taken together, these considerations characterize the potential value of our dataset of student-created LOs and provide a means in general for capturing the potential value of renewable assignments.

Content Quality

We examined two facets of content quality that are especially important to mathematics LOs: accuracy and completeness. Accuracy here refers to the mathematical correctness of the LO. Out of the 52 LOs in the dataset, 32 of them are accurate. The remaining 20 are inaccurate to some degree. Because all inaccuracy does not have the same impact on the value of an LO, we distinguish between minor and major inaccuracy.

One way an inaccuracy is regarded as minor is when the inaccuracy is only peripherally related to the LO's central idea. For example, in a webpage LO that introduces its users to derivatives, some examples of partial derivatives are included as a brief preview of multivariable calculus; one of those examples has mistakenly used a positive sign instead of a negative sign. This inaccuracy is regarded as minor because it is not related to single-variable differentiation, which is the central purpose of the LO. The intended users of this LO are not likely learning multivariable calculus and thus likely would not be affected by this inaccuracy. The other two types of minor inaccuracies are errors that are represented correctly elsewhere in the LO, and inaccuracies in an LO that is meant to be used by someone who already knows the mathematics and thus would be likely to notice and correct the error on their own, such as a lesson plan. Out of the 20 LOs that are inaccurate, 14 contain only minor inaccuracies.

Inaccuracies were deemed *major* if they were likely to negatively impact the learning of the user. An example of a major inaccuracy is found in a practice exam. When it comes to the limit definition of a derivative, its answer key used $f(x) + h$ instead of $f(x + h)$. The answer key of a practice exam is used for students to judge their own performance, and, in some cases, students may even study the answer key to learn a typical way to solve a problem. Having inaccuracies in an answer key could really mislead students that are trying to evaluate their understanding of the material, especially when the inaccuracy looks so similar to the correct answer and actually represents a common error. There are 6 LOs in the dataset that contain a *major* inaccuracy. (It

turned out that no LOs in our data set happened to contain multiple different types of inaccuracy.)

Completeness is another facet of content quality. An LO is considered complete when all relevant mathematics is included in the LO and is explained sufficiently for the central purpose of the LO. Otherwise, the absence of relevant mathematics could be misleading, causing the users to believe that they are sufficiently shown some mathematics, and the LO is considered incomplete. Out of the 52 LOs in the dataset, 38 are considered complete. The remaining 14 were incomplete for one of two main reasons.

One way an LO could deviate from complete content is if some relevant mathematics is excluded from it. For example, in a video LO about derivatives, all the common differentiation rules (power rule, product rule, quotient rule, etc.) were covered except for the sum and difference rules. There were 8 LOs that deviated from complete content.

Another way an LO is regarded as incomplete is if some relevant mathematics is mentioned in an LO but not sufficiently explained to its users. For example, in an LO about why it is impossible to trisect an angle or square a circle using a straightedge and a compass, the meaning of those geometric constructions was explained. But the explanations of the impossibility of those constructions were built on the unexplained contradiction between those constructions and some property of constructible numbers. The student creator simply gave the form of a constructible number and used algebra to show that the number needed for those geometric constructions is not a constructible number. There were 6 LOs that had incomplete explanations.

Learning Goal Alignment

Learning goal alignment is the second consideration for evaluating the potential value of a student-created LO. For this consideration, we examined alignment within learning goals and between learning goals and activities. When an LO has an explicitly stated learning goal in its title or description, we take that as the intended learning goal of the LO. Otherwise, we infer the LO's learning goal from its activity as if we, as mathematics educators, were determining how to use the LO with our students. All these student-created LOs show an obvious alignment between their learning goal and activities, meaning the activities are indeed intended to help users reach the learning goal. However, four of the 52 LOs have multiple activities, and the reason those specific activities are put together is not addressed in the LOs. Furthermore, there is no clear pedagogical reason we could identify for those activities to be put together. One example comes from a video tutorial about trisecting an angle and doubling a cube. Although these two topics may seem connected to someone who is familiar with the classic geometric constructions that cannot be done with a collapsing compass and a straightedge, the LO does not explicitly connect the two problems for the intended users. Therefore, to the intended users, the two geometric constructions are likely just two disjoint problems.

Presentation Design

An LO that looks and feels professional can enhance the learning experience of its users. However, since the student creators of the LOs were not professional designers, we focused our analysis on determining whether these LOs met the basic expectations of professionally created LO. Out of the 52 LOs, 36 of them present a relatively professional design. The remaining 16 display some rudimentary flaws that make the LO seem unprofessional and that could distract their users. For example, multiple video LOs in the dataset feature the student-creators' body or hands as they wrote or drew on a piece of paper or a whiteboard, which often block the users' view. Because of the obstruction, users cannot follow the algebraic manipulation or geometric

constructions as the videos play. An expert might choose to write the mathematics on a digital surface that could be shown to the users without blocking their views.

Interaction Usability

The *interaction usability* of an LO is about how well it performs as an object of its type. Almost all of the LOs in the dataset have a reasonable and expected interaction usability. This is likely a result of the LOs being in well-established formats (e.g., documents, slideshows, videos). There are only 3 LOs in the dataset that could be improved in *interaction usability*, and all 3 LOs are games. Two of those were *Jeopardy!* (Davies, 1964–2024) games that did not have a built-in score tracker, the lack of which severely reduces the interaction usability. The remaining one is the Geometry *Fluxx* card game (Looney, 1997). The original *Fluxx* card game is a game where its rules change as the players play the game. Although the overall structure of Geometry *Fluxx* follows the original *Fluxx* card game, each player needs to construct a geometric model using points and lines in order to win the game. With game rules that change continually throughout the game, and a unique variation of constructing a geometric model, users of the LO could really benefit from some sample plays, but there are no sample plays in the rulebook. This lack of examples clearly reduces the interaction usability of the LO.

Visibility

Lastly, we introduce *visibility* (which emerged from our analysis) as an additional consideration for evaluating the potential value of student-created LOs. Visibility is the likelihood of the intended users of an LO coming across it. Given the vastness of the internet, some LOs may, for all intents and purposes, be invisible to its intended users. We examined two facets of this consideration: (a) whether an LO shows up in a general search of the internet (i.e., a Google search), and (b) whether it shows up in a search on its host platform (e.g., YouTube). We did this by using keywords that someone would likely use if they were searching for resources similar to each of the LOs.

Google. Only two LOs in the dataset showed up in the first page of the Google search results. One is a video about Fano Geometry. Using the search words ["Fano Geometry"], the LO shows up as the second video result on the first page (out of 6,350 results). Thus, this LO was very visible. None of the other LOs showed up within the first three pages of the results and therefore are not regarded as visible to the intended users through a Google search.

It should be noted that there are two collections of LOs in the dataset that each exist online as a collection and could reasonably be searched for as a collection. One of the two is the collection of 13 lesson plans that cover the geometry standards in the Utah core of integrated secondary mathematics II. When a search is done for the lesson plans as a collection, the collection of lesson plans shows up as the third result (out of a total of 775,000 results). A possible reason that these LOs show up so early in the search results could be they are a sizable collection. However, even when a search for the collection was done, the other collection of geometry proofs did not show up within the first three pages of search results.

Host platform. Even though most of the LOs in the dataset do not show up early on a Google search, there are other ways intended users could come across those LOs. More specifically, hyperlinks to these LOs are hosted on some platform for the intended users.[1] For example, all the LOs are hosted on the professor's website, which is where we ourselves found the LOs. However, not all LOs are regarded as visible to the intended users merely by being on

[1] We have considered the difference between a platform hosting the hyperlinks to the LOs and hosting the actual LOs. We decided that although there may be a difference, the visibility of an LO is not affected. Therefore, we do not distinguish said difference hereafter.

the professor's website. All the LOs on this platform are grouped by the class where the assignment to create the LO is given, and thus not necessarily by the intended users. Therefore, as a platform, the professor's website is considered a host platform for the intended users when it comes to some LOs, and not a host platform for the intended users for other LOs. For example, the intended users of the Trigonometric Substitution video tutorial are Calculus II students, but the LO is grouped under Calculus III because that was the class where the assignment was given to the student creator. Although this method of grouping is convenient to student creators who may be looking for resources to help them in completing their own assignment, it lowers the likelihood of the intended users of the LOs coming across the LOs (and thus lowers the visibility and potential value of the LO). Overall, there are 16 LOs whose intended users are students of the same class and are therefore regarded as visible. In fact, most of these LOs are incorporated into the text for the class.

Some of the LOs whose intended users are anyone that is interested in that particular topic in mathematics are hosted on other online platforms such as YouTube (https://www.youtube.com) or Quizlet (https://quizlet.com). Thus, beyond a Google search, these LOs may come up if one searches for that particular type of LO on the platform where it is hosted. For example, an intended user may come across a video LO while searching for videos on YouTube. Of the remaining 36 LOs in the dataset whose intended users are broader than students in the same classes, 32 exist on a host platform. A search for each of those 32 LOs on its host platform was done to see whether it is likely for someone to come across those LO while searching within that subset of the internet. The two LOs that showed up on a Google search (Fano Geometry, The Alternate Interior Angle Theorem) also showed up in a search on YouTube, which is the host platform for these two LOs. The same 13 lesson plans that showed up as a collection on a Google search also showed up as a collection on their host platform (TeachersPayTeachers). In addition, seven of those 32 showed up within the first three pages of the search results, which makes it likely for someone to come across them if they were searching on those host platforms. Since the remaining 10 LOs did not show up, it is unlikely that their intended users would come across those LOs on their own even if a search is done on the specific host platform.

Lastly, there are four LOs not hosted on any platform for their intended users: the webpage Learning to Derive, whose intended users are students of calculus I; two slideshows created in the form of *Jeopardy!* (Davies, 1964–2024), whose intended users are teachers of calculus and geometry; and a children's book in document format Trinity Discovers Infinity, whose intended users are caregivers of children. These LOs exist only on the professor's webpage, where these LOs are grouped under either Calculus III or Modern Geometry rather than by their intended users. Therefore, despite being on the professor's website, these LOs are not hosted on a platform for their intended users. We also note that the professor's website has changed from being publicly available (in May 2021) to requiring university credentials for access (in March 2025). Therefore, in addition to these four LOs being not very visible, the public access to them is virtually extinct, meaning there is not a way for the public to access these four LOs anymore.

Discussion

We now further discuss the five considerations for capturing the potential value of student-created LOs. Content quality ensures the accuracy and completeness of the mathematics; learning goal alignment examines whether the learning activity serves the learning goal; presentation design looks at the aesthetics of an LO; interaction usability is about how well an LO functions; and visibility captures the likelihood of an LO being discovered by its intended users.

With respect to the content quality consideration, we note that there are not many major mathematical inaccuracies in the LOs (only 4 out of 52). In other words, most of the student-created LOs are mostly correct. This overall accuracy might be explained, at least in part, by the fact that about half of the LOs focus on content that is considered prerequisite to the course the student creators were in. However, this observation may still be somewhat surprising considering the fact that the student creators were still novices themselves in the mathematical content of the other half of the LOs.

The LOs in our data set had strong learning goal alignment. In fact, in all of the LOs, the learning activities are aligned with the learning goal. The only issue we identified is the alignment between multiple learning goals not explicit in an LO. This overall strong alignment shows that these student creators can accurately describe the mathematics they choose to present.

One thing to consider when it comes to presentation design is the creation of video LOs. This is especially true when it comes to non-scripted videos showing the student creators writing, either on a piece of paper or on a whiteboard and talking about some mathematical concepts. This kind of LO could be appealing to the student creator as it is a straightforward way for them to directly present their understanding. However, the mathematical content on these videos are not always easy to see because the users' view is sometimes blocked by the student creator's hand or body. The presentation of the mathematics is also somewhat difficult to follow because of the nature of real time non-scripted human speech coupled with the fact that the student creators usually do not have an expertise in verbally presenting mathematics.

Most of the LOs do not have an issue with interaction usability. Moreover, the few issues we do see are minor in terms of their impact to the LOs' usability. Although most of these LOs are in standardized formats (e.g., document, video) and are used directly on well-established platforms (e.g., YouTube), this shows that these student creators possess the technical skills needed to create usable LOs in the digital world.

In the visibility consideration, we showed how unlikely it is for someone to come across most of the LOs in our dataset through a Google search. This is especially true for LOs that cover a topic that is broad and common such as derivatives or integrals. There are already many high-quality resources online with various scopes that are freely available to anyone for virtually any general purposes. On the other hand, an LO created to extend or supplement the current curriculum does not need to rely on showing up on a Google search for its potential value to be realized. Since the curriculum being extended is already used by the teacher, there are already users almost guaranteed to view the LOs because those LOs are incorporated into the course materials. As a result, and as can be seen in Croft et al.'s (2013) study, the student creators of LOs designed to extend the current curriculum would not need to worry about increasing the visibility of their LOs because the intended users would have access to them through the class, such as on the class website or the learning management system that hosts the class material. As reported earlier, one study found evidence that student-created LOs in a chemistry course were quite valuable to their peers (Jordan et al., 2016). A key finding of our study aligns with Jordan et al.'s finding: the LOs in our data set seemed to be more valuable when they were created for future students in the same class.

Conclusion

Despite the fact that many of the LOs measure up well according to the first four considerations, not all of them are likely to be discovered by their intended users because of low visibility. Therefore, it is evident that visibility is an essential consideration when it comes to the potential value of an LO designed to fulfill a renewable assignment, where the vision is to

"provide a lasting benefit to the broader community of learners" (Wiley & Hilton, 2018, p. 137). Given how unlikely it is for someone to actually find LOs with low visibility, let alone use them and benefit from them, the potential value of these LOs is significantly lowered. This matters because a reason students find renewable assignments meaningful is their belief that their LOs could actually help others (Allan et al., 2017). Therefore, if an educator wishes to implement renewable assignments into their practice (and thus help their students bring value to the broader community), in addition to the first four considerations, they should carefully consider with their students the visibility of the LOs to the intended users.

There is evidence that student-created LOs in mathematics can bring value to a broader community. To increase the likelihood that an LO would actually be used, however, teachers may need to focus the students' effort to fill a specific need in the course that the students identified. This would help students create an LO that is more user-specific because of their own recent learning experience in the same course. Moreover, students can be more confident that the intended users of their LOs would come across the LOs because the intended users are future students taking the same course from the same teacher. In short, there is a trade off between the likelihood of the value of the LO being realized and the broadness of the community that could potentially benefit from the LO. And it may be more practical to help students bring value to a community, through renewable assignments, that they are more familiar with than one that is too broad.

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TEACHERS' WORK WITH GENERATIVE AI: INSTRUMENTAL GENESIS, PROVISIONAL RESOURCES, AND DOCUMENTATIONAL GENESIS

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Mathematics teachers increasingly use generative artificial intelligence (GenAI) to generate instructional materials, yet existing research provides limited theoretical guidance for understanding this professional work. This paper proposes integrating instrumental genesis and documentational genesis to understand how teachers interact with GenAI systems and transform AI-generated materials into instructional resources. I introduce the construct of provisional resources to describe AI-generated instructional artifacts whose epistemic status is indeterminate and must be established through teachers' professional judgment. By focusing the dual genesis process, this framework examines how teachers interact with GenAI systems and how they evaluate, adapt, and stabilize AI-generated materials. This approach extends the theories of instrumental genesis and documentational genesis to include generative technologies, emphasizing the crucial role of teachers in AI-supported mathematics teaching.

Keywords: Technology, Cognition, Teacher Educator

Generative Artificial Intelligence (GenAI) is rapidly entering the professional landscape of mathematics teaching and learning. Mathematics teachers increasingly use GenAI systems to generate mathematical tasks, draft explanations of mathematical ideas, and support lesson planning (e.g., Gurl et al., 2025; Pepin et al., 2025). Currently, teachers' engagement with GenAI systems seems to occur primarily outside classroom instruction, such as lesson preparation and assessment-related work, where AI-generated materials are produced, adapted, and refined prior to instructional use. As a result, GenAI may shape both the production of instructional resources and teachers' professional activity around planning, assessment, and instructional decision-making.

Despite the growing presence of AI in mathematics education, existing research offers limited theoretical guidance for understanding how mathematics teachers work with such systems. Much of the literature on AI in education has emphasized issues of adoption, acceptance, and readiness, often foregrounding teachers' beliefs, attitudes, or technological competencies rather than their instructional and epistemic work (e.g., Xue et al., 2025; Ng et al., 2023). This framing is particularly limited in the context of mathematics education. Unlike mathematics-specific technologies (e.g., computer algebra systems and dynamic geometry environments) whose mathematical behavior is largely deterministic, GenAI systems produce variable and non-deterministic outputs. As a result, teachers cannot assume the mathematical correctness, representational appropriateness, and instructional suitability of AI-generated materials. They must engage in substantial professional judgment to evaluate, adapt, and contextualize AI-generated tasks, explanations, or examples before these materials can function as viable instructional resources. Understanding teachers' work with GenAI thus requires theoretical perspectives that attend to their mathematical and pedagogical decision-making through which AI-generated materials are transformed for instructional use.

Meanwhile, emerging discourse in fields such as human-computer interaction and information systems focuses on the different roles that GenAI can play, ranging from tool-like automation to more collaborative or partner-like forms of interaction (Dellermann et al., 2019).

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Such descriptions tend to foreground the technology itself rather than teachers' professional work through which GenAI systems and their outputs are integrated in practice. In mathematics education, where epistemic responsibility and instructional accountability are central, there is a need to develop theories that explain how GenAI becomes integrated into teachers' work without attributing agency or authority to the technology itself.

This paper argues that understanding teachers' engagement with GenAI requires moving beyond role-based frameworks and focusing on the detailed analysis of teachers' evolving professional activities over time. It requires distinguishing between teachers' interaction with the GenAI system itself and their subsequent work with the materials generated through that interaction. Without this distinction, we might underestimate the epistemic demands placed on teachers when working with AI-generated resources and therefore obscure our understanding of teachers' work with GenAI. To this end, this paper introduces a theoretical framework that examines teachers' work with GenAI.

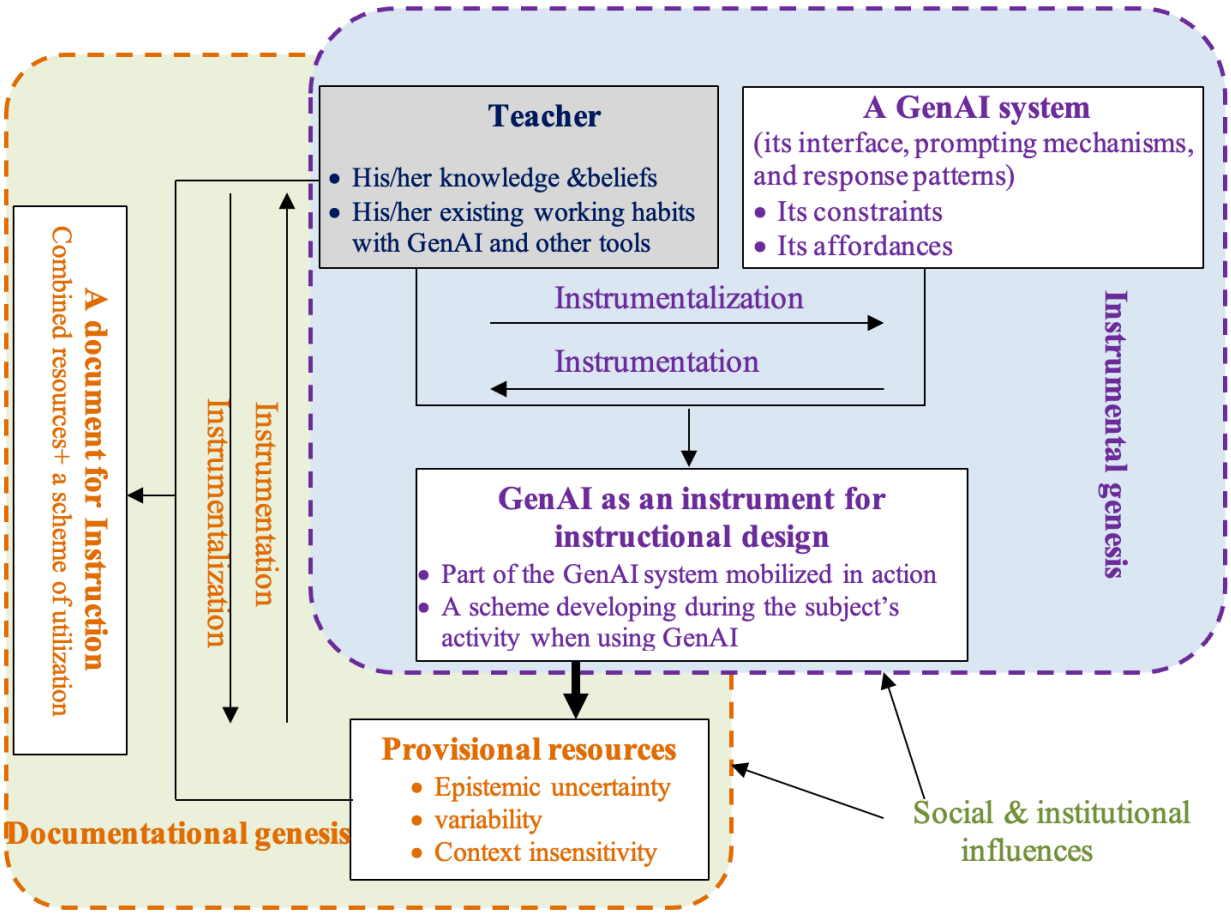


Figure 1: A Theoretical Framework for Understanding Teachers' Work with GenAI Through Instrumental and Documentational Genesis

A Theoretical Framework for Understanding Teachers' Work with GenAI

Integrating constructs in instrumental genesis and the documentational approach to didactics, this paper proposes a theoretical framework that conceptualizes teachers' work with GenAI as a dual genesis process: instrumental genesis and documentational genesis. Through instrumental

genesis, teachers develop utilization schemes that transform a GenAI system from an artifact into an instrument for instructional use. This instrumental activity produces AI-generated artifacts, conceptualized here as *provisional resources*, which require further professional work before instructional use. Through documentational genesis, teachers evaluate, adapt, and stabilize these provisional resources into instructional documents to be imbedded in their evolving resource systems. Figure 1 provides a visual representation of this dual genesis process.

Instrumental Genesis with GenAI

Instrumental genesis, originating in the theory of instrument-mediated activity (Rabardel & Beguin, 2005), describes the process through which an artifact becomes an instrument as users develop and appropriate utilization schemes for acting with it (Trouche, 2004; Trouche, 2018). An instrument is thus not the artifact itself, but the artifact together with the schemes that organize its use. Similar to many other digital technologies, a GenAI system, including its interface, prompting mechanisms, and response patterns, can be perceived as an artifact. Teachers' sustained interaction with this artifact gives rise to instrumental genesis, through which the GenAI system becomes an instrument for instructional design.

Instrumental genesis unfolds through the dialectical interplay of instrumentation and instrumentalization (Artigue, 2002). In instrumental genesis with GenAI, instrumentation refers to the ways in which the affordances and constraints of the GenAI system shape teachers' activities. The conversational interface of GenAI encourages iterative interaction and supports teachers' engagement in cycles of input, interpretation, and re-prompting. The non-deterministic response patterns of AI systems further shape teachers' activity by introducing variability and uncertainty, which requires teachers engage in comparison, selection, and evaluation of multiple outputs. Additionally, the speed and abundance with which GenAI produces outputs reorganize the temporal structure of instructional planning work. Teachers are less likely to be constrained by the need to create initial drafts and more engaged in activities, such as evaluating the mathematical correctness of AI-generated output, its pedagogical suitability, and its alignment with instructional goals. Through instrumentation, interaction with a GenAI system shapes how teachers engage instruction preparation without determining the outcomes of that work.

Instrumentalization refers to the process through which teachers appropriate a GenAI system to serve their instructional aims. In instrumental genesis with GenAI, instrumentalization primarily involves the development of utilization schemes for using a particular GenAI system. When interacting with a GenAI system, teachers learn to write prompts in which they might specify instructional goals, the focal mathematical concepts in a lesson, the types of representations to be used, and the levels of cognitive demand of mathematical tasks. They also develop strategies for how to handle AI-generated outputs, such as deciding when to accept responses, when to refine prompts, and when to abandon a line of interaction. Over time, teachers may establish techniques for using a specific GenAI tool for instruction, which might further shape how the GenAI system functions as an instrument in their instruction. It is also worth noting that when working with the same GenAI system different teachers may develop different utilization schemes.

The relationship between instrumentation and instrumentalization is dialectical. As GenAI shapes teachers' activities (instrumentation), teachers respond by adapting how they interact with the system (instrumentalization). These adaptations, in turn, modify how the system shapes teachers' subsequent activities. Instrumental genesis thus unfolds as a dynamic process in which teachers' utilization schemes and the functional use of GenAI co-evolve, which leads to the formation of GenAI as an instrument for instruction.

Emergence of Provisional Resources

The use GenAI introduces a unique challenge for examining technology use in mathematics education. Unlike mathematics-specific technologies such as computer algebra systems, graphing tools, and dynamic geometry software, GenAI does not merely mediate mathematics teaching and learning activities by providing computational capacities or multiple representations of a concept. Instead, it generates instructional artifacts that may later be used by teachers in their instruction. This generative nature is fundamentally important. From the perspective of instrumental genesis, a GenAI system initially functions as an artifact with which teachers may interact. Through sustained engagement, teachers develop utilization schemes that transform the system into an instrument for instruction purposes. When the artifact is generative, however, the outputs produced through interactions become possible resources for instructional use. These resources may include, for instance, mathematical tasks, lesson plans, mathematical explanations, or simulated student responses. These resources are neither authored in the traditional sense nor inherently pedagogically validated. They are produced through a probabilistic generative process whose internal mechanisms are a black box to its users.

I conceptualize these AI-generated instructional materials as *provisional resources*. They are resources in the sense that they are available for instructional use, but they are provisional because their epistemic status is yet to be established by teachers. Teachers need to engage in further professional work to determine their mathematical correctness, pedagogical appropriateness, and contextual relevance. The term "provisional" emphasizes that these resources are tentative and their legitimacy is subject to teachers' subsequent evaluation. The notion of provisional resources foregrounds the *epistemic uncertainty* that characterizes AI-generated materials. Unlike textbooks or curriculum-aligned digital resources, provisional resources do not carry implicit warrants of mathematical correctness, pedagogical appropriateness, and contextual relevance. Their legitimacy is not embedded in their origin but must be established through teachers' professional evaluation. As a result, provisional resources increase the demands on teachers' capacity to evaluate mathematical accuracy, pedagogical appropriateness, and contextual fit. In this sense, the notion of provisional resources makes visible forms of epistemic labor that often remain implicit when teachers work with resources from other sources (e.g., textbooks and resources from websites).

Provisional resources are also characterized by their *variability*. Because GenAI can generate multiple versions of a task or a lesson plan in response to the same or similar prompts, this variability might influence teachers' engagement with resources by encouraging comparison, selection, and refinement. The variability of provisional resources thus contributes to a shift in teachers' work from resource production or acquisition toward resource evaluation and curation.

Another defining feature of provisional resources is their incomplete alignment with instructional contexts. AI-generated resources are typically produced without access to knowledge of students from a particular classroom, classroom norms established by a teacher and students, curricular pacing and sequencing of a particular school, or institutional constraints. As a result, provisional resources often require substantial contextualization before they can be used productively in a particular classroom. This work of *contextualization* further underscores the provisional nature of these resources and the central role of teachers' professional knowledge.

To summarize, provisional resources are tentative instructional artifacts whose value and legitimacy are realized only through teachers' subsequent professional activity. This conceptualization of AI-generated artifacts provides a language for analyzing the distinctive

epistemic and professional challenges posed by teachers' use of GenAI and clarifies why teachers' documentation work becomes especially significant in AI-supported teaching.

Documentational Genesis with Provisional Resources

Documentational genesis provides a lens for analyzing how provisional resources are evaluated, adapted, and stabilized into instructional documents embedded in teachers' evolving resource systems. It refers to the long-term developmental process through which teachers transform resources into documents, which can be understood as the combination of resources, schemes of use, and the professional knowledge that underpins those schemes (Gueudet & Trouche, 2009; Trouche et al., 2018). Documentation work is not limited to the selection of materials for a particular lesson with a particular group of students. It encompasses the ongoing organization, adaptation, and re-use of resources across instructional contexts over time. When teachers work with AI-generated provisional resources, this documentation work becomes particularly visible, as the legitimacy of these resources cannot be assumed and must be established by teachers. Documentational genesis in the context of GenAI is shaped fundamentally by the epistemic uncertainty of provisional resources and thus the call for professional judgments to establish their instructional legitimacy.

As in instrumental genesis, the processes of instrumentation and instrumentalization in documentational genesis are closely connected, through which provisional resources are accepted, refined, or discarded. Instrumentation at the documentational level focuses on how provisional resources shape teachers' activities. The use of GenAI calls for teachers to foreground their professional evaluation and scrutiny of AI-generated materials. Given that AI-generated materials sometimes contain mathematical inaccuracies, inappropriate or incorrect representations, or misalignments with instructional goals, teachers need to assess their reliability and instructional usefulness. AI-generated materials thus call for teachers to focus their work on evaluating the epistemic legitimacy of provisional resources, such as checking mathematical correctness and considering their contextual relevancy for a particular group of students within a specific curricular sequence. In this sense, the characteristics of provisional resources (e.g., variability, opacity of generation, and lack of institutional validation) actively shape how teachers engage in documentation work. Instrumentation is also evident in how provisional resources invite comparison. Teachers may generate multiple AI-produced versions of a task or a lesson plan and compare them with one another, with existing curricular materials, or with the resources they have accumulated in the past. Such comparative work influences how teachers assess the quality of provisional resources. Through instrumentation, provisional resources shape both what teachers attend to and how they reason about instructional resources.

Instrumentalization at the documentational level describes how teachers actively transform provisional resources into documents for instructional use. For example, a teacher might revise an AI-generated task, rephrase a mathematical explanation, or modify a visual representation to better fit their classroom needs. Teachers may choose parts of a provisional resource, keeping what's useful and leaving out what isn't, or blend AI-generated content with materials from textbooks, previous lessons, or other sources. These choices shape how provisional resources are woven into classroom instruction. Instrumentalization also involves the development of schemes for working with provisional resources. Over time, teachers may establish stable ways of deciding when a provisional resource is sufficiently reliable for use, when further adaptation is needed, or when it should be discarded.

As in instrumental genesis, instrumentation and instrumentalization in documentational genesis are interrelated. Provisional resources shape teachers' documentation activity by

demanding evaluation and judgment, while teachers' actions reshape how those resources are handled. Through this dialectical interaction, some provisional resources are transformed into documents and therefore become part of teachers' resource systems, while others remain short-lived or are discarded completely. Documentational genesis explains how AI-generated materials move from tentative artifacts to durable components of instructional planning practice.

By analyzing documentational genesis with provisional resources, the framework clarifies how teachers' work with GenAI goes beyond interaction with the GenAI system itself. Instrumental genesis explains how teachers learn to work with the GenAI system as an instrument, while documentational genesis explains how teachers subsequently work with the products of that interaction. Together, these processes provide a coherent account of teachers' professional activities with GenAI.

It is important to recognize that instrumental genesis and documentational genesis are shaped by social and institutional conditions beyond each individual teacher's interactions with GenAI. The ways teachers use GenAI are shaped by school norms around curriculum, assessments, and technology policies, as well as by professional conversations within schools and teaching communities. Similarly, documentational genesis happens within shared resource systems, such as common instructional materials, teamwork, and group standards for instructional quality. Decisions about whether and how provisional resources are evaluated, adapted, and stabilized are therefore not only individual decisions. Instead, they are influenced by institutional expectations, accountability structures, and collaborative practices. Attending to these social and institutional dimensions highlights that teachers' work with GenAI is often situated within broader social contexts that shape how their utilization schemes emerge and develop over time.

Discussion and Conclusion

The framework proposed in this paper offers a theory-driven account of teachers' work with GenAI that centers professional activities such as the evaluation, adaptation, and contextualization of AI-generated materials rather than merely technological categorization. The framework conceptualizes teachers' engagement with GenAI through instrumental genesis and documentational genesis, offering a lens to examine how GenAI systems become meaningful in practice and how AI-generated materials are transformed into instructional documents. In this section, I elaborate on the theoretical contributions of the framework and outline implications for future research.

The framework extends instrumental genesis to account for the generative nature of GenAI. Much of the mathematics education literature on instrumental genesis has focused on tools that mediate mathematical activities without producing instructional materials themselves, such as computer algebra systems or dynamic geometry environments (e.g., Artigue, 2002; Trouche, 2004). In these cases, instrument-mediated activity primarily reshapes how users act on mathematical objects. GenAI differs in a fundamental way in that AI-mediated activity produces instructional artifacts that persist beyond the immediate interaction. By explicitly theorizing these outputs as provisional resources, the framework extends instrumental genesis to the use of GenAI. Provisional resources are understood as an outcome of instrumental genesis when the artifact is generative. The construct of provisional resources makes visible a form of professional work that is central to AI-supported teaching.

The documentational approach to didactics has emphasized that teachers' professional development is inseparable from their documentation work with resources and the evolution of their resource systems (Gueudet & Trouche, 2009). The framework proposed in this paper builds on this insight by showing how documentational genesis is reshaped when resources are

provisional rather than institutionally validated or pedagogically stabilized. AI-generated resources require teachers to engage in more intensive epistemic work, including validating mathematical correctness, determining pedagogical appropriateness, and assessing contextual fit. By articulating instrumentation and instrumentalization at the documentational level, the framework clarifies how provisional resources reorganize documentation work and foregrounds teachers' professional responsibility for instructional materials. In this sense, GenAI does not necessarily simplify teachers' work. Instead, it increases the need for teachers' professional judgment and makes documentation work more visible.

The framework also reframes contemporary discourse about the roles of AI in education. Recent literature has increasingly described GenAI as functioning in roles such as tool (Sarkar, 2023), hybrid actor (Dellermann et al., 2019), or partner/agent (Krakowski, 2025). While such descriptions capture aspects of how GenAI may be used in practice, they should not be interpreted as ontological claims about what GenAI systems are or can become. The framework developed here offers an alternative interpretation grounded in teachers' professional work. From this perspective, perceived roles of GenAI systems are not their inherent properties. Rather, they emerge from teachers' schemes across two levels of genesis: how teachers interact with the GenAI system through instrumental genesis and how they work with AI-generated materials through documentational genesis. GenAI may be experienced as a tool when teachers direct how they interact with the GenAI system and their documentation schemes emphasize rigorous validation and adaptation. It may also act as a hybrid actor when teachers allow GenAI outputs to shape instructional direction while retaining evaluative authority. It may be also perceived as an agent when teachers use AI-generated materials with limited scrutiny. In all cases, however, the locus of agency and responsibility remains with teachers' schemes of action. What changes is not the ontology of the GenAI system but the AI-mediated activity through which the system and its outputs are taken up. This reinterpretation allows the framework to critically examine GenAI use in mathematics teaching while maintaining a clear focus on teacher agency and epistemic responsibility.

The framework also provides important implications for empirical research. By theorizing teacher's work with GenAI as a dual genesis process and conceptualizing AI-generated materials as provisional resources, it provides a structured lens for designing studies that examine teachers' use of GenAI for instruction over time. Researchers can investigate, for example, how teachers' utilization schemes for interacting with GenAI systems develop, how these schemes shape the production of provisional resources, and how documentation schemes evolve as teachers evaluate, organize, adapt, and stabilize AI-generated materials. Longitudinal designs are particularly well suited to capturing how teachers' schemes change and how different dual genesis activities give rise to different patterns of GenAI use.

The framework invites a more cautious and theoretically grounded approach to claims about the transformative potential of GenAI use in mathematics education. By situating GenAI within established theoretical constructs, the framework resists both technological determinism and uncritical enthusiasm. It emphasizes that the educational significance of GenAI depends not on the capabilities of the technology alone but on how teachers' utilization schemes evolve in response to it. In doing so, the framework contributes to a growing body of scholarship that seeks to understand AI use in mathematics teaching and learning.

In closing, this integrated framework reframes teachers' engagement with GenAI as a form of professional work that unfolds through a dual genesis process. By conceptualizing AI-generated materials as provisional resources, the paper highlights the epistemic and pedagogical

judgments required to establish the instructional legitimacy of these materials. In doing so, it shifts attention away from technological characterizations of GenAI and toward the evolving schemes through which teachers evaluate, adapt, and stabilize instructional resources, which opens new directions for research on mathematics teaching and learning in AI-rich environments.

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CONSTRUCTING MATHEMATICAL MEANING IN BLOCK-BASED PROGRAMMING: AN EXPLORATION OF HARD-CODING AND PARALLELISM

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While programming is advocated for fostering computational thinking in mathematics teaching and learning, students often face challenges when using programming tools such as Scratch to solve mathematical problems. This study analyzed task-based interviews with two sixth-grade students regarding the modelling of arithmetic sequences in Scratch. The findings reveal friction between mathematical intuition and computational concepts. Students resisted abstraction by manually calculating loop termination values rather than utilizing relational operators to define generalizable rules and struggled with computational parallelization. We conclude that a working Scratch program does not necessarily imply conceptual mastery, and educators should support students in unpacking the logical structures and mathematical relationships underlying programming artifacts.

Keywords: Computational Thinking, Computing and Coding, Problem-Solving, Algebra and Algebraic Thinking

Background

Constructionism and Learning with Object

This study is grounded in Papert's (1980) theory of constructionism. According to Papert's constructionism, learning is optimized when a learner is engaged in the construction of an external public entity, object, for example, a computer program or the produced outputs. A computer program can serve as an object-to-think-with that enables and supports students in applying and practicing abstract mathematical concepts and problem-solving strategies. In the Scratch programming environment, for example, students can have access to the program scripts and outputs on the interface at the same time, and they can manipulate, reuse, and remix the scripts to achieve certain mathematical goals. Programming also supports a safe learning space where mistakes are common and acceptable. When a student's program does not yield the desired outcomes, the visual feedback is instantaneous and non-judgmental for the student and motivates them to improve the program. Students are encouraged to explore alternatives, modify unproductive choices, and improve mathematical reasoning and understanding during the problem-solving process (Resnick et al., 2005).

Moreover, Papert (1980) introduced the concept of syntonic learning, which refers to learning that resonates with the learner's self-perception. For example, a programming tool such as LOGO (a robotic or on-screen turtle that draws lines) facilitates body syntonicity, allowing students to project themselves physically into the computational object. By imagining, "If I were the turtle, how would I move?", students engage their own spatial and bodily awareness to navigate mathematical problems, thereby grounding abstract mathematical reasoning in that computational object. The object simultaneously represents a "resonance" between external forms and concepts and what students know about themselves (Papert, 1980; Watt, 1998).

Challenges in Integrating Computational Thinking and Mathematical Reasoning

Computational Thinking (CT) in K-12 education is increasingly conceptualized as an interplay, a reciprocal relationship in which CT is a vehicle for mathematical meaning-making

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

(Weintrop et al., 2016). However, Cui and Ng (2021) conducted an empirical study involving eight students in Grades 5–6. They demonstrated that the challenges faced by primary students were compounded by the necessity to learn in a block-based programming environment while simultaneously applying mathematical concepts and problem-solving skills within that context. Cui and Ng (2021) also found that a programming-enhanced environment does not guarantee mathematical engagement in a programming-enhanced environment. They concluded this as a potential conflict in which the student engaged in trial-and-error or tinkering with the computer interface in the programming environment. This behavior is not a mathematical reflection of the deep-thinking strategy; rather, it is a superficial CT strategy that only partially replaces it. This means that the challenge for the teacher is to prevent students from exceeding this superficial tinkering stage in real mathematical teaching when incorporating programming-enhanced mathematics problem design.

Despite increased advocacy for the inclusion of CT in mathematics classrooms, the specific cognitive mechanisms that promote or hinder this confluence remain under-represented. Ye et al. (2023) performed a systematic analysis of previous empirical studies and found that in the fields of mathematics and STEM, more empirical research is needed to prescribe ways that would help educators incorporate CT in formal K-12 curricula. Research on CT is in its early stages, with few studies specifically addressing the importance of programming in helping particular mathematical learning content. Therefore, this study aims to clarify the underlying difficulties that can stem from mathematics and CT in the programming setting. Thus, this study addresses the following question: What challenges do upper-primary students perceive in their mathematical reasoning in a block-based programming environment?

Theoretical Framework

To analyze the specific challenges students face, this study draws on Brennan and Resnick's (2012) computational thinking framework, focusing on the unique affordances and demands of block-based programming environments. Given that visual and structural features significantly shape how learners engage in computational thinking on programming platforms, we operationalized the interplay by focusing on the interplay between two different practices drawn from Brennan and Resnick's framework dimensions, including computational concepts and computational practices, and recontextualizing them for a block-based programming environment. The first is testing, debugging, and "hard-coding." Testing and debugging are core computational practices (Brennan & Resnick, 2012). For this study, the students used computational models, such as the Scratch program, to understand abstract mathematical concepts and used their problem-solving strategies to find and test solutions by reusing and remixing the program while applying mathematical reasoning. Cui and Ng (2021) argued that during troubleshooting and debugging processes, students normally use mass mental calculations to execute the arithmetic and simply treat the computational tool as the output display. Therefore, building on Cui and Ng (2021), we define "hard-coding" as a problematic debugging practice in which students bypass computational logic.

Furthermore, how students transform mathematical reasoning into computational concepts, such as iteration, variable, and parallelism, is still under study (Brennan & Resnick, 2012). For example, in traditional mathematics, a variable denoted as x serves as an abstract placeholder. This abstraction often poses challenges for beginners who struggle to shift their perspective from arithmetic, where math is a series of calculations, to algebra, where math involves abstract structures and relationships. Conversely, in block-based programming, a variable serves as a tangible container that can be instantiated, invoked, and visually traced instantaneously on the

Scratch interface, especially when students remix the programs. Dragging variable blocks into the coding script converts the abstract notion of a variable into a concrete, manipulable visual container. Unlike mathematics, the Scratch programming environment integrates the concept of parallelism, enabling the execution of sequences of instructions concurrently. This capability facilitates parallelism across various objects, such as scripts for different sprites within the same program (Brennan & Resnick, 2012). Because parallelism is absent from traditional mathematics curricula, we utilized this construct to analyze the cognitive hurdles students face when translating linear mathematical reasoning into concurrent computational concepts.

Methods

Research Design

This study used a qualitative case study method to examine how students reasoned mathematically and computationally while solving programming-based mathematical tasks. Two semi-structured task-based interviews were conducted with a pair of sixth graders (aged 11–12). They worked together in the interview, collaborating to complete tasks that required them to create two arithmetic sequences and find the point at which the two sequences intersected. The researcher used a semi-structured interview protocol to support students' exploration and collaborative problem-solving rather than providing direct solutions. Open-ended questions like "Why do you think so?" and "Can you show me what you mean?" were used to encourage students to think aloud and clarify their thought processes during discussions.

Participants

The study involved a pair of students, Zoe and Daniel (pseudonyms), from a public primary school in the Chinese mainland. We used purposeful sampling to select students who fulfilled the profile of novice programming learners, who had basic fluency with Scratch, which they learned through regular computer classes at their school, but who lacked advanced computational concepts and problem-solving practices. The selection criterion was to ensure that the study could apply mathematical reasoning and problem-solving strategies to learn arithmetic sequences in a programming-enhanced environment, rather than trying to test already known ideas.

Task Design

The main task was structured into three sub-activities to progressively scaffold students' understanding from operationalizing simple recursive rules to modelling and simulating arithmetic sequences (see Figure 1).

Activity 1: Technical scaffolding for variables and lists. A warm-up task was conducted to reduce the interference of technical challenges. Students were introduced to the variable and list blocks needed to log data: the variable blocks were used to store the current value, and lists were used to store a history of variable values as the program ran. The student dyad was asked to create a simple sequence of integers from 1 to 10. This stage ensured students were convinced that a loop needs an initialization, using the blocks "set [variable] to [1]" and "change [variable] by [1] step."

Activity 2: Mathematical scaffolding for describing the first term and the common difference. The students were asked to remix their program to generate specific sequences of 3, 6, 9, … and 3, 8, 13.… The researcher explicitly guided students to map the mathematical concept of the first term a_1 and the common difference d to the coding. Furthermore, to challenge the robustness of their model, students were tasked with creating a sequence with an initial term greater than 30 and ending with the first negative number. Students needed to conceptualize the pattern as a negative integer value using the "change variable by [ ]" block to bridge the gap between subtraction and adding negative integers in the loop.

Activity 3: Influencer battle simulation. This activity asks students to simulate a competition game that models a social media scenario. The scenario is presented as follows: two videos compete for "likes:" one is the "sleepy cat," who starts with high initial popularity with 500 likes but grows 10 likes per hour. Another is the "dancing cockroach," who begins with low initial popularity of 50 likes but grows 55 likes per hour. Students were asked to create a Scratch program that simulated the hourly growth of both videos and determined the number of hours when the two videos had the same number of likes. Through their exploration and discussion, students were expected to recognize that while the initial value of the "dancing cockroach" video is lower, its greater rate of change will enable it to catch up with the "sleepy cat." To support peer collaboration, the students worked in pairs and were encouraged to use a think-aloud approach in the interview. This open collaborative environment allowed them to freely discuss their mathematical reasoning, co-construct computational logic, and jointly solve problems, which facilitated the subsequent exploration of their collaborative routines.

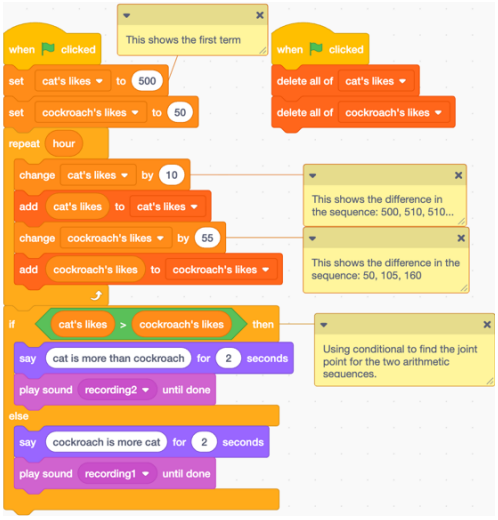


Figure 1: The Sample Script for the Increasing and Decreasing Sequences

Data Collection and Analysis

In the task-based interview, the children's programming activities on the computer were recorded on screen, and their written work was captured by another camera next to their desk. After the interviews, the videos were transcribed word-for-word, including all gestures, on-screen actions, and interactions. We analyzed the students' reasoning and practices using NVivo, along with researchers' reflective notes to observe changes in their thinking processes. We used thematic analysis (Braun & Clarke, 2006) to analyze students' discourses and problem-solving practices. We focused on moments of confusion, troubleshooting, and debugging, where the program had issues and students worked together to fix them. The coding process had two phases: first, creating initial codes based on students' expressions, and second, developing initial themes. Inductive coding revealed new insights into students' real problem-solving experiences. Deductive codes were based on Brennan and Resnick's (2016) framework, which includes computational practices, such as using computational models for testing and debugging, reusing, and remixing, and computational concepts, such as sequences, loops, and parallelism.

Results

We report on the interaction between Zoe and Daniel, who worked on three programming-based mathematical activities. We chose this case because their exchange showcases how their mathematical reasoning conflicted with the computational concepts and practices in a block-based environment. In this section, we outline their reasoning process through two separate excerpts from the task-based interview.

Excerpt 1: The Cognitive Gap Between Arithmetic Operations and Algorithmic Thinking

In Activity 2, students were asked to create a decreasing arithmetic sequence from a number greater than 30 to the first negative number. As shown in Figure 2, students realized that every term should be smaller than its previous term by their mathematical intuition and thus required a negative common difference. However, while establishing a termination condition for the conditional loop, they hard-coded a specific termination value of "–4" by calculating it rather than using a logical operator in the loop. Hence, the program went into an infinite loop because the computer was unable to stop the process if there was no –4 in the result list. The excerpt is shown below:

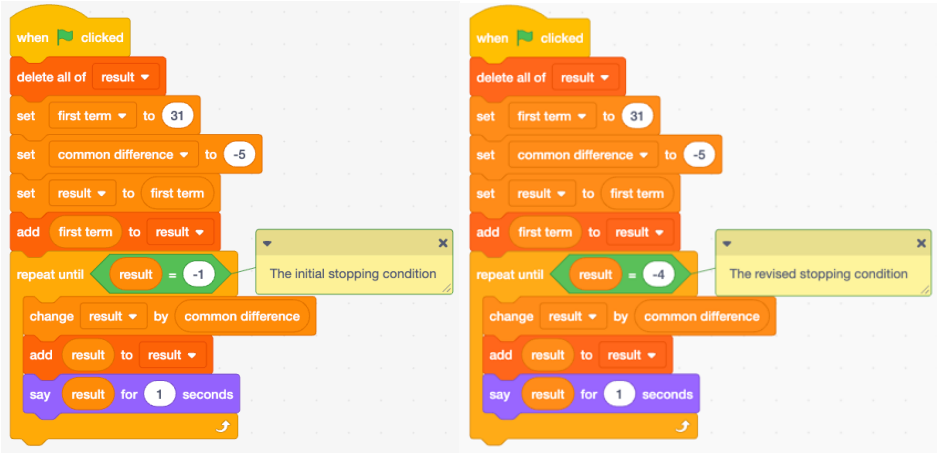


Figure 2: Hard-Coding the Decreasing Arithmetic Sequence

[Daniel sets the common difference to –5, the first term to 31, and sets the repeat loop until result = –1]

Daniel: Hmm? No, that's wrong. I need to set it to –4.
Interviewer: It must end with the first negative number.

[Students were working together to debug the program.]

Interviewer: Is that correct? I thought Daniel was doing fine just now. At first, This repeat block was set to end with –1. Why did you change it to –4?
Daniel: I made a very basic mistake at the beginning.
Zoe: No, he just calculated it wrong.
Zoe: Because it has to repeat until the first negative number.
Interviewer: So the list we see must go up to the first negative number.

This excerpt illustrates the clear interaction between mathematical reasoning and computational thinking. The students correctly identified the inverse relation needed for the

sequence. However, Daniel used a mental calculation to set the loop stopping condition to –1, but the outcome mismatched with what he intended, which illustrated a lack of abstraction in the loop stopping condition. When Daniel realized that he had made a calculation error, he carried out the calculation mentally to identify that –4 was the first negative number and used this to set up the stop repeat condition. Instead of leveraging the computer's work to automate the process, the student hard-coded a solution. He relied on his own mental math to determine the exact stopping condition, essentially doing the work for the computer rather than letting the program calculate it.

Excerpt 2: From Parallelism Confusion to Serial Logic

In Activity 3, Zoe and Daniel modelled the growth in likes for the "sleepy cat" and "dancing cockroach" competition game. Initially, the students created two separate programs triggered by the same "when this sprite clicked" event (see Figure 3). In the following excerpt, the participants attempt to synchronize the arithmetic progression of two sprites.

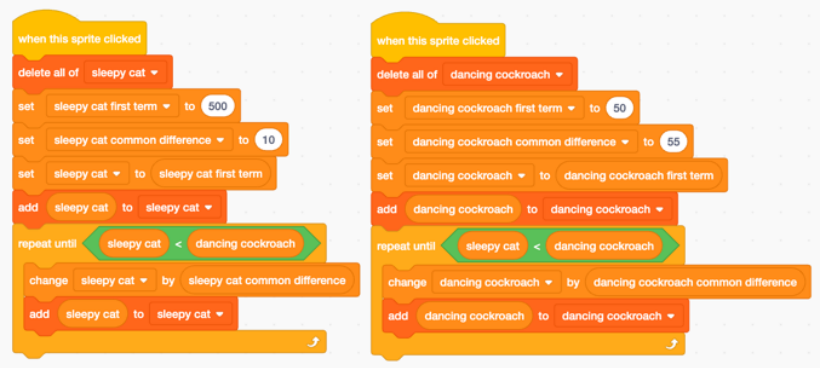


Figure 1: Two Separate Programs Created by Students in the Activity 3

Zoe:	It can keep moving, but it won't stop. It won't stop.
[…]	
Interviewer:	Is there a stop at the very bottom?
Daniel:	No, the [list] length is 200,000.
Interviewer:	Why? Because you did not define the loop [condition].
Zoe:	Right.
Daniel:	You can't scroll to the bottom.
Interviewer:	If you find that running these two in parallel is causing issues, you can try merging them. But be careful.
Zoe:	Oh, merge! Anyway, these three blocks need to move over. The first term is wrong here, this is for dancing cockroach. These three definitely need to be overcome; they are the initial information. Bring them here… and then set the sleepy cat… I'm trying, okay? [look at Daniel]
Daniel:	There's an empty spot down here.
Interviewer:	Does it stop now?
Zoe:	It stopped.

[Zoe and Daniel were very happy because it worked this time.]

Interviewer:	Look, the cat starts at 510, ending at 610. Okay, dancing cockroach stopped at 655. So that means it took 12 hours for it to stop.
Zoe:	I am so awesome!
Daniel:	Why were there so many problems before?
Zoe:	You mean why was there so much stuff before?
Daniel:	I mean, when we did those two separately earlier, why were there so many problems?
Zoe:	Yeah, why couldn't we succeed when they were separated? [Both students were confused on the parallelism problem.]

This excerpt highlights an important disparity between students' syntactic success and their conceptual understanding. Although they successfully fixed the bug by following the interviewer's hint to "merge" the two scripts into one, their final discussion revealed that they did not actually understand how parallelism worked. They remained confused about why their original, separate scripts conflicted and failed to work together.

Daniel's last question, "I mean, when we did those two separately earlier, why were there so many problems?" demonstrated that their decision to switch from parallel to serial execution was based on trial and error rather than a deep understanding of the codes. The students struggled to conceptualize how two independent loops interacted in real time when evaluating a shared condition, such as whether the sleepy cat was smaller than the dancing cockroach. The parallel scripts were viewed as two independent worlds rather than considering the need for synchronization of the state of the shared variables when two or more threads relied on them.

Transitioning from Arithmetic to Algebraic Thinking through Block-based Programming

More importantly, we argue that the programming environment gradually served as a mediator in the transition from arithmetic logic to algebraic rationale. When confronted with a query regarding the long-term result, the pupils abandoned the immediately prior emphasis on total quantity calculation and adopted Zoe's own use of the code as an epistemic practice, directing participants immediately to the script itself: "[The] cat begins high (points to set to 500), but it's slow (points to change by 10). [The] cockroach begins low [but] this block (change by 55) equals much bigger." This suggests that the visual design of block-based programming acts as a helpful mediator between arithmetic and algebra. Specifically, the clear distinction between the "set to" block (representing the starting value, or intercept) and the "change by" block (representing the growth rate, or slope) helps students move beyond simple mental arithmetic calculations to understand the mathematical principles governing the system.

Discussion and Conclusion

This study examines the interaction between mathematical reasoning and computational thinking during modeling simulation tasks. The findings indicate that while block-based programming environments like Scratch facilitate simpler algorithmic expression, they expose a specific challenge when students' mathematical fluency can interfere with computational thinking when the two domains require different types of reasoning. First, mathematical fluency can hinder computational thinking if certain CT practices are missing. For example, in the arithmetic sequence task in Excerpt 1, students identified the mathematical relationship for a negative difference correctly. However, they relied heavily on mental calculations, which prevented them from engaging in abstract thinking (Weintrop et al., 2016). As Papert argued that ''computer presence could contribute to mental processes not only instrumentally but in more essential, conceptual ways, influencing how people think even when they are far removed from

physical contact with a computer'' (Papert, 1980, p. 4). By hard coding the loop termination value, students viewed the computer as a static tool rather than an interactive problem-solving partner. This aligns with research showing that novice programmers struggle to make the cognitive shift from calculating the arithmetic answer to defining the rule (Noss & Hoyles, 1996), a crucial step for both computational algebraic thinking. Second, challenges with computational parallelization were noted in Excerpt 2 (Brennan & Resnick, 2012). In mathematics, two functions change simultaneously and continuously, but in computing, simultaneous execution is hidden in the programming environment. Daniel's frustration with why parallel scripts failed and merged scripts worked indicates that students didn't grasp the parallelism issue. This suggests that without explicit instruction bridging mathematical and computational thinking, students might see parallel scripts as separate mathematical entities rather than interacting computational processes. These findings point to a gap that teaching practices need to explicitly address directly, specifically how students move from mathematical formulas to algorithmic thinking. Instead of assuming that writing code will automatically develop algebraic thinking, educators should focus on using relational operators like "smaller than" or "bigger than" instead of specific values to define loop-stopping conditions. Additionally, teachers should differentiate between how mathematics works simultaneously and how code runs sequentially. Clarify this helps students stop relying on trial and error and guessing and instead integrating mathematical and computational thinking.

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IN-SERVICE MATHEMATICS TEACHERS' PERCEPTIONS ABOUT USING AI IN MATHEMATICS EDUCATION

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As generative artificial intelligence (Gen AI) tools such as ChatGPT, Gemini, and Copilot become increasingly integrated into K-12 classrooms, understanding teachers' perceptions is critical. Guided by the TPACK framework, this action research study examined how six in-service mathematics teachers engaged in a professional learning community (PLC) focused on AI use in mathematics education. Data included reflections, AI-generated lesson artifacts, and PLC transcripts. Findings indicate a shift from cautious optimism to a more nuanced view of AI as a productive yet incomplete collaborator. While teachers valued AI for efficiency and idea generation, they emphasized the need for professional judgment, AI literacy, and sustained professional development to ensure rigorous and equitable mathematics instruction.

Keywords: Professional Development, Technology, AI Integration

Purposes of the Study

Artificial intelligence (AI) is already being integrated into K-12 education through tools—such as ChatGPT, Gemini, and Copilot—that teachers use to support instructional planning, develop instructional materials, and promote student engagement (Holmes et al., 2019; Pedró et al., 2019; U.S. Department of Education, 2023). Some examples include generating lesson plans, crafting contextualized mathematical problems, producing mathematical stories, and creating visual representations. These can inform how teachers design tasks, represent mathematical ideas, and connect mathematics to meaningful contexts.

As AI becomes more prevalent in classrooms, scholars and policy makers emphasize the importance of developing AI-literacy skills among K-12 students (Long & Magerko, 2020; Pedró et al., 2019; Touretzky et al., 2019; U.S. Department of Education, 2023). AI literacy involves understanding how AI systems generate outputs, and critically evaluating AI limitations, ethical implications, and potential biases (Long & Magerko, 2020; Pedró et al., 2019). In mathematics classrooms, AI literacy is especially relevant when students' interact with AI-generated responses that may influence mathematical reasoning and sense-making.

There is a need for teacher professional development (PD) to help teachers support ethical and effective AI-use in K-12 classrooms which can directly influence students' development of AI-literacy skills (Ding et al., 2024). Learning about teachers' perceptions of AI in education can help to facilitate their learning (Aga et al., 2024). Further, there is a technology divide not just due to school resources but also due to teacher knowledge of technology, in particular AI use (Tan & Tang, 2025). To learn about teachers' mindsets and skills towards AI use we investigated how 6 in-service teachers engaged in conversations about AI use in mathematics education and developed activities for AI based mathematics instruction. Our research question was: *What are mathematics in-service teachers' perceptions about using AI in mathematics education?*

Theoretical Framework

The Technology, Pedagogy, and Content Knowledge (TPACK) framework, developed by Koehler and Mishra (2005, 2009), extends Shulman's (1986, 1987) notion of pedagogical content knowledge by incorporating technology into teachers' professional knowledge. Rather

than treating technology as an add-on, TPACK conceptualizes teacher knowledge as an integrated system where technology, pedagogy, and content dynamically interact. This framework is relevant for examining mathematics teachers' perceptions of AI, as AI tools increasingly influence instructional decision-making (Holmes et al., 2019), student engagement (Steenbergen-Hu & Cooper, 2014), and sense-making (Drijvers, 2015; Pedró et al., 2019).

Within TPACK, content knowledge (CK) refers to teachers' understanding of mathematical concepts and practices; pedagogical knowledge (PK) encompasses instructional strategies, discourse facilitation, assessment, and support for diverse learners; and technological knowledge (TK) includes familiarity with digital tools and ability to adapt to emerging technologies. In AI contexts, TK includes operational skills, understanding how AI systems function (e.g., automated feedback, generative problem creation, and adaptive learning) and recognizing their limitations. TPACK represents teachers' integrated understanding of how AI can support mathematically meaningful and pedagogically sound instruction.

The intersection among these domains is foundational to understanding TPACK: technological content knowledge (TCK) involves how AI transforms mathematical representations and problem-solving pathways, while technological pedagogical knowledge (TPK) addresses how AI reshapes instructional practices, including scaffolding and formative assessment (Koehler et al., 2013).

A defining feature of TPACK is its emphasis on context. Teachers' perceptions of AI are shaped by curricular demands, classroom norms, ethical concerns, and beliefs about student agency and reasoning (Selwyn, 2019). Research suggests that teachers may perceive advanced technologies as undermining the integrity of mathematical reasoning or diminishing teachers' instructional role unless they are supported in examining how technology intersects with pedagogy and content (Graham, 2011). TPACK is a useful theoretical lens for investigating mathematics teachers' perceptions of AI integration by highlighting the complex, situated knowledge required to leverage AI in ways that promote deep mathematical understanding.

Methods

We employed action research methodology (Stringer & Aragón, 2020) to examine and improve practice through systematic inquiry within a professional learning context. Action research supports practitioners in investigating their own settings and generating actionable knowledge to inform instructional improvement (Mertler, 2009). The meetings took place virtually during the Fall 2025 semester and were facilitated by a mathematics teacher educator (MTE) at a public university in the Mid-Atlantic U.S.

Our participants included six in-service teachers (five at the high school level and one at a community college) who engaged in four PLC meetings to discuss AI use in mathematics education. We analyzed four PLC meeting recordings, written reflections about AI in mathematics education, and teacher-developed AI activities. We conducted open coding (Corbin & Strauss, 2014) of PLC meetings and reflections to look for emergent themes regarding perceptions about AI use in mathematics education. We also employed thematic analysis (Guest et al., 2011) to examine patterns in teachers' pedagogical use of AI.

Findings

Initial Ideas about AI in Mathematics Education

PLC data revealed a pattern of cautious optimism toward integrating generative AI (Gen AI) in mathematics education. Many teachers described using AI to develop lesson plans, engaging tasks, examples, datasets etc. AI was frequently characterized as a time-saving tool and a helpful

"jumping off point" for instructional ideas. Participants also viewed AI as a potential personalized tutor providing students with alternative explanations and additional practice.

Participants expressed significant concerns about students' overreliance on AI. A dominant theme was the tension between AI as a support versus a shortcut. Teachers worried about student AI use to obtain answers without developing conceptual understanding. Issues of cheating, diminished critical thinking, and unclear boundaries for acceptable use were repeatedly noted. Some participants expressed hesitation about becoming dependent on AI in their own teaching. Overall, they perceived AI as a promising yet complex tool, valued for efficiency and support, but raised concerns about maintaining deep mathematical thinking and responsible AI use.

Engagement in Using GenAI to Develop Lesson Plan or Activity

Engagement in PD (e.g., the calculator activity) was eye opening for teachers and they learned about the predictive nature of AI (PLC meeting 1). Teachers worked in teams to develop mathematics activities throughout the semester. They used caution when considering student engagement with AI, as a thought partner. Teachers in their teams decided on learning goals and math standards, considered implementation strategies and questions they would need their students to answer. They considered if their students would use AI to learn a new concept or critique AI work using existing knowledge. Such questions allowed the teachers to consider what features can allow students to engage with AI as a supportive collaborator (PLC Day 3).

Teachers also learned how to use AI to develop comics with the goal to develop students' mathematical identity. Teachers used the comic generation capacity of GenAI tools to consider ideas such as using comics to discuss social justice issues, helping students learn about practical applications of mathematics, financial literacy, using comics to help students engage in mathematical discussions, and making word problems more accessible (PLC Day 4).

Reflection on Using GenAI After Developing Lesson Plan or Activity

After using AI to generate a mathematical activity, teachers described AI as a collaborative partner and idea generator, rather than a replacement for teacher expertise. Participants reported using AI iteratively. They began with an initial prompt and refined outputs by adding specific conditions, parameters, and revisions. Several characterized AI as a "starting point" or "curriculum developer" that helped structure lessons, suggest pacing, generate differentiation strategies, and propose student engagement ideas. Teachers appreciated AI's efficiency, its ability to generate ideas they had not previously considered, and its responsiveness to revisions. Participants noted that AI was helpful for brainstorming, organizing lesson components, and quickly producing a draft that could be improved through professional judgment.

At the same time, teachers emphasized the need for critical evaluation and adaptation. Common concerns included lack of detail, limited alignment with specific learner needs, superficial engagement with mathematical content, and unrealistic classroom assumptions. Participants reported that AI-generated materials were often "thin" in instructional depth and required substantial refinement to ensure the integrity of conceptual understanding and contextual relevance. Several also noted that AI lacked knowledge of their particular students' interests and classroom dynamics. Overall, teachers positioned AI as a productive but incomplete collaborator, which is valuable for efficiency and idea generation, yet dependent on teacher expertise to ensure quality, relevance, and meaningful mathematical learning.

Questions, Concerns, and Ideas from PLC Meetings

Teachers shared that schools are in the process of developing AI use policies. Some teachers mentioned having access to AI chatbots in schools, but student AI use is not allowed. Districts are considering pros and cons of AI aided education and proceeding with caution (PLC Day 3).

Teachers mentioned using AI to generate ideas for planning instruction (e.g., differentiation, real world activities, etc.) (PLC day 2). They also mentioned using AI for administrative tasks like writing letters of recommendation, generating assignments for students (PLC meeting 1).

We found a variation in teacher knowledge, skills and their familiarity with AI tools. Not all teachers were at the same level of comfort or experience using AI based educational tools or using GenAI in general. Some teachers were scared of using AI and mentioned that they are not "tech people". All teachers needed guidance to engage with AI. The PLC allowed an opportunity for teachers to discuss AI use in a safe space. Some teachers compared it to calculators; others compared it to search engines such as Google (PLC meeting 1). There is a need to provide PD for teachers to level the playing field for students. During the PLC, teachers worked in collaboration with other teachers. Such collaboration can help them brainstorm ideas and critique AI work. In particular, the access divide may decrease when teachers scared of technology, work with teachers who are experienced and accepting of new technology (PLC Day 2).

Teachers also provided recommendations for student AI use. They brainstormed what AI skills students need (PLC meeting 1) and expressed caution about student AI use (PLC Day 2). Teachers perceived challenges about students using AI to learn math. This included language barriers for ELL students, if they would be able to communicate as well as native English speakers, if students' literacy skills would cause a barrier in developing prompts, time and pacing concerns, and student motivation concerns. Teachers stressed that students are not used to learning mathematics by reading from books, they see worked out solutions in classroom or work with their peers to discover math concepts. However, in all these cases the teacher is present for guidance. Independent student engagement with AI would depend on many factors that can develop an access and ability divide both for learning math and using AI (PLC Day 3).

Discussion and Conclusion

Findings indicate that teachers' perceptions of GenAI in mathematics education shifted from cautious optimism to a more nuanced view of AI as a productive yet incomplete collaborator. Initially, teachers valued AI for efficiency, idea generation, and differentiation, while expressing concerns about cheating, overreliance, and reduced conceptual understanding. In subsequent lesson design experiences they positioned AI as a drafting partner that required iterative prompting, critical evaluation, and professional judgment to ensure conceptual understanding and alignment with student needs. PLC discussions also revealed variability in teachers' confidence and familiarity with AI, suggesting a divide rooted more in knowledge and mindset than access. The collaborative PLC structure helped mitigate this gap through experimentation, dialogue, and reflection, supporting intentional design of AI-supported tasks that emphasized learning goals, equity considerations, and use of AI as a thought partner rather than a shortcut.

Overall, teachers perceived AI as a promising instructional tool for planning, differentiation, and engagement, but also as an instructional tool that requires careful oversight to maintain deep mathematical thinking and equitable access. These findings carry important implications for policy and professional learning. As districts develop AI-use policies, guidance should move beyond restriction toward clarifying responsible, instructional use for both teachers and students. Sustained and collaborative PD is essential to strengthen teachers' AI literacy and reduce disparities in readiness. Moreover, since teachers identified concerns related to language barriers, prompt-writing skills, motivation, and overreliance, AI integration should include explicit instruction in student AI literacy. Without intentional scaffolding and aligned policy support, the use of AI risks widening existing inequities in mathematics learning rather than enhancing it.

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TEACHING IN SPACE: DREAMING BIG DREAMS ABOUT THE MATHEMATICS CLASSROOM OF THE FUTURE

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In this report, we describe how seven secondary mathematics teachers recruited from public and private schools across a rural northeastern state imagined integrating an immersive virtual reality environment into their teaching practice. During an interactive focus group, secondary teachers tested a multiplayer immersive virtual environment and then were interviewed about their explorations of that environment. Participants were told that the goal of the session was to dream big dreams – i.e., they were urged to set aside concerns about time, resources, and curricular constraints, and to focus on the mathematics that would be possible to teach in the environment. In this report, we provide a design overview of the environment (TriO), describe teachers' activities within it, and report teachers' initial ideas about the mathematical learning opportunities it afforded.

Keywords: Technology, Geometry and Spatial Reasoning

Introduction

New technologies continually transform the kinds of mathematical explorations we can conduct in and outside of the classrooms. From the introduction of digital calculators in the 1970s, to the emergence of computer algebra systems, dynamic geometry software such as GeoGebra and Desmos, teachers and students adapted to the changing landscape of possibilities. In the past three decades, we have seen the rise of extended reality (XR) technologies being used to explore mathematical ideas from new lenses (Kaufmann et al., 2000; Dimmel et al., 2021; Walkington et al., 2025). Here, we imagine the possibility of transforming how we understand and explore space, an endeavor that has been elusive in classroom mathematics. Action-based design is a genre of embodied design that builds on learners' perceptuomotor capacity to develop new routines for solving a sensorimotor task (Abrahamson, 2014). By creating perceptuomotor schemas to engage in the embodied task, learners connect their intuitive actions to the formal concepts. *TriO* (Dimmel et al., 2025) is a virtual environment that tasks three participants with coordinating joint, distributed and asymmetrical action. Distributing the cognition across multiple people and a shared artifact can yield a combined cognition that no single person holds (Hutchins, 1995).

Environment Design

TriO extends the design of OЯTHO—a two-player, touch-based interactive digital game, wherein players work together to navigate a digital ball around a digital track by controlling either its vertical (Player 1) or horizontal (Player 2) location (Potega vel Żabik et al., 2024) to encompass 3-dimensional space. It is a fully three-dimensional, visually immersive, interactive

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

environment that was designed for simultaneous exploration by multiple immersed learners. This virtual environment features a small widget made out of three orthogonally intersecting discs, such that the position of the widget is controlled by the three color-coded handles. The goal is to coordinate the widget to touch the numbered starbursts to activate them. (Figure 1). Each handle controls a single degree of freedom of the widget's location in space along the x, y, or z axes (Figure 2). Three players working collaboratively can freely move the widget along any (continuous) path in R3. Participants could also activate 2D rectangular walls (Figure 3), which are sections of a plane that is orthogonal to the axis they controlled. Each wall is the same color as its corresponding handle, is semi-transparent, and is etched with coordinate lines.

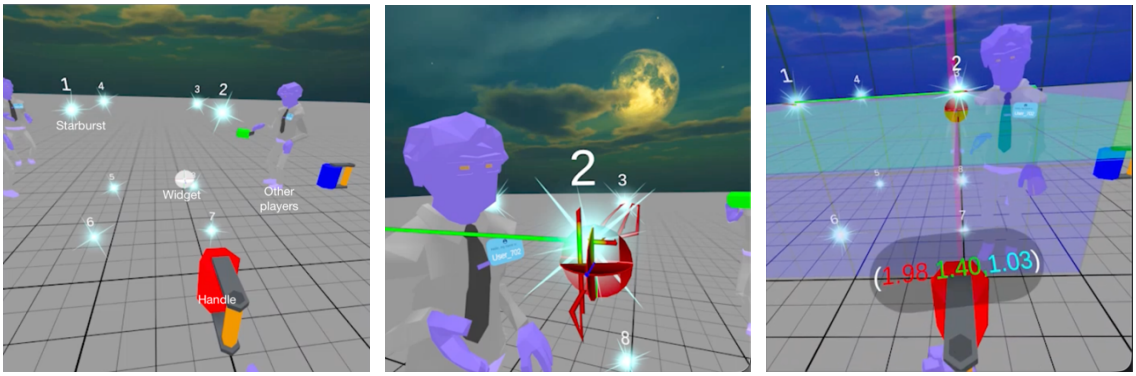


Figure 1 [Left]: The Elements of TriO
Figure 2 [Middle]: Trail Showing the Optimal Path Between Two Starbursts
Figure 3 [Right]: Semi-Transparent Walls and the Coordinates of the Widget

Data Collection and Analysis

We invited seven teachers for a three-day workshop (August 2025) in a rural northeastern state. This workshop marked the second year of our data collection and playtesting sessions we conducted with teachers, and only one teacher attended both years. The teachers participated in a series of focus groups in virtual environments. A member of the IMRE lab was present in the focus group to guide the discussion. The teachers were positioned as participant co-designers of the environments. The focus group sessions were both audio and video recorded.

Authors watched and highlighted portions of the video clips where teachers had productive struggles, made remarks on how to modify the environment for classroom use, or demonstrated understanding of mathematical mastery. We here selected two slices from a holistic view of the activities in *TriO* and their potential as a learning tool.

Teacher Experiences

Discursive Challenges in *TriO*

One of our teacher participants, Mike, noted the struggle of developing a common language for collaborating in *TriO.*

> Mike: I don't know if you've ever been driving with someone and they say, "Turn that way." You're like, "What way?" And they point out the windshield and say, "That way" and try to communicate. Well from my perspective, I don't know if we're pointing to the right or left. So that's very similar here and much lower stakes. We can maybe develop some

positive or negative perspectives because if I say left, left doesn't mean the same from standing across from perspective. So being able to develop that type of communication skills and perspectives is very important for these puzzles.

Mike is aware that saying "go right" or "go left" can be confusing to his teammates as they might be facing a different direction. Instead, he said "The blue plane needs to come towards me," while placing his finger on the widget and pointing towards him. When we investigated the footage closely, we saw as Mike got closer to the widget to gesture at it, his line of vision got blocked by the trails left behind by the widget. Additionally, due to identical looking avatars, it wasn't immediately clear who controlled which handle. Given these constraints, Mike didn't opt for simply giving cardinal directions; instead, he distributed the spatial reference across himself and the widget to guide his teammates. He elaborated on this communication challenge:

Mike: I think what would be an addition as far as helping communication is maybe if we had negative symbols. So, you could say, hey blue, go positive this much or this much.
Nigel: Just north, south, east, west.
Mike: Some sort of indication of direction.
Nigel: You could put a compass on the floor in the middle of like, hey this needs to go north.
Mike: If you just had plus minus, you have a center as origin, and you could say, "Hey, go positive, go negative."

The need for negative/positive direction labels in the coordinate system wasn't just a request from the teachers, it emphasized how traditional math diagrams come with default viewing angles (head-on, isometric view). This isn't obvious to the viewer but it shapes how we perceive the diagram. In our design we didn't anticipate this issue as diagrams of 2D coordinate systems don't label negative/positive directions explicitly. It is implicitly understood that up/right is the positive direction while down/left is the negative direction in a 2D coordinate system. But in *TriO*, one can move *around* the diagram, essentially flipping the orientation of positive and negative continuously. This shows that the semiotics of classroom diagrams are often taken for granted. A gentle change in dimensionality and perspective challenged what we might confidently know about coordinate systems.

Challenge of Simultaneous Coordination

In another exercise, teachers were prompted to coordinate drawing a helix around a rod, but technical difficulties didn't allow the widget to move smoothly. That's when Nigel suggested that they should try to draw a basketball hoop. To our surprise, this self-defined goal turned out to be very successful as three teachers coordinated to draw a basketball hoop without any visual reference. Their coordination was mediated by simple phrases such as "now we need to draw the backboard" and "we need a circle here." This moment underlined that joint action doesn't need explicit instructions to be successful. The teachers' coordination emerged from the shared idea of a basketball hoop and the artifact's support for joint, coordinated drawing in 3D space. This marked a clear instance of cognition distributed across the players and the artifact.

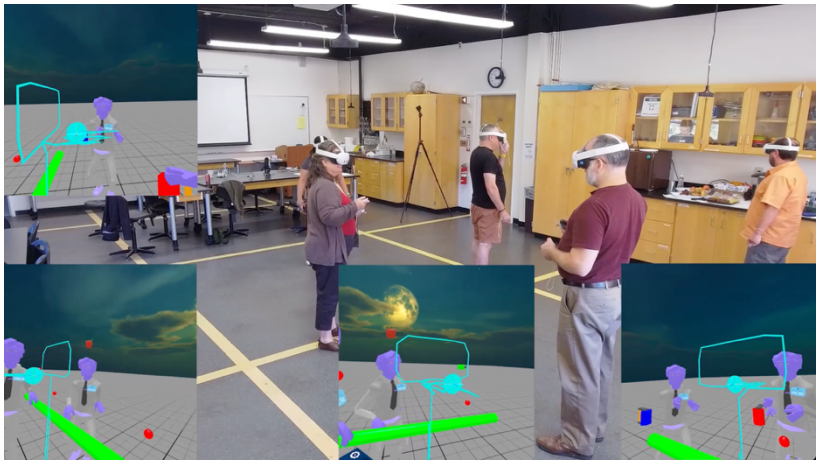


Figure 4: The Completed Basketball Hoop with a Backboard

Closing Remarks and Future Direction

What makes *TriO* a unique math tool is that the artifact is embodied, social, and distributed across subjects. Mathematical activity unfolds through coordinated movement, gesture, language, and perception. The space itself becomes a shared medium for reasoning. As participants engage with TriO, they develop sensorimotor schemes for orienting, aligning, and transforming within 3D space, alongside linguistic schemes for naming, describing, and negotiating spatial relationships. Meaning emerges through this dynamic interplay between action and discourse.

In many classroom settings, geometry remains confined to static diagrams and predominantly two-dimensional representations. Opportunities to reason within fully three-dimensional coordinate systems are comparatively rare. *TriO* addresses this curricular gap by offering a task that is accessible in its basic premise yet intellectually demanding in its mastery. The simplicity of the goal invites participation from all backgrounds, while the depth of spatial coordination required sustains inquiry and growth.

We see strong potential for *TriO* in K-12 education as a tool for cultivating spatial reasoning, embodied coordination, and precise spatial language. Beyond learning specific geometric content, students can have opportunities to experience mathematics as collaborative, enacted, and lived in space.

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LINGUISTIC PLASTICITY IN SPATIAL LEARNING ENVIRONMENTS

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Spatial and immersive technologies promise embodied connections between abstract mathematical entities and physical experience, yet interdisciplinary synthesis struggles due to terminological plasticity and hardware-centered descriptions. This systematic review uses a dual-layer coding scheme to index relevant articles from the ERIC database, identifying affordances studied masked by vague labels like "Immersive Virtual Reality" (IVR). Grounded in inclusive materialism, we argue that the failure to precisely index these digital architectures limits our ability to theorize the intra-actions of learners and digitally rendered realizations of mathematical objects.

Keywords: Technology, Mathematical Representations

Spatial and immersive digital technologies have promised embodied connections between abstract mathematical entities and our physical presence in the world. However, across educational research, the theoretical affordances of these technologies are inconsistently described in relation to their use in empirical studies of learning environment design across disciplines. This disconnect creates a barrier for researchers to compare and synthesize research with a broader educational research community.

Conventions in Spatial Display Nomenclature

The terminology for describing digitally rendered spatial environments is varied and plastic, with common references to underlying software and hardware technologies (i.e., Immersive Virtual Reality, Stereoscopic Virtual Reality, Spatial Computing, Augmented Reality, Mixed Reality, and Extended Reality). These catch-all terms mask fundamental differences in the affordances that are supported by different hardware and software technologies, the affordances that are included in a designed learning environment, and the ways in which these affordances might support learner agency over the digitally rendered environment. These terminologies also prioritize the reporting of the brand (Meta Quest, Apple iPad) or a generic term for the underlying technology (stereoscopic head-mounted display, tablet) over the specific spatial properties (e.g., scale, multimodal rendering, spatial navigation) of the "intra-actions" (de Freitas & Sinclair, 2014) that link the embodied use of a digital environment to mathematical meaning making.

The alignment with device marketing terminology serves as a challenge for the PMENA community to interface with researchers in other disciplines with shared interests in technology and learning environment design. In this brief report, we look across educational research disciplines, from medicine to vocational training, and ask how are these spatial properties of intra-actions with spatially realized digital environments reported in relation to teaching and learning?

Systematic Literature Review Method

To understand how affordances are described and studied across a broader set of educational literature, we conducted a systematic literature review following the PRISMA 2020 framework (Page et al., 2021). We identified 5,517 items in the ERIC database, which has an English

language bias but substantially reduces the amount of literature without relevance to teaching and learning captured by our query. Our query was broad in scope, searching for title and abstract matches for commonly used language to describe spatial rendering technologies.

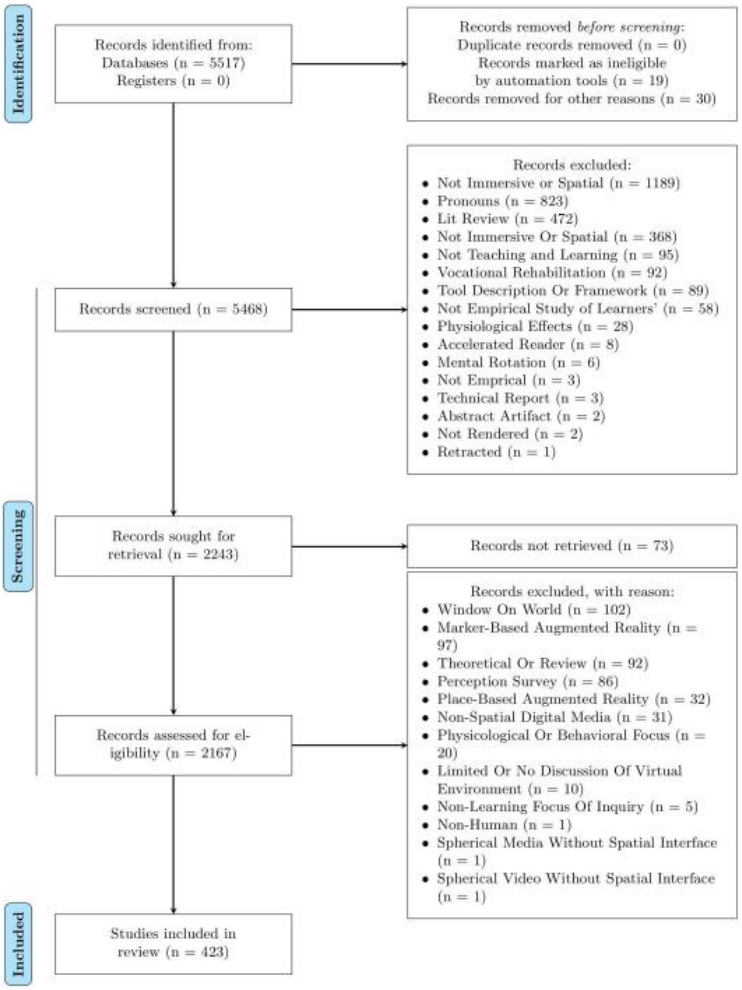


Figure 1: PRISMA Flow Diagram

Selection Process

We conducted an abstract review using ASReview (van de Schoot et al., 2021), where records are sorted in order of projected relevance every record's inclusion or exclusion decision is made by the human reviewer. At this stage, many records were excluded for use of initialisms or abbreviations for other meanings (e.g., "Mental Rotation," or "Mr.") or use of digital technologies without a spatial interface (e.g., descriptions of fixed window-on-world environments similar to *SecondLife* as "immersive virtual reality"). We focused our search to identify digital environments with spatial renderings or spatially embodied interfaces where *learners* could see, feel, or hear properties of the content that they were interacting with through time and space. Many items passed the abstract review phase for further full text screening due to limited descriptions that describe an intervention only in terms of ambiguous terms for the class of technology (e.g., the effects of Augmented Reality on learning).

We were able to retrieve 2243 records for full text eligibility analysis in Catchii (Halman and Oshlack, 2024), without any machine learning automation. For this preliminary analysis, 900 texts were randomly selected for eligibility analysis. We identified 423 records with some spatial element to either the realization of digital objects through either rendering or user interface technologies. Most records are excluded at this phase for using (1) a fixed window-on-world configuration with only two-dimensional rendering and interface, (2) target-based (often marker-based) augmented reality without a described a spatial component to the learner's interactions, and (3) placed-based augmented reality implementations that use GPS (or other spatial positioning) to trigger non-spatial digital media. Only 10 records did not have a detailed enough description of the designed environment to determine if there was a spatial component to their implementation. Some included records had limited descriptions but included figures of participants' use of the environment, or reference commercial or open-source software that documented spatial capacities, or reference other studies of the same designed environment with more thorough descriptions of the spatial capacities.

Coding Schema: Design Reporting and Spatial Affordances

The eligible reports were coded to identify broad trends in patterns of the affordances and learning-related constructs studied across disciplines. Additional descriptive codes were included capture how the authors represented their designed environment.

To contextualize each report, metadata about the publication date, sub-discipline, content area, population, methodology, and leaning-related constructs or outcomes were captured. Additional data was extracted from authors descriptions, or inferred from figures or references for the visual, auditory or haptic renderings system (e.g., Binocular Stereoscopic Displays). These were further described in terms of the Degrees of Freedom (DOF) of supported by the interface; we only considered 3DOF environments supporting rotational movement of the learner or 6DOF environments supporting both rotational and translational movements of the learner. This data supports additional temporal and cross-sub-discipline quantitative analysis, that will be conducted in a future stage of the project.

We captured the level of agency that students had over the representation, from a passive video viewing experience to collaborative interactions with digital objects. We also captured the types of interaction, from taking different perspectives, to manipulation of objects, to creation of new digital objects (Roussou et al., 2006). Finally, we captured the analytical granularity of the designed environments' use in the report's method, to distinguish studies treating the designed environment as a "black box" from studies identifying or isolating specific affordances.

Often information included in these codes was not directly stated in the reports but was inferred from the hardware used or elements of the design of the environment. We captured the language that the researchers used to describe the type of digital technology that they used for their designed environment. We also coded for the presence of a detailed qualitative description of the designed environment, references to a commercially marketed digital environment, the availability of source code, the inclusion of images or videos of the designed environment, or the inclusion of an a priori analysis of the environment.

A secondary round of meta-codes applies the 4E cognitive science framework (Anderson, 2025) to each record's representation of the learning environment as embodied, embedded, extended, and/or enacted, through their theoretical framing, methodology, results, and discussion. The 4E cognitive science framework provides a bridge between a diverse set of theories and disciplinary focus areas to strong strands of research into enactivism, new materialism and embodied cognition in the PMENA community. The 4E lens supports this

bridge with a focus on how processes of learning and knowing are studied, rather than passing trends in consumer hardware, emerging and disappearing software platforms, or disciplinary silos.

Future Analysis: Temporal and Disciplinary Trends

In the next stage of this analysis, we will conduct a temporal analysis of trends across modalities of digital-spatial renderings and the presence of descriptions of the affordances of the reported digital environments. A Bayesian changepoint analysis will be conducted using RBeast (Zhao et al., 2019; McNeill, 2022) and c2z (Skaar, 2023). The Bayesian changepoint analysis is expected to identify changes in research activity around availability of consumer hardware for underlying technologies. We are interested to identify any temporal changes in the specific affordances studied or the use of the underlying technology as a methodological black box.

We will also conduct a cross-discipline analysis to better how to study of the affordances of digital-spatial rendering technologies are distributed across educational research. This will identify affordances relevant to the teaching and learning of mathematics that may have been understudied in the mathematics education literature due to differences in linguistic conventions across disciplines.

Preliminary Findings: Linguistic Ambiguity

Many keywords used to identify literature relevant to spatial, digitally rendered spatial arguments. For example, "Immersive Virtual Reality" presents often throughout this dataset. These environments typically are rendered in a way that includes digital objects but not stimuli from the surrounding physical world, an affordance shared by most implementations of virtual reality. The immersiveness points to a sense of presence, but the implementation can vary widely. NeoTrie (Rodriguez et al., 2021) is a dynamic geometry-like environment in a virtual spatial environment rendered with consumer-grade stereoscopically rendered environment that uses 6DOF tracking for both rendering and embodied interactions with mathematical manipulatives. Other dynamic geometry environments with similar spatial affordances and lack of access to stimuli from physical surroundings may use the word immersive as a qualitative descriptor but not use IVR as a keyword (Walkington et al., 2025).

Astatke (2024) contrasts an implementation of immersive virtual reality as a 3DOF stereoscopic rendering of 360 video with "DVR" (Desktop Virtual Reality) using a monitor, mouse and keyboard to display and navigate a 360 video, in the context of "STREAM-learning." However, the same monitor, mouse and keyboard interface is often described as immersive virtual reality in literature excluded from our analysis during the full-text eligibility phase. These ambiguities were not recently developed. Slater (2003) cautions against confusing immersion and presence in understanding the relationships between affordances of digital environments.

We find similarly ambiguous language across variants of augmented reality, mixed reality, extended reality and other spatial rendering modalities. However, salient affordances cross-cut these technologies. The type of spatial presence and embodied spatial manipulation possible in NeoTrie may be better related to a shAR dynamic geometry implementation (Walkington et al., 2021), though it may be difficult to differentiate in a simple query with a substantially different set of digital-spatial interface affordances available in the tablet-based GeoGebra AR (Gustei et al., 2025).

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NOTICING STUDENTS' THINKING IN DYNAMIC GEOMETRY ENVIRONMENTS: PRESERVICE TEACHERS' DEVELOPING TPACK

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This study examines preservice mathematics teachers' learning as they engaged in dynamic geometry tasks and structured analysis of students' interactions within dynamic geometry environments (DGEs). Drawing on professional noticing, TPACK, and the Technology-Based Geometric Thinking Framework, the study explored how PSTs interpreted students' geometric thinking and how their reflections reflected TPACK subdomains. Data included written reflections, analyses of students' DGE activity, and pre/post surveys. Findings suggest PSTs increasingly framed DGEs as environments for exploring relationships and invariants. Noticing students' DGE activity emerged as a mechanism for supporting PSTs' TPACK development.

Keywords: Preservice Teacher Education, Teacher Noticing, Technology

Introduction

Technology is an essential part of a student's educational experience, especially in mathematics classrooms where it can deepen conceptual understanding, facilitate visualization, and provide multiple entry points to a task. However, we know that the use of technology alone will not improve learning. Rather, the decisions teachers make about how to use technology determine whether it positively impacts student learning. It is expected that "Well-prepared beginning teachers of mathematics are proficient with tools and technology designed to support mathematical reasoning and sense-making, both in doing mathematics themselves and in supporting student learning of mathematics" (Association of Mathematics Teacher Educators, 2017). Research on technological pedagogical content knowledge (TPACK) has documented growth in preservice teachers' knowledge through coursework and interventions (Mishra & Koehler, 2006; Niess, 2005; Harvey & Caro, 2017) as well as the importance of professional noticing in learning to attend to and interpret students' mathematical thinking (Jacobs et al., 2010). However, these areas remain largely disconnected. In particular, preservice teachers (PSTs) are rarely supported in learning to notice students' thinking as it emerges through interactions with digital tools (Dick et al., 2021; McCulloch et al., 2021).

While many researchers have documented growth in PSTs' TPACK through instructional interventions (e.g., Harvey & Caro, 2017), helping future teachers notice and interpret students' mathematical thinking when technology is involved has not been a central focus in teacher preparation programs (McCulloch et al., 2021). Although learning to notice students' thinking is widely recognized as an essential skill for teaching (Jacobs et al., 2010), relatively little research has examined how PSTs develop this skill within technology-mediated environments (Dick et al., 2021). This gap is particularly important in dynamic geometry environments (DGEs), which offer opportunities for making students' geometric reasoning visible by examining their tool interactions (Arzarello et al., 1998).

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Dynamic geometry environments (DGEs) function not only as tools for constructing mathematical representations but also as mediational spaces where students' geometric thinking becomes visible through their actions, representations, and problem-solving strategies (Arzarello et al., 1998; Dick et al., 2021). Students' dragging, measuring, constructing, conjecturing, exploring, and testing within DGEs can reveal their reasoning about geometric relationships, invariance, and structure (Hoyes et al., 2025). Supporting PSTs in learning to attend to these actions is central to developing their capacity to interpret students' geometric thinking in DGE contexts.

The purpose of this study is to examine how engaging PSTs in dynamic geometry tasks as both learners and emerging teachers, within an instructional sequence intentionally focused on noticing students' thinking, supports the development of TPACK. Specifically, this study investigates how opportunities to analyze students' interactions with DGEs influence PSTs' conceptions of the role of technology in teaching and their reasoning about instructional decision-making. This study is guided by the following research questions:

- RQ1. How and to what aspects of students' geometric thinking do PSTs attend when analyzing video of students interacting with DGEs?
- RQ2. How does engaging in analysis of students' actions with a DGE activity shape PSTs' reasoning about teaching and learning geometry?
- RQ3. How do PSTs conceptualize the role of DGEs in supporting students' geometric thinking?
- RQ4. In what ways do PSTs' reflections and survey responses reflect TPACK subdomains following participation in the instructional sequence?

Theoretical Frameworks

This study draws on technological pedagogical content knowledge (TPACK), professional noticing, and the Technology-Based Geometric Thinking (TBGT) Framework (Hoyes et al., 2025) as complementary perspectives for understanding how PSTs learn to interpret students' geometric thinking in DGEs. TPACK provides a framework for understanding the integrated knowledge required to teach mathematics with technology, emphasizing the interplay among technological, pedagogical, and content knowledge domains (Mishra & Koehler, 2006). Professional noticing describes how teachers learn to attend to, interpret, and respond to students' mathematical thinking (Jacobs, Lamb, & Philipp, 2010). In this study, these perspectives are brought together to examine how PSTs' reasoning develops as they analyze students' actions and geometric thinking in DGEs.

To support PSTs' engagement and potential growth, the instructional sequence incorporated the TBGT Framework, which was designed to guide PSTs' attention to key aspects of students' engagement and geometric thinking in the DGE activity. The framework directed PSTs to consider students' interactions with the DGE, their verbal and written explanations, and the mathematical ideas reflected in those interactions. In this way, the TBGT Framework positioned noticing as an interpretive practice grounded in examining students' actions within DGEs rather than evaluating correctness or task completion.

Within this study, the TBGT Framework supported PSTs in learning what to attend to when analyzing students' interactions with DGEs and what those interactions might reveal about students' geometric reasoning. The TBGT framework-oriented PSTs toward interpreting students' actions and explanations rather than focusing on correctness or task completion. In

doing so, it created opportunities for PSTs to consider how technology, pedagogy, and mathematical content intersect when making sense of students' thinking.

Methods

This study was conducted within a geometry unit of a technology course for secondary PSTs; fifteen PSTs from that course agreed to participate in the study. The course instructor designed an instructional sequence aimed at supporting PSTs' engagement with DGEs and their analysis of students' geometric thinking. During the sequence, PSTs first worked through geometry tasks using DGEs as learners. The TBGT Framework was introduced to support PSTs in attending to students' interactions with the DGE, their written and spoken work, and the mathematical ideas reflected in those interactions. They then analyzed video recordings of secondary students engaging with the same tasks. Through these experiences, PSTs engaged in the practice of noticing students' thinking in technology-mediated environments.

PSTs also completed a pre- and post-TPACK survey (Zelkowski et al., 2013) and written reflections about their work on the DGE tasks, experiences analyzing students' thinking, and developing perspectives on teaching with technology. Quantitative survey data were analyzed using paired samples *t*-tests to examine patterns of change in PSTs' self-reported TPACK. All 15 PSTs completed the pre-survey, and only eight completed all questions in the post-survey. Paired analyses were conducted on matched responses ($n = 8$). Given the small sample size and attrition, results are interpreted descriptively and in relation to qualitative findings.

Qualitative data were analyzed in two complementary ways. First, responses were examined to identify how and to what aspects of students' geometric thinking PSTs attended when analyzing video of students interacting with DGEs. Second, PST reflections were categorized using subdomains of the TPACK framework (Mishra & Koehler, 2006), including technological knowledge (TK), pedagogical knowledge (PK), content knowledge (CK), and their intersections (e.g., TPK, TCK, PCK, and TPACK). Coding was iterative and focused on identifying patterns across PST responses rather than individual trajectories. This analytic approach allowed for examination of both the substance of PSTs' noticing and how their reasoning reflected integrated technological, pedagogical, and content considerations.

Findings

Analysis of PSTs' reflections and survey responses revealed meaningful shifts in how they understood and used DGEs. Three themes emerged that illustrate these shifts: changes in how PSTs conceptualized the role of DGEs, growth in what they noticed about students' geometric thinking, and increasingly integrated instructional reasoning grounded in students' interactions with technology.

Theme 1: From Construction Tool to Reasoning Environment

Initial responses in reflections positioned DGEs as tools for visualization, efficiency, and checking constructions. Over time, some PSTs described dynamic manipulation (especially dragging) as a way to surface invariants, test conjectures, and generalize relationships. In these moments, DGEs were framed less as instruments for producing accurate figures and more as environments that make geometric reasoning visible and support sense-making.

Theme 2: Shifts in What PSTs Notice About Students' Geometric Thinking in DGEs

As PSTs analyzed students' work with DGEs, their attention moved beyond correctness toward how students reasoned through interactions with the tool. PSTs referenced students' strategies (e.g., testing cases through dragging), emerging conjectures, and limitations (e.g., reliance on visual evidence without justification). These descriptions reflect increasingly

differentiated noticing of mathematical properties, invariants, and relationships, as well as attention to students' approaches and misunderstandings.

Theme 3: Instructional Reasoning Informed by DGE Task and Activities

PSTs began to connect what became visible in students' DGE interactions to instructional decisions, including when to press for justification, how to scaffold vocabulary, and how to design tasks that invite comparison and generalization. Reflections increasingly integrated technological, pedagogical, and content considerations. For example, PSTs began using actions in DGEs to reveal structure while planning questions that move students from visual confirmation toward geometric explanation.

Taken together, these themes suggest that when PSTs attended to students' actions within DGEs, they were more likely to conceptualize technology as supporting sense-making and to articulate instructional moves grounded in students' technology-mediated reasoning. This interplay was reflected in language consistent with TCK (technology revealing mathematical structure), TPK (technology shaping instructional choices), and, at times, integrated TPACK.

Conclusions

Preliminary findings suggest that PSTs' shifts in how they conceptualized DGEs, what they noticed about students' geometric thinking, and how they reasoned about instruction were interconnected and reflected movement across TPACK subdomains. When PSTs described DGEs as environments that reveal geometric relationships and invariants rather than tools for producing constructions, their reasoning reflected technological content knowledge (TCK), positioning technology as a means for making mathematical structure visible. As PSTs attended more closely to students' strategies, conjectures, and misunderstandings while interacting with DGEs, their reflections increasingly reflected pedagogical content knowledge (PCK) and technological pedagogical knowledge (TPK), particularly in anticipating student difficulties and considering how technology might shape instructional moves. Moments in which PSTs connected students' DGE activity to specific teaching decisions (e.g., when to press for justification, how to scaffold explanations, or how to leverage dynamic representations to support generalization) demonstrated more integrated technological pedagogical content knowledge (TPACK). In these cases, noticing students' thinking within DGEs functioned as a "bridge" linking mathematical reasoning, instructional decision-making, and technology use.

At the same time, variability in PST responses indicates that integration across subdomains was uneven. Some reflections remained focused on efficiency and visualization, suggesting continued emphasis on technological knowledge (TK) without sustained connections to pedagogy or content. These differences highlight the importance of teacher preparation experiences that explicitly position DGEs as resources for interpreting student thinking and for informing instructional decisions, not solely for constructing or visualizing mathematical ideas. Altogether, these findings suggest that structured opportunities to analyze students' actions in DGEs may support PST movement from isolated TK or TCK perspectives toward more integrated TPACK reasoning, particularly when noticing is foregrounded as a central practice in learning to teach with technology.

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CUSTOMIZED AI ASSISTANTS IN SOLVING MATHEMATICS COMPETITION PROBLEMS

ASISTENTES PERSONALIZADOS CON IA EN LA RESOLUCIÓN DE PROBLEMAS DE CONCURSOS DE MATEMÁTICAS

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This study explores the design and implementation of customized artificial intelligence assistants, developed through ChatGPT, to support mathematics competition problem solving among high school students. The study examines student–assistant interactions by conceptualizing these assistants as mediating artifacts grounded in a problem-solving perspective. Analytical episodes illustrate how the assistants can foster exploration, strategic reformulation, and mathematical argumentation during the resolution of non-routine problems. Preliminary findings suggest that customized AI assistants may contribute to students' development of mathematical knowledge when they are intentionally designed and integrated into cognitively demanding mathematical activities. Implications for the design of AI-supported learning environments in mathematics education are discussed.

Keywords: Problem-Solving, Technology, High School Education

Introduction

Within mathematics education, non-routine problem solving has long been recognized as a central means for promoting the development of mathematical knowledge, particularly in contexts characterized by high cognitive demand, such as mathematics competitions. At the same time, recent advances in generative artificial intelligence (AI) have introduced new opportunities for supporting students' mathematical activity and learning processes (Mishra et al., 2023). However, important questions remain regarding how these technologies can be incorporated into educational settings without replacing students' reasoning processes or diminishing their opportunities to construct mathematical understanding.

This study is situated at the intersection of problem solving and digital technologies. Specifically, it investigates the potential of customized AI assistants developed through ChatGPT as mediators of mathematical learning (Santos-Trigo, 2007). The focus is placed on high school students engaged in the resolution of non-routine mathematics competition problems, a context that remains relatively underexplored within the emerging literature on artificial intelligence in mathematics education.

Purpose and Research Questions

The purpose of this study is to design, refine, and investigate the use of a customized assistant developed in ChatGPT to support mathematics competition problem solving among high school students. The study pursues two complementary goals: (a) generating knowledge about how such assistants can be configured to promote cognitive actions associated with mathematical problem solving (Santos-Trigo, 2007), and (b) producing empirical evidence regarding the problem-solving processes that emerge when students interact with the assistant while working on non-routine mathematical tasks.

The study is guided by the following research questions: (1) How does the use of a customized assistant influence students' processes of exploration, representation, argumentation,

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

and validation while solving mathematics competition problems? (2) What patterns of interaction emerge between students and the assistant during problem-solving processes, particularly regarding the coordination of representations, the use of invariants, generalization, and metacognitive control? (3) What design principles and interaction rules emerge that support students' autonomy, and how are these principles refined through successive cycles of design and implementation?

Conceptual Framework

In this study, problem solving is conceptualized as a knowledge-generating activity through which students explore mathematical situations, formulate conjectures, validate results, and reflect on their strategies (Santos-Trigo, 2007). From this perspective, the emphasis is not limited to obtaining a correct solution; rather, attention is directed toward the cognitive and metacognitive processes that emerge throughout mathematical activity.

Previous research has highlighted the role of non-routine problems as privileged contexts for examining how students construct and reorganize mathematical knowledge, as well as how they make and revise strategic decisions during problem-solving processes (Santos-Trigo & Barrera-Mora, 2007).

Non-Routine Problem Solving and Cognitive Demand

Within the framework of this study, a non-routine problem is understood as a task whose solution cannot be obtained through the direct and mechanical application of a previously learned algorithm. Instead, such problems require students to engage in exploration, conjecture generation, strategic decision-making, and metacognitive monitoring. Because of these characteristics, non-routine problems constitute a central component of mathematics competitions, where exploration, argumentation, and justification are valued more highly than the mere production of answers.

The problems selected for the design and refinement of the assistant were chosen precisely because they provide opportunities to investigate problem solving as a complex mathematical practice. In these contexts, mathematical knowledge develops through the coordination of representations, the production and refinement of conjectures, the construction of arguments, and the communication and revision of results (Santos-Trigo, 2007).

Scaffolding, Agency, and Validation Control in AI-Supported Problem Solving

A major challenge associated with the integration of generative artificial intelligence into mathematics education is preserving students' agency and their capacity to construct mathematical knowledge (Mishra et al., 2023). When AI systems provide plausible explanations or detailed solution procedures, there is a risk that mathematical activity may shift from knowledge construction toward answer seeking or uncritical acceptance of generated responses.

For this reason, the present study focuses on designing the assistant as a form of conversational scaffolding that makes mathematical decisions and justifications visible while avoiding the replacement of students' intellectual work (Zhou et al., 2025). Operationally, the assistant privileges questioning, requests for explanation, and prompts for verification while restricting the provision of complete solutions.

Digital Technologies as Mediating Tools

Digital technologies can function as cognitive tools when they are intentionally designed and integrated into mathematical activity. Beyond automating procedures, such technologies can support exploration, visualization, experimentation, and reflection, thereby expanding opportunities for mathematical inquiry (Santos-Trigo, 2007a). From this perspective, assistants based on generative artificial intelligence—particularly ChatGPT—represent a new class of

digital instruments whose educational value depends critically on both their design and the theoretical principles that guide their use (Haçat & Kepceoğlu, 2025).

Their role is not predetermined by the technology itself; rather, it emerges from the interactions established between students, tasks, and the assistant. In this study, customized AI assistants are conceptualized as mediators of mathematical reasoning.

Research Design

The study adopts a Design-Based Research (DBR) approach focused on the development and refinement of a customized assistant implemented through ChatGPT. Design-Based Research is particularly appropriate for examining the integration of emerging technologies into mathematics education because it supports iterative cycles of design, implementation, analysis, and refinement (Santos-Trigo et al., 2016).

Design Cycles and Design Conjectures

The study is organized around three iterative design cycles. During each cycle: (a) non-routine mathematical problems with the potential to generate multiple solution strategies are selected, (b) students interact with the customized assistant while solving the tasks, (c) interaction data are collected and analyzed, and (d) both the assistant and the tasks are refined based on empirical findings.

This iterative logic is central to design experiments, where the objective extends beyond determining whether an intervention is effective. Instead, the emphasis is placed on understanding how learning processes unfold, when particular forms of support become productive, and under what conditions the intervention may be transferable to other contexts (Cobb et al., 2003). The study is conducted within a mathematics competition preparation environment, where students engage with non-routine problems in sessions that provide sufficient time for exploring strategies, constructing arguments, and discussing alternative approaches.

Task Selection and a Priori Analysis

The mathematical tasks are selected through an a priori analysis that considers four complementary dimensions: (a) the central mathematical ideas involved and their possible curricular connections; (b) potential solution pathways and alternative approaches; (c) anticipated obstacles and difficulties that students may encounter; and (d) opportunities for promoting exploration, representation, argumentation, and validation.

This criterion is important because it reduces the likelihood that interactions with the assistant become limited to the reproduction of procedural steps. Each design cycle generates two complementary forms of outcomes. The first concerns learning and process outcomes, including changes in students' strategies, argumentation practices, and metacognitive control. The second concerns design outcomes, including principles, interaction rules, and specific assistant configurations that prove productive or problematic during implementation.

Context and Participants

The study is conducted with high school students who participate in mathematics competition preparation activities. Participants work in guided problem-solving sessions in which they engage with non-routine mathematical tasks supported by a customized assistant based on generative artificial intelligence.

Desing of the Customized Assistant

The assistant was designed based on principles derived from problem-solving research. Its interventions prioritize recurring conversational actions such as asking students to explain their interpretation of a problem statement, encouraging the construction of representations, prompting

conjecture generation and justification, proposing verification strategies, and inviting students to consider alternative approaches. When students directly request a solution, the assistant redirects the interaction toward graduated hints and reflective questions rather than providing complete answers (Zhou et al., 2025).

The assistant functions as a personalized tutor guided by the following principles: (a) guide rather than solve; (b) follow Pólya's four phases of problem solving; (c) avoid advancing without evidence of understanding; (d) treat errors as opportunities for reflection and learning; and (d) provide support during moments of impasse through minimal and strategically targeted hints.

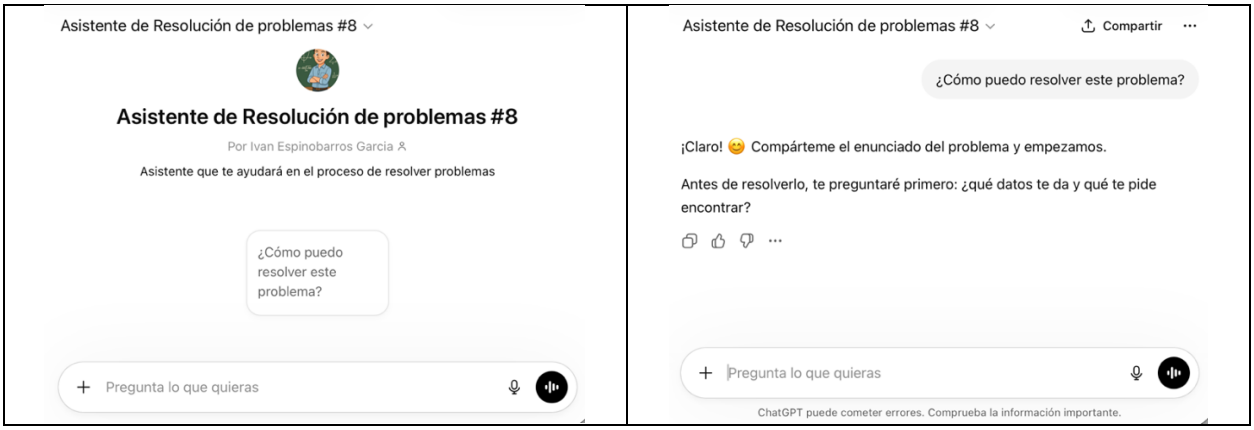


Figure 1: Assistant Interface and Example of an Initial Interaction

Data Collection and Analysis

The data sources include records of student–assistant interactions, students' written productions, and researcher observations. The analysis focuses on: (a) the problem-solving strategies employed by students, (b) the ways in which the assistant functions as cognitive support, (c) changes in students' approaches to solving problems, and (d) evidence of mathematical argumentation and metacognitive control.

Data Sources and Documentation Procedures

Three complementary sources of data are documented throughout the study: (a) complete conversation records between students and the assistant, including questions, responses, and reformulations; (b) students' mathematical productions, such as drafts, sketches, notes, and written justifications; and (c) field notes describing moments of impasse, strategy shifts, and collective discussions when these occur.

Analytic Strategy and Quality Criteria

The analysis combines episodic readings of critical moments with the coding of problem-solving actions and assistant interventions. Initial categories are derived from both the theoretical framework and the a priori analysis of tasks. These categories include: problem interpretation, case exploration, conjecture construction, justification and validation, and metacognitive control.

Preliminary Findings

The analysis of interaction episodes suggests that students tend to use the assistant primarily as a resource for reorganizing their thinking rather than as a source of immediate answers. Several episodes reveal that the assistant's interventions support the reformulation of conjectures, the exploration of alternative representations, and the explicit articulation of

mathematical reasoning. Over time, however, students appeared to adjust their expectations and increasingly recognized the assistant as a mediator of reasoning rather than a problem solver.

Analytical Interaction Vignettes

To illustrate how the assistant functions as a mediator of mathematical activity, a representative interaction episode is presented. The problem analyzed corresponds to Problem 43 from the booklet Introductory Problems for the 35th Mexican Mathematical Olympiad (García Velázquez et al., 2021, p. 25), which states: "Using the numbers from 1 to 26, form 13 fractions such that each number is used exactly once as either a numerator or a denominator. After simplifying all fractions, what is the maximum number of integers that can be obtained?"

The four tables that follow constitute an analytical excerpt that allows the interaction to be interpreted through the lens of Pólya's problem-solving phases. Table 1 illustrates the stage of understanding the problem. During this phase, the assistant directs the student's attention toward two critical constraints: divisibility and the requirement that each number be used exactly once. Rather than providing guidance toward a specific solution, the assistant encourages the student to articulate the conditions embedded in the problem statement.

Table 1: Pólya's First Phase

Student	Problem solving assistant #8
Problema 43. Con los números del 1 al 26 se quieren formar 13 fracciones, usando los números disponibles como numerador o denominador de manera que cada uno de ellos se use solo una vez. Luego, cada fracción se simplifica. ¿Cuál es la máxima cantidad de enteros que se pueden obtener después de la simplificación?	¡Buen problema! 🧩 Antes de pensar en cálculos, vamos con la Fase 1: comprenderlo. Cuéntame con tus propias palabras: ¿Qué números se pueden usar, cuántas fracciones hay que formar y qué significa que "después de simplificar se obtengan enteros"?
Se pueden usar 26 números, del 1 al 26, se pueden formar 13 fracciones pero tal vez no todas cumplen con los requisitos del problema	Bien, ya identificaste los datos principales ✅ Afinemos una parte clave: el problema dice que los números del 1 al 26 se usan como numerador o denominador, y que cada uno se usa solo una vez. Entonces, ¿crees que al final deben usarse los 26 números exactamente una vez para formar las 13 fracciones?

Table 2 corresponds to the design of a plan. The assistant invites the student to identify potentially productive numerical relationships and introduces strategic criteria for constructing fractions that simplify to integers. In this way, the assistant supports the development of a plan while preserving the student's responsibility for making mathematical decisions.

Table 2: Pólya's Second Phase

Student	Problem solving assistant #8
	Ya entendimos el problema, así que pasemos con cuidado a la Fase 2: diseñar un plan. Queremos maximizar la cantidad de fracciones que terminan siendo enteros. Piensa en esto: Si eliges una pareja (a, b) para una fracción, necesitas que uno sea múltiplo del otro. Con los números del 1 al 26, ¿qué números te parecen "fáciles" de emparejar porque tienen muchos múltiplos disponibles? 🤔
Se me ocurren el 6, el 8, el 12, el 16, el 20 y el 24	¡Buena observación! Esos números grandes pueden funcionar muy bien como numeradores, porque varios tienen divisores entre 1 y 26 ✅ Ahora pensemos al revés: si quieres que una fracción sea entera, conviene formar parejas tipo: $\frac{\text{múltiplo}}{\text{divisor}}$

Table 3 reflects the implementation of the plan. The student proposes concrete pairings; the assistant responds by encouraging reflection on the mathematical validity of those choices.

Table 3: Pólya's Third Phase

Student	Problem solving assistant #8
12/4, 16/8, 20/10, 24/6	Ahora piensa estratégicamente 🤔 El problema pide el máximo número posible de fracciones que se simplifican a enteros. Como hay 13 fracciones en total, una pregunta útil es: ¿Crees que es posible lograr que las 13 sean enteros? ¿O habrá algunos números "problemáticos" que obliguen a que alguna fracción no lo sea? Mira especialmente los números primos grandes como $13, 17, 19, 23$. ¿Qué múltiplos tienen dentro del 1 al 26?
Para los números 13, 17, 19, 23; se pueden formar las fracciones 26/13, 19/17, 23/1	Interesante razonamiento 💡 Veo una idea importante ahí, pero quiero que revisemos algo con calma. Propusiste: $\frac{26}{13}, \frac{19}{17}, \frac{23}{1}$ Pregúntate una por una: ¿$26/13$ se convierte en entero? ¿$19/17$ se convierte en entero? ¿$23/1$ se convierte en entero? Y además: ¿en $19/17$, el denominador 17 divide al numerador 19? 🤔

Finally, Table 4 represents the stage of looking back and validating the solution. The assistant prompts the student to verify whether all numbers have been used exactly once and whether the proposed construction satisfies the requirements of the problem. Through these interventions, the assistant encourages error detection, validation, and strategic revision.

Table 4: Pólya's Fourth Phase

Student	Problem solving assistant #8
	Ahora prueba a emparejarlos como "múltiplo / divisor." Mira algunos candidatos: $\frac{14}{7}, \frac{15}{5}, \frac{18}{9}, \frac{21}{3}, \frac{22}{11}$ ¿Qué número quedaría sin pareja con esa elección? ¿Y con qué podrías emparejarlo para formar la sexta fracción?
14/2, 15/3, 25/5, 21/7, 18/9, 22/11	¡Muy bien! 👏 Estás construyendo las parejas con mucha estrategia. Tus nuevas fracciones son: $\frac{14}{2}, \frac{15}{3}, \frac{25}{5}, \frac{21}{7}, \frac{18}{9}, \frac{22}{11}$ Ahora hagamos una revisión cuidadosa (Fase 4 🔍): no quiero avanzar solo por cortesía. ¿Puedes revisar si en todas las fracciones que has construido hasta ahora cada número del 1 al 26 aparece exactamente una vez? Mira especialmente el 2, el 3, el 5 y verifica que ninguno se haya repetido ni que haya quedado alguno sin usar.

For purposes of analytical transparency, the complete interaction can be accessed at: https://chatgpt.com/share/6a16319c-e204-83e8-9b84-662ff1c25e01, the version used during the interaction corresponded to GPT-5.5 (instant).

Discussion

The findings suggest that generative artificial intelligence can function as a mediating tool for mathematical problem solving when its design is oriented toward supporting students' learning processes rather than optimizing answer production. In particular, the interaction episodes illustrate how conversational moves such as requesting conditions, eliciting justifications, and proposing verification strategies can foster mathematical monitoring practices and contribute to the reconstruction of mathematical knowledge during problem solving.

Contributions to Mathematics Education Research

This study contributes to the growing body of research on artificial intelligence in mathematics education by demonstrating how customized AI assistants can be intentionally designed to support mathematical problem solving without diminishing students' intellectual responsibility. Rather than conceptualizing AI systems primarily as providers of explanations or solutions, the study positions customized assistants as mediators of mathematical activity. This perspective shifts attention from the technological capabilities of AI systems to the design principles that shape students' interactions with them.

Limitations and Future Work

As a design-based research study, the findings are necessarily influenced by the context of implementation, the characteristics of the selected tasks, and the interaction rules embedded within the assistant. Consequently, the results should not be interpreted as universally generalizable but rather as contextually grounded design knowledge.

An additional limitation concerns the inherent variability of generative AI models. Because responses may vary across interactions and system updates, continuous monitoring is required to ensure consistency and alignment with educational goals (Abu Daqar et al., 2025).

Future research will expand the range of mathematical problems incorporated into the design cycles and will examine students' interactions over longer periods of time. Particular attention will be devoted to understanding how students internalize practices of verification, argumentation, and strategic reflection beyond their direct interactions with the assistant (García Navarro et al., 2025).

Implications for Practice and Assessment

From a practical perspective, the findings suggest that integrating customized AI assistants into high school mathematics education requires explicit agreements regarding what constitutes mathematical progress and meaningful learning. Rather than focusing exclusively on final answers, assessment practices should attend to students' problem-solving processes, including the quality of their justifications, their capacity to verify mathematical claims, and their strategic flexibility when confronting unfamiliar tasks. One promising avenue for future work involves the development of assessment rubrics centered on problem-solving actions such as exploring, representing, justifying, and validating, as well as examining their coherence with AI-mediated interaction records.

Conclusions

This study highlights the importance of intentionally and theoretically grounded designs for customized assistants based on generative artificial intelligence. Such designs are essential for avoiding purely instrumental uses of AI that reduce students' mathematical activity to the production of immediate answers.

The findings suggest that when AI-based assistants are conceived as mediating instruments aligned with a problem-solving perspective, they can support exploration, the explicit articulation of reasoning, and metacognitive control without displacing students' opportunities to develop mathematical knowledge. From this perspective, artificial intelligence does not function as a substitute for mathematical thinking but rather as a resource that can contribute to its reorganization and enrichment within cognitively demanding environments.

The customized assistant developed in this study incorporates carefully designed interventions that prioritize guiding questions, requests for justification, verification prompts, and representational shifts instead of complete solutions. This design enables students to maintain responsibility for evaluating the validity of their inferences and the coherence of their strategies, thereby supporting the progressive development of mathematical knowledge.

The significance of this tool for mathematics education lies in its potential to expand opportunities for formative feedback within complex mathematical tasks. Unlike traditional resources that provide fixed answers or standardized procedures, the customized assistant enables dynamic interactions that adapt to students' evolving understandings and support reflection in real time. This feature is particularly relevant in educational settings where individualized support is limited but where conceptual depth and rigorous argumentation remain central goals.

Within the specific context of mathematics competitions, the assistant assumes a distinctive role. Rather than functioning as a generator of shortcuts or model solutions, it operates as a mediator that supports strategic exploration, the identification of invariants, the analysis of extreme cases, and the comparison of alternative solution methods. Its purpose is therefore not to accelerate answer production but to strengthen the mathematical practices that characterize successful engagement with non-routine problems.

Ultimately, this work opens an important line of inquiry regarding how artificial intelligence can be integrated into mathematics competition preparation while preserving the formative and

cognitive nature of problem-solving activity. More broadly, it suggests that the educational potential of AI lies not in replacing mathematical thinking, but in creating conditions that encourage students to engage more deeply with it.

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HOW CHILDREN USE DIRECTION AND ABSOLUTE VALUE ON NUMBER LINE TASKS: EVIDENCE FROM FIRST AND THIRD GRADERS

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Children often struggle to coordinate direction, magnitude, and movement simultaneously on integer number line tasks, and language used in prompts may influence their reasoning. We randomly assigned 29 first and 28 third graders to either numbered or unnumbered lines, and they completed five tasks where they indicated movement to go "up/down" or "add/subtract" in a one-on-one interview. We analyzed data statistically and qualitatively through coding responses. Findings only showed a significant difference between first and third graders on numbered lines. Students usually used direction-based reasoning, except for third graders, who shifted to absolute value reasoning when the prompts asked them to "add/subtract." The findings contribute to research on how language may impact children's understanding of directed magnitudes and may inform how they make sense of integer operations.

Keywords: Number Concepts and Operations, Elementary School, Cognition, Mathematical Representations

Although formal instruction on integers in the United States typically begins in middle school, many students continue to struggle with integer operations (Hawthorne et al., 2022; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010). Evidence shows that young children may already possess meaningful intuitions about direction, magnitude, and movement of integers that transfer from their experiences with whole numbers (Bofferding, 2014; 2019; Hawthorne et al., 2022; Thompson & Dreyfus, 1988; Wessman-Enzinger, 2019). These early intuitions could support earlier integer instruction to better prepare students for operations with integers and help them avoid the rules that expire (Karp et al., 2014) that confuse students when they learn about negative integers long after learning about whole number operations.

Researchers have studied how young children reason about direction, magnitude, and movement of integers on a number line and how they coordinate these concepts from different starting positions on a number line (Bofferding, 2014; Thompson & Dreyfus, 1988; Wessman-Enzinger, 2019). For example, Thompson & Dreyfus (1988) experimented with integer addition and subtraction as directed transformations of integers (e.g., [30–60] means start at 30 and move 60 spaces in the negative direction). Yet Bofferding (2019) suggested that this conception of integers works for addition but not as well for subtraction and offered an alternative interpretation of integers based on language that young students naturally use to signify directed operations. For instance, addition can be interpreted as "getting more" positive or negative and subtraction as "getting less" positive or negative from what one already has (Bofferding, 2019, p. 264). For example, a student might explain that –8 is more negative than –6. In this interpretation, numbers exist on a continuum with "more positive" meaning "less negative" (Bofferding, 2019, p. 264). Extending this perspective, "more negative" would correspond to "less positive." For example, in Bofferding's (2019) work with children in second grade, she found that students could solve a problem like 3 – (–1) = 4 by understanding that subtracting a negative or "getting less negative" meant moving in the positive direction (p. 264).

Additionally, an investigation into how the presence or absence of negative and positive numbers on a number line influences children's coordination of direction, magnitude, and movement on a number remains mostly unexplored. Studying how children reason with or without the presence of integer labels on a number line can illuminate their strategies for integrating negative number concepts into their existing knowledge base.

The present study examines how early elementary students respond to *directed magnitude* language that directs upward or downward movement with a specified magnitude on a vertical number line in contrast to *directed operations* language (e.g., adding or subtracting a specified magnitude). Additionally, it investigates how the presence or absence of positive and negative number labels on a number line either supports or constrains students' performance on these tasks. For clarity, the term *directed magnitude* will be used to refer to movement that considers direction and quantity simultaneously. The term *absolute value reasoning* will refer to strategies that attend only to magnitude, while ignoring or inconsistently incorporating direction.

Research Questions

1. How does the use of directional language (e.g., "up/down") and operational language (e.g., "add/subtract") influence students' responses to vertical number-line tasks?
2. To what extent does the presence or absence of numbers on a number line significantly affect students' movement on the number line?

Theoretical Framework

Case's (1996) Central Conceptual Structures for Number (CCSN) offers a productive framework of whole number development that can be extended to explain how children integrate integer ideas into their conceptual structures (Bofferding, 2014; 2019; Case, 1996). Sometimes referred to as a "mental counting line" or "mental number line," Case (1996) explains that children construct a numerical structure in which moving from one position to the next represents adding or subtracting one unit. Case (1996; Okamoto & Case, 1996) further explains that movement along this mental number line also involves differing magnitudes. These concepts form the basis for integer reasoning, where both direction and magnitude must be coordinated.

Bofferding (2019) suggests that integers can be interpreted as *directed displacements*, *directed magnitudes* or vector-like movements on a number line (p. 263). As seen in Thompson and Dreyfus' (1988) report on a computerized turtle's movement along a number line, two sixth graders initially struggled to distinguish between the starting position of the turtle and its displacement. Drawing on Vergnaud's analysis of conceptual fields, the authors noted that the students interpreted the integer as a static number—the turtle's new location—rather than as a transformation involving direction and magnitude. Thompson and Dreyfus (1988) explain that understanding an integer as displacement required that the students have a multiplicative understanding of magnitude and direction—a structure that students eventually developed.

Most studies have used magnitude-based language (e.g., which number is greater: –5 or –6?); or order-based language (e.g., which number is closer to 0? –1 or –2?) (Bofferding, 2014; 2019; Lamb et al., 2018); yet few studies have looked at the influence of operational language (e.g., add or subtract 2) on students' integer reasoning on a number line. Research also shows that operational signs (e.g., +/–) cue whole number strategies when students solve integer addition and subtraction problems (Aqazade & Bofferding, 2021). It is possible, then, that just as students use whole number strategies when they "see" an addition or subtraction sign, they might do the same when they "hear" instructions to add or subtract. This suggests that operational prompts in number line tasks may lead to whole number reasoning rather than directed magnitude reasoning.

A common strategy in whole number reasoning is treating negative numbers by their absolute values (Aqazade, 2022). For example, in Aqazade's (2022) work with fifth graders' and their interpretation of temperature changes, one student, interpreted a change from –20 to 15 degrees as "more cold," explaining, "Because twenty is like going less and then to fifteen, it got more cold" (p. 166). The student understood that –20 meant it was cold because the temperature was in the negatives, but viewed 15 degrees as being less than –20, which makes sense if they are viewing the numbers from an absolute value perspective. Related to absolute value reasoning is a whole number strategy for adding integers, where students will add the absolute values of the numbers but keep the negative sign in the sum (Aqazade & Bofferding, 2021). For example, –1 + 7 will be added as 1 + 7 = 8, and the final answer will be –8, to acknowledge the negative sign (Aqazade & Bofferding, 2021, p. 88). Termed the "add make negative" strategy, Aqazade and Bofferding (2021) found that students consistently used this strategy when adding integers.

Methods

Participants and Design

This study took place in an elementary school in the Midwest during the first two months of Fall 2019. The participants were 29 first grade students and 28 third grade students. There were 16 male and 13 female students in first grade and 11 male and 17 female students in third grade. No other demographic data was collected.

The data were obtained from one section of a pre-test that students took one-on-one with a researcher during a video-taped interview as part of a larger study. The students were randomly assigned either a test form either with or without numbers on a vertical number line (see Figure 1). Half of the first and third graders were randomly assigned to numbered lines and the other half to unnumbered lines. The five questions were:

1. Show where the ant would end up if it goes down 2. [Ant is at –3]
2. Show where the ant would end up if it goes up 2. [Ant is at –2]
3. Show how the ant moves if it adds 2. [Ant is at –3]
4. Show how the ant moves if it subtracts 2. [Ant is at –2]
5. Show where the ant would end up if it goes up 2 then down 2. [Ant is at 0]

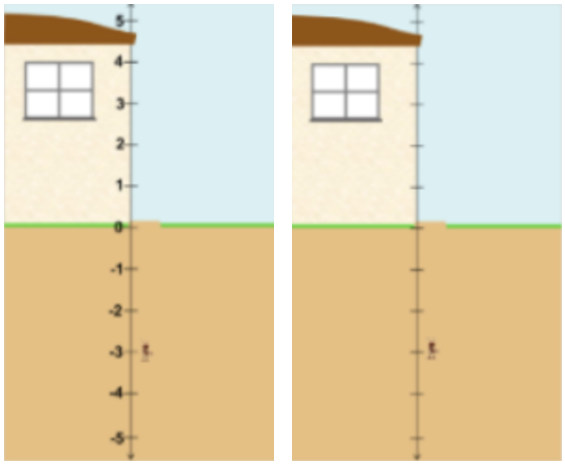


Figure 1: Student Response Forms With (left) and Without Numbers (right)

Data Analysis

First, we analyzed the data quantitatively in two groups using an independent samples t-test: Group A (first and third graders who received numbered lines) and Group B (first and third graders who received unnumbered lines). Neither group passed Levene's test of equal variances; therefore, we used Welch's t-test.

Next, we conducted a qualitative analysis of all 55 student responses to identify patterns in errors and correct responses. We coded based on whether the direction the student moved was correct, the distance moved was correct, or whether they used distance without direction, or direction without distance. Responses in which students primarily used direction were coded as *directed magnitude*, while responses in which students used distance without direction were coded as *absolute value reasoning*. We also created a subcode of *miscount* in cases where students clearly used directed magnitude or absolute value thinking but either counted the initial number in their count or moved their finger too fast when counting, going a tick mark too far. For example, in the former case, if the ant was at –3 and the student was asked to move it down 2, they might count the –3 (the ant's location) as part of the count and move the ant to –4. In the latter case, they might move the ant from –3 to below –5, when trying to count down by 2. Additional patterns also emerged that diverged from the direction and distance categories, and these were grouped under the code *other*. We analyzed errors by number line format (numbered or unnumbered lines), by grade level, and by two separate question groups: 1) questions 1–2 ("down/up") versus questions 3–4 ("add/subtract"), and 2) questions asking, "up vs. add" versus "down vs. subtract." We consulted the relevant video segments to check all responses. The codes used are shown in Table 1.

Table 1: Codes and Descriptions

Codes	Description
	Directed Magnitude
Directed Magnitude (DM)	Uses both direction and magnitude successfully to get the correct answer
DM Miscount	Moves in the correct direction with a slight miscount (i.e., counting the initial number in their count or moving their finger a little too fast and going a tick mark too far).
Direction (Random Amount)	Moves in the correct direction but with a random magnitude (i.e., when the amount moved seems more than a miscount).
	Absolute Value Reasoning
Absolute Value (AV)	Interprets starting number as positive and adds the movement number while ignoring the negative signs; uses magnitudes only.
Absolute Value Miscount	Interprets starting number as positive and adds the given number as a positive but miscounts the final number.
Absolute Value, Random	Interprets starting number as positive, adds the given number as a positive but final response appears random.

Other

Includes: Add as Positive (treating all numbers as positive and adding them for a positive sum), Location (moving to the location of the directed number), Random (did not move in the correct direction or move the correct amount), Didn't Move (kept the ant at its starting location), and Pattern (used the starting location as a repeating pattern to predict the answer).

No response/skip Student did not answer

Results

For numbered lines, third graders outperformed first graders (60% vs. 44% correct) (see Table 2); showing a statistically significant difference, t (22.289) = –2.133, p = .044, d = .801 (large). On unnumbered lines, third graders answered 67% of items correctly in contrast to 55% for first graders; this difference was not statistically significant, t (24.254) = –.691, p = .496, d = –.256 (small).

Overall, third graders answered over 60% of the questions correctly across both number line formats, compared to about 50% of first graders. The predominant response pattern across both grade levels and number line formats was the use of direction, where students moved in the correct direction but not always with the correct distance. For example, when asked where the ant will end up if it goes down two (starting position at –3), one first grader indicated that it would go down to –4. Their response suggests they knew that they were moving down (correct direction) but were not accurate in the distance moved. In contrast, third graders with numbered lines showed more absolute value reasoning particularly on the "add/subtract" questions, where they treated the ant's starting position as a positive number and added the absolute value of the directed number. For example, when one third grader was asked where the ant would end up if it "adds two" (starting position at –3), they settled on –4, adding the absolute value of the ant's starting point (–3) plus the amount it moved (i.e., |–3| + 2), but with a slight miscount.

Table 2: Percent Correct for Questions 1–4

	Numbered Lines		Unnumbered Lines	
	1st	3rd	1st	3rd
Directed Magnitude (DM)	44%	60%	55%	67%
Direction (Random Amount)	35%	25%	33%	53%
DM Miscount	20%	13%	22%	24%
Absolute Value	10%	38%	15%	12%
Absolute Value Miscount	0%	21%	0%	0%
Absolute Value (Random)	0%	0%	11%	12%
Other	31%	0%	4%	0%
No response/Skip	0%	4%	11%	0%

On Questions 1 and 2, across both grade levels and number line formats, directed magnitude responses were the most frequent, ranging from 67–100%. For example, one first grader answering Question 1 (go down two from –3), can be seen in the video looking at –4 with their eyes and telling the interviewer that the answer would be 4 (referring to –4). Thus, the student moved in the correct direction but miscounted the distance. On Question 2, another first grader moved the ant up from –2 to 2, when asked to make it go "up two." The video segment showed

that the student seemed focused more on where it could move ("to" two), rather than a specific amount of movement. A breakdown of scores by question are shown in Table 3.

Table 3: Frequency of Strategies for Questions 1–4

	Numbered		Unnumbered		Numbered		Unnumbered	
	1st	3rd	1st	3rd	1st	3rd	1st	3rd
	Q1 (down)				Q2 (up)			
Directed Magnitude	71%	87%	60%	62%	43%	73%	60%	87%
Direction (Random Amount)	50%	50%	83%	80%	50%	75%	50%	50%
DM Miscount	25%	0%	0%	20%	13%	0%	33%	50%
Absolute Value	0%	0%	0%	0%	0%	0%	17%	0%
Absolute Value Miscount	0%	0%	0%	0%	0%	25%	0%	0%
Other	25%	0%	0%	0%	38%	0%	0%	0%
No Response/Skip	0%	50%	17%	0%	0%	0%	0%	0%
	Q3 (add)				Q4 (subtract)			
Directed Magnitude	43%	40%	47%	39%	43%	40%	53%	85%
Direction (Random Amount)	22%	20%	25%	38%	25%	0%	0%	50%
DM Miscount	11%	10%	25%	25%	38%	22%	29%	0%
Absolute Value	22%	40%	13%	25%	13%	56%	29%	0%
Absolute Value Miscount	0%	10%	0%	0%	0%	22%	0%	0%
Absolute Value (Random)	0%	0%	13%	13%	0%	0%	29%	50%
Other	33%	0%	13%	0%	25%	0%	0%	0%
No Response/Skip	0%	0%	13%	0%	0%	0%	14%	0%

For Questions 3 and 4 (add/subtract), directed magnitude responses were most frequent for first graders on numbered lines (47%) and for third graders on unnumbered lines (60%). For example, on Question 3 (add), one student asked, "Is that up or down?", then chose to go down but returned to it after hearing that the next question was about subtraction and said, "I think this one's up (Question 3) and this one's down (Question 4)." In contrast, absolute value responses were most prominent for third graders on numbered lines (68%); whereas, first graders' responses on unnumbered lines showed a balanced pattern, with direction and absolute value each at 40%. For example, on Question 3 (adding 2 to the ant's position at –3), one third grader chose –5 and said, "Cause if you add 2 to 3, it's 5," treating all the numbers by their absolute values.

For the second analysis, where questions were analyzed by comparing how students did on "up vs. add" and "down vs. subtract," their responses were more varied. On the "up vs. add" questions, all students used direction, except for third graders on numbered lines—they used direction for "up" and absolute value for "down." For both the down and subtract questions, first graders on numbered lines and third graders on unnumbered lines used direction. The unusual finding, however, was when both first graders on unnumbered lines and third graders on numbered lines used direction for "down" and absolute value for "subtract" (see Table 3, Q1 and Q4).

Finally, Question 5 was analyzed separately as it contained two steps: "up two, then down two." The codes that emerged were like the previous codes, where directional responses were

dominant but with a lot more variation, such as a correct response on the first step then moving to the location of the number on the second step (e.g., moves from 0 to 2, then to –2). Students seemed to also demonstrate an understanding of additive inverses, where they would give a single correct response combining both steps (e.g., 0). Another interesting finding was the possible use of zero as a reference point. Several students moved up from zero for the first step and seemed to come back down to zero before moving down for the second step. To add to the interesting conceptions of the number line, one third grader interpreted the middle of the number line as 50 (on his unnumbered line) and said the ant could go either way, such as 90 or 10.

Taken together, students' reasoning reflected a consistent preference for direction, but their absolute value thinking appeared—especially for third graders with numbered lines and for first graders with unnumbered lines—when the prompts used operational language (e.g., add or subtract).

Discussion and Conclusion

A key finding that emerged was the third graders' shift from using direction on the up/down questions to absolute value on the add/subtract questions on numbered lines. This finding suggests that the language of the operational prompt together with the presence of numbers on the number lines may have cued whole number reasoning. Aqazade and Bofferding (2021) noted a similar pattern in fifth graders when presented with integer addition problems. For example, one fifth grader explained that for −9 + 2, adding two meant that he would be staying in the negatives and so he would end up at –11 (Aqazade & Bofferding, 2021). A similar pattern was seen in a fifth grader's explanation (Aqazade, 2022) that a change in temperature from –20 to 15 was a drop in temperature because she viewed it as a change from 20 to 15 degrees, which, in absolute value terms would be a decrease. Students' sensitivity to operational prompts suggests that providing opportunities to practice with operational prompts while supporting their directional reasoning may enhance their thinking with integer operations.

At the same time, for first graders across both number line formats, the results on the add/subtract items in particular suggest that they have an easier time thinking about addition as going up and subtraction as going down on the number line and that they can preference that reasoning over considering the absolute value of the numbers. This suggests that first graders may be ready to incorporate distance with direction using directed magnitude prompts, such as "more high" (moving up) or "less low" (also moving up).

The predominant reasoning types in this study align with prior reports on early understanding of direction and magnitude. Bofferding (2014) identified direction-biased reasoning where students considered the direction separately from magnitude (absolute value) as well as magnitude-aware students who could focus on either direction or magnitude but not at the same time. Thompson and Dreyfus (1988) reported a similar pattern with older students on integer displacement tasks.

The overall responses of first and third graders on unnumbered lines were surprising. Both grade levels shifted from directional reasoning to absolute value reasoning on the "subtract" question, which did not match our expectations, especially for first graders. We expected that first graders on unnumbered lines would default to their sense of direction, rather than absolute value. Further studies on students' reasoning on unnumbered lines with a larger sample size, and a variety of questions and interview prompting may yield clearer results on these groups.

Together, these patterns align with Case's (1996) account of how children coordinate movement, magnitude, and direction for whole number. They also confirm evidence that children treat these concepts separately when integrating negative numbers before they

assimilate them into a cohesive framework (Bofferding, 2014; Thompson & Dreyfus, 1988). Coupled with our findings on how prompting language cues students' directional or absolute value thinking, there may be value in considering how using different types of prompts on number line tasks may prepare students for operations with integers.

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BEYOND CORRECTNESS: DESIGNING LARGE LANGUAGE MODELS AS CO-MATHEMATIZERS THROUGH A UNITS-IN-MATHEMATIZING LENS

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Current research on Large Language Models (LLMs)-based AI focuses on problem-solving accuracy through adaptive feedback, often neglecting the student's interpretive process of mathematizing. This study proposes the Units-In-Mathematizing (UIM) lens, a framework for designing LLMs as co-mathematizers. While recent AI-related studies emphasize procedural correctness, the UIM lens attends to students' interpretative process of mathematizing by emphasizing what they identify and coordinate in the process. Through an empirical case and a comparative scenario, we demonstrate how UIM-informed LLMs could reveal and support students' mathematizing process. This framework provides a principled foundation for LLM design that prioritizes sense-making over correction on problem-solving.

Keywords: Technology, Modeling, Problem-Solving

Recent advancements in Large Language Models (LLMs)-based AI have prioritized personalized mathematics learning (Barrera et al., 2024; Liu et al., 2025) through real-time, adaptive feedback (e.g., LLM-based chatbots acting as intelligent tutors) (Goel & Ghanta, 2025; Scarlatos et al., 2025). While these approaches improve problem-solving fluency, focusing solely on correct answers risks reducing mathematics to a "closed system" (Freudenthal, 1968, p.7) of rules rather than an open-ended process emphasizing sense-making (Reusser, 1988). This study addresses a critical question for the mathematics education community: *How should LLMs be designed to serve as instructional partners for meaningful mathematics rather than as evaluators of correctness in problem-solving?* To address this, we foreground *mathematizing*—the critical transition from a real model to a mathematical model (Ferri, 2006; English, 2006), as conceptualized in the research body on mathematical modeling. We argue that AI must move beyond error correction to support students' interpretive work in mathematizing situations. Accordingly, we propose the *Units-in-Mathematizing* (UIM) lens as a conceptual and analytic foundation for future LLM-based instructional design. Rather than reporting the development or evaluation of a specific LLM system, this paper develops UIM as a lens for specifying what an LLM-based instructional partner should attend to, infer, and respond to when engaging with students' mathematizing. In the following sections, we define the UIM lens, illustrate its analytic value through empirical student data, and discuss its implications for future LLM-based interactions.

Theoretical Perspective

Units-In-Mathematizing (UIM)

In this section, we bridge studies focusing on *units* and those highlighting *mathematizing*, from which we develop the UIM lens. First, unit and unitizing have been addressed in a substantial body of research related to multiplicative reasoning (Behr et al., 1994; Zwanch & Wilkins, 2021) and fractions (Boyce et al., 2024). These studies illustrate that how students define and identify a unit is critical for their ability to think multiplicatively (e.g., Götze & Baiker, 2021) and to apply fractions flexibly (e.g., Steffe &Olive, 2011). Notably, most mathematical situations examined in this line of research include explicitly given units of one,

such as a single bar (e.g., Hackenberg, 2007; Norton et al., 2015). Consequently, the current literature primarily focuses on the cognitive processes students engage in when forming composite units based on pre-given units of one. These studies show that what students identify and count as *units* plays an important role in students' multiplicative and fractional reasoning.

Along a similar line of inquiry, though often without explicitly using the term *unit*, research in mathematical modeling shows that when individuals mathematize situations, they begin by organizing the situation around particular entities and subsequently represent them mathematically (e.g., Chamberlin & Moon, 2005; English, 2006). For instance, Maaß (2006) demonstrates that when a student interprets a problem situation in a distorted way, resulting in an invalid real model (Ferri, 2006), this interpretation inevitably leads to the construction of a distorted mathematical model. This finding underscores the importance of examining how students interpret and structure problem situations and how these interpretations shape their transitions from real models to mathematical models. However, this aspect of students' mathematical activity remains underexplored (Galbraith and Stillman, 2006). As a result, we lack in-depth explanations regarding how students independently form real models, both valid and invalid, in the first place.

To address this gap, we bridge units with the mathematizing process. We propose the *Units-In-Mathematizing* (UIM) lens, in which we define units as the referents that students take up from situations and coordinate in their mathematizing process. In this lens, units are not limited to mathematically conventional units of one or quantitative units. Rather, they include any referents, whether mathematically relevant or not, that students use as anchors for interpreting and organizing a situation. These units may emerge multimodally, including through verbal explanations, numerical expressions, gestures, or interactions with digital environments. Hereafter, UIM will be used to refer to the referents that students pick up from a situation and use to make sense of it, which are viewed as units in their mathematizing process through the UIM lens. In the following sections, we illustrate how the UIM lens can be applied to an empirical dataset. The empirical excerpt is used not as evidence of the effectiveness of an LLM system, but as an illustrative case for showing the analytic value of UIM.

UIM applied in empirical data

The excerpt presented below is drawn from a larger empirical dataset and is used here to illustrate the UIM lens. Deborah (pseudonym), a fourth-grade student, engaged with a mobile game designed around fractions and was asked to explain how she interpreted the game situation. In her explanation (Figure 1), she described how the game used color-coded numbers to represent different aspects of the situation: green numbers indicated the number of wholes, and blue numbers indicated how those wholes were partitioned into pieces.

D: Deborah & R: Researcher - 1st Interview [17:40–20:19]	
R: Okay, so I'm supposed to have one whole, but because I have two in the blue box (Picture A), it got split up like this way, but this part is still visible and the other part is not solid? (Picture B) D: Yeah, like they're still both solid but they're just split. R: Oh, so you still have two, right? But this one is solid and this one is transparent? D: No, like it's like we have a whole here, right? R: We have a whole here, what do you mean here? Where is the whole?	 Picture A

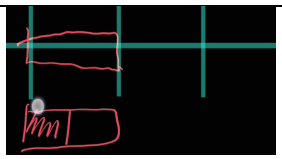
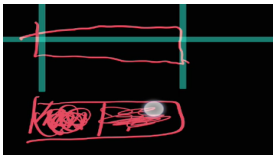
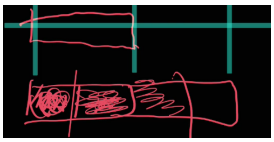
D: So if I press the button (green box) and press 1, it makes this thing a full whole (Picture A), right? And then nothing on this will turn into transparent, we just split it, it didn't turn in transparent. R: But this part is transparent (pointing at the part from ½ to 1 in Picture A)? D: (a) Yeah I don't like, this is what I said, this could be easier if this right here (the whole) you can make a whole you can split it, but none of them turn transparent. Okay, so it's a whole right there, and you split it and like both of them are still solid (Picture C), they just split, they're still connected but split. Then there is another whole, if I can make another whole here and split this one, this one is solid and this one is transparent (Picture D)	 Picture B  Picture C  Picture D

Figure 1: Deborah's Explanation about the Blue and Green Numbers

Analyzed through the UIM lens, Deborah's explanation reveals a rigid UIM structure. Among various referents she recognized such as the color of the two numbers (green and blue), the changing number of green whole sticks, and the half of a green stick turning transparent, the referents she took up and used in her interpretation were 1) the number of green whole sticks, 2) the green numbers, 3) the fact that the green sticks were divided into pieces, and 4) the blue numbers. Drawing on these specific four UIMs in her mathematizing process, she consistently mapped the number of whole sticks to the green number*s*, while mapping the number of pieces into which each whole stick was divided to the blue number*s*. Based on this structure, the combination of green 1 and blue 2 should represent one whole stick divided into two pieces, with both pieces remaining solid in her interpretation. But she did not take up the half of the green stick becoming transparent as a UIM, even though she recognized it. She ended up rejecting it ("*I don't like*") because her UIM structure did not incorporate transparency as a UIM.

This case demonstrates that Deborah's challenge was not a lack of procedural skill but rather the rigidity of her UIM structure in interpreting the situation. Supporting her requires co-mathematizing to help her recognize and reorganize UIM structure, rather than simple corrective feedback. From this perspective, UIM can inform the future design of LLMs, whether through prompt engineering, fine-tuning, or other model-training approaches, by guiding them to attend to the UIMs students identify and coordinate in their interpretations of situations and to pose questions that invite reconsideration of what may count as a unit in a given context.

Units-In-Mathematizing (UIM) as a Framework for LLM Design

The UIM lens foregrounds students' interpretations of situations as central to understanding their mathematical thinking beyond correctness or procedural performance. In a UIM-informed design, the primary object of interpretation is not only the student's final answer, but also how the student organizes the situation: what is treated as a UIM, how UIMs are related, and how these relations change during mathematizing. Thus, rather than specifying a particular technical implementation, UIM lens offers a conceptual basis for future LLM-based instructional partners by clarifying what such systems should attend to, infer, and respond to in students' mathematical activity. Specifically, a UIM-informed system would attend to the units students foreground,

infer whether their difficulties involve procedural errors, unstable unit choices, or rigid UIM structures, and respond with questions that invite students to reconsider, coordinate, or reorganize UIMs. In this sense, the LLM is conceptualized less as a correctness evaluator and more as a co-mathematizing partner that engages with students' interpretive processes.

Illustrative Scenario: Problem-solving Focused vs. UIM-Based LLM Interaction

To illustrate this distinction, consider how two LLM-based interactions might differ. In a problem-solving-focused interaction, a student typically inputs a problem or attempts to solve a given task, and the LLM evaluates the correctness of the response and provides step-by-step feedback. This orientation explains why many studies focus on the accuracy of LLM-generated feedback in response to given inputs (e.g., Goel & Ghanta, 2025; Iqbal et al., 2026; Liu et al., 2025; Scarlatos et al., 2025). While such interactions can support procedural fluency and accuracy, they are not designed to examine how students make sense of the situation itself. As long as students produce the expected answer with an acceptable procedure, their underlying interpretation of the situation may remain unexamined, an issue long recognized in mathematics education research (e.g., Reusser, 1988).

In contrast, a UIM-based LLM interaction foregrounds identification of UIMs and UIM structures as central to the interaction. Rather than treating the student's answer as the primary object of evaluation, the system would treat the student's interpretation of the situation as the object of inquiry. For example, the LLM might introduce a situation that admits multiple interpretations: There are three baskets, each of which can hold five apples. If there are twelve apples, how full will the baskets be? Students' responses may vary, such as "two and two-fifths baskets," "twelve out of fifteen," or expressions of uncertainty. Each response may reflect a different choice of what counts as one whole: one basket, all three baskets together, individual apples, or another referent in the situation.

A UIM-informed LLM would use such responses not merely to determine whether the answer is correct, but to infer how the student is structuring the situation. It could then pose follow-up questions that make the student's unit choices explicit and open to reconsideration, such as "What does 'one' mean in your answer?" or "How would your explanation change if all three baskets together were considered one whole?" Such questions are not intended to lead students directly to a predetermined procedure, but to support their mathematizing by helping them examine and reorganize the units through which they interpret the situation.

Discussion and Limitations

This study proposes *Units-In-Mathematizing* (UIM) as a conceptual lens for rethinking the role of LLMs in mathematics education. Rather than positioning LLM-based AI as a tool for delivering personalized feedback or guiding students through procedural steps, UIM foregrounds students' mathematizing processes, specifically how they identify, coordinate, and reorganize units when interpreting situations mathematically. By illustrating how UIM makes visible aspects of students' mathematizing that may remain hidden in correctness-based assessments, this work contributes a theoretically grounded lens for designing future LLM-based instructional partners. Specifically, UIM shifts the focus of AI-mediated interaction from evaluating students' answers toward clarifying what such systems should attend to, infer about students' interpretations, and respond to in ways that support students' sense-making.

At the same time, this study is intentionally conceptual in nature and has several limitations. The framework has not yet been instantiated in an actual LLM through fine-tuning or prompt engineering, nor has its impact been empirically tested in authentic interaction settings. As a result, the present work does not provide evidence of learning outcomes associated with UIM-

informed AI design. Future research is therefore needed to operationalize this framework in concrete LLM implementations and to examine how UIM-based interactions shape students' mathematizing over time and support deeper mathematical understanding beyond procedural performance.

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MULTIMODAL MICROANALYSIS: EXTENDING PROFESSIONAL NOTICING THROUGH EMBODIED COGNITION

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Professional noticing frameworks have significantly shaped how mathematics educators attend to, interpret, and respond to student thinking. However, traditional approaches often privilege verbal and written forms of expression, potentially overlooking tacit and embodied dimensions of reasoning. In this methodological brief, we introduce multimodal microanalysis, an approach that integrates professional noticing, embodied cognition, and microanalytic methods to systematically examine gesture, gaze, posture, and material interaction alongside speech. Using a cognitive interview with a second-grade student reasoning about even and odd numbers, we illustrate how sequencing attention to embodied and verbal modes reveals dimensions of mathematical thinking that are often missed. This brief highlights the methodological design and analytic insights that emerged from closely examining multimodal data. We argue that multimodal microanalysis advances professional noticing by offering a structured and replicable method for capturing tacit, embodied knowledge in mathematics learning.

Keywords: Teacher Noticing, Research Methods, Mathematical Representations, Teacher Educators

Professional noticing has been conceptualized as the interrelated practices of attending to students' strategies, interpreting their meanings, and deciding how to respond (Jacobs et al., 2010; van Es & Sherin, 2008). These practices have deepened the understanding of teacher learning and instructional responsiveness. Yet, much noticing research foregrounds what students say or write, privileging explicit knowledge while potentially overlooking embodied dimensions of reasoning. As a result, important evidence of student thinking may remain unexamined when it is expressed primarily through action rather than language.

Research in embodied cognition challenges this limitation, positing that mathematical reasoning is distributed across the body and environment (Nemirovsky & Ferrara, 2009; Nathan, 2021). Gesture, gaze, posture, and interaction with materials are not merely expressive but constitutive of cognition (Alibali & Nathan, 2012; Nathan, 2021). Students often demonstrate understanding physically before articulating it verbally. Despite this, professional noticing, embodied cognition, and microgenetic analysis have largely developed as parallel lines of inquiry rather than integrated methodological approaches. Bringing these traditions together creates opportunities to reconsider what counts as observable evidence of knowing in mathematics classrooms.

This brief report shares multimodal microanalysis as a methodological bridge among these traditions. The guiding question for our work is: How does multimodal microanalysis expand what mathematics teacher educators notice and interpret as evidence of student mathematical thinking? We found that by sequencing attention to embodied and verbal modes and by using

fine-grained video analysis, mathematics teacher educators can broaden what counts as evidence of knowing. In doing so, the approach reframes noticing as an inherently multimodal practice.

Theoretical Framing

Professional noticing involves attending to students' strategies, interpreting their mathematical meanings, and deciding how to respond instructionally (Jacobs et al., 2010). More recent scholarship has emphasized that noticing is shaped by teachers' lenses, identities, and sociopolitical contexts (Louie et al., 2021; Rubin & van Es, 2023). However, conventional noticing often prioritizes verbal articulation as the primary site of evidence, which may inadvertently position spoken explanation as the clearest indicator of understanding.

Brady et al. (2022) distinguish between explicit knowledge, which can be verbalized, and tacit knowledge, which is enacted through action and often remains unarticulated. When teachers attend primarily to spoken or written language, tacit dimensions of understanding may remain invisible. Embodied cognition offers a theoretical grounding for recognizing gesture, posture, and material engagement as integral to reasoning rather than peripheral to it (Nemirovsky & Ferrara, 2009; Nathan, 2021). From this perspective, the body is not simply a vehicle for communication but a site where mathematical ideas are formed and refined.

Multimodal microanalysis provides tools for examining learning at fine-grained time scales (Parnafes & diSessa, 2013). This approach captures moment-by-moment conceptual shifts that may occur prior to the more obvious linguistic and cognitive features. Multimodal microanalysis builds on microanalytic traditions while centering embodiment as a primary lens of interpretation. This integration supports a more comprehensive account of how mathematical thinking unfolds across time and modalities.

Multimodal Microanalysis

Multimodal microanalysis is a structured approach to analyzing video data in which embodied and verbal expressions are treated as equally central evidence of reasoning. The approach is characterized by intentional sequencing of modalities and slowed, repeated viewing. Following Radford (2009) and Sinclair (2024), we muted the audio during initial viewing to foreground gesture, gaze, and bodily action before interpreting speech. This sequencing was designed to prevent spoken language from prematurely shaping interpretation and to surface forms of reasoning that might otherwise be overshadowed by verbal explanation.

Video segments were analyzed at multiple playback speeds, including half speed and three-quarter speed, to detect subtle shifts in posture, hand movement, and gaze. Using ELAN software (Lausberg & Sloetjes, 2009), we created temporally aligned annotation tiers that distinguished learner embodiment, learner speech, instructor embodiment, and instructor speech. This layered structure allowed us to examine how modes overlapped, shifted, and interacted across time, making visible the coordination between bodily action and emerging verbal ideas.

Through collaborative coding and discussion, we developed shared interpretations of embodied utterances and their relation to verbal explanations. This process employed multimodal microanalysis as a methodological tool and a professional learning practice for mathematics teacher educators. Collaborative interpretation further supported analytic rigor by making possible researchers' assumptions explicit and open to negotiation.

Case Example

The case analyzed was a cognitive interview with a second-grade student asked to classify the number six as even or odd using blocks (Musser et al., 2013). Six mathematics teacher educators first engaged independently in traditional professional noticing. They documented the

student's verbal responses and observable strategies, establishing a baseline for comparison with the multimodal analysis.

We then selected a forty-second segment for multimodal microanalysis. Two researchers conducted initial ELAN coding, followed by collaborative refinement with the full research team. The aim was to examine what dimensions of reasoning became visible when attention was systematically directed toward embodiment before speech, thereby contrasting insights generated through conventional and multimodal microanalysis noticing.

Findings

Initial professional noticing captured that the student correctly identified six as even and used a pairing strategy with the blocks. The instructor's follow-up prompts were also noted. However, multimodal microanalysis revealed several dimensions that had not been documented in the original observations, demonstrating how analytic focus shapes what is ultimately recognized as evidence.

When first prompted to explain using blocks, the student's brow furrowed, and his gaze dropped toward the table. These embodied cues suggested uncertainty that was not reflected in his verbal assertion. Before manipulating the blocks, the student tapped the table three times, a subtle gesture that may indicate counting or cognitive processing. None of the researchers recorded this action during initial noticing, highlighting how quickly such movements can be overlooked in real-time observation.

The student engaged in two strategies. He first separated the blocks into two groups of three without verbal explanation. This fair-sharing attempt was often missed or uninterpreted during traditional noticing. He later grouped the blocks into three pairs and articulated a teacher-taught definition of evenness. Across these moments, gesture size and fluidity shifted. When uncertain, gestures were smaller and closer to his torso. As his explanation became more fluent, gestures expanded outward and speech became smoother. These changes in movement provided evidence of shifting engagement and understanding, suggesting a dynamic relationship between physical action and conceptual stabilization.

Multimodal microanalysis also surfaced features of the instructor's embodied behavior. Extended moments of stillness supported the student's independent reasoning. In certain instances, the instructor mirrored or echoed the student's gestures, creating embodied coordination that was not initially documented. These observations reframed the interaction as a coordinated, multimodal exchange rather than a purely verbal explanation, emphasizing the co-constructed nature of mathematical sensemaking.

Discussion

Multimodal microanalysis enabled us to broaden what counted as evidence of knowing. By foregrounding gesture, posture, and gaze, the research team recognized tacit dimensions of reasoning that speech alone did not reveal. What initially appeared as confident recall was reinterpreted as partially memorized knowledge supported by embodied experimentation, illustrating how multimodal evidence can refine instructional interpretations.

The structured sequencing of modalities proved essential. By muting audio first, researchers attended to embodied expression without being guided by verbal explanation. Slowed playback allowed for detection of subtle shifts that occurred too quickly to notice in real time. Layered annotation supported systematic comparison across modalities, strengthening the transparency and replicability of the analytic process.

Rather than replacing professional noticing, we assert that multimodal microanalysis extends it. The attending, interpreting, and deciding framework remains central, but what is attended to is expanded. The combination allows embodied utterances to shine as legitimate and analyzable evidence of mathematical offering deeper intuition reasoning. positioning the body as a meaningful source of data rather than contextual background.

Methodological Contribution

This work contributes a methodological extension of professional noticing. Multimodal microanalysis integrates embodiment and microanalytic precision building onto teacher noticing research. It offers procedural guidance for sequencing attention, structuring analysis, and collaboratively interpreting multimodal data. By doing so, it clarifies how mathematics educators can systematically capture mathematical knowledge expressed through bodily action, thereby supporting richer accounts of student thinking and interaction. Figure 1 shows a conceptual framework for this methodological contribution that advances professional noticing by systematically incorporating embodiment into the interpretation of student thinking.

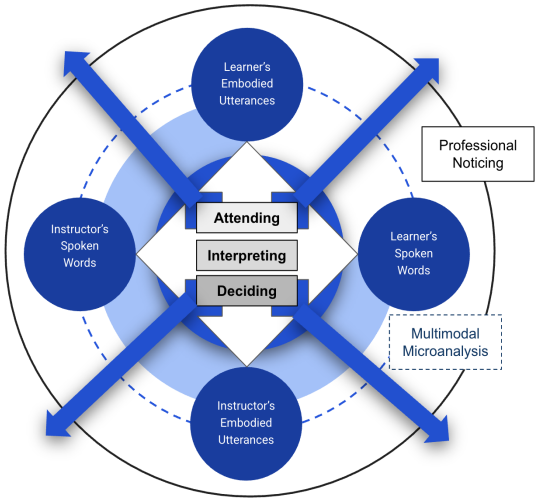


Figure 1: A Conceptual Framework for the Interplay of Professional Noticing and Multimodal Microanalysis Supported by Embodiment

Limitations and Future Directions

Multimodal microanalysis is time-intensive and best suited for focused case studies or short video segments. Interpretive bias remains a consideration, requiring collaborative triangulation and reflexivity. Future research might explore how elements of multimodal microanalysis can be adapted for teacher professional development contexts where time constraints are significant, as well as how the approach might inform the design of noticing-focused learning experiences for educators.

Conclusion

Mathematics learning involves coordinated representations, spatial reasoning, and bodily engagement (Lesh et al., 1987; NCTM, 2014). Research methodologies must likewise account for these dimensions. Multimodal microanalysis extends professional noticing by systematically incorporating embodied cognition into analytic practice. By broadening what counts as evidence

of mathematical thinking, this approach enhances research on teacher learning and supports more comprehensive interpretations of student reasoning, ultimately encouraging mathematics educators to notice not only what students say, but also what their bodies reveal about emerging understanding.

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TEACHERS' PERCEPTIONS AND PREPARATIONS FOR GENERATIVE ARTIFICIAL INTELLIGENCE: SURVEY RESULTS FROM MISSOURI AND TEXAS

PERCEPCIONES Y PREPARATIVOS DE LOS DOCENTES PARA LA INTELIGENCIA ARTIFICIAL GENERATIVA: RESULTADOS DE UNA ENCUESTA DE MISSOURI Y TEXAS

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This brief report examines K–12 mathematics teachers' responses (n = 811 from Missouri and Texas) to a survey about their preparedness for and perceptions of generative AI (GenAI). The survey was a mathematics adaptation of the LEAP-AI instrument (von Gillern et al., 2025) and generated quantitative and qualitative data. Findings indicate moderate confidence with general technology but lower confidence with GenAI for instruction and mixed views on mathematics teachers' responsibilities relative to students' learning of GenAI. Teachers identified content generation and lesson planning as primary affordances of GenAI, alongside concerns about student overreliance and academic dishonesty. State-level differences were observed.

Keywords: Teacher Beliefs, Technology

As technology companies race forward into the artificial intelligence revolution, educators and education scholars are trying to understand the potential benefits and the likely dangers of these technological advances with regard to students. Of particular importance is generative artificial intelligence (GenAI), which is AI designed to produce outputs (text, images, audio, etc.) for a user (e.g., ChatGPT (OpenAI, 2023), Gemini (Google, 2004), and Claude (Anthropic, 2024). GenAI's capabilities go well beyond prior technological revolutions in mathematics education such as calculators (Penglase & Arnold, 1996), dynamic geometry software (Oner, 2009), or problem-solving smartphone apps (Webel & Otten, 2015), though GenAI is still unreliable and requires a diligent user (McGalliard & Otten, 2024; Walkington, 2025). Nevertheless, diligent and non-diligent students seem to be using GenAI to a noticeable degree, as evidenced by sharp declines in GenAI usage after the school year concludes (Burman, 2025).

As these technologies rapidly impact students, it is important to consider and listen to teachers as those who are on the front lines of the revolution, using GenAI themselves and guiding students in its ethical use. In this brief report, we share findings from a survey of 811 K-12 mathematics teachers across Missouri and Texas. The survey pertained to their confidence with technology and their opinions regarding GenAI, including open-ended questions about their opinions on benefits and concerns about GenAI in mathematics education.

Background Literature and Theoretical Perspective

Although we focus this work on teachers, students' perspectives are also worthwhile. Kibici (2025), for example, found a position relationship among university students between digital literacy skills and positive attitudes toward GenAI, as one might expect, and although GenAI was viewed favorably overall, the students had concerns about creative works and ethical use of GenAI. At the middle school level, Adisa and Adefisayo (2025) found that students can have quite nuanced opinions about the use of GenAI in school, shaped by their desires for ownership over learning (or not) and their desires for relationships with human educators (or not).

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Although students may use GenAI of their own volition, for the field of mathematics education broadly it will be crucial to understand teachers' perspectives. Research in this area is still emerging, but relevant studies have been conducted in other areas of education. Working at the university level, Alenezi (2024) found that instructors generally see positive potential for GenAI in education but had relatively low intentions to actually adopt GenAI. At the school level, in literacy education, von Gillern et al. (2026) identified several concerns raised by teachers about GenAI, including the unreliability of the outputs and the potential for cheating by students. They also saw benefits, however, such as time efficiency for themselves as teachers and the potential of automated feedback for students and preparation for an unknown future. Within mathematics, specifically, Busuttil and Calleja (2025) found that teachers who value student discovery and meaningful connections did tend to see a potential role for GenAI in students' learning, but professional development may be centrally important in helping teachers discern how to incorporate the technology into their teaching.

Much work remains to be done on the ethics of GenAI in education (Akgun & Greenhow, 2022) and pedagogical theorizing around the integration of GenAI into teaching and learning (Erduran & Levrini, 2024), but we hope to contribute an illumination of the current landscape in terms of mathematics teachers' perceptions of GenAI. In particular, we seek to answer the following research questions drawing upon a survey of K-12 mathematics teachers:

1. What level of agreement did mathematics teachers express with regard to their preparedness to use GenAI and their responsibility to teach students about GenAI?
2. What benefits of and concerns about GenAI did mathematics teachers describe?

Method

Participants in this study were respondents to a survey distributing electronically to email lists of all certified teachers in Missouri and Texas, specifically, all teachers of grades K–5 and all teachers with mathematics certification in grades 6–12. We received 811 responses (459 from Missouri, 352 from Texas) and although response bias is likely, with the explicit topic of GenAI in the survey invitation and with approximately 4 times as many secondary teachers responding than elementary teachers, a direct inspection revealed reasonably similar responses on all Likert-items between the two subgroups, and the respondents exhibited demographics relative to age, teaching experience, and race that were fairly consistent with state demographics.

Drawing upon survey design methodology (Dillman, 2011), this study involved an adaptation of a GenAI-perception survey for literacy teachers (von Gillern et al., 2025), Literacy Educators' Attitudes and Perceptions on Artificial Intelligence (LEAP-AI), revising the items to relate to mathematics teaching and learning. Validation of this revision with regard to the factor structure is ongoing, but descriptive statistics of key single items are reported here, as shown below, using only the responses of teachers who completed at least 85% of the survey (n = 758).

With regard to the open-ended items, the survey included the following questions: "In what ways do you think AI could be beneficial to math instruction?" and "What are you biggest concerns about AI related to math instruction?" These responses were open coded within the Qualtrics survey software and then two co-authors met to reach an acceptable level of inter-coder agreement. Subsequently, a second round of thematic coding took place to collapse and clarity the results of the first round. Due to space limitations, only the predominant themes are reported below, with data excerpts presented but with no distinctions between subgroups of respondents.

Findings

Mathematics Teachers' Feelings of Preparedness to Teach with GenAI

Table 1 displays results for key items from the LEAPS-AI survey adapted for mathematics teachers, all answered on a 5-point Likert scale (1 = strongly disagree, 5 = strongly agree). Note that teachers in Texas tended to have higher levels of agreement than teachers in Missouri. This was true across all Likert-scale items.

Table 1: Teachers' Level of Agreement to Various Statements About Technology and GenAI

Survey Item	In general, I am confident in my ability to effectively integrate educational technologies.	I am confident in my ability to effectively use GenAI for personal purposes.	I am confident in my ability to effectively use GenAI for educational purposes.	It is important for math teachers to understand how to create effective prompts for GenAI.	It is important for math students to understand that GenAI texts often include inaccuracies.	Math teachers should be responsible for teaching students to use GenAI in different educational scenarios.
MO (*n*=482)	3.02	2.70	2.45	3.11	3.27	2.61
TX (*n*=276)	3.96	3.46	3.20	3.92	4.04	3.54
St. Dev. on item overall	0.88	1.11	1.13	0.90	0.93	1.11

Moving left to right in Table 1, the teachers tended to not feel unconfident about their technology skills in general, with Texas teachers reporting fairly high levels of confidence, but the confidence levels related to GenAI were markedly lower, which makes sense given the relatively new emergence of GenAI. Teachers in both states tended to be slightly more confident about personal use of GenAI than educational use. Texas teachers tended to agree that teachers should understand effective prompting with GenAI and that students should be aware of potential inaccuracies from GenAI (Missouri teachers did not disagree), but results were mixed on the teachers' responsibility. Texas teachers tended to slightly report an agreement that mathematics teachers should bear responsibility in developing students' GenAI skills, but Missouri teachers tended to slightly disagree with this statement.

Teachers' Descriptions of the Potential Benefits of GenAI in Mathematics Instruction

A majority of teachers reported GenAI could be a beneficial tool for creating mathematical situations (e.g., word problems, activities) they could use in their classrooms. Many teachers further specified that GenAI could be helpful in creating content that contained real world connections for students or that differentiated the activities for specific students. For example, a Missouri teacher stated that "AI can help generate problems when we are looking for some more practice […] like, coming up with an equation and putting a back story to it to create a word problem." A Texas teacher also reported actually taking advantage of these sorts of benefits:

> We use AI to diagnose students' learning levels and then tailor instruction to them at that level or slighter harder. It also learns their likes and dislikes to help offer them subjects/topics they prefer compared to non-preferred subjects/topics.

Overall, 390 teachers (51%) specifically mentioned content creation as a benefit of GenAI.

The other predominant theme in the survey responses was the related but distinct notion of lesson planning. A total of 168 teachers (22%) described how GenAI could be beneficial for lesson planning in general or for specific aspects of the planning process, beyond the creation of just individual elements (captured above). For instance, one Texas teacher wrote that GenAI could be used to not only generate problems but "differentiation [the problems] according to each student's needs" as part of the lesson overall, and assisting with the accompanying instruction because the "AI could help demonstrate the steps used to solve each problem."

Other benefits mentioned less often had to do with translating or supporting English learners, providing mathematical support at home, or generally enhancing interest in mathematics in some manner. It should be noted, however, that even on an item about GenAI benefits, 77 teachers (10%) expressed that there might not be benefits or were adamant that GenAI should not be used in mathematics education, which brings us to the next section of results.

Teachers' Descriptions of Concerns about GenAI in Mathematics Teaching and Learning

Most of the concerns teachers had about GenAI in mathematics instruction had to do with their perceived effects on students. There were two primary themes, having to do with a potential negative impact on students' learning, due to their overreliance on the capabilities of GenAI, and also the concern over cheating or academic dishonesty. The two themes may be implicitly related as cheating may undermine a students' depth of learning but can also be a concern in its own right. An example of such concerns as stated by a Missouri teacher is as follows:

> One of my biggest concerns about AI in math instruction is the potential for over-reliance on it, which might reduce critical thinking and problem-solving skills in students. While AI can generate quick solutions and detailed explanations, there's a risk that learners might depend too much on AI-generated answers without fully understanding the underlying concepts.

In total, 239 teachers (32%) expressed concern about detriments to student learning and 220 (29%) expressed concerns about academic dishonesty (and some teachers expressed both and are included in both counts). Note that here, as in the benefits section, the item was open ended and so teachers did not necessarily write about *all* of their concerns (or perceived benefits).

Teachers also expressed concerns related to GenAI's effect on their own work in teaching. One teacher from Missouri wrote, "My biggest concern is that AI is becoming another technology that teachers are not trained to use and therefore cannot appropriately teach their students to use." This response suggests that there may be a beneficial way to use GenAI with students, but teachers may not be equipped to do so and thus, with teachers as the conduit, it is an area of concern. A Texas teacher alluded to the relational work that is central to teaching, stating, "My biggest concern is getting away from the personalized connections formed between teachers and students." Others expressed that it was troublesome to think about the time it would take to learn GenAI when they are already overburdened with existing responsibilities. Overall, however, these teacher-focused themes were not as prevalent as the student-focused ones, with only 71 teachers (9%) expressing concerns rooted in teaching or teachers.

Discussion

Amidst a frantic shift toward GenAI in various aspects of life, including education, we took a moment to collect and consider the perspectives of mathematics teachers from two states in the U.S. The teachers expressed mixed levels of agreement about aspects of teaching and using GenAI, with substantial individual variation and also marked differences between Missouri and Texas (perhaps because of contextual differences between the states or because of a response bias with relatively fewer responses from Texas). Teachers described various benefits and

concerns about GenAI in mathematics education, exhibiting their consistent concern for students above all else. These findings corroborate past research and provide insight into teachers' mindsets as we promote, critique, or design professional development related to GenAI.

Acknowledgments

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PRESERVICE TEACHERS' TPACK DEVELOPMENT WHEN INTEGRATING DIGITAL MATH GAMES INTO FRACTION LESSON PLANS

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This Brief Research Report examined how 22 elementary preservice teachers developed their technological pedagogical content knowledge (TPACK) when integrating digital math games into fraction lesson plans. Data from pre- and post-surveys and a fraction lesson plan were analyzed by summarizing survey data, using stacked bar graphs, and categorizing lesson plans by TPACK developmental levels. Findings showed preservice teachers' lesson plans had three distinct developmental levels: Recognizing, Accepting, and Adapting. These results suggest that preparation programs need to bridge the gap between preservice teachers' reported confidence and their practical integration of digital math games.

Keywords: Preservice Teacher Education, Technology, Elementary School Education

As educational games become common in K-12 classrooms, preservice teachers (PSTs) must learn to effectively integrate these tools into instruction. Integrating digital math games (DMGs) requires synthesizing content, pedagogy, and technology, a process analyzed through the Technological Pedagogical Content Knowledge (TPACK) framework (Mira & Koehler, 2006). This Brief Research Report examines PSTs' TPACK development as evidenced in their self-reported survey responses and fraction lesson plans. This mixed methods study (Creswell & Plano Clark, 2017) aims to provide insights into how to better support PSTs in their TPACK development and improve their abilities to integrate DMGs into mathematics lessons.

Literature Review

Many new teachers are entering the profession underprepared to integrate technology into their instructional practices (Gudmundsdottir & Hatlevik, 2017; Meaney et al., 2023). To better prepare these educators, teacher preparation programs must incorporate frameworks and tools, like DMGs, which effectively support mathematics understanding (Bullock et al., 2020; Falloon, 2013; Kärki et al., 2021; Ke & Abras, 2013; Moyer-Packenham et al., 2019). Specific design features within a DMG actively supports students' mathematics learning (Bullock et al., 2020; Falloon, 2013; Ke & Abras, 2013; Moyer-Packenham et al., 2019). Among the eight mediating design features identified by Moyer-Packenham et al. (2019), standard instructional scaffolds like hints and feedback were reported to be essential, but linked representations and linked physical actions uniquely support students' mathematical understanding. Because these particular features allow students to directly interact with mathematical concepts through physical movement or representational connections, teacher must recognize them when choosing and evaluating DMGs for instruction. In this study, these design features guided the learning module as the specific technology game knowledge PSTs need to effectively integrate DMGs.

This specialized knowledge forms a key component of the Technological Pedagogical Content Knowledge (TPACK) framework (Mishra & Koehler, 2006), a widely used lens for examining the dynamic relationships among a teacher's technology, pedagogy, and content knowledge (Gutiérrez-Fallas & Henriques, 2021; Lachner et al., 2021; da Silva Beuno & Niess 2023). Hsu et al. (2013, 2017, 2020) extended TPACK to include game-specific dimensions (TPACK-G): game knowledge (how to play); game pedagogical knowledge (how to teach with

games) and game content knowledge (how the game represents content). This study used TPACK-G to analyze PSTs' development from pre- to post-survey.

Furthermore, research suggests that teachers' TPACK progresses through five levels of development (Niess et al., 2010; Niess & Gillow-Wiles, 2013): (1) *Recognizing* how technology aligns with content topics, (2) *Accepting* beliefs (positive or negative) about teaching with technology in relation to specific grade levels, (3) *Adapting* tasks by choosing appropriate technologies for instruction, (4) *Exploring* how to effectively implement appropriate technologies in their content areas, and (5) *Advancing* their integration of technology by reflecting on their instructional practices and making changes based on that reflection. While previous studies have utilized this development framework (da Silva Bueno et al., 2023; Kartal & Çinar, 2024; Lyublinskaya & Kaplon-Schilis, 2022), few focus on DMG integration in mathematics. Integrating DMGs is different from using traditional classroom technologies because games have their own internal rule systems and structures. As a result, the DMG context introduces a notable tension between how a teacher wants to guide instruction and how the game's specific design features shape how students interact with the game (Hsu et al., 2013). Therefore, this study uses these levels to categorize PSTs' TPACK-G within their lesson plans, tracking their progress from basic awareness of games for instruction to advance ability to leverage specific design features as core pedagogical tools.

Methods

As part of a larger convergent mixed methods study (Creswell & Plano Clark, 2017), this Brief Research Report addresses the research question: How do preservice teachers TPACK-G survey responses, before and after a learning module, relate to the development levels of TPACK development level demonstrated in post-module their lesson plans? With two guiding sub-questions: (1) What changes, if any, are observed in preservice teachers' TPACK-G survey responses from pre- to post-survey? and (2) How do preservice teachers' lesson plans demonstrate levels of TPACK development?

Participants and Procedures

Participants included 22 elementary PSTs recruited from four semesters of an asynchronous online math methods course from one university. Participant demographics were not collected; however, the university's population is predominantly white (72%), with approximately 57% identifying as female and 43% identifying as male, providing context for the participant sample. Over a three-week instructional module focused on TPACK and DMGs, participants completed a pre-survey, engaged in a learning module on evaluating and integrating DMGs, developed an inquiry-based lesson that integrated a DMG; and completed a post-survey.

Data Sources and Analysis

Data sources for this brief report included PSTs pre- and post-survey responses and their fraction lesson plans. The surveys included 13 Likert scale items measuring agreement, adapted from established TPACK-G literature (Hsu et al., 2013, 2017, 2020). For example, Hsu et al. (2017) used the statement, "I can identify the knowledge related to the subject matter in digital games." In this study, this was adapted to, "I can identify the knowledge related to the mathematics in digital math games." The survey also included two open-ended questions related to game familiarity and game design features. The post-survey included the same items as the pre-survey, but in a randomized order. Finally, PSTs' lesson plans followed the "Launch, Explore, Discuss" framework (Markworth et al., 2015), chosen to shift PSTs focus from explicit instruction toward inquiry-based planning that supports student exploration with DMGs.

To analyze the quantitative data, stacked bar graphs summarized the percentage distribution

and frequencies of PSTs Likert scale ratings (Cooksey, 2020; South et al., 2022) to visually track shifts from pre- to post-survey. Deductive protocol coding (Saldaña, 2021) was used to qualitatively analyze PSTs' lesson plans. The operational definitions for Niess et al.'s (2010) levels of TPACK development (e.g., recognizing, accepting, adapting) served as the coding scheme. Each lesson plan was segmented by structural components (launch, explore, discuss) and analyzed for evidence of game-based mathematical pedagogy. To maximize interpretive reliability, a detailed codebook was used. Survey results and lesson plan codes were compared to explore how changes in self-reported TPACK-G related to PSTs planning decisions.

Results

Analysis revealed positive shifts in PSTs' self-reported TPACK-G from pre- to post-survey, as shown in Figure 1, by a shift from disagreement (red/orange bars) to agreement (blue/green bars). For example, for the statement "I have adequate training to integrate DMGs into mathematics instruction," pre-survey responses favored the disagree portion of the Likert scale (59%, n = 13 Disagree/Strongly Disagree; 27%, n = 6 Neutral; 14%, n = 3 Agree/Strongly Agree). However, on the post-survey PSTs' responses noticeably shifted toward the agree portion of the Likert scale (0% disagree; 5%, n = 1 Neutral; 96%, n = 21 Agree/Strongly Agree). Eight other survey items showed a similar positive trend. Three statements maintained high neutral-to-agree ratings on pre- and post-survey, indicating that positive views towards DMGs were already established at the beginning of the study. In contrast, the statement, "I have no idea how to integrate DMGs into my curriculum," was the only item to shift toward the disagree portion of the scale (95%, n = 21 Disagree/Strongly Disagree; 5%, n = 1 Neutral); however, this statement is negatively phrased, which actually represent a positive shift in self-reported confidence. These combined shifts demonstrate a clearer self-reported understanding of how to integrate DMGs and indicates a broader trend of growth in PSTs' overall self-reported confidence.

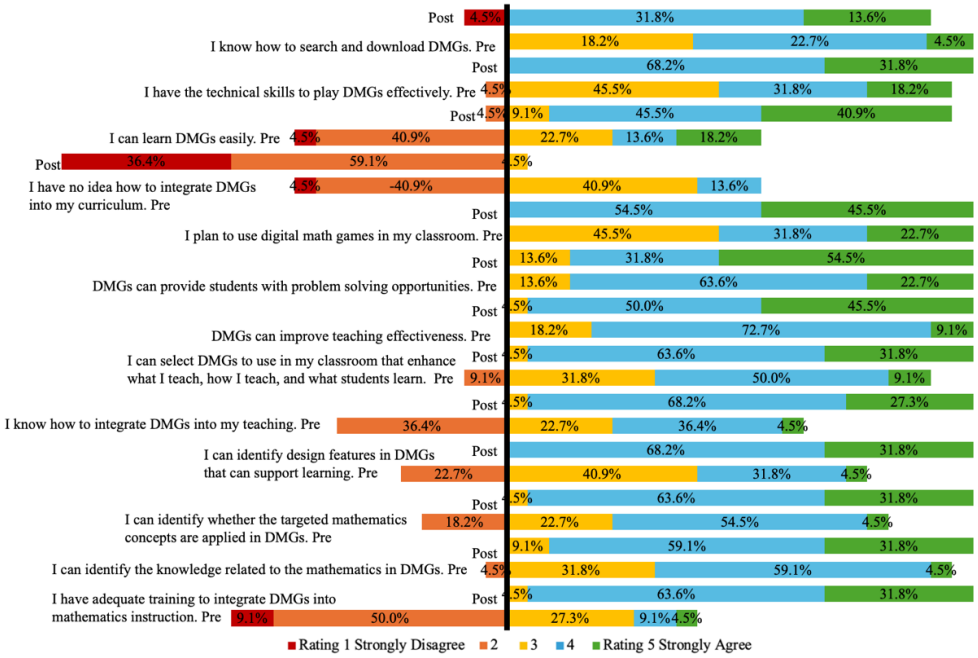


Figure 1: PSTs TPACK-G Pre-Survey and Post-Survey Responses

PSTs lesson plans had three distinct developmental levels that emerged: Recognizing, Accepting, and Adapting. Several PSTs (41%, $n = 9$) demonstrated the Recognizing level by showing awareness of the DMGs, but using the games solely as practice-based or motivational support. For example, one PST described the integration of the game as "Students will practice (by using the game) counting by ones, tens, and hundreds using base ten blocks." Thirty-six percent of PSTs (n = 8) reached the Accepting level, using DMGs to support learning, but limiting student learning through direct instruction or teacher-led demonstrations. For example, one PST wrote "I will use explicit instruction to show students how to navigate the website." Another PST wrote, "Teacher guides step, by step during play." Few PSTs (23%, $n = 5$) showed the Adapting level when they thoughtfully integrated DMGs and asked meaningful questions to support student learning. For example, one PST asked, "Who can explain how it went about finding three fractions that were equivalent in the game?" Another PST asked a specific question related to the game when they asked, "Look at this Triplet. How do we know these three go together?" Comparing these levels to the post-survey results (Figure 1) revealed a disconnect between perception and practice. Many PSTs at the Recognizing and Accepting levels reported high confidence in their ability to integrate DMGs and identify design features, yet their lesson plans reflected limited implementation of DMGs for instructional practice.

Limitations

While this study offers valuable insights, several limitations must be acknowledged. First, the findings may not be generalizable beyond the participants in this study due to the study's short duration and small sample size. Additionally, self-reported data may not fully reflect PSTs actual knowledge or skills for integrating DMGs into mathematics instruction. Finally, the use of a single coder for qualitative analysis may introduce subjectivity and limit the reliability of interpretations. Future research should include larger cohorts and multiple coders to help address these boundaries.

Discussion and Conclusion

While findings indicate a moderate increase in PSTs self-reported TPACK-G following the learning module about DMGs, this perceived growth was not yet reflected in their actual lesson plans. This suggests that while the learning module successfully fostered TPACK-G confidence, translating that confidence into practice remains a challenge. These findings extend the TPACK-G framework by demonstrating that its knowledge dimensions do not develop simultaneously, suggesting misalignment between PSTs' evolving game-related pedagogical beliefs and their ability to translate those beliefs into effective lesson planning.

This structural discrepancy suggests that the framework's developmental progression includes a critical transition point between the "Accepting" and "Adapting" levels. Therefore, this study concludes that while the learning module was an effective tool for shifting PSTs perspectives about DMGs, it must be paired with a deeper, scaffolded practice to translate these gains into practice. In practice, this scaffolding could include providing lesson plan templates with prompts for DMG design features, or utilize peer co-planning sessions where PSTs collaborate to plan lessons that integrate a DMG. Ultimately, teacher education need to include designing experiences that not only build PSTs confidence with integrating DMGs but also ensure the growth is reflected in observable teaching practices. Future research should continue to examine the specific, practical support necessary for PSTs to successfully shift from positive perspectives about DMGs to high-quality DMG integration.

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EXAMINING PEDAGOGICAL DECISION-MAKING IN AI-GENERATED MATHEMATICAL FEEDBACK

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AI-generated feedback can help teachers focus on students' mathematical reasoning by annotating student work to promote engagement and mathematical thinking. Using the Math Forum data corpus, a dataset of mentor feedback to student work and reasoning, we built a Retrieval-Augmented Generation (RAG) model to generate feedback using the noticing and wondering framework. We evaluate this AI system using our dataset against a similar AI system without this added training data, and against an AI system fine-tuned with the Math Forum training data, to determine the viability of a RAG approach to generating feedback. The RAG model tended to focus responses on procedural correctness, while the fine-tuned model centered on student reasoning and process. Our findings suggest that a fine-tuned approach may better direct AI generations but should be coupled with human supervision.

Keywords: Technology, Teacher Noticing

As recent advances in generative AI have emerged, there is greater attention focused on how teachers might integrate this evolving technology into their classroom practices (Holmes et al., 2019; Zafari et al., 2022). We have seen success with these tools as direct-to-student tutors, however they often lack the pedagogical sophistication needed to provide feedback that builds on student thinking rather than simply correcting errors (Stein et al., 2008). Our work investigates whether AI might serve as a support tool that helps teachers draft and refine feedback more efficiently while also retaining their role as the primary mentor.

This study draws on a valuable resource: the Math Forum's thirty-year collection of mentor feedback on students' mathematical thinking across a wide range of "Problems of the Week." This archive extends beyond pairs of student work and mentor responses. It includes rich, paragraph-length explanations of student reasoning spanning multiple grade levels and mathematical standards, capturing both correct and incorrect approaches. Using this unique dataset, we ask whether integrating historical mentor feedback through Retrieval-Augmented Generation (RAG) can enhance the quality of AI-generated feedback.

We compare three AI models in this work: a baseline model with no specialized training, a RAG model informed by Math Forum data, and a fine-tuned model trained on the same data with additional instructional guidance. Through qualitative analysis, we examine whether the draft feedback from each model provides teachers with material they can meaningfully refine and whether it reflects the discourse patterns characteristic of high-quality mathematical mentoring.

Theoretical Framework/Conceptual Framework

This study draws on two key frameworks: Hattie and Timperley's (2007) model of effective feedback and Stein et al.'s (2008) framework for productive mathematical discourse. Together, they define high-quality mathematical feedback as feedback that clarifies goals, responds to

students' current thinking, and supports next steps without lowering cognitive demand. Effective feedback builds on students' reasoning, including partially correct or incorrect ideas, while maintaining their agency as sense-makers (Stein et al., 2008).

Our interdisciplinary team of mathematics educators and AI researchers works within a pipeline designed to generate and analyze AI-produced feedback. Providing strong feedback requires instructional judgment that goes beyond checking correctness: teachers must identify productive ideas in flawed reasoning, anticipate misconceptions, and determine what to address in the moment. These judgments are highly contextual and cannot be reduced to fixed rules.

Guided by these frameworks, we investigate whether Retrieval-Augmented Generation (RAG) can help AI systems approximate aspects of expert feedback by grounding responses in mentor exemplars. Using the Math Forum's rich corpus, we examine whether AI-generated drafts reflect core principles of effective mathematical feedback. Our goal is not to replace teachers, but to evaluate whether theoretically informed training data can produce AI drafts that meaningfully support teacher refinement.

Literature Review

Research on AI in mathematics education shows mixed outcomes. Some scholars argue that AI can ease teachers' workloads by reducing time spent on grading and feedback, potentially allowing more time for planning and individualized support (Owan et al., 2023). Yet despite performing well on many undergraduate-level tasks, AI systems often struggle with more complex mathematical reasoning and frequently miss students' underlying misconceptions (Frieder et al., 2023). As a result, AI-generated feedback tends to prioritize procedural accuracy over the reasoning-focused, adaptive feedback known to support deeper mathematical learning (Stein et al., 2008).

A key limitation in current research is the limited attention to the pedagogical sophistication of AI outputs. Many studies assess correctness or efficiency rather than examining how AI responds to different forms of student thinking (Holmes et al., 2019). In contrast, effective mathematical feedback relies on recognizing productive ideas within incorrect reasoning, anticipating likely misconceptions, and making moment-to-moment decisions about what to address, skills that require specialized pedagogical knowledge (Mishra & Koehler, 2006). Although AI can produce fluent, teacher-like language (Case et al., 2025), this surface-level appropriateness may conceal misunderstandings or lead to unproductive moves such as excessive praise, undermining student agency, or reducing cognitive demand (Dweck, 2007).

This study examines whether Retrieval-Augmented Generation (RAG), which draws on relevant historical examples, can improve the pedagogical quality of AI-generated feedback when trained on expert mentor responses. Using Encompass, a professional development platform that supports teachers in analyzing student thinking through structured noticing and wondering, we compare feedback from three AI models. Our goal is not only to determine whether AI feedback is mathematically accurate, but whether it demonstrates the kinds of pedagogical moves that build on student thinking, sustain cognitive engagement, and position learners as capable mathematical thinkers.

Methodology

We conducted a qualitative, memo-based analysis of AI-generated feedback on 100 samples of student mathematical work across three model conditions: a base model, a RAG model, and a fine-tuned model. For each condition, AI feedback was paired with the corresponding student work (see Figure 1, Adapted from Problem of the Week: Birthday Trip, by National Council of

Teachers of Mathematics), and analytic memos were created through close reading to capture patterns in mathematical accuracy, attention to student reasoning, pedagogical stance, and instructional usefulness.

Our approach followed an iterative, inductive process grounded in memo-based and constant comparative methods (Miles, Huberman, & Saldaña, 2014; Charmaz, 2014). Memos were grouped into higher-level patterns without predefined codes, and memo titles were retained as analytic artifacts to maintain proximity to researchers' observations. Throughout the process, team members discussed emerging themes, using disagreement and surprise as productive analytic tools rather than indicators of unreliability (Erickson, 1986). This allowed us to identify stable failure modes, shifts across model conditions, and design-relevant tensions in how AI systems interpret and respond to student mathematical thinking.

Jade's family is planning to travel to her Aunt Mazie's house to celebrate Aunt Mazie's 102nd birthday. On the first day of the trip they'll drive halfway there and then stop to set up camp for the night. On the second day of the trip they'll drive two-thirds of the remaining distance before stopping for some sightseeing and camping. On the last day they'll have 145 miles left to drive.



Question: How far is the trip to Aunt Mazie's house?

Extra: If the family averages 50 miles per hour while driving on the return trip, do you think the family will make the trip in one day? Why or why not?

Figure 1: Birthday Trip Problem

We used a llama3-based large language model (LLM) and augmented it with a RAG system built from the Math Forum training corpus. The corpus was embedded using the intfloat/e5-large-v2 model in 512-token windows with a 50-token overlap to maintain contextual coherence between student reasoning and mentor responses. For each student submission, we embedded the problem statement, all student messages, all mentor feedback, and associated metadata such as timestamps and rubric scores. The fine-tuned model was trained on NousResearch's Nous-Hermes-2-Mistral-7B-DPO (Teknium et al., 2024), which supports parameter-efficient fine-tuning. Both the RAG and fine-tuned models were trained on a dataset containing math problems, student solutions and resubmissions, mentor feedback, and rubric evaluations.

Results

Across all three conditions, a central challenge persisted: models struggled to make instructionally appropriate judgments about student work, particularly distinguishing between students who were mathematically incorrect and those who were correct but insufficiently explicit in their explanations. Changes in model configuration altered how errors manifested, but did not eliminate the underlying difficulty of deciding when to correct, probe, extend, or conclude.

Baseline Model: Fluent but Obscuring

Baseline feedback often sounded teacher-like and supportive, yet masked mathematical or interpretive weaknesses. The model frequently reframed student work into generic instructional extensions, which diluted attention to the student's actual reasoning needs (e.g., clarity of explanation) and could obscure key misunderstandings.

RAG Model: More Directive, Higher-Risk Distortions

RAG changed the character of errors. Feedback more often included overt mathematical inaccuracies (e.g., mislabeling correct work as incorrect) and sometimes introduced irrelevant interpretations. These failures were frequently paired with a more corrective tone, increasing the risk of undermining student agency and teacher trust even when the language appeared "instructional."

Fine-Tuned Model: Improved Fidelity with Persistent Overreach

Fine-tuning produced notable improvement in recognizing mathematically valid reasoning and adopting a more sense-making stance (e.g., questioning rather than correction). However, outputs frequently introduced unnecessary strategy naming or probed without sufficient specificity, revealing continued difficulty deciding when feedback should stop.

Cross-model Case Illustrating AI Instructional Challenges

One illustrative case (student 71531) highlights how the same incorrect solution elicited meaningfully different responses across conditions. The student concluded the trip was 435 miles by treating 145 miles as one-third of the trip and multiplying by three, a move that reflected a misinterpretation of the problem context rather than a computational error. The baseline model affirmed the multiplication and extended into unrelated ratio talk, masking the core issue. The RAG model adopted a corrective stance but reframed the issue as a procedural misunderstanding, mischaracterizing the student's shared thinking. The fine-tuned model recognized the proportional reasoning as productive yet still did not name the incorrect conclusion, shifting instead to reflective prompts. Across all three, the central instructional issue, misalignment between the student's reasoning and the problem context, was not clearly surfaced. Model modifications changed the form of feedback and the visibility of errors, yet did not resolve the difficulty of deciding whether to correct, probe, extend, or conclude.

Significance, Limitations, and Conclusions

This study reframes evaluation of AI feedback on student thinking from correctness and fluency toward a more pedagogically sound decision-making process. By comparing baseline, RAG, and fine-tuned configurations using qualitative memo analysis, we surface design-relevant failure modes (e.g., fluent masking, directive distortions, task overreach) that can inform more responsible teacher-facing tools.

Results support a teacher-in-the-loop framing: AI is most plausibly useful as a drafting support that teachers refine, not as an autonomous feedback provider. Even when trained on a corpus of mentoring data, models may fail in ways that require human pedagogical intervention, not merely error-checking.

Three limitations surfaced in this study: (1) RAG retrieves only a small number of exemplars, which may not match a student's underlying conceptual need; (2) text-generation models may produce mathematically inaccurate statements; (3) some outputs drift toward generic praise or unnecessary extension. Future work will test retrieval strategies that prioritize high-quality mentor exemplars and explore workflows that make teacher review efficient while preserving pedagogical intent.

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QUALITY OF MATHEMATICS CONTENT AND PROBLEM CONTEXT IN AI-GENERATED WORD PROBLEMS

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For this study, the researchers used three generative AI models to create 25 K-12 mathematical word problems, some with images, that they then evaluated for mathematics quality and problem context. Numerous concerns appeared in both categories. The problems tended to be mathematically simplistic for the intended grade level, with key words appearing in more than half the problems. Images often presented confusing or conflicting information in relation to the textual problem, and incidents of social class, racial, and gender bias appeared in the text or images of a number of problems. Using GenAI to create grade-level-appropriate word problems bears promise but must be viewed critically at this point in time.

Keywords: Problem-Solving; Technology; Equity, Inclusion, and Diversity

Objectives of the Study

The purpose of this research was to assess the quality of GenAI-generated word problems for K-12 students. Specifically, the research question was: What is the fit to Common Core State Standards – Mathematics, the mathematics quality, the problem-context role, and the contribution of images in K-12 word problems generated by three popular AI systems?

Conceptual Framework

Use of generative artificial intelligence (GenAI) in mathematics teaching and learning is a relatively recent and growing phenomenon that is particularly pertinent to problem solving (Schorcht et al., 2024). Still in its infancy, GenAI is being used in many ways in mathematics education, including generating tasks and solutions to tasks (Walkington, 2025). In general, word problems created by GenAI models to date have tended to lack variety in problem type and richness (Qin et al., 2024). Of course, different generative AI models, as well as versions within those models, and the specific prompts used influence the quality of mathematical word problems created (Schorcht et al., 2024).

While GenAI shows potential for adapting word problems to student interests and needs that include use of personalized topics and readability levels (Luik et al., 2025; Walkington, 2025), this technology has demonstrated cultural and other biases in its outputs (Walkington, 2025). For example, AI-generated text can misrepresent or minimally include people from different racial/ethnic backgrounds, use gender-stereotyping concepts and language, and homogenize cultural groups (Ho et al., 2025; Rettberg & Wigers, 2025; Shieh et al., 2026). AI-generated images have also been found to include biased representations, as well as physical inaccuracies (York et al., 2024). Biases and limitations result from GenAI's specific training and the information to which it has access. Wei et al. (2025) note, "Trained on extensive internet text data, LLMs often mirror societal biases, stereotypes, and cultural assumptions, thereby influencing decision-making, entrenching stereotypes, and perpetuating inequalities" (p. 7).

Methods

We four researchers specialize in mathematics teaching and learning, two previously at the elementary level and two at the secondary level. Currently in higher education, we have advanced expertise in mathematics education and equity and diversity issues. This background made us especially suited to conduct this research.

Given that teachers have access to using GenAI to create mathematics content, such as word problems (e.g., Cheng, 2024), we aimed to gauge its effectiveness in creating word problems from prompts with the intent to evaluate the mathematics quality and the context ("storied" part) of the problems.

We generated 25 problems as follows:

- Three each using text only for Grades 1, 3, 5, 7, and high school using ChatGPT (GPT-5.2), Gemini (Gemini 3 Flash), and Claude (Sonnet 4.5)
- Two each using text and images for Grades K (oral), 2, 4, 6, and 8 using ChatGPT (GPT-5.2) and Gemini (Gemini 3 Flash; Claude cannot generate useable images)

Example prompts are:

- Write a math word problem between 30 and 40 words that is appropriate for third-grade students according to the Common Core State Standards for Mathematics. Include at least two people. (first group)
- Write a math word problem between 50 and 60 words that is appropriate for eighth-grade students according to the Common Core State Standards for Mathematics. Include at least two people and an associated image. (second group)

We first reviewed the 25 problems for fit to the Common Core State Standards for Mathematics. All but two that fell in a gray/uncertain area were aligned. However, all of the problems were computational and on the lower (more simplistic) end of the spectrum of problems that could fit the intended grade-level standard. We each then reviewed a sheet that summarized bias issues in GenAI before evaluating the generated problems individually, taking notes on our observations about mathematics quality (structure, richness, accuracy, use of key words or extraneous information, and text-to-image alignment, where relevant) and problem context (familiarity, interest, authenticity, readability, writing mechanics, and equity/diversity matters). Finally, we met to discuss our observations and generate findings that all four of us agreed on.

Results

Dominant findings across the problems are as follows:

- Rich problems were lacking (all problems were computational and somewhat simplistic for the intended grade level; unknowns in three-part equations always appeared last).
- Slightly more than half the problems used key words, which appeared mostly at the lower grade levels.
- Most contexts were familiar (e.g., sharing apples, making cookies).
- Most contexts were uninteresting for young people in terms of the "story," and some had a purpose (e.g., saving money for a bicycle) while others did not (e.g., giving another child apples for no articulated reason).

- Most contexts were real-world ("could" happen) but were not very authentic (i.e., they were contrived).
- Much repetition occurred across problems and across models (e.g., 5 problems were about apples, 4 problems used the name Sarah/Sara)
- The images or information in some images either conflicted with the textual problem or were leading/directive.
- All problems used one boy and one girl as the characters.
- Evidence of bias appeared in some problems. Social class issues appeared most often, increasingly so with advancing grade levels. Examples include saving money for a new bicycle and preparing snacks for a field trip. We also identified several incidences of gender bias, as in the boy making a better financial decision than the girl in one problem, and the boy tipping more than the girl in another. Further, in the images, fair-skinned people appeared disproportionately more often than dark-skinned people (16 of 20).
- Confusing language appeared in the text of some problems in terms of syntax, meaning, and specific words that might be unfamiliar, especially to multilingual learners.
- Writing mechanics issues (e.g., spelling, punctuation) only occurred in the images and only for the Gemini model.

Discussion and Conclusions

Generative AI can support the work teachers do in preparing mathematics tasks. Walkington (2025) says, "We need to better ensure GenAI is actually making teachers' lives easier and better supporting their students" (p. 7). However, a number of mathematics and context-related issues occurred in the problems we generated that made them confusing or simplistic. For example, use of key words is a problematic approach to solving and learning how to solve word problems (e.g., Hardy et al., 2025). Issues have also been noted regarding AI-generated problem contexts in that they tend to be inauthentic, which deviates from recommended practice (Walkington et al., 2026). In relation to social concerns, the incidents of bias found in this research must be given serious consideration.

To use AI properly, therefore, teachers must have more, not less, knowledge of mathematics and mathematics pedagogy (National Council of Teachers of Mathematics 2023). Kim et al. (2025) suggest that preservice teachers learn about GenAI use in mathematics teaching and learning from practical and critical standpoints. In essence, both teachers and students must exercise a healthy skepticism toward AI outputs, analyzing it carefully for mathematical errors and bias in the problem contexts generated (National Council of Teachers of Mathematics 2023; Walkington, 2025).

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ENHANCING KINDERGARTEN SPATIAL REASONING THROUGH ROBOTICS INTEGRATION

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This study examined the effects of a curriculum-aligned Blue-Bot robotics intervention on kindergarteners' spatial reasoning. Eighteen children completed ten weekly robotics sessions integrated with mathematics instruction. An embedded mixed-methods design combined pre–post assessments with a teacher interview. Although overall gains were not statistically significant, students showed improvement in navigation and sequencing skills aligned with robotics tasks. The teacher also reported greater use of spatial language and persistence in problem solving.

Keywords: Early Childhood Education, Technology, Geometry and Spatial Reasoning

Spatial reasoning is central to young children's mathematical development and later success in mathematics and STEM. Yet early geometry often becomes "static," emphasizing shape names and properties while offering fewer opportunities for dynamic spatial thinking involving movement, transformation, perspective taking, and mental rotation (Bruce & Hawes, 2015). Scholars have therefore called for learning environments that support spatial reasoning through object manipulation, representations, and coordinated action in space. Early intervention is especially important because spatial reasoning is not only associated with mathematics achievement but is also malleable and responsive to instruction. For example, Cheng and Mix (2014) found that brief mental-rotation training improved some mathematics outcomes for children ages 6–8. Bruce and Hawes (2015) similarly identify mental rotation as an important but often underdeveloped skill and argue that young children benefit from experiences that make spatial transformations visible and discussable.

Within this broader agenda, educational robotics offers a promising and developmentally appropriate context because it combines movement, planning, spatial language, sequencing, and debugging. Children do not simply use a tool; they engage in embodied problem solving by anticipating movement, testing a plan, observing outcomes, and revising. Francis et al. (2017) argue that robotics tasks support spatial and design thinking because they require attention to direction, orientation, and coordinated action in space. This makes robotics a potentially valuable context for strengthening spatial reasoning. A broader view is provided by Rapti and Sapounidis' review of kindergarten robotics research from 2012 to 2023. Across 59 studies, robotics was commonly examined through outcomes such as critical thinking and problem solving, often using small samples, short interventions, and Bee-Bot or Blue-Bot platforms (Rapti & Sapounidis, 2023). Although the review identified generally positive cognitive outcomes, it also highlighted the need for clearer construct definitions and more systematic research designs.

These concerns extend to how robotics is implemented. Research on early computational thinking emphasizes that instructional design and teacher mediation shape children's engagement and learning (Misirli & Komis, 2023). However, much of the literature has examined short-term, researcher-designed interventions rather than sustained integration into everyday classroom instruction (Rapti & Sapounidis, 2023). Consequently, less is known about how robotics functions within ongoing curricular routines and teachers' pedagogical decision-making. Effective integration requires structured task progression and explicit connections between robot movement and mathematical ideas such as directionality and sequencing (Papadakis & Kalogiannakis, 2020). When Bee-Bot or Blue-Bot activities are embedded within classroom routines, children engage in repeated cycles of prediction, execution, and revision that mirror processes central to spatial reasoning (Misirli & Komis, 2023). Studying robotics in authentic classroom contexts may therefore clarify both instructional processes and domain-specific learning outcomes. A related challenge concerns measurement. Many early childhood robotics studies report broad cognitive or engagement outcomes without clearly defining spatial constructs such as perspective taking, navigation, sequencing, or relative position (Rapti & Sapounidis, 2023). Although validated instruments exist for older students, including the Spatial Reasoning Instrument developed by Ramful et al. (2017), fewer developmentally appropriate measures are available for kindergartners. Adapting validated measures for younger learners may strengthen methodological rigor and clarify how robotics supports specific spatial skills. Addressing these instructional and measurement gaps, the present study examines an instructionally aligned Blue-Bot intervention developed with a kindergarten teacher and coordinated with the existing mathematics curriculum. It focuses on defined spatial reasoning domains and combines quantitative assessment with teacher reflection to examine both learning outcomes and instructional processes. The study addresses two questions: (1) How does participation in an instructionally aligned Blue-Bot robotics intervention influence kindergarten students' spatial reasoning skills? (2) How does the kindergarten teacher perceive and reflect on the integration of robotics alongside her mathematics curriculum?

Method

Participants, Setting, and Procedures

The study included 18 kindergartners, ages 5–6, from a mixed-gender classroom at a university-affiliated early childhood center in the Midwestern United States. Before the intervention, each child completed an individual spatial reasoning assessment. Five pictorial multiple-choice items were adapted from the Spatial Reasoning Instrument (Ramful et al., 2017), selected for alignment with the Blue-Bot lessons, and modified for kindergartners. The assessment measured mental rotation, perspective taking, spatial navigation, and relative position. Each item was scored as correct (1) or incorrect (0), producing a total score from 0 to 5. The intervention consisted of eight weekly, 20-minute lessons integrated into regular classroom activities. Blue-Bot, a transparent floor robot programmed through buttons and fixed stepwise turns, was used to support spatial reasoning, autonomy, and directional understanding. Its design makes command sequences and movements visible for discussion and allows it to function as a digital mathematics manipulative (Berson et al., 2023). Each session included a whole-class introduction, teacher demonstration, work on large grid mats, and reflection. Lessons aligned with the CCSSM and Ohio technology standard K–2.ICT.1.a . Tasks progressed from positional language, single-step commands, and grid navigation to planning, testing, composing, and decomposing multistep paths (Berson et al., 2023). After each session, the researcher and teacher discussed student engagement, challenges, and possible adjustments. After the intervention,

children completed the same assessment using the same scoring procedure, allowing direct comparison of pretest and posttest spatial reasoning scores.

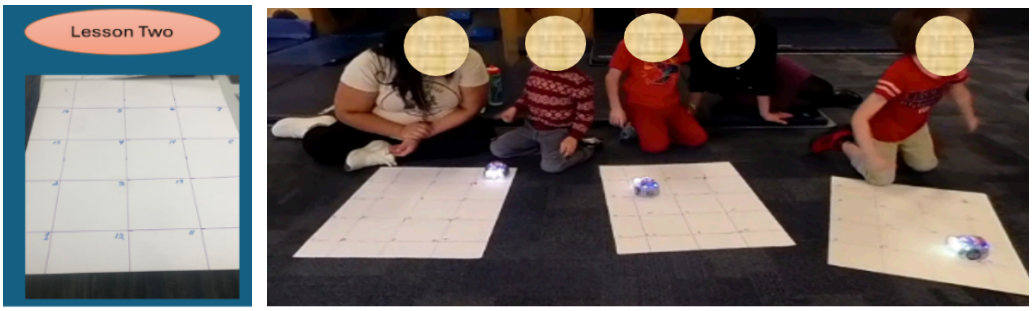


Figure 1: Students Guide the Blue-Bot from 1 to 12 on a Numbered Grid

Data Analysis and Finding

This study used an embedded mixed-methods design, with quantitative pre–post assessment data as the primary strand and a teacher interview used to contextualize the findings (Creswell & Plano Clark, 2011). Given the small sample (N = 18) and limited score range (0–4 after item exclusion), nonparametric tests were used. A Wilcoxon signed-rank test examined changes in total spatial reasoning scores, while McNemar tests examined changes in correct and incorrect responses for individual items representing perspective taking, sequencing, navigation, and relative position. Following the quantitative analysis, the classroom teacher participated in a semi-structured interview (Zazkis & Hazzan, 1999). Questions addressed her confidence using Blue-Bot, observed changes in students' spatial reasoning and directional language, curricular alignment, and possibilities for future integration. The transcript was thematically analyzed for patterns related to learning, engagement, differentiation, and instructional challenges.

A Wilcoxon signed-rank test compared total pretest and posttest scores. Six students improved, five declined, and seven showed no change (see Table 1). The difference was not statistically significant, $Z = -0.98$, $p = .32$, with a small-to-moderate effect size ($r = .23$). These results indicate considerable variation in students' responses to the intervention. To further examine changes across specific spatial reasoning constructs, McNemar tests showed no statistically significant change for any individual item (all $p > .05$). However, descriptive patterns differed across spatial reasoning domains. The strongest improvement occurred on navigation item QD, with eight students improving and four declining. On QC, which involved following a sequence of directions, three students improved and none declined. Perspective taking showed modest improvement, with four students improving and one declining. Relative position showed a less favorable pattern, with three students improving and six declining. Although the quantitative results were not significant, the descriptive trends suggest greater improvement in navigation and sequencing skills, which closely aligned with the Blue-Bot tasks.

Table 1: Changes in Students' Spatial Reasoning Responses from Pretest to Posttest (N=18)

Spatial Reasoning Item	# Students Improved	# Students Declined	# Students No Change	McNemar P Value
QB: Perspective taking	4	1	13	.37
QC: Navigation	3	0	15	.25
QD: Navigation	8	4	6	.38
QE: Position taking	3	6	9	.50
Total	6	5	7	

Note. Improved = change from incorrect at pretest to correct at posttest; Declined = change from correct at pretest to incorrect at posttest; No Change = same response at pretest and posttest.

Teacher reflections supported these patterns. Judy's confidence in using Blue-Bot increased during the intervention, although she initially noted that she was not yet an "expert" and still had logistical questions. As the sessions progressed, she observed stronger directional language, sequencing, and flexibility in route planning. Students increasingly used terms such as "forward," "backward," and "turn," and one child independently revised a path by programming the robot backward when the original plan failed. Judy viewed this as evidence of spatial problem solving rather than memorized movement, consistent with the descriptive gains in navigation and sequencing. She also reported greater engagement and persistence. Students who often became frustrated during mathematics were more willing to revise and try again during Blue-Bot activities. Judy attributed this partly to the hands-on, open-ended nature of the tasks, which created a lower-pressure environment and allowed children to work at different levels of complexity. By the end of the intervention, she viewed Blue-Bot as a flexible instructional tool that could support mathematics and other areas, including literacy, such as coding sight words.

Discussion

This study contributes to research on instructionally aligned robotics for domain-specific spatial reasoning in early childhood. Responding to calls for clearer constructs and stronger alignment between instruction and assessment (Rapti & Sapounidis, 2023), it targeted defined spatial domains within mathematics instruction. Although gains were not significant, descriptive patterns showed greater improvement in navigation and sequencing, skills closely reflected in Blue-Bot activities. Programming required children to predict, execute, evaluate, and revise movements, processes central to spatial reasoning (Misirli & Komis, 2023). Teacher observations of stronger directional language, flexible path revision, and growing confidence further suggest that curriculum-connected robotics may support both spatial reasoning and sustainable classroom integration.

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TEACHING AND LEARNING MATHEMATICS IN THE AGE OF AI: PROSPECTS, PRIORITIES, PROCESSES, AND PROBLEMS (4PS)

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The use of Artificial Intelligence (AI) in education, in general, and in mathematics education in particular, is reshaping how the concepts, procedures, and applications are taught and learned. In this poster, we discuss teaching mathematics meaningfully in the age of AI with reference to prospects, priorities, and processes, and problems (4P) associated with using those tools. This study aims at examining the 4Ps of AI augmentation in mathematics education drawing on current research.

The *prospects* of AI in mathematics education extend beyond simple automation toward meaningful augmentation of teaching and learning. AI tools can support mathematical reasoning, generate multiple solution strategies, provide immediate feedback, and personalize instruction based on students' needs and interests. AI-augmented platforms allow both students and teachers for self-paced learning and individualized support. In addition, AI augmented tools can assist teachers in designing lessons, tasks, and formative assessments, contributing to ongoing professional growth (Martinovic & Danesi, 2025). When used thoughtfully, AI has the potential to deepen conceptual understanding and encourage reflective thinking. The research community in PME-NA are also focusing on the potentials and issues related to AI in mathematics education (Belbase et al., 2025; Zhu et al., 2025; Zhuang et al., 2025), the research interest in this area is growing over the last few years.

At the same time, clear *priorities* must guide the integration of AI tools in the teaching and learning of mathematics. Conceptual understanding and mathematical reasoning should always remain the central focus of instructional practices. Teachers should spend considerate amount of time discussing the mathematical concepts critically, verifying the knowledge sources, and refining AI-generated responses rather than simply accepting them as is. Ethical use, academic integrity, and data privacy are essential considerations in AI-supported environments. Effective implementation requires thoughtful scaffolding and teacher preparation to ensure that AI enhances human interaction rather than replaces it (Noakes et al., 2024; Richard et al., 2022). The AI-augmented tools can help mathematics educators and teachers to plan what to teach and what is not needed to teach because AI can help students solve those problems. Hence, the new priorities of teaching mathematics are in the horizon that mathematics education community needs to identify, explore, and priorities so that mathematics can be interesting, teaching of this subject can be rewarding, and use of AI in this direction can be an added value in TPACK and STEM education.

The *processes* of mathematics teaching are also shifting very rapidly. AI-augmented instruction can promote learning-by-teaching models, tiered task design, and structured peer collaboration. Human–machine–human interaction cycles encourage students to explore ideas, evaluate AI outputs, and strengthen their reasoning. Teachers can use AI-generated insights to identify patterns in student thinking and adjust instruction accordingly (Noakes et al., 2024; Ramadhani & Ramadani, 2025). In this way, classroom learning can shift from simply seeking answers to evaluating and reasoning about solutions. Hence, AI-augmented tools may shift the process of teaching and learning mathematics by shifting its role to support both teachers and learners through the processes of idiosyncratic questioning, receiving feedback, and reposing new questions.

However, several emerging *problems* must be addressed as we progress toward over reliance on the AI tools. AI systems may occasionally generate inaccurate or misleading outputs, particularly in mathematics, that appear convincing. Overreliance on AI may reduce perseverance and engagement with high-cognitive-demand mathematical tasks. Concerns regarding equity of access, privacy, and reduced human-to-human interaction also remain significant. Therefore, while AI offers strong promise, mathematics educators must adopt a critical and balanced approach to ensure meaningful, ethical, and equitable learning experiences (Richard et al., 2022).

In conclusion, AI can support mathematical reasoning, personalization in learning, and reflective practices. However, its impact depends on thoughtful and ethical integration into classroom teaching through policies and classroom practices to ensure that AI enhances conceptual understanding rather than replacing critical thinking. A balanced approach can help sustain meaningful, equitable, and high-quality mathematics education in the age of AI.

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BRIDGING REPRESENTATIONS: INTEGRATING COMPUTATIONAL ACTIVITIES INTO AN INTRODUCTORY LINEAR ALGEBRA COURSE

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Although computation plays a growing important role in modern STEM practices and research, current research and pedagogy has not yet fully leveraged how computation can support students' mathematical thinking rather than simply providing technical skills (Lockwood & Mørken, 2021). When computation is added to undergraduate mathematics courses, it is often disconnected from course goals or treated as an add-on, which misses its potential, especially given that many students experience coding as creative and exploratory while viewing mathematics as comparatively rigid (Isomöttönen et al., 2020, Selbach-Allen et al., 2020). This work focuses on developing a structure for integrating computation into a standard linear algebra course to support conceptual learning and foster creative engagement with mathematical ideas.

In developing this course, we started by establishing the foundational pieces needed for a full implementation in Fall 2026. We gathered all existing linear algebra syllabi and course materials from the past five years to identify departmental topic priorities and key learning objectives. These were compared to the objectives in our previously developed computational modules (Castle, 2023a) to ensure alignment and content coverage. Because the modules were designed from the perspective of computation as pedagogy, we mapped all objectives across the semester and identified where conceptual scaffolding or coding support was needed. This work led to the development of paired mathematical and computational learning objectives for each major topic. This was an explicit choice to ensure computation was positioned as a pedagogical tool for broadening the forms of representation and exploration available to students. The mapping process also clarified where departmental expectations naturally supported computational approaches and where additional support structures would be needed to prepare students.

These paired objectives guided our development of a recurring learning cycle organized around the big mathematical ideas in the course. Each cycle begins with tasks that introduce a core concept and allow for computational exploration. Students then work in computational modules to test examples, visualize structures, explore patterns, and refine initial predictions. Post-module tasks help students articulate what the computational work revealed, and formalization tasks then connect that exploratory work back to established linear algebra structures and definitions. The design draws on theoretical perspectives that position computation as a mediating tool for mathematical creativity and is informed by prior pilot work showing that prediction, experimentation, and reflection in computational environments can reshape how students view mathematics and their relationship with it (Castle, 2023b).

This work offers a structured, intentional model for integrating computation into undergraduate linear algebra while meeting departmental and college learning standards. Through our framework, objectives, and examples, we aim to contribute to ongoing conversations about how computation can be meaningfully integrated into undergraduate mathematics in ways that support conceptual understanding, broaden students' views of the discipline, and create space for more creative mathematical activity in STEM courses.

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DESIGN PRINCIPLES FOR CLASSROOM ARTIFICIAL INTELLIGENCE TO SURFACE AND SUPPORT STUDENTS' PROPORTIONAL REASONING OVER TIME

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As AI-driven technologies become more common in mathematics classrooms (e.g., Borba et al., 2024), understanding how they support development of deep mathematical understanding in collaborative classroom settings remains limited (Hwang & Tu, 2021; Mohamed et al., 2022). Most research and classroom use of AI in mathematics focuses on either generative tools like ChatGPT, which provide explanations but can be unreliable or misleading, or intelligent tutoring systems, which offer individualized and adaptive assistance but are rarely designed for collaborative classroom learning (Wardat et al., 2023; Engelbrecht & Borba, 2023). Therefore, there is a need for AI tools that support inquiry-oriented, problem-based mathematics, helping teachers gain real-time, actionable insights into students' collaborative learning over time.

Our project leverages a digital collaborative platform with an embedded middle grades problem-based mathematics curriculum, where students document and share their reasoning through different mathematical representations and tools (e.g., Edson, 2024; Edson & Phillips, 2021) Teachers and researchers tag student work using a proportional reasoning rubric, creating a curated dataset that will train AI models to interpret students' reasoning (Edson et al., 2024). As teachers tag work, the platform's analytics immediately organize, sort, and filter student contributions, providing real-time insights that help teachers monitor reasoning patterns, facilitate discussions, and reflect on instruction. By integrating multimodal student work, collaborative documentation, and teacher-generated tagging, the platform provides a classroom-embedded approach for accessing actionable insights into students' mathematical thinking.

This poster presents design principles, guidelines derived from iterative design-based research (e.g., Barab & Squire, 2004), that make student reasoning visible and actionable for both students and teachers. The platform supports collaborative problem-solving, enabling learners to build on each other's work while preserving individual strategies. Teacher-facing analytics provide real-time, actionable insights through sorting, filtering, and comparison of student contributions to guide instruction and discussion. Multimodal representations (text, equations, tables, graphs, and drawings) capture diverse approaches and support meaningful analysis. Together, these principles illustrate how tools can integrate student-centered learning, collaboration, and AI-powered analytics to enhance classroom practice.

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MAPPING DYNAMIC GEOMETRY ENVIRONMENT RESEARCH: A SYSTEMATIC THEMATIC REVIEW OF TECHNOLOGY-MEDIATED GEOMETRY LEARNING

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Over three decades, research on dynamic geometry environments (DGEs) has expanded across geometry learning, reasoning, pedagogy, and student engagement. Despite this growth, there has been limited systematic synthesis of the thematic directions organizing empirical DGE scholarship. This study presents a systematic review of 233 peer-reviewed empirical journal articles published between 1991 and 2023 to identify the major thematic trajectories in DGE research. The guiding research question is: What research themes have emerged across three decades of empirical research on dynamic geometry environments?

Following PRISMA screening procedures, this study employed a hybrid structural topic modeling (STM) approach combining R-based STM with AI-assisted topic generation using Copilot and ChatGPT (Chen et al., 2020; Farrell, 2016; Roberts et al., 2014). Candidate topic structures generated using Copilot and ChatGPT informed, but did not determine, a researcher-led deductive thematic analysis. Researchers completed all final coding and theme refinement (Braun & Clarke, 2006; Fereday & Muir-Cochrane, 2006).

Five major themes emerged: Geometry Content and Conceptual Understanding, Cognitive and Learning Processes, Individual Differences and Accessibility, Pedagogical Implementation, and Student Outcomes and Engagement. Geometry Content and Conceptual Understanding was the dominant research focus (36.9%), whereas Individual Differences and Accessibility remained comparatively underrepresented (9.4%). The review also revealed a longitudinal shift from early investigations emphasizing geometric reasoning, dragging, and proof toward increasing attention to pedagogy, classroom implementation, learner diversity, and engagement.

The resulting thematic map reveals how DGE scholarship has evolved alongside changes in educational technology and instructional priorities. This synthesis provides a framework for positioning future studies within established and emerging areas of DGE research.

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LEARNING EXPONENTIAL FUNCTIONS WITH IMMERISVE VIRTUAL REALITY

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Mathematics teachers and students alike want the ideas in their classrooms to be meaningful and relevant. To do so, many teachers and curricula craft real-world contexts for students to explore grade-level mathematics (e.g., Leonard et al., 2010). Educational technologies, such as immersive virtual reality (IVR) have the potential to support students in seeing how the mathematics they are studying applies to real-world situations. PrismsVR is one IVR supplemental mathematics curriculum that has the potential to support students seeing mathematics as interesting and worthwhile, by taking students on virtual field trips to explore real-world phenomena, which are then mathematized. In this poster, we investigate the impact of one of PrismVR's modules, exponential functions, on students' mathematics achievement.

The affordances and outcomes of IVR are identified in the Cognitive-Affective Model of Immersive Learning (Makransky & Petersen, 2021). In this model, *presence* and *agency* are two key technological affordances which influence the affective and cognitive factors (e.g., interest, motivation, self-efficacy) that contribute to learning. *Presence* is the feeling of "being there" (Ijsselsteijn & Riva, n.d.). IVR hardware and software can provide users an "illusion of being physically present in a different world or setting" (Engelbrecht & Borba, 2024, p. 287). *Agency* within the context of IVR is the degree to which users can control their own actions within the virtual world.

We designed a randomized controlled trial to measure the impact of the PrismVR's exponential functions module on students' mathematics achievement. We recruited 26 teachers teaching Algebra 1 (or its equivalent) and their students (262 treatment and 252 control students) to participate in the study. Teachers were randomly assigned to the treatment or comparison group using blocked randomization. Treatment and control teachers taught their courses in their business-as-usual manner; however, the treatment condition also experienced a PrismsVR module as an introduction to three units of study: linear functions, systems of linear functions, and exponential functions. We used a 27-question researcher-designed pre- and post-assessment to measure mathematics achievement with regard to exponential functions (Cronbach's $\alpha = 0.83$).

We analyzed students' learning of exponential functions by evaluating pre-test differences between the treatment and control groups to determine that, in fact, there were low baseline differences (–0.02 standard deviations). We then performed a 2-level mixed effects linear regression to determine if differences between treatment and control were present at posttest. The model accounted for nesting of students within teachers (a random effect), and included fixed effects for condition, and pre-test. The results revealed a significant difference in the posttest assessment adjusted means between the treatment and control groups, showing a 2.94-point increase (effect size, $g = 0.53$) between treatment and control groups.

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HUMAN-IN-THE-LOOP AI IN MATHEMATICS EDUCATION: A CONCEPTUAL FRAMEWORK FOR INTEGRATING AI TOOLS WITH TEACHER EXPERTISE

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This poster presents a conceptual framework for integrating artificial intelligence (AI) tools into mathematics teaching in ways that strengthen, rather than replace, teacher expertise. The purpose is to synthesize recent literature on AI in mathematics education (Awang et al., 2025; Grassini, 2023; Nguyen & Pham, 2025; Opesemowo & Ndlovu, 2024; Yavich, 2025) and translate it into a practical, classroom-oriented model for instructional decision-making, formative feedback, and equitable implementation. The following questions guide the study: (1) Where does AI add meaningful value in mathematics teaching and learning? (2) What aspects of mathematics teaching must remain teacher-led to preserve sensemaking, justification, and sound pedagogical judgment? (3) What classroom routines and guardrails can support responsible and equitable use of AI tools in mathematics classrooms?

Theoretical Framework

This project is guided by a human-in-the-loop perspective (Mosqueira-Rey et al., 2023; Munro, 2021) on AI integration in mathematics education, which treats AI as a support tool whose outputs require human interpretation, verification, and pedagogical decision-making. Rather than positioning AI as a replacement for teaching, this perspective emphasizes the central role of teacher expertise in evaluating mathematical validity, interpreting student thinking, and designing responsive instruction (Datta et al., 2022).

Methods

This project uses an integrative review methodology (Whittemore & Knafl, 2005) to synthesize recent scholarship and policy-oriented literature on AI in education and mathematics education. The review focused on literature published between 2019 and 2026 to ensure the study captures both pre-generative AI developments and the rapid expansion of generative AI tools in mathematics education. Sources were identified through high-quality, reliable databases: Scopus, ERIC, EBSCOhost, and professional organization sources such as NCTM (2024). Search terms included AI-related terms, mathematics education terms, and framework-related terms. The criteria for inclusion were based on whether the source addressed AI use in education or mathematics education and contributed to understanding teacher decision-making. The exclusion process removed sources that were not peer-reviewed journal articles, non-English, or technical AI sources unrelated to the study. After screening 843 articles, the selected literature was thoroughly analyzed by identifying common AI affordances, recurring risks and constraints, and instances requiring teacher judgment. These patterns were then synthesized into a human-in-the-loop framework for classroom use in mathematics.

Results and Implications

The integrative review suggests that AI tools can contribute to mathematics instruction in four main ways: generating multiple explanations, identifying possible misconceptions in student

work, offering alternative representations, and helping teachers scale formative feedback. However, the framework positions these AI contributions as provisional rather than authoritative. In other words, every AI output needs to be reviewed by a teacher to ensure mathematical accuracy, conceptual depth, proper representation, and alignment with instructional objectives.

The mathematics-specific contribution is pinpointing crucial "handoff points" where teacher expertise is essential: assessing the validity of solutions generated by AI, interpreting students' unconventional strategies, determining if a representation clarifies or obscures mathematical structure, and guiding students toward justification instead of just answers. Additionally, the framework considers equity as a key design element. For instance, when AI creates contextualized math problems, teachers need to evaluate whether the context, language complexity, assumptions, and accessibility promote inclusive participation and avoid reinforcing biases or increasing disparities.

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MATHBUDDY: DEVELOPING AND PILOTING AN AI COMPANION AGENT FOR MATHEMATICAL WORD PROBLEM SOLVING

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With the rapid development of generative AI, an increasing number of students are using AI technologies to learn mathematics. This trend appears promising, as many pedagogical agents are virtual characters designed to enhance teaching and learning (Siegle et al., 2023). However, how to integrate AI into core subject learning in purposeful, pedagogy-aligned ways and how to cultivate effective human–AI collaboration remains in need of systematic investigation (OECD, 2025; UNESCO, 2024). Moreover, research on student–student interaction shows that conceptual growth is amplified when learners actively interpret their peers' thinking, interrogating, adopting, and building on alternative perspectives of others (Rasmussen et al., 2020; Webb, 2023), yet few pedagogical agent designs foster such engagement with the agents' "thinking." To address these gaps, this poster proposes *MathBuddy*, an AI companion for mathematical word problem solving, and analyzes its piloting with students in Hong Kong.

Research Design

Closely aligned with the Hong Kong local curriculum, MathBuddy is powered by the Grok 4.3 model via API services and operates as a self-hosted chatbot web application. It engages students in meaningful dialogues by harmonizing the dual features of collaborative problem-solving agents and teachable agents, collaborating with students to jointly solve mathematical problems and prompting students to explain mathematical ideas. To ensure MathBuddy's conversational moves both mathematically rigorous and pedagogically powerful, we employ prompt engineering informed by established practices in teacher talk moves (Chen et al., 2020), Socratic teaching (Zhang et al., 2024), and peer discussion (Rasmussen et al., 2020). In addition, the agent's knowledge base comprises (1) a summary of key concepts, (2) a bank of mathematical problems situated in authentic, cultural-relevant contexts, and (3) detailed mathematical and pedagogical descriptions of each problem.

We conducted a pilot study with Hong Kong junior secondary students ($n = 45$) to examine the interaction dynamics and pedagogical potential of MathBuddy. Each participant engaged in an individual 30-minute problem-solving session with MathBuddy, followed by a one-on-one stimulated recall interview lasting 30–40 minutes with a researcher, who presented selected chatlogs and invited the student to reflect on their thinking and interpretations of MathBuddy's prompts. Qualitative thematic analysis is used to analyze the chatlogs and transcripts to identify students' problem-solving strategies and interaction patterns, while video data is used to triangulate findings and capture non-verbal cues.

Findings and Implications

The study is expected to yield interaction patterns between students and MathBuddy, including the characteristics of prompts that students perceive as valuable, as well as the ways in which these prompts influence students' reasoning trajectories and mathematical understandings. These findings will generate empirically informed design principles for curriculum-aligned AI companion agents that support collaborative problem-solving in mathematics.

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STUDENT-CREATED MATHEMATICS WIKIBOOK: BEYOND TECHNICAL ACCESSIBILITY

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Wikibooks sit in the intersection of open textbook and student authorship, which are two open education practices that have received some attention from researchers (Galarza & Perry, 2017; Hilton, 2020; Neill, 2024; Wong & Leatham, 2026). Contribution from diverse students in this process of knowledge creation could further increase the value of the learning resource being created. Our goal is to explore the nature of student contribution in the Wikibooks platform where technical barriers are minimal. Our research question is the following: How do student contribute to a Wikibook? More specifically, we are interested in understanding the ways different student authors engage in the development of a Wikibook as they contribute to it.

Methods

The open textbook we examined is "High School Mathematics Extensions" (HSME, 2025), which is one of the three featured books in mathematics on Wikibooks. The listed authors of this Wikibook identified themselves as students. For this study, we selected the Discussion Pages associated with the *Main Page* and the *Primes and Modular Arithmetic Page* (Primes Page) of the HSME book. There was a total of 80 comments in the two Discussion Pages. Garrison et al. (2001) introduced a framework for practical inquiry to understand the level of cognitive presence in computer-mediated communication. We adapted this practical inquiry framework, which provided four phases of inquiry, to assess the cognitive presence of the contributors of HSME to see how they make meaning through conversations in the Discussion (see Wong & Çaylı, 2026).

Results

Overall, we have found that while the primary author showed a dominance in the Resolution Phase, Wikibook allowed different authors in different positions a chance of contributing. As seen in our analysis, all 15 but two of the comments in the Resolution Phase was made by the primary author. This shows that the primary made most of the decisions around this Wikibook. This also shows that the non-author contributors tend to not make decisions around this Wikibook. On the other hand, the number of comments were comparatively high in Triggering and Exploration. Moreover, the number of distinct contributors in the Triggering and Exploration Phases was much higher compared to Resolution. In other words, many contributors participated in the development of this Wikibook by making inquiries and brainstorming, and not as much in creating ideas to resolve issues. This might indicate that this Wikibook allowed different authors in different positions a chance of contributing.

Conclusion

The Wikibooks platform is considered to have a high level of openness regarding its technical aspects (Hilton et al., 2010). Although this high level of technical accessibility allowed all contributors to contribute in any of the four phases, our analysis showed an uneven distribution of higher-level cognitive presence among the contributors. Even though everyone had the same technical access to the Wikibook, there is something non-technical that could have played a role is this domination.

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DESIGNING OPPOSITE ODYSSEY: A BOARD-AND-CARD GAME FOR INTEGER ORDER AND MAGNITUDE

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Keywords: Algebra and Algebraic Thinking, Number Concepts and Operations

Algebra is regarded as the gateway subject into advanced mathematics and STEM pathways (Cai & Knuth, 2005; National Mathematics Advisory Panel, 2008). Yet, many students encounter algebra in tandem with negative numbers in middle school, a combination that can intensify conceptual difficulty (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Despite longstanding efforts to strengthen algebra readiness in the early elementary grades (e.g., Knuth et al., 2016), students continue to struggle with negative numbers and the multiple meanings of the minus sign (Vlassis, 2004, 2008). These challenges are further compounded by over-generalizations of whole number "rules," such as subtraction makes numbers smaller (see, e.g., Karp et al., 2014). To address this need for early understanding of negative numbers, we are designing six board and card games that support elementary students in exploring integer principles in meaningful and motivating ways (see, e.g., Dickman & Bofferding, 2016–2017; Wessman-Enzinger & Bofferding, 2014).

This poster focuses on Opposite Odyssey, the first game in a design sequence, which targets barriers related to integer order and value. Our game design work follows Cardinot et al.'s (2022) five-step game development framework and highlights three foundational integer principles: the minus sign can designate negative or opposite (Vlassis, 2004), numbers exist on both sides of zero and are ordered relative to zero, and integers involve both linear value and absolute value (Ball, 1993; Bofferding, 2014). These principles are embedded directly into the game's mechanics rather than treated as supplemental explanations. Two to four players construct signed two-digit numbers using digit and sign cards and place tokens on a shared number line. A randomly selected target number (100, 0, −100) requires players to reason about "closeness" in ways that make distinctions between absolute and linear value (Wessman-Enzinger & Bofferding, 2018). Action cards (e.g., switching the digit order, creating opposites) introduce strategic decision-making that encourages comparison, discussion, and justification while maintaining the playful nature of the game. Iterative prototyping and playtesting with advisory board members, educators, and 2nd–4th grade students informed revisions to the game. Data sources include usability notes, focus group interviews, and observations of students' behavioral, emotional, and cognitive engagement.

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WORKING GROUPS AND RESEARCH COLLOQUIA

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

REIMAGINING EARLY CHILDHOOD MATHEMATICS TEACHER PREPARATION

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This new working group aims to reimagine current standards for early childhood mathematics teacher preparation. Often subsumed into elementary teacher preparation or mathematics teacher preparation at large, we believe that early childhood mathematics teacher preparation requires unique pedagogical, developmental, and mathematical expertise and therefore warrants focused recommendations. Together, we will: 1) identify skills that early childhood teachers need to support students' mathematical development; 2) gather information about existing standards and recommendations; 3) analyze that information for similarities, differences, gaps, and aspects to be elevated; 4) begin collaboratively revising recommendations; 5) develop a plan for drafting and disseminating updated recommendations; and 6) establish a shared resource repository to support early childhood math teacher educators.

Keywords: Early Childhood Education, Preservice Teacher Education, Teacher Educators, Standards

Early childhood mathematics education (ECME) is a distinct and evolving niche within the broader field of teacher education (Parks & Wager, 2015; Whyte et al., 2018). Unlike other early childhood preservice teacher (PST) coursework and teaching experiences, ECME is murky, with varying emphasis on Developmentally Appropriate Practices (DAP) (National Association for the Education of Young Children [NAEYC], 2020) in methods courses due to differing certification standards (e.g., early childhood teachers in Illinois and New York are certified to teach through second grade whereas they are certified through sixth grade in Texas), shifting institutional or state priorities (e.g., changes in mathematics coursework requirements to accommodate additional literacy courses in teacher preparation programs), and blurred disciplinary boundaries (e.g., mathematics content is sometimes embedded within integrated science, technology, engineering, and mathematics (STEM) methods courses rather than taught as a standalone mathematics methods course; Sarama & Clements, 2009). Together, this means that early childhood teachers across the United States experience significant variety in their preparation for teaching mathematics.

Theoretical and Conceptual Background

Taking a sociocultural perspective on learning (Lave & Wenger, 1991; Vygotsky, 1978), we understand that teaching and learning are deeply interconnected, not only with one another but also with the identities of teachers and students and the broader school and societal contexts in which they are situated. Given the unique nature of early childhood mathematics education,

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teaching early childhood mathematics requires specialized skills and knowledge. Preservice early childhood teachers need to develop a deep understanding of foundational mathematics concepts so that they are prepared to enter the classroom (Association of Mathematics Teacher Educators [AMTE], 2017). This includes understanding how young children develop number sense, spatial reasoning, and problem-solving skills. Preservice early childhood teachers also need experience with and knowledge of developmentally appropriate practices to support mathematical growth. Another skill preservice early childhood teachers need for teaching mathematics is responsiveness; skilled teachers make responsive, pedagogical decisions to support their students both before, during, and after instruction (Robertson et al., 2015). This may involve considering what students know and who students are to ensure that lessons are culturally sustaining (Paris, 2012), increasing or decreasing challenge, adding relevant examples, or explaining concepts in different ways to ensure that their students achieve the intended learning outcomes (e.g., Cirillo et al., 2009; Dietiker et al., 2018). Finally, collaborating with families is also key to early childhood mathematics education. Preservice teachers need to learn to not only share what learning is happening in the classroom with students' families but also engage families in meaningful ways such that caregivers become true partners in children's early math development (NAEYC & National Council of Teachers of Mathematics [NCTM], 2010).

Working Group Session Goals and Plans

This new working group will provide a collaborative space to critically reflect on the current state of early childhood math teacher preparation and what is needed to strengthen the field. Our group builds on a four-year, ongoing community circle through the Association of Mathematics Teacher Educators. During this time, our group has reported on various topics applicable to early childhood math teacher educators (ECMTEs), such as: the landscape of early childhood mathematics teacher prep, integrating play-based learning in early childhood mathematics methods courses, adapting curricular resources in early childhood mathematics, and pathways for professional development of ECMTEs.

Over the course of the three days, our team will lead participants in reimagining recommendations and core pedagogical practices for supporting preservice early childhood teachers in developing as mathematics educators. The goals of the working group across the three 90-minute sessions are as follows:

- Day 1: Identify a collection of skills that beginning early childhood teachers need to support students' mathematical development. Gather information about standards and recommendations for the preparation of early childhood mathematics teachers that exist across state and national contexts and from various professional organizations.
- Day 2: Analyze the information gathered on Day 1 for similarities, differences, gaps, and aspects to be elevated.
- Day 3: Begin collaboratively revising guidelines and recommendations for professional organizations and develop a structured plan for drafting and disseminating updated recommendations. Establish a shared resource repository to support ECMTEs in implementing the revised recommendations.

As a result of this new working group, we will establish a plan for continued action that will support us in developing recommendations for early childhood mathematics teacher preparation that better echo expertise from both mathematics and early childhood educators.

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PRODUCTIVE STRUGGLE EVIDENCE: STRUGGLE, PROGRESS, CULTURE AND TECHNOLOGY

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In its third year, this working group continues to develop a shared understanding of productive struggle by engaging participants in examining evidence of mathematical struggle in various contexts. In the first session, participants will discuss evidence of mathematical struggle and progress. In the second session, participants will add another layer of evidence by exploring how noticing students' public productive struggle connects to a sense of belonging, intersecting with culture and language differences. In the third session, participants will discuss how technology and tools impact students' access to mathematically productive struggle.

Keywords: Cognition, Instructional Activities and Practices, Research Methods, Teacher Educators

The Productive Struggle Working Group brings together mathematics educators, researchers, and teacher educators to examine existing constructs of productive struggle and to develop new, critical conceptualizations of struggle that connect cognition to affective components of teaching and learning mathematics. The first iteration of this working group sought to develop a shared, consensus definition of productive struggle to address the variety of definitions used by researchers (Kamlue & Van Zoest, 2024). In our second year, we added to our conceptions of productive struggle by incorporating teacher productive struggle (Baker et al., 2024) and the rights of the learner (Kalinec-Craig, 2017). As a group, we workshopped a theoretical framework that described how intentional and routine engagement in productive struggle can help to foster and reinforce students' sense of belonging in mathematics.

For PME-NA 48, we will continue our iterative design cycle of reviewing, revisiting, and refining our definitions and frameworks. Our planned sessions invite participants to dive deeper into the classroom interactions that foster (or hinder) mathematical struggle and a sense of belonging, as we continue to explore the questions and wonderings raised by participants.

Theoretical Background

Our conceptions of productive struggle are informed by Warshauer (2015) and Jarry-Shore and Richardson (2024), which are rooted in Hiebert and Grouws' (2007) definition of productive struggle and Kapur's (2009) definition of productive failure. Both the National Council of Teachers of Mathematics (NCTM, 2014) and the Association of Mathematics Teacher Educators (AMTE, 2017) identify providing students with opportunities to engage in productive struggle as a key component of effective mathematics teaching and learning. We extend the positions of NCTM and AMTE to assert that routine engagement in productive struggle can empower learners and cultivate a sense of belonging in mathematics. In turn, students' growing sense of belonging is imperative for students to take risks, make mistakes, and develop the vulnerability needed to struggle with mathematics (Buli, 2025; San Giovanni et al., 2020). Together, productive struggle and a sense of belonging reinforce students' mathematical authority, agency, and identity (Ramos-Duke & Anthony, 2026).

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Structure of the Sessions

Each session will focus on evidence of productive struggle by: (1) exploring evidence of mathematical struggle and mathematical progress; (2) connecting that evidence to culture, language, and a sense of belonging; and (3) identifying instructional tools and technology that support the cultivation of productive struggle.

Session 1: Evidence of Mathematical Struggle and Progress

The first session will introduce participants to two types of evidence: (a) evidence of mathematical struggle and (b) evidence of mathematical progress. *Evidence of Mathematical Struggle* is defined as "a verbal, written, or physical expression of tension—mathematical confusion, uncertainty, frustration or unsettledness" (Kamlue &Van Zoest, 2026). *Evidence of Mathematical Progress* is exhibited by "learners' verbal or written expressions indicating that a learner has made sense of something mathematical, regardless of whether that something was the task, the underlying mathematics of an artifact, or another mathematical idea introduced to the discussion" (Kamlue & Van Zoest, 2026). Participants will be presented with a case study of a learner who showcases both types of evidence. We invite participants to wonder, when you walk into a classroom (K-16), how would you determine whether someone is experiencing productive struggle or mathematical progress?

Session 2: Noticing: Culture, Language, Belonging, and Productive Struggle

In this session, we will invite participants to consider productive struggle as a social construct that supports a sense of belonging in mathematics (CASEL, 2022; Jarry-Shore & Richardson, 2024; Sherin et al., 2011). A sense of belonging is the feeling of being a member or of fitting into a group or community (CASEL, 2022), and it is imperative for students to take risks, make mistakes, and be vulnerable enough to struggle through solutions (Buli, 2025; San Giovanni et al., 2020). Therefore, we consider the need for teacher noticing of students' struggle through a cultural and linguistic lens not only to "observe, realize and attend to" (Sherin et al., 2011, p. 280) their struggle but also to highlight, celebrate, and create a culture of collective struggle. Evidence of struggle and progress resides in the students' own language. Students walk into a mathematics classroom with their own phrases and feelings about struggle and progress (Moll et al., 2006; Yosso, 2005). Thus, we will invite participants to explore: How might a teacher use noticing to promote productive struggle in linguistically and culturally diverse classrooms? How do we know students are comfortable with productive struggle? How might this practice support a sense of belonging in the whole group and in small groups? How might the components of mathematical identity impact productive struggle and a sense of belonging?

Session 3: Noticing: Technology, Tools, and Productive Struggle

Our final session invites participants to consider the influence of instructional tools and technology on students' opportunities to engage in mathematically productive struggle. The National Council of Teachers of Mathematics (NCTM) states that technology should "elicit and use evidence of student thinking to communicate positive messages about learners' mathematical identities" (NCTM, 2023). Teachers should intentionally use technology tools that promote sense-making and foster classroom discourse. However, while NCTM calls for teachers to support productive struggle in learning mathematics (NCTM, 2014), none of the position statements on technology or AI describe how teachers can use technology to support mathematical struggle (NCTM, 2023; 2024). We invite participants to consider: How might common classroom tools and activities impact students' opportunities to engage in productive struggle? How does the use of technology influence or hinder students' sense of belonging? How do we prepare mathematics educators to leverage these tools in meaningful ways?

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WORKING GROUP REPORT: EMPOWERING STUDENTS WITH PRODUCTIVE STRUGGLE WORKING GROUP

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At PME-NA 47, the Productive Struggle in Mathematics Education Research, Teaching, and Learning Working Group met for its second year, this time focusing explicitly on the link between productive struggle and students' sense of belonging. During our sessions we explored how routinely providing opportunities for productive struggle can lead to a sense of belonging in mathematics and developing tools and practices for supporting teachers in the creation and maintenance of productive struggle. The group has sustained a dedicated group of members who meet regularly online since the end of PME-NA 2024. In this report, we summarize the history of the working group, describe our progress and our plans for the future of the working group.

Keywords: Research Methods, Instructional Activities and Practices, Teacher Educators, Preservice Teacher Education

Our PME-NA working group brings together mathematics education researchers and teacher educators to examine conceptualizations of productive struggle and explore how we might study students' experiences of productive struggle. While mathematics education research has operationalized varying definitions of *productive struggle* (Kamlue & Van Zoest, 2024), our working group utilizes the conceptualizations established by Warshauer (2015) and Jarry-Shore and Richardson (2024), which are rooted in Hiebert and Grouws' (2007) definition of productive struggle. Hiebert and Grouws (2007) describe productive struggle as "students expend[ing] effort to make sense of mathematics," grappling with tasks that are within their reach (p. 387).

In this report, we summarize the outcome of our sessions at PME-NA 47 in State College, PA. Then, we describe the ongoing progress and the long-term plans for the working group. During our time together at PME-NA 47, we continued to build on the working group's shared, consensus definition of productive struggle, by examining established theoretical frameworks and developing our own frameworks. We also explored how teacher productive struggle (Baker et al., 2024) connects to definitions and conceptualizations of student productive struggle. Finally, we invited participants to humanize struggle through the lenses of sense of belonging and the rights of the learner.

Summary of 2025 Conference Activities

Session One

Our goal for the first session was to develop a continuum of student struggle. The intent was to integrate existing research and working group discussions into a diagram that captured the range of student experiences from unproductive struggle to productive struggle, documenting the student behaviors and emotions indicative at each stage.

We began the session by sharing the consensus definitions of *struggle* and *productive struggle* generated by the PME-NA 46 working group (Table 1). All but one participant were attending the working group sessions for the first time and were not familiar with the different conceptions of productive struggle.

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Table 1: PME-NA 46 Participant Responses

What is Struggle?	What is Productive Struggle?
Struggle can appear in non-verbal ways in addition to what is said.	Productive struggle is going somewhere; leading to the correct solution eventually *or* leading to something mathematical along the way.
Struggle ends in failure and a feeling that the efforts are going nowhere.	Productive struggle is from the students' perspective/view.
Struggle is not making headway but giving up.	Productive struggle requires a movement towards either a goal or a greater understanding.
Struggle is not knowing where to go.	Struggle is productive if there is progress made (or engagement is continued) toward resolving a task.

We spent the majority of the session examining formal and informal frameworks of productive struggle (Jarry-Shore & Richardson, 2024; Murdoch et al., 2020; Sengupta-Irving & Agarwal, 2017; Warshauer et al., 2021). As participants scrutinized each framework, we asked them to consider the following:

1. Does the diagram help to illuminate what productive struggle is?
2. Does the diagram help us to identify types of struggle?
3. Is the diagram useful for the work of teacher educators and in-service teachers?

We then invited participants to consider how they might expand, improve, or combine the various frameworks and diagrams. Facilitators charted these conversations on the board (see Figure 1).

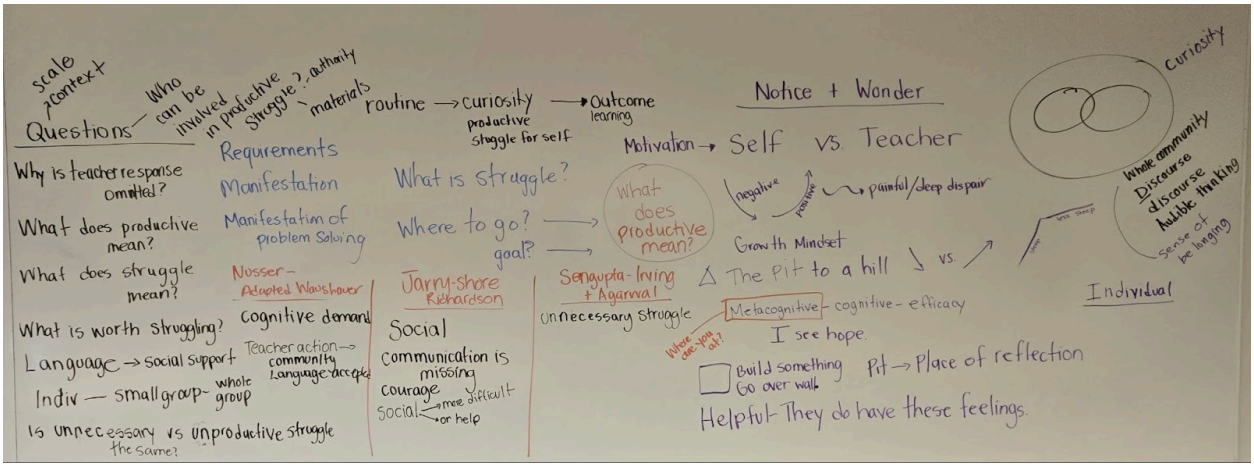


Figure 1: Session One Conversations

As shown in Figure 1, participants raised many of the same questions that the working group explored in our prior sessions at PME-NA 46. We concluded the session by asking participants

to ponder, “Why do the words we use to describe or quantify mathematical struggle matter for students, teachers, and teacher educators?”

Session Two

For this session, participants considered the challenges of *teacher productive struggle* (Baker et al., 2024). Our goal for this session were for participants to consider multiple perspectives. For example, what does it mean to help or intervene as a teacher when intending to support struggling students? From the perspective of mathematics teacher educators or math leaders, how can teachers be supported in embracing the vulnerability necessary to center mathematical sense-making in students’ thinking?

We began by having participants engage in a high-cognitive demand math task. Then, participants formed groups of four where they re-examined the task, each group member approaching the task from one of four perspectives:

4. Anticipating student approaches
5. Brainstorming teacher moves to use when implementing the task
6. What teacher struggles could occur while facilitating this task?
7. How do we support pre-service teachers in facilitating this task and supporting productive struggle more broadly?

Each group shared their responses to the four prompts which we recorded on the board (Figure 2). We then tied our group discussion to Baker and colleagues’ (2024) work on the relationship between student and teacher productive struggle. We found that our identified “teacher struggles” aligned with and extended the experiences identified by Baker and colleagues (2024). As with the previous day, participants posed several questions and wonderings which were documented and explored in the following session.

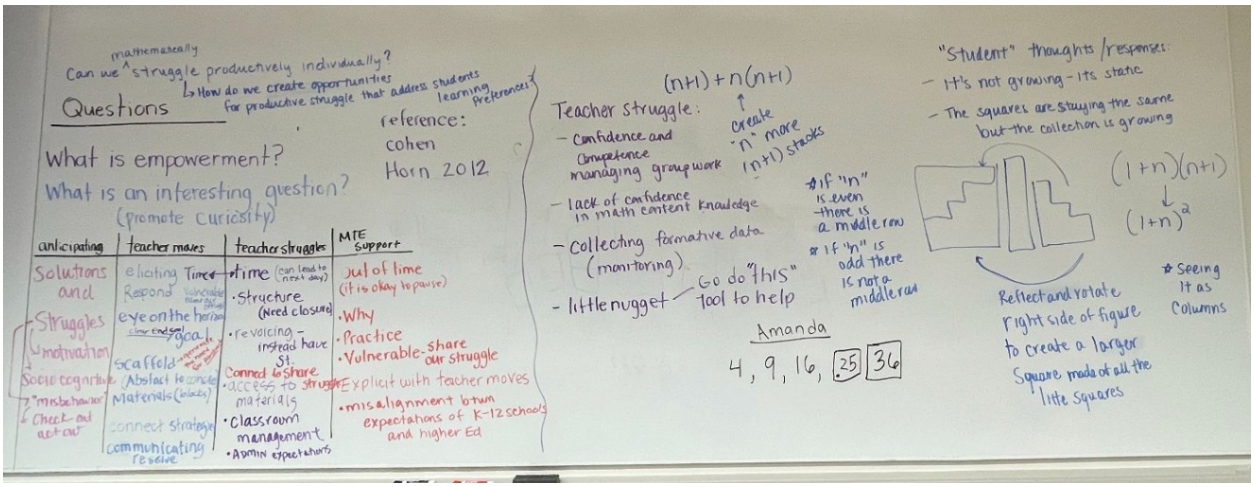


Figure 2: Session Two Conversations

Session Three

In this session, we opened the session with a discussion of, “What does “empowerment” mean to you?” in mathematics. Our goal was to link established research on productive struggle (Hiebert & Grouws, 2007; Jarry-Shore & Richardson, 2024; Warshauer, 2015) with literature on rehumanizing mathematics, to demonstrate how establishing a safe environment for routine

engagement in productive struggle serves to development of a sense of belonging in mathematics (A Pathway to Equitable Math Instruction, 2021; Kalinec-Craig & Rios, 2024). Participants readily identified mathematical identity components. We built upon this discussion to segway into telling a story about students who had their own language and expressions to describe production struggle and frustration.

Between PME-NA 46 and 47, working group members discussed the symbiotic nature of learners' sense of belonging in mathematics and experiences of productive struggle. We brought session participants into this conversation by inviting them to connect the pieces shown in Figure 3 with the theoretical frameworks of productive struggle discussed in Session One.

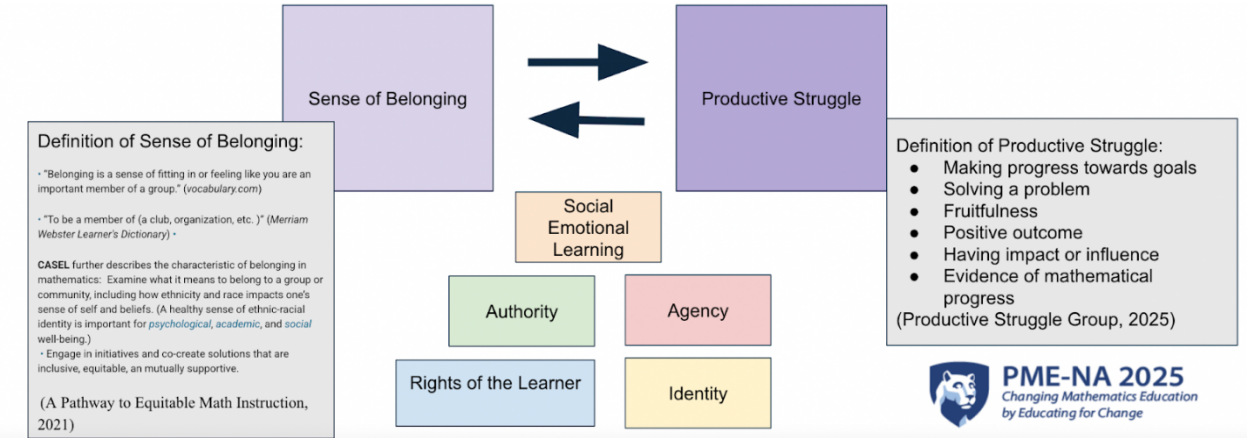


Figure 3: Slide Prompting Participants to Examine the Relationship of Sense of Belonging and Productive Struggle

Participants began working independently, connecting the concepts of agency, authority, identity, social emotional learning with the rights of the learner, sense of belonging and productive struggle, crafting their theoretical models. Some participants added on to the existing frameworks, while others connected the concepts shown in Figure 3 in new ways. Figure 4 shows two of the models that participants created and shared with the group.

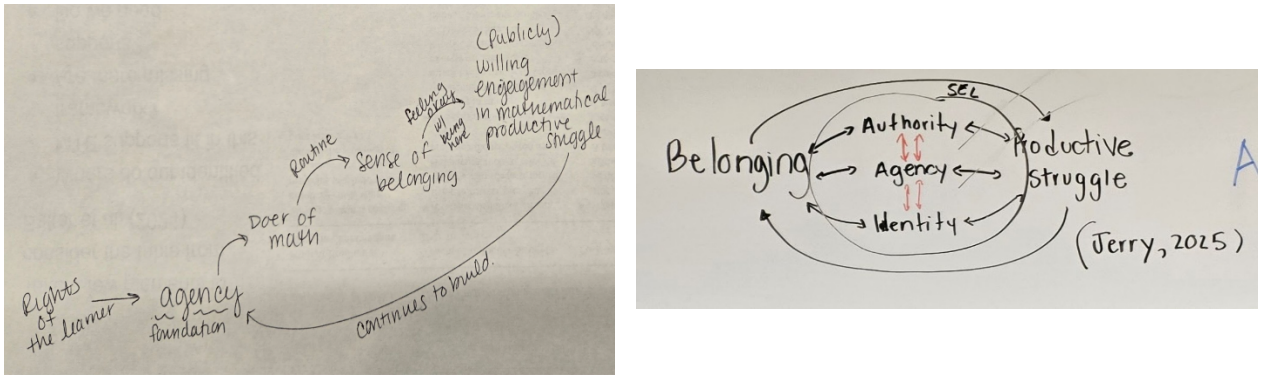


Figure 4: Theoretical Models of the Relationship Between Sense of Belonging and Productive Struggle

Our discussions of the proposed models and frameworks helped to solidify our understandings of the concepts and refine our ideas. We aim to explore these models further in the next iteration of the working group at PME-NA 47.

After the Conference

The productive struggle working group continues to meet online monthly. These meetings are a space for novice and experienced mathematics educators and teacher educators to collaborate. During these meetings we pose wonderings, develop lines of inquiry, share literature on productive struggle, and provide feedback on each other's research and drafts for publication.

Currently, members of the working group are editing a special issue of *Investigations in Mathematics Learning*. This special issue seeks to promote and formalize the ideas explored by the working group. Contributing authors were invited to refine existing definitions and conceptions of productive struggle, examine the social dimension of mathematics learning and its impact on students' experiences of productive struggle, refine methodological practices of studying students' experiences of productive struggle in context, and to explore the roles of individual and collective productive struggle in developing students' authority, agency, and sense of belonging in mathematics.

Next Steps

Continuing to build on our prior efforts, our proposed working group sessions for PME-NA 2026 will: (1) exploring evidence of mathematical struggle and mathematical progress; (2) connecting that evidence to culture, language, and sense of belonging; and (3) identifying instructional tools and technology that support the cultivation of productive struggle.

Additionally, outside of the conference, working group members are in the beginning stages of a research project where they are gathering empirical evidence to support the theoretical framings proposed by the group. We aim to have IRB approved data to explore with participants at PME-NA 48.

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MATHEMATICS EDUCATION FOR CITIZENSHIP: BRIDGING THEORY AND PRACTICE

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This working group met at PME-NA 47 for the first time to build a community around mathematics for citizenship. This year, we gather again to continue examining mathematics for democratic citizenship related to interpreting public claims, engaging with social issues and injustices, and supporting equity and social responsibility. We aim to deepen our understanding of mathematics for citizenship in ways that connect theory to classroom practice as we explore the unique roles of mathematics teacher educators and mathematics education researchers in this endeavor. The working group is participant-driven and aspires to be a place for building community with participants interested in relationships across mathematics, mathematics education, citizenship, and democracy.

Keywords: Social Justice; Instructional Vision; Equity, Inclusion, and Diversity; Curriculum

Given the current political climate in the world, the role of democratic citizens is a topic of increasing interest. This interest has extended to the field of mathematics education through work examining the role of mathematics in citizenship development (e.g., Brass, 2024; Geiger et al., 2023; Skovsmose et al., 2023) and calls from mathematics education organizations emphasizing the importance of mathematics in citizen development (e.g., Association of Mathematics Teacher Educators [AMTE], 2017, 2022; National Council of Teachers of Mathematics [NCTM], 2018, 2020). Yet, despite this increased interest, work connecting classroom practice of mathematics and citizenship integration is scarce. In the second year of this PME-NA working group, we continue to focus on mathematics for citizenship, which we view as the connection between mathematics education and democratic citizenship. We invite participants to explore these connections through mathematical thinking skills and mathematical content integrations that support students' development as citizens and interrogate social issues and injustices. Our goal for this working group is to consider theory in ways that bridge to classroom practice by deepening our knowledge of mathematics for citizenship. As we do this, we will focus on strategies that foster equity and social responsibility to prepare engaged and informed citizens in our diverse society (e.g., Gutstein, 2016; Jablonka, 2020; Weiland, 2019).

Taking a critical perspective, our working group will also explore the formatting power of mathematics in society (Skovsmose, 1994, 2021) and how it contributes to citizens' understanding of the world around them. We draw on Geiger et al. (2023), who emphasize that understanding the role that mathematics plays in formatting is "fundamental for informed and responsible citizenship so that citizens are empowered to engage with the mathematics underpinning societal issues" (p. 926). As our working group explores the formatting power of mathematics, we will examine the ways that mathematics can format social issues through the ways that mathematics is used to depict these issues and the ways that mathematics influences actions to address these issues (Skovsmose, 2021).

This working group seeks to answer these questions through our explorations:

- How is mathematics connected to citizen development?

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

- What connections between mathematics and democratic citizenship can be fostered and enacted in classroom practice? How can this be done?
- What influence does the formatting power of mathematics (Skovsmose, 1994) have on the ways in which citizens engage with the world around them?
- How can mathematics teacher educators and mathematics education researchers foster citizen development and work to defend democracy? What unique aspects of these roles aid or hinder this work?
- What does the absence of mathematics for citizenship from classrooms mean for our field? What impact does this have on society?

2026 Working Group Sessions

Session 1

We will begin by building shared understanding for new and returning participants. We will start with a short framing presentation by an invited scholar, Dr. Vince Geiger, introducing mathematics for citizenship and mathematics for democracy. This short presentation will include key ideas, such as the use of quantitative reasoning for interpreting public claims and using mathematics for engaging with social issues and injustices. Following the presentation, working group members will engage in a whole-group discussion to surface participants' questions, ideas, and connections. Building on this discussion, we will collaboratively articulate a working statement about the role and purpose of mathematics education for citizenship and democracy and begin mapping key dimensions of mathematics for citizenship related to mathematics education (e.g., Standards for Mathematics Practice [NGA Center & CCSSO, 2010], mathematics content standards).

Session 2

A short presentation introducing the formatting power of mathematics as a lens for examining how mathematics shapes what becomes visible, actionable, and considered true in society will be used to begin this Day 2. This will be followed by a whole-group discussion to unpack ideas related to the formatting power of mathematics and how these ideas connect to mathematics education. We will revise our purpose statement and map from Session 1, as needed, considering the formatting lens of mathematics. The remainder of Session 2 will be a working session. Participants will form groups based on shared interests (e.g., quantitative reasoning and public claims, curriculum/task design, teacher education, crises and the formatting power of mathematics, equity and social responsibility, empirical research designs). Groups will share ideas using a jigsaw format, so that participants leave with a range of approaches and examples beyond their initial group's case. This will lay the foundation for Day 3.

Session 3

The session will begin with a short presentation that highlights examples of scholarship and classroom-oriented work on mathematics for citizenship and democracy, with particular attention to tasks and instructional approaches that support students in interpreting and interrogating social issues and injustices. During this session, groups will work to brainstorm and outline next-step products such as a draft study design, a classroom task sequence, a teacher-education module, or a plan for cross-institutional collaboration. These discussions will support ongoing community-building and contribute to future research and teaching collaborations among working group participants. To encourage progress on these collaborations, we will also solidify plans for meeting after the 2026 PME-NA conference during this session.

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WORKING GROUP REPORT

MATHEMATICS EDUCATION FOR CITIZENSHIP: BRIDGING THEORY AND PRACTICE

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This report shares the details of our first working group meeting at PMENA-47. The aim of the first meeting of this working group was to build a community around mathematics for citizenship. The group made progress towards developing a working definition for citizenship and clarifying the role of citizenship for mathematics. Members of this group continue to meet regularly as we continue to solidify our shared understanding of mathematics for citizenship and generate research ideas based on our shared interests. In the report, we summarize the progress made at the PME-NA conference in 2025, provide updates about our shared work after the conference, and share our future plans for the working group.

Keywords: Social Justice; Instructional Vision; Equity, Inclusion, and Diversity; Curriculum

History of Working Group

Building on scholarship that frames statistical literacy as an intersection of statistical and critical literacies (Gal & Geiger, 2022; Weiland, 2019), and that highlights quantitative civic literacy as essential for evaluating mathematical claims in political and social contexts (Raygoza, 2019; Steen, 1999), this group was formed to explore how these ideas can be meaningfully translated into teaching and teacher education. The group's discussions also centered on mathematics as a practice that supports critical thinking, democratic deliberation, and the justification, critique, and negotiation of ideas (Brass, 2024; Geiger et al., 2023; Jablonka, 2020; Maas, 2019; Skovsmose et al., 2023). At the same time, members were motivated by equity and justice concerns, particularly the need to help students recognize how mathematical domains and policies can reproduce or challenge social inequalities while developing agency to question and act on injustice (Gutstein, 2016). In response to increasing calls from organizations such as AMTE (2017, 2022) and NCTM (2018, 2020) to integrate civic dimensions into mathematics instruction and the persistent gap between conceptual scholarship and classroom practice, the working group was established to begin bridging theory and practice and to lay the foundation for future empirical research.

Work at PME-NA 47

This working group formed around a shared interest in understanding and strengthening the relationship between mathematics education and citizenship. Here, we detail the activities and discussions that took place within our working group during the 2025 PME-NA Conference.

Day 1: Learning Working Group Members' Interests

Day 1 focused on sharing responses to guiding questions to better understand working group members' interests, priorities, and experiences. This helped establish a collaborative foundation and highlighted how working group members approached "citizenship" from multiple perspectives, including classroom practice, research design, and community engagement. These were the guiding questions to which working group members responded:

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

- What is the fundamental purpose of mathematics education in a democracy? What structures exist or are needed to operationalize this commitment?
- How do you define citizenship in your own work context or daily life? What dimensions feel more relevant to your practice?
- When and how do you see mathematics intersecting with citizenship in your work or daily life? How does citizenship show up in what students bring to the mathematics classroom?
- Think of a moment when mathematics influenced a civic or social issue that mattered to you. What mathematical concepts were involved, and what impact did they have?
- What mathematical ideas feel naturally or more readily connected to citizenship concepts and why do you think these connections emerge?

Figure 1, Figure 2, and Figure 3 capture our working group members' individual responses to these questions. Working group members engaged in collective discussion about these responses as we sought to understand and clarify different perspectives related to mathematics and citizenship within our group. Through our collective discussion, members of the working group explored the concept of citizenship, emphasized the aspects that we see as essential to citizenship, and clarified the relationship between mathematics and citizenship.

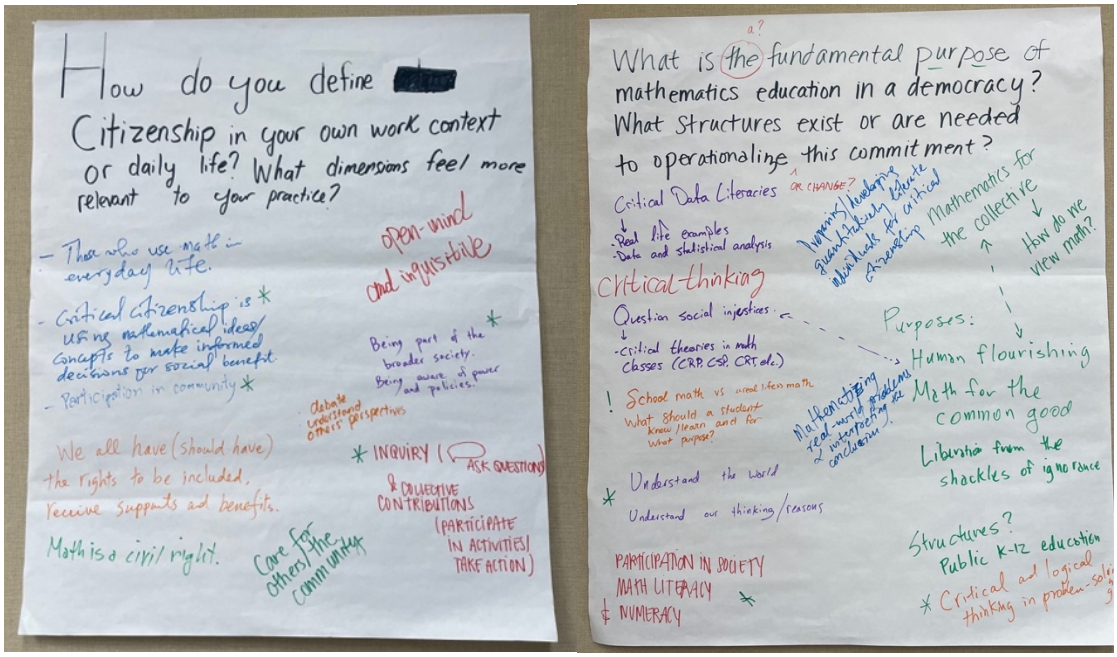


Figure 1: Responses Generated by Working Group Members, Part 1

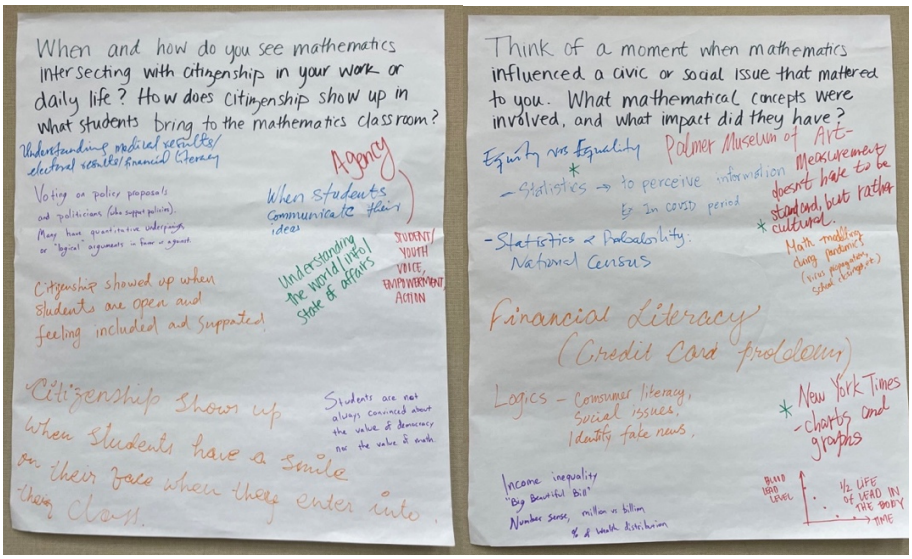


Figure 2: Responses Generated by Working Group Members, Part 2

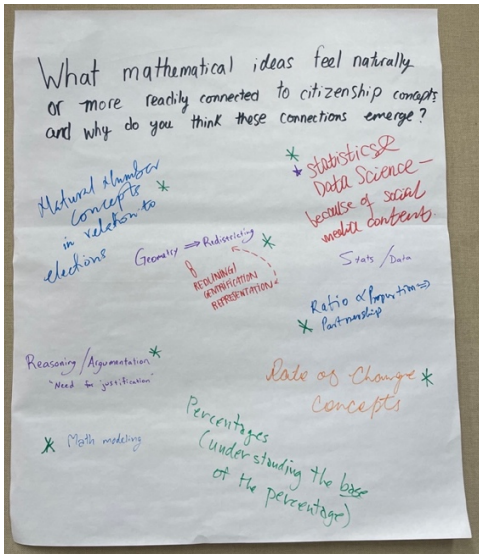


Figure 3: Responses Generated by Working Group Members, Part 3

Day 2: Defining Citizenship and Clarifying the Role of Mathematics

On Day 2, working group members worked to build a common language for mathematics for citizenship, explored how mathematics for citizenship connects to civic participation and belonging, and considered how mathematics can function as a tool for reasoning about public issues. We began our discussions using the shared experience of the *Counted Out* (Abeles, 2024) screening from the previous day of the conference. Working group members discussed the screening and its connection to mathematics for citizenship, using it as a springboard to examine equity, participation, and the role of quantitative information in public life.

Working definition of citizenship. To ground our discussion, we began by talking through widely used lenses for thinking about citizenship. Westheimer and Kahne's (2004) framework helped us distinguish among personally responsible, participatory, and justice-oriented

conceptions of the "good" citizen. Building from that conversation, we also talked through Banks's (2017) complementary emphasis on who is recognized as a citizen and what rights these citizens have, highlighting how some groups experience failed citizenship when they are structurally excluded. We used these frameworks as starting points to collectively co-construct our own working definition of citizenship. Through iterative, whole-group discussion and revision, working group members negotiated shared language and priorities, as you can see in Figure 4.

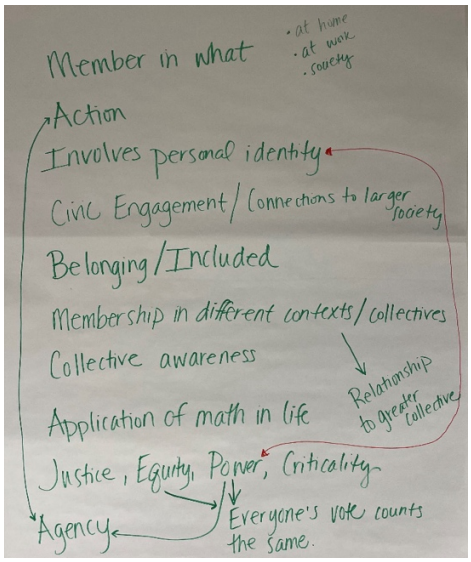


Figure 4: Working Definition of Citizenship

The group explored citizenship as participation in community, the right to be included and supported, and the responsibilities and care that sustain communities. Working group members also emphasized that citizenship involves awareness of power, politics, and decision-making processes and that being a citizen can mean engaging actively, supporting others, debating, and contributing to communal life in "civilized" and constructive ways. An important nuance emerged: citizenship is not only an external designation but also part of personal identity and how one is perceived and accepted by others. This raised key questions about who "decides" citizenship and how belonging and equal rights are negotiated in spaces like schools and communities.

Mathematical domains in relationship with citizenship. A brief presentation was given to discuss the common mathematical domains that are in conversation with citizenship. Quantitative literacy was among the mathematical domains that we discussed, as several members of the working group conduct research focused in this area. We also talked about the use of data, statistical literacy, and mathematical modeling as platforms for emphasizing the relationship between mathematics and citizenship. There were also working group members who conduct research focused in these areas. The working group members also discussed mathematics for citizenship through reasoning, where quantitative reasoning, mathematical thinking, and critical thinking serve as tools for interpreting information, making judgments, and participating in civic life. These discussions led to broader questions: what are the foundations and purposes of schooling, and why do we teach mathematics?

Small-group research conversations. Working group members met in small groups to share research interests and possible directions. This provided a setting for working group members to share their thoughts, current research projects related to mathematics for citizenship, and ideas for future work. After meeting in small groups, we reconvened as a larger group and shared the ideas that were generated. Topics for potential collaborative work included using mathematics as a decision-making tool for public issues (e.g., vaccines and herd immunity), designing horizontal and vertical studies relating mathematics and citizenship across grade levels, linking statistics to social justice contexts, and exploring interdisciplinary approaches that combine mathematics and social science methods with preservice teachers. One of the themes that emerged during these discussions was linked to measurement and impact. Working group members questioned how the effectiveness of mathematics for citizenship can be evaluated, especially since citizenship development is not confined to classrooms. This prompted early thinking about possible study designs that look across years and contexts (e.g., examining representations of citizenship across multiple courses or tracking students' opportunities to use mathematics for civic reasoning over time).

Day 3: Future Directions and Collective Outputs

Due to travel schedules, Day 3 was conducted with a smaller group. Working group members proposed several future directions: (1) developing grade-spanning activities and materials related to civic issues, (2) continuing to refine and make explicit a shared definition of mathematics for citizenship, (3) articulating the role and purpose of mathematics for citizenship with the longer-term goal of a position statement, (4) mapping mathematics for citizenship to established mathematics practices and principles to clarify alignment and support uptake by educators and researchers. Reflections on these future directions continued between conferences and formed the foundation of our working group plans for the 2026 PME-NA conference.

Progress Between Conferences

After the conference, the working group held three meetings focused on community-building and sustainability. These meetings centered on hopes and expectations for the future and reinforced a shared desire to keep the group intellectually active by continuing to share papers and exchange ideas. These post-conference discussions functioned as a platform for clarifying goals, identifying what members need from the community, and maintaining momentum before the submission of the next working group proposal. These discussions also led to several questions the working group would like to address in the future.

Questions to Address in the Future

The questions build on our work from the 2025 conference and provide direction for our future working group sessions:

- What are specific ways in which mathematics is connected to citizen development?
- How can the connections between mathematics and democratic citizenship be enacted in classroom practice?
- How does the formatting power of mathematics (Skovsmose, 1994) influence the ways we engage with mathematics for citizenship?
- What unique roles do mathematics teacher educators and mathematics education researchers have in citizen development and in defending democracy?
- If mathematics is essential for developing citizens, but is absent from classroom practice, what does that mean for our field? What does that mean for our society?

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WORKING GROUP ON UNIVERSAL DESIGN FOR LEARNING IN SECONDARY AND POSTSECONDARY MATHEMATICS

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Mathematics classrooms at the secondary and postsecondary levels are increasingly diverse, challenging assumptions that learning environments are "one size fits all." This Working Group examines Universal Design for Learning (UDL) as an empirically based framework for intentionally designing mathematics instruction across secondary and postsecondary contexts that anticipates learner variability. Although widely used in K–12 and disability-focused contexts, UDL has received limited attention in PME-NA research. Participants will collaboratively interrogate research-based instructional practices through a UDL lens, examining affordances, access friction, and participation—particularly for students traditionally marginalized in mathematics education. The Working Group emphasizes interactive, case-based inquiry and aims for sustained collaboration beyond the conference.

Keywords: Instructional Activities and Practices, Students with Disabilities, Undergraduate Education

Mathematics educators recognize the complexity inherent in teaching and learning environments. Intersecting social, environmental, and instructional factors shape what and how students come to learn new mathematical ideas. As postsecondary mathematics classrooms become increasingly diverse in students' cognitive profiles, identities, and prior experiences, educators are recognizing that learning environments are not "one size fits all." Rather, instructors must be attuned to students' varied needs and intentionally design instructional environments that identify and mitigate barriers to learning.

Theoretical Background

Universal Design for Learning (UDL) is an empirically based design and learning framework that foregrounds learner variability and supports intentional instructional design (CAST, 2024). Rather than responding to students' needs through retrofitted accommodations, UDL emphasizes anticipating variability from the outset. Although UDL has been taken up in K–12 and disability-focused scholarship, it has received limited explicit attention in PME-NA proceedings and secondary and postsecondary mathematics education research. This new Working Group seeks to address that gap by situating research-based instructional practices in secondary and undergraduate mathematics within a broader UDL framework. We aim for this Working Group to offer regular communication and collaboration beyond the conference and to convene at future PME-NA conferences as well.

When framed as an emancipatory approach to ensuring equitable access to instruction, UDL offers a lens through which instructional practices can be interrogated for the benefit of students traditionally excluded from the research literature. In this Working Group, participants will collaboratively examine questions such as: What are the affordances of particular instructional practices, and for whom? Where does access friction arise that limits engagement, sensemaking, or participation? How might instructional design be reimagined to better support students at the margins of mathematics classrooms?

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Structure of the Sessions

Sessions are designed to be highly interactive and inclusive of participants' contributions, case studies, and inquiries. Activities will be planned as means for eliciting input from participants, recognizing that interactions that follow can be anticipated but not fully planned. Flexibility to pursue participants' favored inquiries is key to a successful Working Group.

The goals of the Working Group will remain constant. They are to:

1. Examine how secondary and postsecondary mathematics educators conceptualize and enact instructional design in response to increasingly diverse learners.
2. Use UDL as a theoretical and analytic lens to interrogate research-based instructional practices in undergraduate mathematics.
3. Center the perspectives and experiences of "students in the margins" to better understand how instructional environments enable or constrain access to mathematical learning.
4. Identify tensions, affordances, and limitations of UDL when applied to secondary and postsecondary mathematics contexts.
5. Develop a shared research agenda that supports ongoing collaboration beyond the conference, including joint manuscripts, symposia, and grant proposals.

On Day 1, the focus will be on getting to know the instructional and research contexts of the participants. Additionally, developing a shared understanding of the Universal Design for Learning framework and the principles of active design are critical. Tabletop activities where participants sort sample instructional practices and discuss how they relate to the Universal Design for Learning framework will support participants to get to know each other and dive deep into the UDL considerations. A case study that draws from the organizers' ongoing research on postsecondary mathematics instruction will launch small group discussions that map instructional practices to UDL principles, identifying affordances, access friction, and opportunities for redesign.

Day 2 will zoom out from specific examples and case studies of UDL-aligned instruction to focus on theoretical dialogue. Small group discussions will connect UDL to existing PME-NA research traditions, including work on instructional systems, participation, equity, and learning environments. There will be a focus on amplifying the voices of students in the margins to interrogate instructional environments, and identifying how this approach intersects with other research traditions.

The work on Day 3 will set the stage for ongoing collaboration between PME-NA conferences. Participants will be invited to propose or join on to shared research opportunities that can extend collaboration beyond the formal meetings of the Working Group. The session will conclude after collectively deciding the medium and frequency for ongoing collaboration, and providing online access to the products and working documents for the Working Group.

Ultimately, this Working Group aims to amplify the voices of students in the margins so that their particular needs are not overlooked in the design of mathematics learning environments. Anticipated outcomes include a shared language for discussing instructional design for learner variability, an articulated set of research questions and theoretical tensions related to UDL in postsecondary mathematics, and concrete plans for continued collaboration. Continuing, expanding, and propagating research that centers the experiences of students traditionally excluded from mathematics education research is an aspirational and optimistic outcome for participants in this group.

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) RESEARCH COLLOQUIUM

COLOQUIO DE INVESTIGACIÓN: COGNICIÓN E IMAGINACIÓN MATEMÁTICA CORPORIZADA

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The Embodied Mathematical Imagination and Cognition (EMIC) Research Colloquium offers hands-on activities; demonstrations; and spaces for exploration and discussion of the embodied nature of mathematics education. For PME-NA 48, the EMIC group proposes a session that will allow the broad community of researchers engaged in this work to take stock of where the field stands and plan directions for the future. The session will be organized around a series of provocations, which will be shared on the first day the colloquium convenes, and whose purpose will be to challenge the assembled researchers to grapple with theoretical, methodological, and pedagogical/curricular challenges to the embodied cognition paradigm. The provocations will frame the demonstrations and explorations that will take place on day two and then be revisited on day three, as the group plans future collaborative research projects.

Keywords: Cognition, Learning Theory, Technology, Informal Education

Theoretical Background

A fundamental aspect of mathematical reasoning is that people impart symbolic notational systems and computational procedures with *meaning*. For example, early on, children regard NUMBER as more than just a visual string of numerals. To be meaningful, numbers evoke: spatial location along a line, magnitude, place value, cultural associations, and so on. At their core, theories of embodied mathematical cognition seek to offer accounts of how these meanings form from our sensorimotor interactions in the world and how we can design objects and experiences to cultivate these formations, foster inclusive and effective educational experiences, and reliably test hypotheses and assess educational outcomes of all students (e.g., Abrahamson et al., 2020; Alibali & Nathan, 2012; Arzarello et al., 2009; de Freitas & Sinclair, 2014; Edwards et al., 2014; McKinney de Royston et al., 2020; Radford, 2009; Walkington et al., 2021, 2024).

A Brief History and Motivation of the EMIC Working Group

The EMIC working group started at PME-NA 2015. We have convened annually at PME-NA ever since, organizing events that investigate the embodied nature of mathematics through physical and digital games, arts and crafts, individual and collaborative dance and movement, language, perception, immersive experiences, and more. EMIC members continue to explore new technologies and research methods that acknowledge and leverage the inherently embodied nature of mathematical cognition. Recent work goes beyond scholarly publications to offer novel technologies scaled for classroom use, special education, and multilingual learners; teacher learning for instructional uses of gestures; and webinars for teachers and parents.

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Embodied Learning at a Crossroads: Provocation, Reflection, and Reimagination

With more than a decade of history, the EMIC community now encompasses a broad and expanding network of researchers working within diverse theoretical framings under the 4E cognitive umbrella (Extended, Enacted, Embodied, and Embedded). Over the years, EMIC has become a collaborative space where mathematics education researchers have shared design-based work that has broken new ground theoretically, methodologically, and pedagogically. For example, last year's gathering featured interactive demonstrations illustrating how emerging technologies (e.g., robotics; haptic interaction; author, date), analog tools (SunRules; author, date), children's toys (Speks Toys, Play-Doh; author, date), and ancient, enactive practice (yoga; aiuthor, date) can create contexts where embodied mathematical learning can emerge. Our vision for this year's gathering is twofold. First, we aim to once again feature interactive demonstrations from both long-standing and new members of the EMIC community. Second, we seek to engage collective visioning: taking stock of where we are as a field and imagining a future in which embodied design-based work will continue to thrive.

To frame this collective visioning, we will organize this year's sessions around a series of *provocations* that will be presented during the first day of the working group. We use *provocation* in the sense of a spirited, yet collegial, challenge. Our aim is to stir the collective imagination of participants and identify the big-picture questions that our field needs to wrestle with in the years ahead. We will organize the three days of the colloquium as follows:

On Day 1, we will begin with introductions and an overview of embodied mathematical thinking, teaching, and design principles. Participants will then be invited to share Provocations—broad questions related to theoretical, pedagogical, or practical challenges we're wrestling with in our work. These include questions such as: *How do I respond to reviewers who are looking for evidence that's incompatible with my epistemological commitments? How might we build toward greater scientific consensus when embodiment scholars draw from diverse theoretical perspectives? How might access to new stores of multimodal data from federally funded initiatives shape and advance the kinds of questions and methods we will pursue over the next several years?*

On Day 2, members of the working group will engage participants in demos of activities that span K-16 content areas. The demos will be framed by introductory discussions that (a) call back to the provocations from Day 1 and that (b) foreground theories of enactive and embodied cognition to highlight the research underpinnings of their designs. Following the demos, the session will conclude with a group reflection that examines how various theories of embodied or enactive design were made evident through the implementations of the different activities.

On Day 3, we will reflect on Day 1 and Day 2 experiences in relation to the broader question of how embodied and enactivist approaches to education can enable novel forms of mathematical activity by grounding conceptual understandings of mathematical ideas in sensorimotor and materially agential interactions and experiences. We will follow by returning to previously raised provocations (and perhaps raise new ones), share emerging insights, and plan new collaborations to pursue them over the course of the year.

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PROGRESS REPORT: THE EMBODIED MATHEMATICAL IMAGINATION AND COGNITION WORKING GROUP

INFORME DE PROGRESO: EL GRUPO DE TRABAJO DE COGNICIÓN E IMAGINACIÓN MATEMÁTICA INCORPORADA

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The Embodied Mathematical Imagination and Cognition (EMIC) Working Group was established in 2015 at the annual PME-NA conference. As an interdisciplinary group of researchers, teachers, students, and practitioners, EMIC has strived to advance our understanding of how our physical experiences and body-based resources impact the ways we learn mathematics. Over the past several years, EMIC's founding members have collaborated on initiatives to a) expand this area of research through grants, workshops, and publications, b) share these findings with teachers and practitioners, and c) grow the community through new members and opportunities for participation. This progress report highlights the progress and accomplishments of EMIC and its members through 2025 at PME-NA 47.

Keywords: Technology, Cognition, Informal Education, Learning Theory

A Brief History and Motivation of the EMIC Working Group

The first meeting of the EMIC working group took place in East Lansing, MI during PME-NA 2015. It has a somewhat longer origin, however, growing out of several earlier collaborative efforts to review the existing literature, document embodied behaviors, and design theoretically motivated interventions. One early event was the organization of the 2007 AERA symposium, "Mathematics Learning and Embodied Cognition." This and other gatherings led to a funded NSF "catalyst" grant to explore a Science of Learning Center, which was to involve scholars from multiple institutions and countries. Though unfunded, those SLC efforts shaped a subsequent 6-year NSF-REESE grant, "Tangibility for the Teaching, Learning, and Communicating of Mathematics," starting in 2008. Interest from the International PME community in this topic grew and led to special issues of *Educational Studies in Mathematics* (2009), *The Journal of the Learning Sciences* (2012), and an NCTM 2013 research pre-session keynote panel, "Embodied cognition: What it means to know and do mathematics," along with a series of academic presentations, book chapters, and journal articles, as well as several masters' theses and doctoral dissertations. By now, several research programs have formed to investigate the embodied nature of mathematics (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; de Freitas & Sinclair, 2014; Edwards et al., 2014; Lakoff & Núñez, 2000; Radford 2009), demonstrating a "critical mass" of projects, findings, senior and junior investigators, and conceptual frameworks to support an on-going community of like-minded scholars within the mathematics education research community.

It was within this historical context that approximately 22 members of PME-NA 2015 came together for three 90-min sessions of semi-structured activities. On Day 1, the organizers

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engaged attendees in some of the body-based math activities used in their research on proportional reasoning and geometry. One activity (Figure 1b) engaged participants in a projection activity using Alberti's Window, which helped experience the role of body perspective on form and representation. We discussed how embodied theories are advancing our understanding of mathematical thinking, and how these ideas are shaping a new class of educational interventions. During Day 2, we used hands-on activities to expand our own understanding of topology. We then built on the emerging rapport among the group to hold a facilitated discussion of the potential intellectual benefits of forming a self-sustaining Working Group on embodied cognition, along with the necessary infrastructure it would need to maintain. Several concrete proposals led to the list of Future Steps on Day 3. However, before we tackled those matters, participants began the session doing math games and activities in small groups, including Spirograph, Set, Rush Hour, Tangrams, and Mastermind. We reflected on how some games and activities draw people into rich mathematical thinking and actions, and how we naturally engage in math through these activities. Day 3 culminated in an organized list of Future Steps. We have convened annually since.



Figure 1: A Small Selection of Embodied Activities Created by EMIC Organizers and Experienced by EMIC Participants. *Clockwise from top left*: (a) Experiencing Geometric Transformations; (b) Acting Out Geometry Conjectures; (c) Constructing Icosahedra First as Small, then at Human Scale; and (d) Enacting Topological Relations

Founding Members

The EMIC community has welcomed several new members since its inception in 2015 and the leaders of our working group at PME-NA have shifted each year to reflect the focus of each annual meeting. Below are the founding members of EMIC who have led the charge for many of this group's collaborations within and between PME-NA convenings.

- Candace Walkington, Southern Methodist University
- Caro Williams-Pierce, University at Albany, SUNY / University of Maryland
- David Landy, Indiana University / Netflix
- Dor Abrahamson, University of California, Berkeley
- Erin Ottmar, Worcester Polytechnic Institute
- Martha W. Alibali, University of Wisconsin-Madison
- Mitchell J. Nathan, University of Wisconsin-Madison
- Ricardo Nemirovsky, San Diego State University
- Rogers Hall, Vanderbilt University

Collective Efforts and Accomplishments

Beyond PME-NA, EMIC members have actively collaborated over the past several years to continue advancing this field of research, bridging the gap between research and classroom practice, and expanding the EMIC community. Below, we provide overviews of the progress and accomplishments shared by members of our group (see Table 1).

Table 1: Events Organized by the EMIC Community Since PME-NA 2015

Year	Event
2015	EMIC Working Group at PME-NA 37. Theme: Embodied Mathematical Imagination and Cognition. The launch of the working group and community at PME-NA.
2016	EMIC Working Group at PME-NA 38. Theme: Finding Common Interests for Creating an Online Community of Scholars.
2017	EMIC Working Group at PME-NA 39. Theme: Embodiment at the Crossroads of Mathematical Collaboration. Topics included the power of collaborative gestures.
2018	EMIC Working Group at PME-NA 40. Theme: Extending Theoretical Frameworks of Embodied Cognition to Assessing and Assisting Mathematics Learning.
2019	EMIC Working Group at PME-NA 41. Theme: Co-design of Novel Embodied Instructional Activities for Mathematics Education. Topics included the influences of eliciting movement in physical and video game-based learning activities.
2019	(May) NSF-funded EMIC I Workshop at the University of Wisconsin-Madison.
2020	(May). EMIC Virtual Panel #1. Hosted by embodiedmathematics.com.
2020	(October). EMIC Virtual Panel #2. Hosted by embodiedmathematics.com.
2021	(June; postponed from June 2020). EMIC Virtual Working Group at PME-NA 42. Theme: How Embodied Mathematics Can Bridge Cultural Divides. Topics included varieties of cultural and linguistic ways of expressing mathematical ideas.
2021	(October). NSF-funded EMIC II Workshop at Colorado State University, Fort Collins.
2021	(October). EMIC Research Colloquium at PME-NA 43. Theme: Promoting Inclusive Mathematics Education Research and Practices. Topics included exploring varieties of physical inclusivity in different virtual stations and immersive technology.

2022 Members of the EMIC community hosted visiting scholars, practitioners, and students to their home sites for activities to promote embodied mathematics teaching and learning; including: methodology training, new collaborations, workshops (e.g., dance + math).

2022 (November). EMIC Research Colloquium at PME-NA 44. Theme: Critical Dissonance and Resonant Harmony. Topics included traditional vs. embodied views of mathematics and embodied activities for more socially-just education.

2023 EMIC Research Colloquium at PME-NA 45. EMIC members Karl Schaffer and Mitchell Nathan held a 2-day dance + math workshop in Madison WI with students and faculty from dance, theater, math, and education.

2024 EMIC Working Group at PME-NA 46. Theme: *"Envisioning the Future of Mathematics Education in Uncertain Times"* Experience perspectives across multiple institutions of mathematical activity, teaching, learning, assessment, and design for current uses and challenges of VR/AR to foster inclusive and effective educational experiences.

2025 EMIC Working Group at PME-NA 47. Theme: *"Changing Mathematics Education by Educating for Change."* We further describe these activities below.

Recap of EMIC 2025: Changing Mathematics Education by Educating for Change

At PME-NA 47, the EMIC Research Colloquium built on the conference theme to consider the rapid changes within mathematics education through an embodiment lens. This year we focused on hands-on activities and group discussions around the central questions that frame this year's conference theme. This enhanced the conversations and led to new ideas, perspectives and collaborations. Activities addressed (1) *Inclusion and Epistemic Justice* to meet needs of neurodivergent and neurotypical children (2) *Mathematics as Transformative*. Strategic uses of physical materials and digital holograms in immersive environments using AR/VR technology. (3) *Respond to the Changes in a Dynamic World*. Advancements in sensor and display technologies, research methodologies, and learning theories provide an ability to closely track these rapid changes and guide these developments in furtherance of educational objectives.

On Day 1, after introductions of approximately 20 attendees, participants experienced how embodiment can be a frame for surfacing key dimensions of epistemological pluralism and opening new pathways for inclusive instruction; querying embodiment theory through work with marginalized learners and neurodivergent and neurotypical children. Activities included: *Balance Board Math* (Figure 2) to investigate ways periodic movement (common in "stimming") can support understanding of periodicity and transcendental functions (Tancredi & Abrahamson, 2024); *Play-Doh* to explore integral calculus; and magnets and loops to delve into notions of "*Stretchy Minds*" and how we think about agency and protecting the structures that shape identity within institutional hierarchies of authority.



Figure 2: Play-Doh to Explore Integral Calculus, Balance Board Mathematics, Multiplication by Sunlight with a SunRule

On Day 2, we built *SunRules* (Dimmel et al., 2021) using craft materials and geometry, and then went outside to use the SunRule as a way to explore multiplication and division (Figure 2). People were in small groups and explored trigonometry and geocentric geometry. We debriefed in a whole group discussion.

On Day 3, we engaged in *Math Yoga* and ways of connecting with our bodies in stable and less stable configurations to focus on basic properties of geometry and linear and nonlinear relationships. The last 30 mins of Day 3 were spent debriefing and community building, addressing these prompts:

1. Reflect on what we've learned, what ideas we've generated, lingering questions
2. Future math of embodiment in education
3. How to collaborate with educators and teachers to work together towards embodiment—hurdles to overcome to implement—research to practice and practice to research

Highlights: Funded Grant Proposals and Projects

1. NSF EMIC I. A successful proposal to host a three-day NSF Synthesis and Design workshop, May 20-22, 2019, hosted by University of Wisconsin-Madison. The theme was "The Future of Embodied Design for Mathematical Imagination and Cognition" with 8 members serving as organizers. There were 50 attendees, comprised of international researchers, educational practitioners, and graduate students. All attendees presented. Keynotes from Dr. Brian Bottge and Dr. Maxine McKinney de Royston. This resulted in a white paper (Nathan et al., 2021) and open access article (Abrahamson et al., 2020).
    a. Follow-up collaboration between a WG faculty member (Abrahamson) and workshop participant (Danny Luecke), a Secondary Math Education Developer/Instructor at Turtle Mountain Community College, North Dakota, to create pre-/in-service teacher courses, Ojibwe Math I, II, and III, as well as a course titled Math In Context.
2. NSF EMIC II. A successful proposal to host a second NSF three-day Workshop in 2020 (delayed to Sept. 25-26, 2021), hosted by Colorado State University. The theme was: "EMIC: Professional Development for Undergraduate Mathematics Instructors" with 7

member organizers. Keynotes from Dr. Nathalie Sinclair, Dr. Brooke Ernest, Dr. Francisco Ortega, & Dr. Molly Kelton.
3. Several grants awarded by IES CASL program to study embodied cognition, including: the role of action in pre-college proof performance in geometry (Funded 2016-2025 for Nathan & Walkington); the use of perceptual learning technology to study algebra learning (Ottmar & Landy, 2018), use of augmented and virtual reality (AR/VR) to foster collaborative embodiment in immersive learning environments (Walkington & Nathan, 2020); and developing embodied AI (NSF; Nathan & Swart).

Products and Dissemination

1. Exhibit presented by Abrahamson and Rosenbaum at an NSF-funded WG meeting (EMIC I, Madison, WI) has led to a research project (Palatnik, WG faculty), a journal paper (Benally, Kimiko, Palatnik, Abrahamson, in press), and a Spencer proposal (item).
2. A joint symposium on Embodied Cognition with six members at the 2018 APA Technology, Mind & Society Conference (Harrison et al., 2018)
3. Publication of Nathan (2021), a textbook on embodied cognition for education.
4. A mechatronic embodied design for blind and visually impaired geometry students, in collaboration with the PhET lab and the CHROME lab wins Best Paper both at two annual conferences (Lambert, Fiedler, et al., 2022; Lambert, Tancredi, et al., 2022).
5. Published Walkington et al., (2022) on the cognitive relevance of actions for math ed.
6. Best Design Paper at ICLS for *The Hidden Village*, an embodied video game for improving mathematical insight and geometry proof production (Nathan et al., 2022).
7. Springer publishers solicited a chapter on embodied design for its forthcoming handbook of digital educational resources (Abrahamson et al., in press).
8. iLRN 2025 *Best Doctoral Colloquium Paper "*A Comparison of Teachers and Students in Categorizing and Representing Geometric Conjectures using AR Simulations" (UW).
9. Nathan, M. J. (2026, April 15). "Embodied Learning." Invited presentation to the Science of Learning Podcast Series, Dr. R. Keith Sawyer.

Community Engagement/Broadening Participation

1. Expanding a group website using the Google Sites platform to connect scholars, support ongoing interactions throughout the year, and regularly adding additional resources/activities https://sites.google.com/site/emicpmena/home
2. Several members creating and teaching Embodied Cognition and Gesture seminars at their institutions.
3. Extending the embodied-design agenda into special education in dialogue with Universal Design for Learning (Abrahamson, Flood, Miele, & Siu, in press)
4. Some senior members joining junior members' grant proposals as Co-PIs and advisors.
5. Interest from Head of Education R&D (Dr. Ilona łowiecka-Tańska) of the largest STEM museum in Europe (the Copernicus Hall of Science in Warsaw, Poland) to incorporate exhibits that had been developed for/at an NSF-funded WG meeting (EMIC I, Madison, WI) by Abrahamson and Rosenbaum.
6. EMIC Virtual Workshop #1, May 19, 2020. Theme: "Instructional Gestures for Classrooms and On-Line Mathematics Learning." Designed for an audience of K-16 teachers, parents, and mathematics education researchers. Panelists: Dr. Mitchell J.

Nathan, Dr. Martha Alibali, & Dr. Rebecca Boncoddo. Discussant: Dr. Hortensia Soto. Moderator: Dr. Caro Williams-Pierce. Hosted by embodiedmathematics.com.

7. EMIC Virtual Workshop #1, October 12, 2020. Theme: "Reading Group: The Future of Embodied Design for Mathematics Teaching and Learning." Convener: Dr. Dor Abrahamson, with a panel of authors from the (Abrahamson et al., 2020) paper.
8. Steering Committee, "Shaping the Future of Mathematics Learning and Education: A Scoping Workshop," July 23–24, 2024 (virtual workshop). Organizer: Dr. Soo-Siang Lim, Program Director, Science of Learning and Augmented Intelligence Program, NSF

Conclusions and Looking Ahead

Embodied accounts of reasoning, learning, and instruction are valuable frameworks for understanding and cultivating mathematics learning and teaching, and for informing a new generation of designs for interactive technologies, curricula, and assessments. The true potential for these ideas and practices to lead to transformational mathematics education depends on a thriving, informed, interactive community of scholars and practitioners who engage with these approaches, evaluate their impacts, and improve theory and practice. Thus, we see predictably recurrent events such as the PME-NA working groups and research colloquia as a cornerstone for maintaining a highly educated and engaged core community. We will continue to offer outreach events and pursue external funding for specialized workshops when the resources are available. Future research colloquium topics that offer particular promise to the mathematics education community are: Broadening participation in mathematics and STEM through embodied design for learning, teaching, and assessment; the growing access to augmented and immersive technology, such as AR and VR, as a means to transform learners' collaborative interactions with mathematical objects and phenomena (Walkington et al., 2021); and the potential for multimodal learning analytics that use machine learning methods to integrate complex data streams that include channels of speech, gesture, eye gaze, facial expressions, mouse clicks, and so on, to support more sophisticated descriptions of learners' behaviors that can lead to real-time formative assessment and predictive models of educationally important student outcomes (Abrahamson et al., 2021; Closser et al., 2021).

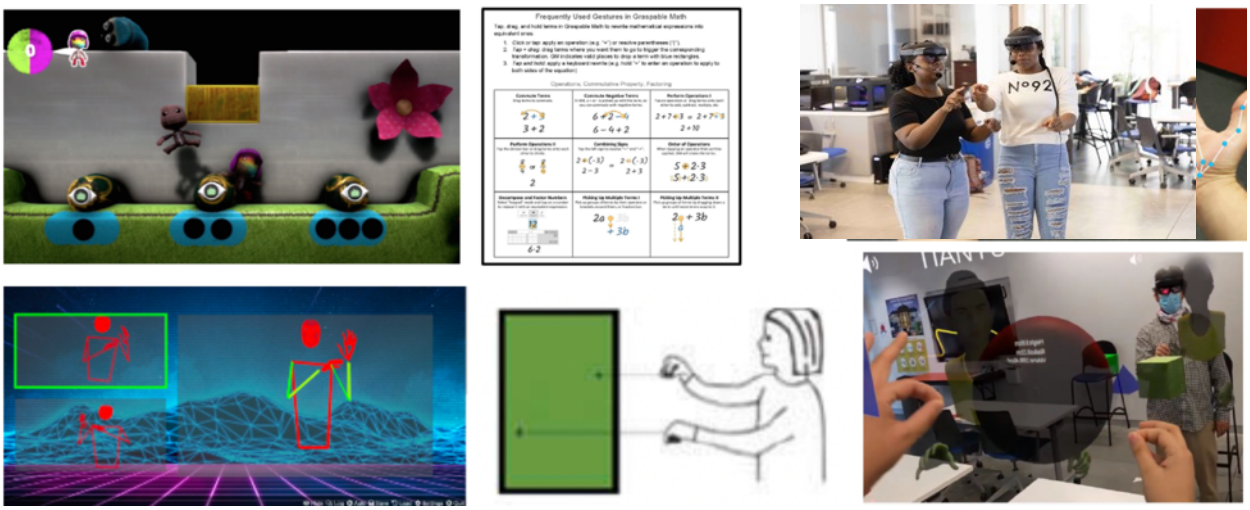


Figure 4: Embodied Technologies Designed by EMIC Members that Engage Body Based Processes in Support of Mathematical Reasoning.

Clockwise from top left: **The Collaborative Video Game, *Rolly's Adventure*; *Graspable Math*; Hand Joints and Holographic Math Objects in the Augmented Reality/Virtual Reality (AR/VR) *shAR Geometry* Environment; *Mathematical Imagery Trainer for Proportion (MIT-P)*; *The Hidden Village-Online***

Acknowledgments

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(RE)TURNING TO RELATIONALITY IN MATHEMATICS EDUCATION

VOLVIENDO A LO RELACIONAL EN LA EDUCACIÓN MATEMÁTICA

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Justice-oriented research and practice are often rooted in relational theoretical frameworks. Yet many of these frameworks contain a critique of representationalism that is nontrivial to understand and internalize. This critique is even more difficult to understand in relation to dominant ontologies of Mathematics. Our working group will connect people who are interested to learn more about research in Mathematics Education involving ethico-onto-epistemological frameworks, the associated critique of representationalism, or interested in helping to develop day-to-day practices of attunement to relationality. Our working group aims to build a community to sustain further research by initiating discussion, creating new communication channels for remaining in touch, and organizing small study groups for participants to engage deeply with each other's scholarship.

Keywords: Feminisms, Post Qualitative Research, Affect, Material Agency

Disciplinary practices cyclically emerge from relationships, and relationships are iteratively shaped by actions and practices. Like other disciplinary behavior, the practices that perpetuate forms of emergent *harm* can also be considered "inherent" to Mathematics (Leyva, 2022; Philip et al., 2018; Prescod-Weinstein, 2020; Vakil, 2020). We are troubled by this.

Anzaldúa (2002) moves us to consider how: "Change requires great heat. We must turn the heat on our own selves, *the first site of working toward social justice and transformation*" (p. xxxviii, emphasis added). We wish to explore how relational theoretical frameworks—and a (re)turn to the relationality inherent to Mathematics—can facilitate our teacher-selves to become better at sensing the ways in which Mathematics can problematically participate in perpetuating ongoing harm. With our individual teaching practices in mind, we feel that a relational (re)turning to self can also initiate profound (re)configurings of ourselves as inseparable parts of a spiritual flow where everything is alive and "full of thought, desire, contemplation and will" (Watts, 2013, p. 23).

For ourselves and for others, we have noticed that relational theoretical frameworks can be very generative and restorative while also being very bewildering. We seek to question "the very nature of mathematics … and how we are affected by that practice" (Gutiérrez, 2017, p. 2), but it is nontrivial to reimagine Mathematics without the familiar and dominant "two-world ontology" (Barad, 2003; St. Pierre, 2021) of representations. Our working group will be a place where we and our participants develop innovative approaches towards reconciling these tensions.

The first goal of our working group will be to document the configurations of disciplinary attunement to relationships that we and our participants have noticed in ourselves. Constrained within each of our institutional and curricular realities, we wish to explore more restorative forms of Mathematics that are compatible with ethico-onto-epistemological theoretical frameworks.

The second goal for our working group is to bring together scholars who are excited to reciprocally shape each other's thinking about response-able pedagogies (Chen, 2023), methodological implications (Rosiek & Snyder, 2020), and daily practices of justice (Everett,

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

2020). Our intentions resonate with Renato Marcone's (2025) recent suggestion to create small study groups of scholars who become deeply familiarized and engaged with each other's work.

Unlike one of the narratives about coming to know someone other than yourself, the two of us have noticed that coming to know each other's work has *not* necessarily involved coming to learn how to "complete" each other's sentences. Instead, we have noticed (Abreu & Gouvea, 2026) that *not*-knowing each other's thoughts (experiences, or detailed backgrounds of familiar literature) has created a fertile grounding for learning to "move-*with* each other's sentences," and learning to "*move* each other's sentences." We hope to facilitate networks of similar working relationships—meaningfully moving and being moved by others.

Outcomes and Organization

Through this working group, we will set up a communally-organized communication channel, and curate a shared Zotero library for references and collaborative commenting on PDF files. If there is sufficient interest among participants, we will organize a proposal for a special issue in an established journal. And most generatively, we will facilitate the organization of small, research-sustaining, groups that read each other's work deeply to collaborate, move each other's ideas, and move-with each other's ideas for progressing this work.

Day 1. Using trans-disciplinary guidelines (Ahenakew, 2016), we intend to begin by sharing different examples in which the intrinsic relationality of Mathematics becomes more palpable. We wish to help re-sensitize ourselves to the relationships that we are *already* a part of. We will document the practices of attunement to disciplinary relationships that are surfaced by our discussion and carry forward into the following days and in our working group report.

Day 2. Each of the three days will include classroom episodes to analyze. We plan for the second day to focus on specific ways to *talk* about mathematics, *teach* mathematics, and *engage in research* that are compatible with relational frameworks and the associated critique of representationalism. We will also engage with the ethical implications that are inherent to defining what others are or know (Warren et al., 2020). We will discuss developing practices of continually re-turning to turn over new understanding (Hamilton & Neimanis, 2018; Murris & Bozalek, 2019). We will summarize our conversations and points of confusion for the working group report.

Day 3. In our final session, our focus will be on daily practices of justice that are appropriate and feasible when working in the reality of our current time and places. Our intention and hope is for the working groups' discussion of day to day practices of justice to meaningfully complement the work of Rochelle Gutiérrez (2017, 2019) describing the concept of Mathematx. One aspect of our work is to find appropriate, accessible, and non-culturally-appropriating ways of being mathematical and relational that honor the intention and spirit of the Indigenous aesthetics that have inspired her formulation of Mathematx. We will be searching for subtle-yet-impactful ways of respecting students' many complicated disciplinary relationships, while also gently moving and moving-with student Mathematical thinking with care and attunement. We will document the highlights of our discussion for our report.

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NEW WORKING GROUP: DESIGNING TOOLS FOR TEACHER ENACTMENT OF EQUITY-ORIENTED MATH INSTRUCTION

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This new working group aims to formalize a collaborative inquiry community of math teacher educators exploring how practical tools can support teachers' enactment of equity-oriented math instruction in preservice and in-service settings. Drawing on Culturally Responsive Math Teaching Frameworks, participants share and analyze practical tools they have created or are developing, identify considerations for tool development and use, and share research approaches they have used to study tool development, use, and impact.

Keywords: Culture and Asset-Based Pedagogies; Equity, Inclusion, and Diversity; Instructional Activities and Practices

Working Group Goals and Strategies to Reach the Goals

The goals of this new working group are to: (1) create a community of math teacher educators (MTEs) who are interested or engaged in designing tools that support pre-service and/or in-service teachers enacting equity-oriented math instruction; (2) share participant-created tools; (3) identify considerations for tool development and use; and (4) discuss research approaches to study tool development, use, and impact. The working group organizers have each engaged in grant-funded work that includes designing tools to support teachers' enactment of equity-oriented math instruction. A critical part of that work has included gathering with MTEs from across the country who are engaged in similar efforts to discuss design principles, implementation efforts, and knowledge generation around tools with an explicit equity focus. A recent half-day workshop involving 35 MTEs surfaced a shared interest in creating opportunities for continued collaboration, leading to the vision for this working group.

To meet the goals outlined above, this working group will provide structured opportunities for participants to analyze tools using Zavala & Aguirre's (2024) Culturally Responsive Math Teaching frameworks and any other equity frameworks shared by participants, share examples of considerations and dilemmas that arise as teachers use tools, and explore research approaches that have been used to study tool development, use and impact. To support sustained collaboration, participants will also discuss opportunities for a special issue focused on the design and implementation for tools that support equity-oriented math teaching.

Theoretical Perspectives

While math educators and researchers have called for equitable and inclusive education for decades (e.g. AMTE, 2017; NCTM, 1989, 2000), the field continues to document inequitable and dehumanizing experiences and outcomes for historically excluded students, especially for those of Black, Latinx, and Indigenous descent (e.g., Berry III et al., 2014; Gutiérrez, 2018; Martin, 2019; NCSM & TODOS Math for All, 2016). In response to the critical need for

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teaching that centers and leverages the contributions of students from marginalized groups (e.g., racial, socio-economic, gender, and linguistic), this working group gathers together and provides examples of tools developed by MTEs to support teachers enacting equity-oriented instruction and problematizes tool use in teacher education programs and professional development settings.

This working group focuses on "practical tools" (Grossman, Smagorinsky, & Valencia, 1999) that support teachers in developing knowledge-*in* and knowledge-*of* practice (Cochran-Smith & Lytle, 1999). Practical tools are "classroom practices, strategies, and resources" that have "local and immediate utility" for teachers (Grossman, Smagorinsky, & Valencia, 1999, p. 14). Such tools may also include "social routines" that "foster productive conversations about teaching and learning, build a shared language of practice and support principled risk-taking" (Thompson, et al., 2015, p. 1), and can prompt teachers to learn about (new) instructional practices and while navigating tensions related to these practices in their own classrooms. This working group will explore how these practical tools make teachers' equity-oriented dilemmas visible and support teachers in making instructional decisions that advance more equitable and just math learning experiences and outcomes for students.

This working group uses Zavala & Aguirre's (2024) Culturally Responsive Math Teaching (CRMT) Framework as a starting point to describe equity-oriented instruction. The framework includes three strands of instructional practices: (1) centering students' knowledges and identities; (2) sustaining high rigor with high support, and (3) disrupting power structures and advancing equitable participation. The working group will initially use this framework to create a shared understanding of equity-oriented math instructional practices and use it to critically analyze the opportunities and limitations in the tools participants have developed.

Working Group Organization and Participant Engagement

Session 1: Sharing participants' tools. Session activities: 1) introductions and session overview (5 min); 2) participants discuss in groups organizer- and participant-created tools, including (a) the strand(s) of the CRMT framework the tools intend to have teachers enact, (b) how the tools help teachers enact these instructional practices, and (c) the knowledge and skills teachers need to use the tools (45 mins); 3) whole group reflection of the tools shared and group responses to the guiding questions (25 min); 4) summary and preview of Session 2 (15 min).

Session 2: Tool development and use. Session activities: 1) overview of sessions 1 and 2 (5 min); 2) participants discuss in groups organizer- and participant-provided cases of tool development and use, including (a) how the tools were developed (including equity-related frameworks that informed design), (b) a comparison of intended versus enacted tool use (given the goals for teachers' enactment of equity-oriented instruction), and (c) equity-oriented dilemmas that surfaced during tool (re)development and use (40 min); 3) whole-group reflection identifying common themes and questions from small-group discussion of guiding questions (30 min); 4) summary and preview of Session 3 (15 min).

Session 3: Studying tool development, use and impact. Session activities: 1) overview of sessions 1-3 (10 min); 2) participants share and discuss in groups their research methods to study the development, use and impact of their tools, including: (a) research methods and frameworks used for data collection and analysis, and (b) what was learned from their study of their tool development, use and impact (in particular related to equity-oriented math instruction) (30 mins); 3) whole group reflection of the research methods shared and group responses to the guiding questions (30 mins); and 4) identify next steps for this working group (e.g. journal special issue, future working group session, reconvene at 2027 PME-NA annual meeting) (20 mins).

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UNDER-RESEARCHED INSTRUCTIONAL ACTIVITIES AND PRACTICES IN PK-12 MATHEMATICS EDUCATION: ILLUMINATING THE BLACK HOLES

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The focus of this working group is establishing and carrying out research agendas on instructional activities and practices in PK-12 mathematics education that have attained critical gravity in the teaching field but do not yet have a significant research base establishing their efficacy. We use the metaphor of Black Holes of research (Matney et al., 2020) to capture the gravitational pull of these activities, the obscurity surrounding whether and how the activities are efficacious, and the related research that has accrued around these practices and suggests their value. This PME-NA will be our group's third official meeting. Current subgroups focus on the instructional phenomena of Number Talks, Which One Doesn't Belong? and adaptive teaching. We invite new members to learn from and contribute to our ever-evolving approaches with the option of connecting with existing subgroups or initiating new collaborations.

Keywords: Instructional Activities and Practices, Research Methods

This working group formed around the common interest in researching instructional activities and practices in PK-12 mathematics education that are widely used and have significant anecdotal support. While research related to aspects of these practices also suggests they are valuable, the practices themselves have not yet been systematically and thoroughly researched. The Black Hole metaphor (Matney et al., 2020) frames our conversations as we consider the need to shed light on how these activities and practices are enacted and how and to what extent they are efficacious.

In this third meeting of our working group, we seek to further define our research agendas, coordinate specific research plans, and integrate new members according to their own research interests. Our goals for this meeting are:

- Welcome new participants, introduce the metaphor of Black Holes of research, and share recent individual and collective work towards illumination.
- Discuss experiences in seeking funding and approaches towards common challenges.
- Within and across focused subgroups, refine definitions of relevant constructs, including efficacy, and continue to develop appropriate methods for capturing those constructs.
- Develop clear timelines for accomplishing research agendas, including applying for grant funding and identifying PK-12 contexts in which to conduct research.

Session Plans

On Day 1 of the working group, we will introduce ourselves and welcome new members. We will orient the group to the metaphor of Black Holes of research surrounding instructional practices and activities and share examples of in-progress work to illuminate specific Black Holes. Each previously formed research subgroup will provide updates on the work they have engaged in over the past year. New members will have the opportunity to meet each subgroup to determine whether they would like to join or begin a parallel effort. Day 2 will be focused time for subgroups to meet and make progress according to the current status of their research efforts. Time will be set aside at the end of the session to reconvene and connect as a whole group. On Day 3, subgroups will develop specific timelines for the next steps of their research agendas.

The remainder of the proposal describes the current subgroups, allowing interested participants to determine whether there is an existing project they would like to learn more about. New members are also welcome to initiate new collaborations around other Black Holes.

Descriptions of the Focused Subgroups

The Number Talks (Humphreys & Parker, 2015; Parrish, 2011) group is exploring the possibility of using Artificial Intelligence (AI) to assist educators in designing and implementing Number Talks for upper elementary students focused on the conceptual foundation of fraction understanding. The group is particularly interested in whether differences exist among AI, mathematics education researchers, and educators in predicting and interpreting the strategies students employ during Number Talks. Findings will then be leveraged to explore potential ways to support educators in constructing Number Talks more effectively. The group will use working group sessions to write a proposal for funding from the National Science Foundation.

The Which One Doesn't Belong (WODB) (Danielson, 2016) group is collecting examples of WODB activities found in curricula as well as those that are teacher-created. The group will be developing a coding scheme to characterize features of the WODB activities which are consistently present and which vary across grade level, content, and source. Plans are to identify the core features of WODB activities as designed and develop a proposal to examine variations within teachers' enactment. We will examine relationships between these variations and a number of student outcomes.

The Adaptive Teaching subgroup is interested in how teachers plan for and enact instructional routines. Drawing from the work of Allen and colleagues (2013), we are interested in examining how teachers' enactment of these routines may deviate from provided curriculum materials or written lesson plans, demonstrating how teachers use their professional judgment to meet the needs of their students in the midst of instruction. Using this definition as a foundation, we are working to conceptualize what efficacious enactment looks like in context of the current instructional routines. In this capacity, we will discuss research methods capable of capturing these constructs integrated with the Number Talks and WODB subgroups.

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Description for conference program: Our working group strives to plan and carry out research on under-researched instructional activities and practices in PK-12 mathematics education (also known as practices surrounded by "Black Holes"). In this third meeting of our working group, we will share progress of research projects in development and continue to collaborate to move these projects forward. Current projects plan to investigate Numbers Talks, Which One Doesn't Belong? and Adaptive Teaching. New participants are welcome to connect with existing groups or initiate new collaborations around activities or practices of interest. Please see our website at blackholesofmathematics.wordpress.com/.

PME-NA 47 REPORT: UNDER-RESEARCHED INSTRUCTIONAL ACTIVITIES AND PRACTICES IN PK-12 MATHEMATICS EDUCATION: ILLUMINATING THE BLACK HOLES

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In our working group's second meeting, we welcomed several new members and made significant progress towards defining specific research foci. First, we provided an overview of the Black Holes of research metaphor (Matney et al., 2020) and discussed examples. The metaphor describes instructional practices in PK-12 mathematics education that have attained critical gravity in the teaching field but whose efficacy has not been thoroughly and systematically researched. Then, we formed flexible subgroups interested in researching specific instructional practices or activities in PK-12 settings: Number Talks, open routines, and adaptive teaching. Within and across our subgroups, we explored the construct of efficacy and the multitude of ways it can be defined and measured. Each subgroup set research goals and has gathered virtually since the conference to prepare for conversations with grant program officers.

Keywords: Instructional Activities and Practices, Research Methods

Our working group is excited to report considerable progress made at PME-NA 47. Building on our work at and since PME-NA 46, we worked toward the following goals:

- Welcome new participants, introduce the metaphor of Black Holes of research, and share examples that build on this working group's productivity from PME-NA 2024.
- Identify challenges to gaining access to and researching in PreK-12 settings and brainstorm strategies for addressing them.
- Determine the theoretical and conceptual layers of the under-researched instructional activities and practices, identifying commonalities, priorities for research, and appropriate research methods.
- Consider variations in teachers' interpretations and implementation of the activities and practices.
- Produce a one-page summary as a tool to begin conversations with a grant program officer as we look towards developing a grant proposal.

As described in the day-by-day report below, as we addressed each goal, we shifted flexibly between whole-group and small-group discussions. The small groups formed organically based

on instructional practices and activities of interest. Foci and members of the small groups shifted throughout the three days as a result of conversations within and across groups. By the conclusion of PME-NA 47, three defined subgroups had formed, each focused on one of the following activities or practices: Number Talks (Humphreys & Parker, 2015; Parrish, 2011), Which One Doesn't Belong? (Danielson, 2016), and Adaptive Teaching (e.g., Allen et al., 2013; Gallagher et al., 2022). These three subgroups have continued to meet separately since PME-NA 47 to further define relevant constructs, conduct preliminary research, and strategize for grant applications. The whole group has also convened at several points to discuss common issues, consider points of intersection with future grant applications, and ensure connectedness across our approaches even while we pursue parallel avenues. We aim to use these opportunities to develop a coherent program of research that each grant and study can address.

Day 1: Orienting to the Metaphor and Forming Initial Groups

The first day of our working group focused on orienting new members to the group and to the metaphor of a Black Hole of research in mathematics education. Following introductions of existing members and new participants, Gabriel Matney provided background on the metaphor and how and why he created it (as first described in Matney et al. [2020]). Gabriel began by posing a question to our group: When, where, and how do research and instructional practice intersect? Gabriel and his colleagues had been doing Number Talks (Humphreys & Parker, 2015; Parrish, 2011) as practitioners of mathematics teaching for many years. When asked by one of their partnering districts to share research articles about Number Talks' efficacy with K-12 students, they believed that it would be easy to produce the evidence. Gabriel told the district he would have the articles to them by next week.

However, this led to a year-long peer-reviewed manuscript search across 17 databases, 576 unique articles, and only one article that provided efficacy evidence about Number Talks with K-12 students. As Gabriel and his colleagues reflected on this, they realized that Number Talks, which are a positive and meaningful instructional practice, were not a "hole" of knowledge. Number Talks has developed a large gravity in the teaching field because of its value to so many practitioners. Furthermore, the practice itself uses many elements of teaching, such as discourse, questioning, and number sense that have been researched, but the instructional practice itself has not. Hence, it was within this research setting that the Black Holes of research metaphor was born. Number Talks had attained a critical gravity in the field, like a Black Hole. Although the efficacy of the practice remained unresearched, there was other research pointing to its value, as the accretion disk of a Black Hole reveals to all that it is there.

Challenged by these circumstances, Gabriel began to work with his colleagues to think of the things in research that have garnered gravitational pull in the mathematics education field, and the Black Holes paper (and metaphor) provided a way to share that same challenge with others. He then shared several questions about Number Talks that remain unclear, and that could be asked of many other instructional practices and educational phenomena, including: What are the variations of Number Talks in the field? What are the purposes of these variations, and in what ways are they beneficial or not to students? What variations offer greater equity? What professional noticings and teacher moves contribute to making Number Talks effective, and how do we define, from research evidence, what makes a Number Talk highly effective? Gabriel concluded the presentation by sharing that Number Talks are not alone among educational and research phenomena. Other Black Holes exist. He shared his enthusiasm to explore with this working group what other Black Holes were out there and to figure out how to illuminate them.

Since the initial publication of Matney et al.'s (2020) literature review, researchers have

begun to shed light on the case of Number Talks. Dawn Woods shared about the work she has been doing as a result of her National Science Foundation (NSF) CAREER grant entitled *Sparking "Number Talks" to Strengthen Mathematical Identities.* This is a five-year participatory research project where we are identifying the funds of knowledge and the funds of identity that students, families, and communities bring to mathematics learning spaces. Then, teachers use the information to design Number Talks that may impact positive mathematical identity development. Early findings are showing that teachers are using Number Talks to highlight family rituals, play situations, and the world around them to give students moments to shine as they share the mathematics that is relevant to them. Further, student self-portraits are showing evidence of productive mathematical dispositions and other identity elements related to productive mathematical identity.

As an example of another Black Hole that has been identified but not yet systematically researched, Sarah Hartman shared about the instructional practice of adaptive teaching. The black hole metaphor aids in conceptualizing the complex nature of adaptive teaching, as it is not well conceptualized, and there are many literature bases that inform our current understanding of the skill of adaptability. Following a composite view of adaptive teachers presented by Randi (2022), Sarah discussed a mathematics education-specific conceptualization of related literature fields. The conceptualization involved a tetrahedral model where each edge represented the supporting competencies adaptive teachers need to be adaptive, faces of adaptive action that are further supported by the edges, and vertices that represent boundary navigation in the classroom. In McMillan and Jess's (2021) discussion of boundary navigation in the classroom setting, they describe three types of boundaries teachers must navigate: Individual boundaries, environmental boundaries, and task-related boundaries. Sarah discussed these related literature bases and explained a conceptualization of the literature informing adaptive teaching. Sarah concluded this session by answering any questions attendees of the working group had about adaptive teaching.

Following these examples of activities and practices to which the metaphor applies, we invited participants to form subgroups according to common research interests. To form these groups, we first brainstormed as a whole group about different areas we were interested in exploring. Resulting from this brainstorm, we decided that there were four distinct groups of interest and expertise in the group of attendees. As such. four groups formed around these initial foci (see Figure 1): Number Talks, open routines, content, and adaptive teaching.

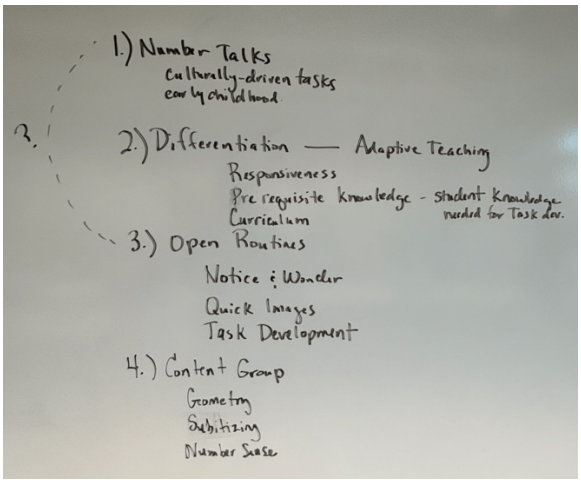


Figure 1: Initial Formations of Small Groups

After the first day, the open routines group merged with the Number Talks group because they only had two members. The Number Talks group began by discussing constructs related to efficacy, specifically how to measure whether and how Number Talks work as an instructional practice. We discussed constructs such as agency and elicitation that could provide insight into how Number Talks work as an instructional routine.

The content-focused group began by discussing the need to understand instructional routines and activities' impact on K-12 students' mathematical thinking, mathematical knowledge, and mathematical problem-solving ability. The group further discussed teachers' (in-service and preservice) development of mathematical pedagogical content knowledge related to the ambitious goal of having mathematically proficient students with prowess in these three domains: mathematical thinking, mathematical knowledge, and mathematical problem-solving ability. Following these discussions, the content subgroup agreed that the broader conversation should be narrowed to begin with research for a particular routine. The routine of main interest was Which One Doesn't Belong? (WODB), an activity popularized by Danielson (2016). In a WODB activity, learners are presented with images of four mathematically interesting items and asked to choose which one does not belong and why. All of the four items can be justified in multiple ways. This conversation spurred the content group's eventual merger with the open routines group to further work on WODB research.

The adaptive teaching group began by discussing different aspects of adaptive teaching we could explore. To get the group started, Sarah compiled a list of adaptive practices discussed in the literature. From this list, the group began working to understand what adaptive teaching is in the moment, what adaptive teaching is during planning, and the specific aspects of such processes. Using the group's varied expertise areas, a variety of ideas emerged, including cognitively demanding tasks, differences in adaptive teaching in elementary, secondary, and higher education levels, as well as differences in adaptive teaching with novice or veteran teachers. Afterwards, the group decided that a more specific aim was necessary to begin investigating adaptive teaching, and the group discussed what data could be collected and how adaptive teaching could be measured within some of the instructional routines that were presented previously (number talks and open routines).

Day 2: Fleshing out the Multi-Faceted Construct of Efficacy

To begin our second day, Gina Kling and her colleague Sarah Larison shared about their current research on the Black Hole surrounding the "How Many?" routine (Danielson, 2018). In the How Many routine, the class is presented with an image of a carefully curated collection of dice, dominoes, or other real-world objects, and is simply asked to describe how many they see (see Figure 2). Because of the open nature of the prompt, the task offers multiple entry points (National Council of Teachers of Mathematics [NCTM], 2014), as students can reasonably respond with the number of colors, the number of dice, or the number of dots, the latter of which offers opportunities for rich conversations of computational strategies. For example, for the first image in Figure 1, some children might begin with counting by 1s, skip counting, or combining the addends in order as they move across the image. But others might utilize helpful features of the structure, and attend to the double facts of 2 + 2 and 3 + 3 first, or even combine 2s and 3s to make 5s, and then 5s to make 10s. As students interact with the image and share their strategies with the class, an opportunity to assimilate more efficient strategies invented by their peers arises (Vygotsky, 1978).

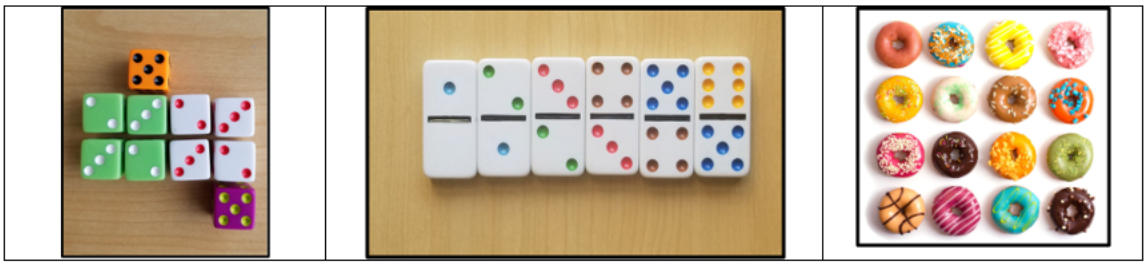


Figure 2: "How Many" Images from https://multiplicitylab.northwestern.edu/

Gina and Sarah shared the results of a recent study in which a three-week, nine-lesson How Many interventions in second and third grade produced statistically significant growth in student fluency as defined by flexibility, accuracy, efficiency, and appropriate strategy use (National Research Council [NRC], 2001). Despite the brevity of the intervention, the results of the study yield both practical implications in terms of a ready-to-use How Many curriculum and scholarly implications in terms of the introduction of a fluency framework that can measure fluency in the context of multiple addends, something lacking in the field. In particular, the latter might help address an outstanding question of the Black Holes Working Group—how might we measure the efficacy of a number sense routine?

Following the sharing of this example, given the focus of our group on systematically researching the *efficacy* of instructional practices and activities in PK-12 settings, it was critical to discuss what efficacy can actually mean. To frame our efforts, Sarah Hartman shared an overview of the family resemblance approach. The Family Resemblance Approach (FRA) (Dagher & Erduran, 2016) is a set-theoretic framework of the nature of science that uses the idea of overlapping similarities to describe how things are related. The goal of using this approach was to acknowledge the features of efficacy across each subgroup that could be used to create a common set of characteristics to study efficacy, which ultimately allowed us to contemplate defining and operationalizing efficacy.

To begin our discussion of efficacy, we returned to our subgroups, prefacing with a brief description of each one so that new members could decide where they might want to connect. Because the content-focused group had pivoted to talk about content with the context of open routines, the original members of the open routines group shifted to join them. This expanded open routines group identified several key properties of efficacy common across these instruction practices. First of all, we felt it was necessary that all students have access to engage meaningfully with the routine. In particular, the visual nature of some of our routines of interest, such as WODB and How Many, likely increases the accessibility of the routine. Secondly, this engagement is collective; that is, it is a social activity where students have opportunities to pick up on each other's thinking. This raised an interesting dimension to measuring efficacy, thinking about the collective growth of the class as a result of engaging in the routine as compared to the individual growth at the student level. We also considered efficacy in terms of facets that are valued within the mathematics education community but not necessarily at a policy level (e.g., agency, access, and ownership) as well as measures that are typically of more concern to administrators and policymakers (straightforward concept and skill assessments).

The number talks subgroup defined efficacy as the extent to which sustained participation in Number Talks supported students' mathematical development and engagement over time. The subgroup outlined a proposed design that followed the same cohort of third- and fourth-grade

students for two years and compared classrooms implementing Number Talks with business-as-usual (BAU) classrooms. The conceptual framing centered on three interconnected constructs: number sense, mathematical identity, and agency/engagement. The subgroup noted that classroom interactions during Number Talks—particularly teacher elicitation practices and student participation—were expected to support power shifts, increase points of engagement, and strengthen students' identities as mathematical thinkers. Efficacy was discussed in terms of both proximal and distal outcomes. Proximal measures focused on in-the-moment features of the Number Talk, including students' engagement, participation, and teacher–student interactions. Distal measures addressed longer-term learning outcomes, such as whole-number and fraction development, estimation, and mathematical identity, using established assessments, identity tasks, and stimulated-recall interviews. Overall, the subgroup framed efficacy as growth in both mathematical understanding and students' agency, engagement, and mathematical identity, highlighting the cognitive and affective outcomes associated with sustained participation in Number Talks.

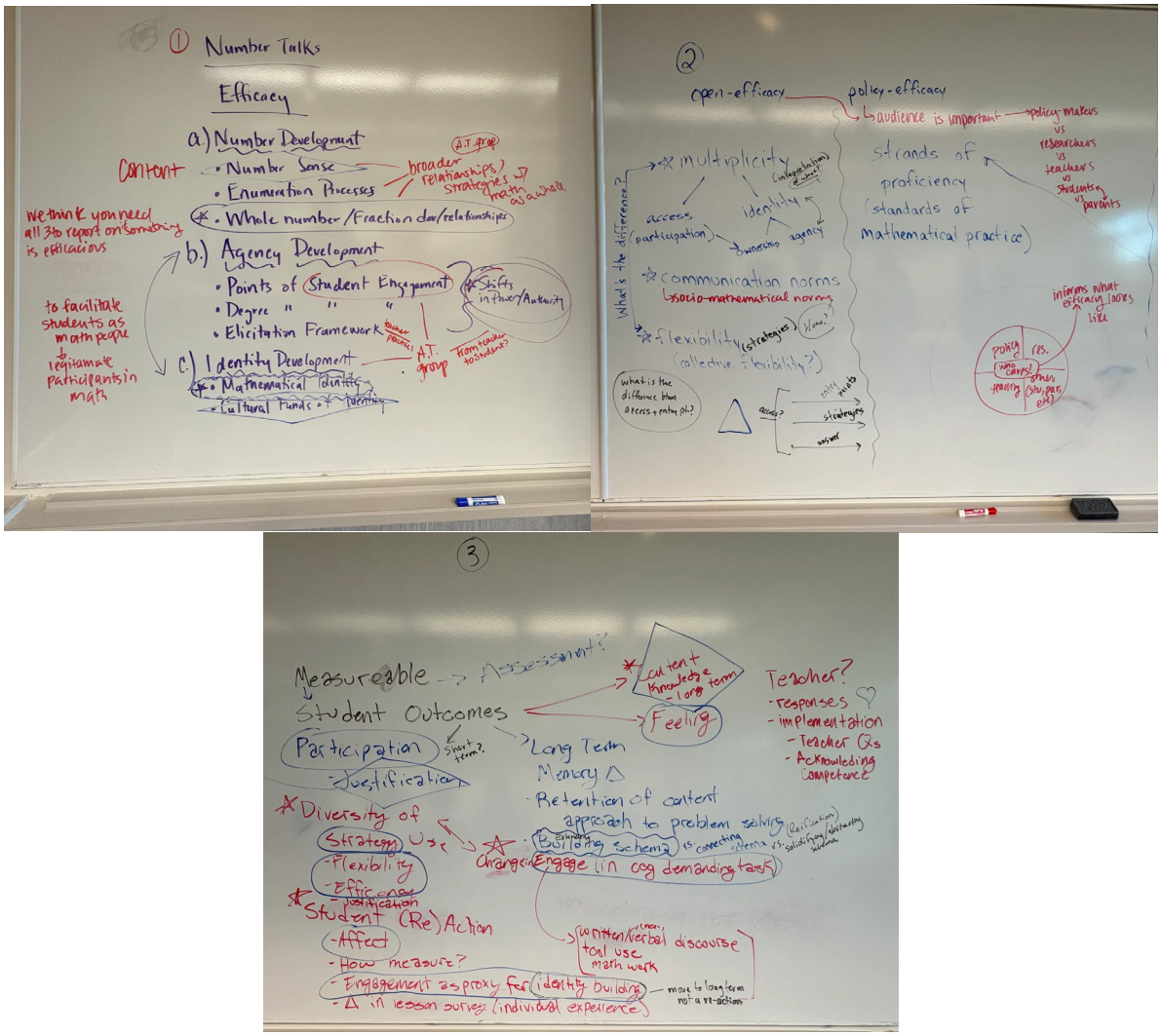


Figure 3: Whiteboard Recording of Efficacy Discussions

In the adaptive teaching subgroup, discussions of efficacy revolved around measurable outcomes, student outcomes in the form of content and affect, and teacher outcomes. For example, student outcomes included participation, justification, long term memory, retention of content, building schema, engagement, approaches to problem solving, diversity of strategy use with flexibility and efficiency, and student action and reaction. Efficacy for teachers and outcomes included attitude changes, responses, and implementation. Overwhelmingly, the adaptive teaching subgroup focused primarily on measurable student outcomes and who the audience is when discussing efficacy. For example, what is important and makes something efficacious for policy-makers is different from researchers, teachers, students, or parents.

Each subgroup recorded highlights of their own on the room's whiteboard, then rotated to read and comment on the other groups' thinking (see Figure 3). Looking across the reflections of the different groups, themes of efficacy in terms of student dispositions and student mathematical competency emerged. We felt that student agency and mathematical identity could be positively impacted by these activities, such as Number Talks, open routines, and through adaptive teaching, particularly as mathematical power and authority in the classroom is shared and students are seen as legitimate participants in mathematics. Pairing with these affective traits are considerations of gains in mathematical competency. Efficacy through this lens may be hallmarked by flexibility and efficiency in thinking, diversity in strategy use, and, in general, behaviors consistent with those described by the Strands of Mathematical Proficiency (NRC, 2001) and the Standards for Mathematical Practice (NGA & CCSSO, 2010).

Day 3: Drafting One-Pagers

For our final meeting, each subgroup met to further refine its focus, set specific goals, and draft one-page summaries to use in seeking funding opportunities. Beth MacDonald provided each group with a template for an NSF one-page narrative. This provided attendees from Days 1 and 2 who could not attend Day 3 an opportunity to contribute to the continued work and contribution towards a one-page narrative of a grant application. The one-page narrative allowed some groups an opportunity to explicate what they wanted to examine, their problem statement, and how the study they want to conduct will broadly impact society and intellectual merit in our field.

On the final day of our meeting, the content-focused/open routines group decided on a focus of one open routine in particular: Which One Doesn't Belong (WODB)? In keeping with the idea that the metaphor of a Black Hole involves a critical gravity that has attracted practitioners to a practice, we noted that WODB's critical gravity can be clearly established through its pervasive presence online (#WODB) and increasing inclusion in published curricula. The activity is also used widely across grades PK-12. Furthermore, WODB has a defined structure that is conducive to identifying instances of it, making it an appealing choice for a first routine to explore. As an initial research question, we decided to seek out examples of WODB as enacted in order to describe how teachers and students are engaging in WODB and begin to identify variations. We aimed to use this first question to build a foundation for researching how variations in implementation impact the efficacy of the routine.

The adaptive teaching subgroup created an outline for a one-page proposal. This began by identifying open questions in the field of adaptive teaching, such as how preservice teachers learn about differentiation and adaptive practices, and how to support their learning in those areas. Another question that was generated is how differentiation and adaptive practices are enacted. Furthermore, the group contemplated working towards a professional development

aimed at the enactment of adaptive teaching and outcomes for students. Finally, some draft research questions were created focusing on how teachers adapt and the impact of adaptations.

All work over the course of the three days was documented on our website (https://blackholesofmathematics.wordpress.com/, see "Group Meetings").

Continued Work

Following PME-NA, each subgroup has met to continue its work. The Number Talks group met four times since the PME-NA meeting. In this time, we have explored several funding agencies, determined the breadth of the work on Number Talks in the mathematics education field, and brought on people with expertise to discuss the means of using particular tools and exploring phenomena tangential to Number Talk routines. From this, we settled on the NSF STEM K-12 funding solicitation. The remainder of this spring and summer semester, the group plans to use this information to begin to form a comprehensive problem statement, research focus, methods, a broader impact statement, and a statement delineating the intellectual merit we think this study will contribute to. Once we have a one-page narrative developed, we plan to meet with program officers (i.e., Eric Knuth, Margret Hjalmarson). This will provide us enough information for our subsequent working group meeting at PME-NA 48 to fully develop aspects of a grant narrative for this solicitation. By developing this narrative at the working group, our subgroup will better understand how to clearly articulate the program of research we plan to develop in relation to the proposed study.

The WODB subgroup–which formed from the original open routines and content-focused subgroups–is working on surveying existing WODB tasks in curricula and teacher-created materials to identify variations in task design and suggestions for implementation. We also met with author Christopher Danielson (who formalized and popularized WODB) to gain his perspective on WODB task design and how we can describe and assess the efficacious outcomes that result from interaction with and discussion of a WODB task. Two interesting ways of thinking about the efficacy of WODB were defining learning as "having new questions to ask" and whether a task prompts the conversation the designer wants it to prompt. As we work to organize the features and variations of WODB as designed, our next steps are to identify classrooms in which to conduct pilot studies to understand how these variations play out in enactment (and what additional variations arise in enactment). With these findings in hand, we plan to seek grant support to support a more comprehensive study.

The adaptive teaching subgroup is currently focused on investigating where and how adaptive teaching occurs when enacting a number talk or an open task, such as WODB. The adaptive teaching group is focused on a different area of these open routines, adaptive enactment. Adaptive enactment is a central term we use to describe decision-making across various practices. Rather than focusing explicitly on what the efficacy of a practice is, we are working to understand what it means to enact a routine efficaciously. This complements other subgroups' work by focusing on different aspects of the Number Talk routine and the WODB routine. Next steps for the group involve collaborating with other groups to begin conceptualizing data collection to accomplish this goal and reading relevant literature on this subject.

The working group as a whole continues to be in contact virtually and in person as we plan future working group sessions, share progress, and consider ways to connect our work. We are proudly a group that is productive, inclusive, and joyful–three adjectives that, perhaps unsurprisingly, also apply to the practices and activities on which our research seeks to shed light.

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A NEW WORKING GROUP: SUPPORTING TEACHER EDUCATORS WITH IMPLEMENTING, RESEARCHING, AND COACHING BUILDING THINKING CLASSROOMS

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This working group seeks to produce innovative ways to support teachers' implementation of 'Building Thinking Classrooms' at the university and K-12 levels. Communities of learners will be created for the following three main groups: Research, Coaching, and Implementation.

Keywords: Professional Development, Preservice Teacher Education, Classroom Discourse, Problem Solving

Introduction

Since it was first introduced to the mathematics education community, *Building Thinking Classrooms* (BTC) (Liljedahl, 2021) continues to grow in relevance among teachers and teacher educators across North America. We see this growth in regional conferences, professional learning programs, and learning communities across social media platforms. The relevance of this initiative has grown so much that the first annual International Building Thinking Classrooms Conference was held in Indianapolis in 2023. While BTC is a research-based initiative, the resources for teachers are mainly limited to the book and online forums for sharing ideas. For teachers to successfully implement, critique, and supplement these practices, guidance is needed to support their work. Our goal as mathematics teachers and mathematics teacher educators is to study how the practices and their implementation are supporting effective teaching and learning. Our aim is to support teachers to draw on the practices, along with other research-based initiatives that align with increasing student thinking in the classroom, as they make professional decisions in their classroom.

The authors of this working group have engaged in various initiatives centered around BTC. One author has created an Instructional Circle (Melville, 2022) to provide personalized and deprivatized feedback to teachers as they implement BTC. There are three stages to implementation of BTC that become more difficult to implement in a classroom (Liljedahl, 2016). The instructional circle is a form of professional learning where teachers can find support from experienced teachers and knowledgeable others about ways they can improve their instructional practices. Another author developed an initiative with a local school district to provide instruction and materials for teachers wanting to implement BTC but lacking the support or materials to be successful. Lastly, one author's team is engaging in an initiative looking at specific aspects of BTC including task writing and "thin slicing" and how teachers are utilizing them in their implementation of BTC (Frazee et al., 2023).

As mathematics teachers and mathematics teacher educators, we feel it is worthwhile to support the implementation of BTCs in a variety of contexts. This instructional practice supports ambitious teaching as defined in *Principles to Actions: Ensuring Mathematical Success for All* (National Council of Teachers of Mathematics [NCTM], 2014). The creators of BTC also understand professional learning principles that state incremental changes to teacher practices are

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more sustainable long-term than sudden transformational professional developments (Otten et al., 2022).

Goals of the Working Group

The goals of this working group are two-fold. First, we want to serve as a forum for authors and participants to share a variety of research projects centered around *Building Thinking Classrooms* (Liljedahl, 2021). This will inform the conversations about how to create a more effective implementation and support structure for teachers using this instructional practice in both K-12 and university levels. Second, instructional circles (Melville, 2022) will be created for the participants of the working group to form professional learning communities for support around their specific interest with BTC beyond this conference. Specifically, these contexts may include using BTC in math content courses, math methods courses, in-service professional learning, and ways in which research projects can be designed to study various aspects of these settings. We believe that these instructional circles will be a valuable resource as we implement and study the effects of BTC on student learning and teacher education and the growth of mathematics teacher educators.

We have had a previous iteration of this working group that has since been dissolved. However, with the interest and outreach from previous attendees, we feel that there are sufficient need and interest for us to hold this working group again. Changes that we will make are a monthly newsletter and meeting for attendees who are interested and need further support. Then, in future iterations of the working group, we will be able to integrate new members and allow space to delve deeper into different problems of practice and research instead of remaining at a surface level.

Organization and Presentation Plan

The first session will start with presentations from the authors about what they are currently doing regarding coaching, implementing, and teaching BTC in methods courses and professional learning contexts. The authors alone have a variety of initiatives around BTC. One example is that BTC is being studied as a vehicle for task assessment and professional learning of teachers around planning (Frazee et al., 2023). Another example is through an Instructional Circle (Melville, 2022) the first author gave personalized feedback to teachers as they implemented BTC. The last initiative studies how BTC is being used in a more traditional professional learning course for teachers who are currently implementing or would like to implement these instructional strategies.

The second session will include reports from non-author participants about their current initiatives that utilize BTC or that could benefit from using BTC. These participants will volunteer during the first session for this activity. The second session will end with forming break out groups according to participants' interest in utilizing BTC (i.e., math content classrooms, math methods classrooms, professional learning settings, or researching the effects of BTC in these other contexts).

The third session will continue with the breakout groups for the different contexts that participants are interested in utilizing BTC. During these breakout sessions, we plan to help organize the participants into instructional circles to provide support and fresh ideas to implement or study BTC in different settings. This session will also include each group sharing their current ideas and goals with the whole group.

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TEACHING AND LEARNING WITH DATA INVESTIGATION ACROSS DISCIPLINARY CONTEXTS AND BOUNDARIES

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In our eighth year as a working group, we will continue to discuss current recommendations and research for teaching and learning with data investigations with a focus on cross-disciplinary contexts, the role of artificial intelligence, and political implications in the current state of STEM education research. We will also foster dialogue around ongoing and future possible collaborations around researching the teaching and learning of data investigations. We invite returning and new members to join us in collaboratively researching how to best support meaningful data investigations across disciplinary contexts and boundaries.

Keywords: Data Analysis and Statistics, Integrated STEM / STEAM

Background

Our working group's collaboration on examining teaching and learning statistics and data science concepts and practices through data investigations has led us to key emergent questions over the years. For example, as we considered the interdisciplinary nature of data investigation, we encountered the issue of disentangling the notion of content and context. What we consider as content may not be content from other disciplinary points of view. More recently, the emergence of artificial intelligence and machine learning (AI/ML) is reshaping what data investigation means, moving beyond traditional statistical analysis. This change also prompted us to rethink teaching and learning with data investigation with emergent data investigation tools and methods. In addition, over the past year, members of the working group became keenly aware of the changing landscape of research on teaching and learning, driven by changes in policies and the political climate, which drives our considerations for meeting in 2026.

Theoretical Background

While considering the current issues shaping our work, the working group continues to focus on integrating existing recommendations for teaching statistics and data science in K-12 education (Bargagliotti et al., 2020; NCTM, NSTA, ASA, NCSS, & CSTA, 2024). Most real-world data investigations draw from concepts, knowledge, and practices beyond mathematics and statistics. Consequently, this working group seeks to connect projects spanning diverse disciplines and frameworks for data investigations (Lee et al., 2022; Wong et al., 2025). Notably, we include socioscientific (Zeidler et al., 2019; Smucker et al., 2024), sociopolitical (Miller et

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

al., 2024; Casey et al., 2022), and data science and AI/ML (Data Science 4 Everyone, 2025; Sanusi et al., 2023; Tseng et al., 2021; Touretzky et al., 2025) contexts for data investigation.

Organization of Working Group Activities

Over the three sessions at PME-NA, we plan to engage with participants through discussions, brief demos, and activities around the above mentioned three key emergent themes detailed below (Figure 1).

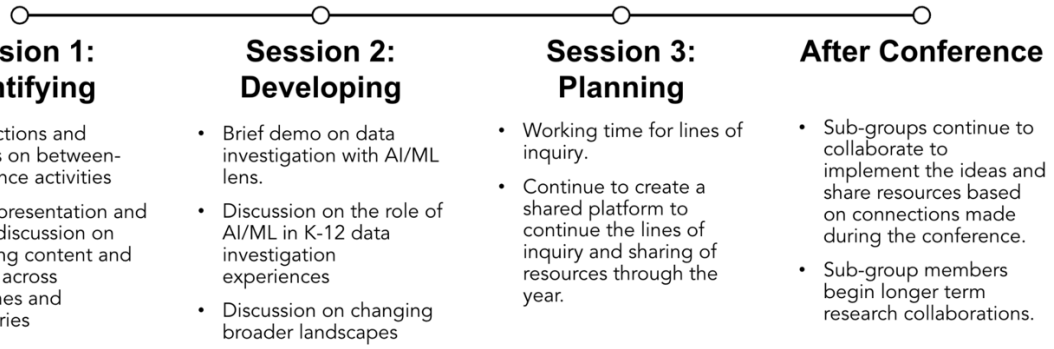


Figure 1: Outline of Activities for PME-NA Working Group Sessions

1. Theorizing Content and Context Across Disciplines and Boundaries

One of the key features of teaching and learning with data investigation is the consideration of context. As the working group delved into conceptualizing data investigation, we repeatedly encountered conceptual ambiguity between the methods and approaches of data investigation (i.e., content) and the issues or topics the data investigation aims to illuminate (i.e., context). From the perspective of another discipline (e.g., social studies education), the context of data investigation is the content that defines its discipline's learning. We will further disentangle this conceptual ambiguity by considering available frameworks in multidisciplinary settings.

2. The Role of AI/ML in Data Investigation

We position AI/ML as a lens for examining data practices in K-12 mathematics and statistics classrooms and in teacher education. To spark this discussion, working group participants will engage in a brief demo on training and testing simple ML models. We introduce web-based ML models that do not require students to write code but instead foreground data-handling practices and core ML concepts. We will also explore how ML-based investigations could reshape core data practices and instructional approaches in K-12 settings.

3. Statistics and Data Science Education in a Broader Landscape

Our work is constantly shaped by broader policy shifts and technological developments. Inconsistent policy regarding K-12 data science curricula and the rise of generative AI have raised concerns about ethical use, misinformation, and learning impacts. At the same time, our research activities are impacted by grant terminations, limits to federal data access, and increasing political tensions. We aim to exchange our perspectives on these pressing issues.

As momentum continues to grow, the group actively welcomes new members interested in advancing teaching and learning with data investigations across disciplinary contexts and boundaries. The working group activities are organized so that new participants can get an overview of the group's history and prior work before participating in discussions on the current focus around data investigations.

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TEACHING AND LEARNING WITH DATA INVESTIGATION: WORKING GROUP REPORT FROM 2025

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The Teaching and Learning with Data Investigation Working Group continues to be a source of community and production. With many projects finished, near completion, and underway, this group is very active. Herein we summarize the background of the group, activities at the 2025 PME-NA Conference, and current and future activities.

Keywords: Data Analysis and Statistics, Integrated STEM / STEAM

Background

This working group, which first convened in 2019 at PME-NA, has steadily sharpened its focus and expanded membership each year. Initially, the group explored a newly published framework for meaningful statistical learning environments (Ben-Zvi et al., 2018), sparking discussions on different design aspects. In 2020, the focus shifted to statistics education across boundaries, bringing together scholars from the US and Mexico with bilingual sessions in Spanish and English. By 2021, the group began examining the role of data science education in statistics and mathematics curricula. In 2022, attention turned to the growing movement to integrate data science into K-12 education, emphasizing data investigations as a means to address broader data literacy needs rather than treating statistics and data science as separate fields (Education Development Center, 2015; Lee et al., 2022). Additionally, work was done by a subgroup to create a publication for the practitioner journal *Mathematics Teacher: Learning and Teaching PK-12* to consider how to translate the 5 practices for orchestrating productive discourse (Smith & Stein, 2018) to teaching statistics in the mathematics classroom (Smucker et al., 2024). In 2023, the group identified investigations of socio-scientific and socio-political data as productive vehicles for inter- and transdisciplinary learning. In 2024, subsets of our working group continued to meet throughout the year to work on a conference grant proposal, a book chapter for the *Mexican Society for Research in Mathematics Educatio*n (Weiland et al., in press), a Statistical 5 Practices lesson plan and activity for *NCTM Illuminations*, various *ZDM* journal article submissions for a Special Issue one of our members is helping to edit, and to plan for the 2025 PME-NA conference. In 2025, we welcomed several new members to this working group at Penn State University and continued ongoing work from the previous year.

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

Overview of the Working Group Activities

At the 2025 PME-NA conference, we had three main goals for the working group: (1) to continue to explore how the context of data (its collection, organization, and social relevance) intersects with the context of the students, their experiences, curricular goals, local settings, and disciplinary perspectives in mathematics, social studies, and sciences, (2) examine the affordances and challenges of transdisciplinary data investigations across instructional contexts, and (3) investigate the role of artificial intelligence in data science education, particularly in fostering students' critical evaluation of AI-generated artifacts. Based on these goals we planned interactive and engaging activities across the three days of the conference. For each session, we had around 20 people participate each day. Summaries of the activities are described by day, in the sections that follow.

Day 1

The first day began with a data talk led by Franziska Peterson, where participants investigated information from the Pennsylvania *Climate Impacts Assessment.* This task provided an opportunity to quickly engage in conversation regarding our Working Group theme of Transdisciplinary Data Investigations connected to the context of the PME-NA 2025 conference location. After the data talk and group introductions, in person participants gathered around poster papers and online participants joined a Zoom breakout room to discuss questions our group felt were salient to the state of the field of statistics and data science education:

- What contextual examples from your experience and/or work provide opportunities for transdisciplinary data investigations?
- What are the challenges to enacting or providing transdisciplinary data investigation experiences?
- What role can AI play in transdisciplinary data investigation experiences? Have you used AI as a tool for working with data? If so, how?

Participants circulated around these questions and shared their experience with these topics and which they were interested in delving into more deeply (see Figure 1 for poster examples). Regarding the first question, participants highlighted several real-world contexts that offer strong opportunities for transdisciplinary data investigations. These included traffic stop data, school records, and school funding data, with two members emphasizing the depth and complexity required to meaningfully analyze these data. Others shared experiences with place-based learning focused on climate change, particularly examining environmental factors and equity issues within local communities, as well as youth participatory action research in which students engaged in graphical interpretation of data to investigate and act on issues relevant to their communities. Across examples, the common theme was using authentic, socially relevant data to connect multiple disciplines and uncover deeper systemic patterns.

The discussion around the second question revealed several key challenges to enacting transdisciplinary data investigations. Limited access to technology restricts the ability to work with large, real-world datasets, and many educators lack familiarity with data visualization tools. Addressing controversial topics—such as race and racism—can also be challenging. Significant planning time is required to design and implement meaningful data investigations, raising questions about how to better support teachers. Gaps in contextual knowledge can shape how data is interpreted, prompting the need for strategies to help teachers build relevant background knowledge. Finally, the siloed structure of STEM education presents systemic barriers, as schools often lack structures that support collaboration across disciplines.

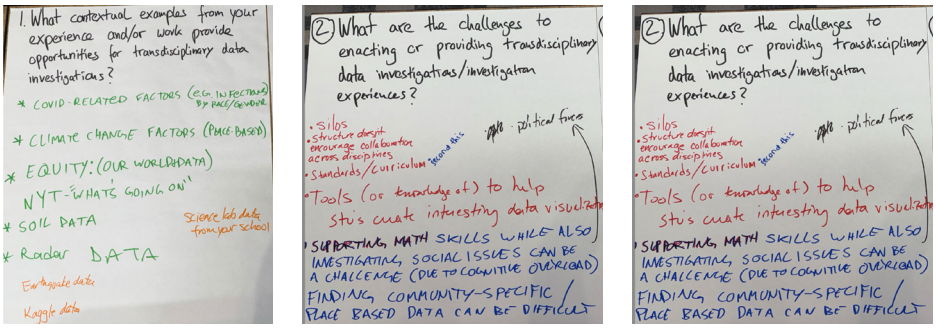


Figure 1: Day 1 Discussion Group Poster Examples

Ideas shared around the third question included using AI to clean and analyze datasets, generate Python code for statistical investigations, and locate useful datasets for instructional purposes. AI can streamline technical processes and expand access to more complex analyses, including practices already common in university settings. However, effective use requires significant background knowledge, raising questions about how to prepare both teachers and students to use AI thoughtfully and effectively. Ethical concerns around AI-generated analysis also remain important. Participants emphasized the need for lessons and resources that help students understand how AI works, ensuring it is used critically and responsibly rather than as a black box tool.

This activity verified that all participants were interested in thinking together about how to support students and teachers in these aspects of transdisciplinary data investigations, which we would continue to explore in a variety of ways throughout the conference working group sessions. Day 1 ended with confirmation that there was much interest in more deeply examining the affordances of two technologies (CODAP and AI) the next day, along with an invitation to share other research interests associated with data investigation.

Day 2

Travis Weiland began Day 2 with an introduction to data from the *American Community Survey* which could efficiently be examined via the Census Microdata Plug-in in *CODAP* (see Figure 2 below*).* We were asked while exploring to ask ourselves, "What is a question that would be interesting to investigate with this data?"

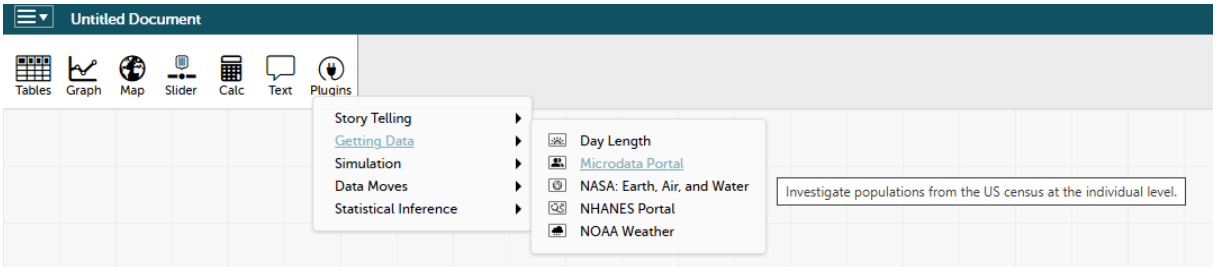


Figure 2: CODAP Microdata Portal

In addition to CODAP, the working group devoted considerable time to examining the affordances of other available technologies (e.g., Google Teachable Machine) and data source for teaching, learning, and research. Karie Smucker and Francisco Sepúlveda next shared an

overview of the forthcoming *NCTM Illuminations* lesson plan and activity, which utilizes this data and technology, which they have been developing with other working group members. Again, in order to facilitate reflection and conversation, participants shared some of their thoughts and reflections regarding these tasks via butcher paper (see Figure 3 below).

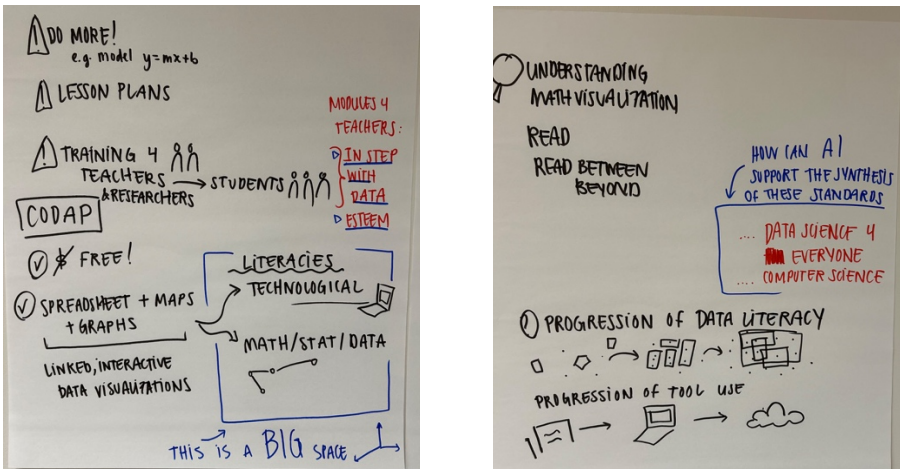


Figure 3: Day 2 CODAP Microportal Reflections

Figure 3 posters were generated by working group newcomers to visually represent their initial ideas about how CODAP microportals could be integrated into their current projects, curricula, and research contexts.

Finally, Day 2 ended with Sunghwan Byun leading the group through ways that he and his colleagues are utilizing AI prompt templates to support students and teachers in deepening their understanding of how to investigate data with R technology (see Figure 4 below).

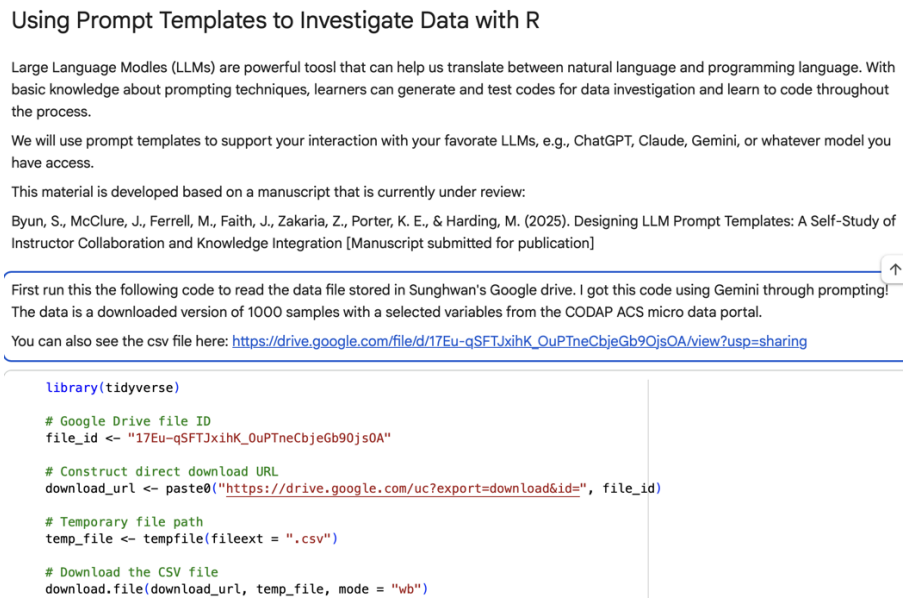


Figure 4: Presentation by Sunghwan Byun Regarding Using AI and R for Data Investigation

Day 3

Day 3 began with Anthony Fernandes demonstrating instructional resources and research findings associated with the *Statistical Investigations of Systemic Racism* project (see Figure 5 below). This led to rich discussion regarding strategies to support students and teachers in exploring data and discussions around controversial topics like policing of traffic stops in the U.S. The goal of the project is to prepare future teachers to develop their statistical understanding and awareness of how racism acts as a system.

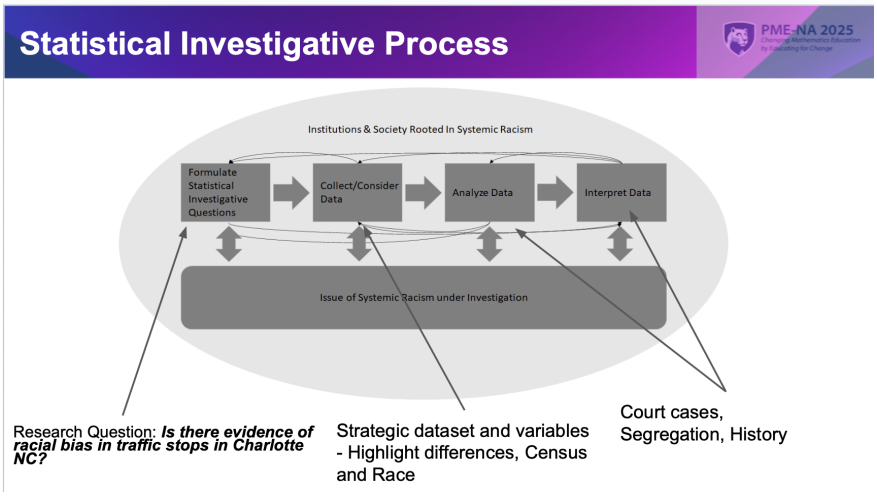


Figure 5: Screenshot from Statistical Investigations of Systemic Racism Project

Next, Mathew Beckman, from the PME-NA host institution highlighted research he is involved in associated with utilizing AI in support of teaching large (statistics) classes and the CAUSE community supported data science education researchers. Day 3 concluded with reflection on the many topics examined across the three days and discussion of collaborative 'next steps'. These reflective discussions set the stage for the post-conference working group activities outlined below.

Post-Conference Working Group Activities

After the 2025 PME-NA Conference, the Teaching and Learning with Data Investigation Working Group has continued to welcome new members and meet to coordinate research groups. In order to support this work, the working group leaders synthesized the ideas generated on day three of the conference into three key lines of inquiry, which are listed below:

1. Exploration of the conception and interface of context and content
2. Investigating how students make sense of non-traditional data visualizations
3. Investigating teaching and learning with CODAP and its design features

Each of these subgroups has met since the conference and their progress moving each theme forward is summarized below:

1) Context and Content

Eleven members met to discuss their conceptions around context and content. Given that we were approaching this topic from different experiences, we spent some time discussing our

thinking about content and context. For example, context involved the context of our work—with students, preservice teachers, and teachers, and also the context of the statistical investigations—climate and race. Some time was spent discussing how these contexts intersected with the content that was needed at a particular level, along with the use of the appropriate technology. We also discussed how defining context and content can be challenging when pursuing interdisciplinary investigations; for example, what is considered context from a statistical perspective may be the focus content for a science or social science teacher. In closing, we agreed to continue to meet to refine our definitions and think about possible common areas for collaboration, in particular the potential for working on a theoretical publication which considers these issues and began generating literature of related publications to share and discuss.

2) Non-traditional Data Visualizations

Seven interested members have met twice to discuss research interests and goals around the topic of the teaching and learning of data visualizations. The group has decided to start by conducting a systematic review of existing literature in this area in order to become familiar with the literature and to create a manuscript to report on the results of the review. Members have begun to identify other relevant reviews, possible research questions and possible boundaries for the search and are exploring possible keywords, databases and journals to review.

3) Teaching and Learning with CODAP

Eight members met and discussed possible common interests around the idea of investigating aspects of teaching and learning with CODAP. Group members' interests vary and are broad in scope, so the group is continuing to meet to identify possible overlap in interests for continued collaboration. Some potential areas for collaboration included seeking grant funding to expand the sampling tool for the American Community Survey CODAP microportal, developing tasks using the microportal for use with students, and potential study designs to investigate how students and/or teachers utilize CODAP to approach data investigations.

We have continued to meet in subgroups around these three areas, above, prepare this report, work on our 2026 proposal, complete the activities highlighted at the beginning of this report, and identify other areas of synergy. We are very excited and grateful to have such a collaborative and productive community of teachers, researchers, and teacher educators focused on moving our careers and our field forward.

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GENDER AND SEXUALITY WORKING GROUP: CHALLENGING THE NEUTRALITY MYTH IN MATHEMATICS EDUCATION

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Since 2018, the Gender and Sexuality Working Group has convened annually at PME-NA. The past 2 years, our focus has been on concealable stigmatized identities, culminating in the presentation and exploration of data last year from a study involving pre-service teachers that was conducted by the leadership team. Building on this study's findings, wherein many pre-service teachers questioned the need to consider gender and sexuality in mathematics teaching and learning, in this year's working group, we will explore the neutrality myth in mathematics education—that mathematics is unbiased, objective, and value-free—with a focus on gender and sexuality. During the working group sessions, participants will have the opportunity to explore how the neutrality myth impacts mathematics and mathematics education, both in terms of teaching/learning and research, through discussions and hands-on activities.

Keywords: Gender; LGBTQIA+; Equity, Inclusion, and Diversity

More than perhaps any other subject area, mathematics is associated with myths surrounding its purported neutrality. Arguments are made that mathematics is "just numbers" and "has one right answer," so it is not impacted by sociocultural factors that play a more obvious role in other subject areas, namely the humanities and social sciences (Ernest, 2021; Lue & O'Neal Turner, 2020). Scholars working in the diversity, equity, and inclusion spaces have questioned the presumed neutrality of mathematics, mathematics education, and mathematics education research, for decades (e.g., Burton, 1995; Ernest, 1998; Smith, 1994). However, due to the pervasive nature of this neutrality myth, there is pushback—from parents, teachers, administrators, and the general public—on inclusive pedagogical practices and topics such as social justice mathematics (Gómez Marchant et al., 2023; Mamolo & Pinto, 2015).

Theoretical Background

The neutrality of mathematics, with regard to gender and sexuality, has been challenged in the discipline of mathematics, in the field of mathematics education, and in mathematics education research (e.g., Frankenstein, 1983; Yankova, 2024). This challenge can be framed using Rands' (2009) four forms of gender education. Although Rands focuses on gender, these ideas can be equally applied to sexuality. The view of mathematics as neutral often leads to *gender-free* or *gender-blind education*, whereby it is believed that "gender can and should be ignored in educational contexts and … is irrelevant to education" (p. 425). However, sometimes the harmful impacts of mathematical neutrality (e.g., as Eurocentric or hegemonic) can be represented in *gender-stereotyped education*, where binary gender stereotypes are embedded within educational systems (e.g., implicit bias against girls in mathematics). Educators can work to counteract the previous two forms of education through *gender-sensitive education* or *gender-complex education*. In mathematics, gender-sensitive education could entail addressing issues of

gender in content or pedagogy directly, but often still adhering to the gender binary. In contrast, gender-complex education goes beyond the gender binary and addresses multiple intersecting levels of privilege and oppression. In mathematics, gender-complex education might entail questioning how gender is represented in mathematics problems in textbooks (e.g., Parise, 2021).

History and Goals of the Working Group

During the past two PME-NA conferences, the Gender and Sexuality Working Group (GSWG) concentrated on concealable stigmatized identities (CSIs; Hall et al., 2024; Piatek-Jimenez et al., 2025). In 2024, the focus of the GSWG sessions was intersectionality and CSIs. Participants delved into these topics, both generally and with respect to their own experiences, and hypothetical case studies reviewed during the sessions. Based on suggestions from participants at the GSWG sessions, the leadership team conducted a small research project in which they collected questionnaire data from 95 pre-service teachers in Australia, Canada, and the U.S. Specifically, participants were asked to consider how four CSIs—gender, sexual orientation, mental health, and socioeconomic status—impacted their own mathematics learning, as well as how these identities might influence their future students' mathematics learning.

During the 2025 GSWG sessions, initial findings from this study were shared as aggregate overviews and six case studies. Although nearly all questionnaire participants could readily provide examples of ways that mental health and socioeconomic status may play a role in mathematics education, a large proportion questioned how gender and/or sexual orientation would be relevant to mathematics teaching and learning. Sharing these findings led to discussions in the group about the study participants' views of mathematics as neutral, asocial, and competitive. After reflection, the leadership team determined that the conception of mathematics as neutral would be a worthwhile topic for further exploration in the 2026 GSWG.

Description of This Year's Working Group Sessions

The GSWG sessions will follow a three-part structure, where various aspects of mathematics education will be addressed with respect to the neutrality myth, focusing on gender and sexuality. In *Session 1*, the focus will be mathematics itself. Members of the leadership team will give a brief presentation to acquaint participants with the topic of neutrality in mathematics, as well as explain the impetus of this year's GSWG theme based on last year's group discussions. The remainder of the session will be devoted to small-group discussions regarding the reification of this myth in popular media and amongst the general public, with examples provided to analyze. *Session 2's* focus will be examining the neutrality of mathematics teaching and learning. Participants will be provided with practical hands-on examples of various types of mathematics problems (e.g., word problems) for analysis and discussion. For *Session 3*, the focus will then extend to challenging the neutrality of mathematics education research. We will share examples for analysis and facilitate discussions on future avenues of research. We anticipate that these discussions will lead to scholarly collaborations amongst the GSWG participants.

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GENDER AND SEXUALITY WORKING GROUP REPORT: INVESTIGATING PRE-SERVICE TEACHERS' EXPERIENCES WITH AND VIEWS OF CONCEALABLE STIGMATIZED IDENTITIES

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The Gender and Sexuality Working Group has convened at PME-NA each year since 2018. For the past two years, this working group has focused on the topics of intersectionality and concealable stigmatized identities (CSIs) in relation to gender and sexuality in mathematics education. During PME-NA 2025, the working group met and discussed data that were collected prior to the conference that related to pre-service teachers' experiences with and beliefs about the learning and teaching of mathematics in relation to their own and students' CSIs. Participants in the working group had the opportunity to explore and discuss both quantitative data and case studies.

Keywords: Gender; LGBTQIA+; Equity, Inclusion, and Diversity

History of the Working Group

The concept of the Gender and Sexuality Working Group emerged from conversations about equity in mathematics education during the 2017 PME-NA conference in Indianapolis, IN and the first official sessions were held during the 2018 PME-NA conference in Greenville, SC. Since then, the group has convened each year at PME-NA to support ongoing and important dialogue related to gender and sexuality in mathematics education. At these conferences, the working group leaders and participants addressed a range of topics, including theoretical and empirical work (Przybyla-Kuchek et al., 2022), conceptual and methodological approaches (Jackson et al., 2021), strategies for data collection (Ataide Pinheiro et al., 2023), and concealable stigmatized identities (CSIs; Hall et al., 2024; Piatek-Jimenez et al., 2025). This working group has consistently fostered an inclusive and supportive environment for graduate students and early-career scholars, offering spaces to present research, assume leadership roles, and collaborate with researchers from the United States, Canada, Brazil, and Australia.

Theoretical Underpinnings of This Working Group

Three theories form the foundation for the scholarly endeavors of this working group. *Intersectionality theory* (Crenshaw, 1989) is the understanding that social identities (e.g., gender, sexuality, race, social class) do not operate in individuals' lives independently but rather interact in meaningful ways in classrooms and other settings (Smooth, 2013). A second theory that explains educational phenomena and relates to this working group is *social identity theory* (Tajfel & Turner, 1979). The essence of this theory is that people tend to perceive themselves as being members of specific social identity groups and hold subjective beliefs about different groups and their relationships, which influences their perceptions and behavior (Ellemers & Haslam, 2012). Finally, *minority stress theory* is integral to this working group's efforts (Brooks, 1981). Through employing this theory, we acknowledge the role of the "excess stress burden that

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places sexual and gender minority people at greater risk for negative health outcomes [e.g., mental health] compared with cisgender straight people" (Frost & Meyer, 2023, p. 2).

Progress Made During and In Between Conferences

During the Gender and Sexuality Working Group sessions at the 2024 PME-NA conference, discussions centered on the multiple identities that individuals bring with them to the classroom and the influence that these identities can play in mathematics teaching and learning. We explored the role of a variety of identities, but discussion homed in on concealable stigmatized identities. Although having concealable identities allows students to have more control over which identities they share with others, the choice to do so often engenders stress and can result in feelings of invisibility and exclusion when students with CSIs are unable to bring their entire selves to the classroom (Keblbeck et al., 2024; Vaccaro et al., 2024). Many questions arose during the working group sessions surrounding these topics, including the roles that teacher education programs could play in assisting K–12 teachers to better understand the intersectional identities of their future students and relationships between these identities and classroom teaching and learning.

Between the 2024 and 2025 conferences, the leadership team of the working group designed and carried out a study in which we collected data from 95 pre-service K–12 mathematics teachers on their own personal experiences with learning mathematics in terms of four specific CSIs (gender, sexual orientation, socioeconomic status, and mental health) and their perspectives on how those same four CSIs might affect the mathematics learning of their future students. The participants were enrolled in four different universities in three countries (the United States, Canada, and Australia). We brought our initial findings from this study, in the form of both quantitative results and case studies, to the 2025 working group sessions for the participants to discuss.

2025 Working Group Sessions and Efforts Made to Include Participants

During the 2025 working group sessions, our goal was to delve into conversations with the working group participants about pre-service teachers' experiences with and knowledge of the roles of CSIs in mathematics teaching and learning. By exploring both CSIs that are pertinent to the topic of this working group (i.e., gender and sexual orientation) and CSIs that are not the direct focus of this working group but are underexplored in mathematics education research (i.e., socioeconomic status and mental health) and are important identities that interact with gender and sexuality, we aimed to better understand students' intersectional experiences. Throughout the working group sessions, we found that the working group participants were able to more deeply analyze societal views of mathematics as a field, and of gender and sexuality in general and in relation to mathematics, by exploring these macro-level identities. A day-by-day breakdown of our 2025 working group sessions follows.

Session 1

During Session 1, we began by having each attendee, including the members of the leadership team, introduce themselves and their academic interest in topics related to gender and/or sexuality in mathematics education. Members of the leadership team then gave a 15-minute presentation to acquaint participants with the topics of intersectionality and CSIs, including the history of the concepts and examples from mathematics education research. We also shared the Wheel of Power and Privilege (see Figure 1) with the group. The remainder of the session was devoted to small-group discussions regarding participants' experiences with CSIs in their research on gender and sexuality, as well as in their teaching. Some other CSIs that

participants discussed included learning disabilities, motherhood within the context of being a scholar, and non-motherhood in contexts outside of academia.

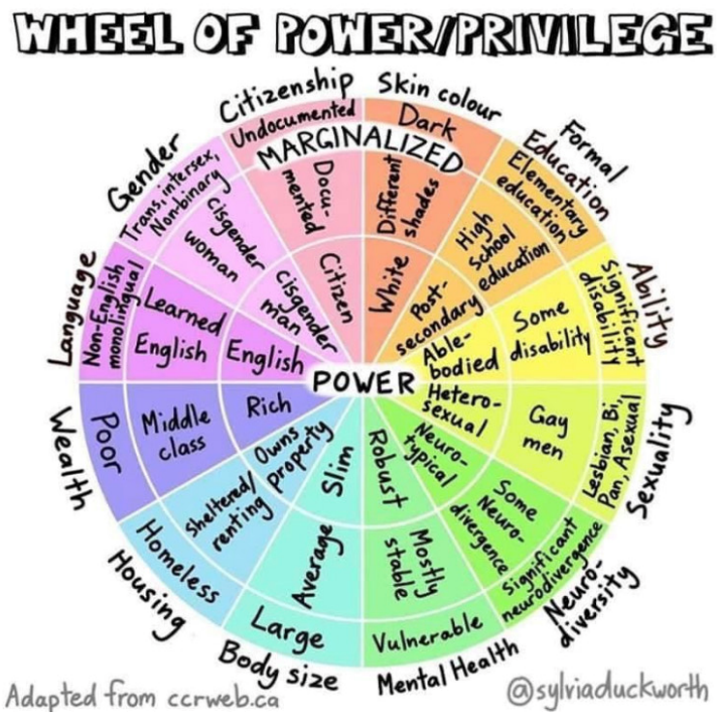


Figure 1: The Wheel of Power/Privilege (Duckworth, 2020)

Session 2

Session 2 began with a brief explanation of the data collection that was undertaken by the leadership team since the previous year's conference. The leadership team also shared several salient quantitative results that were identified during data analysis. This presentation of the data and main themes led to a fruitful and engaging whole-group discussion.

Following the whole-group discussion of these broader themes, participants divided into groups of three to four people each. The leadership team then distributed six case studies of research participants to each group to discuss. Small-group discussions focused on notable aspects of each case, implications for teacher education programs, and implications for future research on intersectionality and CSIs in relation to mathematics teaching and learning. At the conclusion of the session, each small group shared with the whole group highlights from their conversations.

Session 3

We began Session 3 with introductions of new participants. It was also noted that none of the small groups had completed their discussions of the case studies during Session 2, and working group participants demonstrated an interest in continuing this activity. Therefore, the first 30 minutes of Session 3 were spent discussing the case studies in small groups again. During the final hour of the session, a whole-group discussion took place in which participants shared over-arching themes their small groups discussed during the previous two sessions and ideas for future work in this area.

Some of the main themes noted were that many of the pre-service teacher study participants

saw socioeconomic status and mental health as being salient to the learning of mathematics; however, they viewed gender and sexual orientation as irrelevant to mathematics and/or to the mathematics classroom. Mathematics was viewed by most of the study participants as a subject that is "neutral" and therefore participants stated that gender and sexual orientation should not affect the teaching and learning of mathematics. These discussions led to working group participants making connections with previous research they had read, including Rands' (2009) gender-complex education perspective on mathematics education.

Questions to Address in the Future

The discussions held during the 2025 PME-NA Gender and Sexuality Working Group sessions prompted conversations about pre-service teachers' views on the nature of mathematics as being "neutral," which led to many pre-service teachers in the study to claim that a student's gender and sexual orientation would not and should not affect their learning in a mathematics classroom. However, this perspective is not supported by the literature (e.g., Makowski et al., 2025; Suárez et al., 2024). Holding this viewpoint, endorsed by the pre-service teachers who completed the questionnaire, may make them unaware of their own implicit biases (e.g., stereotypes) toward girls and women in mathematics, whose self-concepts and career intentions can then be negatively influenced (Lindner et al., 2022). Further, these same pre-service teachers recognized that when a student has low socioeconomic status or mental health concerns, these identities could directly affect their mathematics learning, which is inconsistent with their beliefs that other student macro-level identities are irrelevant to mathematics learning. These findings have direct implications for teacher education programs.

Based on the discussions held during the 2025 Gender and Sexuality Working Group sessions, the working group leadership team has identified two topics that we plan to address further during future working group sessions. First, we intend to explore more deeply the current research literature on mathematics as a neutral domain and its implications for who feels welcome and included within the field of mathematics and in the mathematics classroom. We also plan to explore more deeply how pre-service teachers' views of mathematics as being gender- and sexuality-neutral affect those who are underrepresented in mathematics and how teacher education programs can help to ameliorate the issue.

Acknowledgments

The working group leadership team would like to acknowledge the pre-service teachers who participated in our study and the 2025 PME-NA attendees who participated in the working group sessions.

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DEVELOPING THE NEXT GENERATION OF ELEMENTARY MATHEMATICS TEACHER EDUCATORS

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There is a need to understand: 1) what steps PhD programs are currently taking to prepare the next generation of elementary mathematics teacher educator (EMTE) s; 2) what type of preparation and support EMTE require to develop expertise (Goos, 2025); and 3) to generate ideas for PhD programs or for targeted professional development for future or induction EMTE who have had little to no opportunity in their graduate program to develop practices in teaching K-6 mathematics teaching. Thus, we propose a new working group addressing the topic of Developing the Next Generation of Elementary Mathematics Teacher Educators.

Keywords: Mathematics Teacher Educators, Doctoral Education, Elementary School Mathematics Education

Theoretical Background

Existing research has identified significant differences in the work of teaching mathematics and teaching mathematics teaching (Shaughnessy et al., 2016). These differences are especially evident for elementary mathematics teacher educators (EMTE) who are preparing to work with prospective elementary school mathematics teachers. EMTE teaching mathematics teaching are often challenged by the limited theory and resources devoted to teaching mathematics teaching to preservice K-6 teachers. While hypothetical learning trajectories exist for mathematics concept development, such trajectories for pedagogical concept development (Simon, 2018), a central element to designing opportunities for children's mathematics learning, are limited. In addition to the challenge of theory and resources, a more basic challenge lies in the development of MTEs in general (Chapman, 2008). PhD programs are primarily designed to support future faculty as researchers (Reys, 2018), leaving expertise in teaching mathematics teaching to develop through experience (Chauvot, 2008; Wilson, 2006). There is an ongoing need to support the development of MTE as teachers of teaching, a role most PhD students will assume (Shih et al., 2020).

While there is little research insight into the preparation of future MTEs (Chinen & Din, 2024), descriptions and studies of MTEs learning to teach teaching have grown and illustrate the significant challenges novice MTEs face (e.g., Simpson, 2019) as they begin their work with preservice elementary teachers with little to no preparation. Goos (2025) noted that MTE expertise and development is an under researched field particularly given its complexity and the scrutiny of policy makers regarding content and learning opportunities programs provide for future elementary teachers. In addition, the goal of EMTE preparation should reach beyond "surviving" to developing expertise in teaching prospective elementary mathematics teachers. This next generation will be situated to draw from tools prior generations of EMTE have not conceptualized as part of elementary mathematics education including the integration of AI, data science, and computer science. These findings suggest that developing a research agenda

including researchable questions and professional learning opportunities leading to a knowledge base that can inform advocacy and development efforts for the next generation of EMTEs is a critical need.

Title, Topic, and Goals

There is a need to understand: (1) what steps PhD programs are currently taking to prepare the next generation of elementary mathematics teacher educators; (2) what type of preparation and support EMTE require to develop expertise (Goos, 2025); and (3) to generate ideas for PhD programs or for targeted professional development in K-6 mathematics teaching for future or induction EMTE who have had little to no opportunity in their graduate program to develop this practice. Thus, we are proposing a new working group addressing the topic of Developing the Next Generation of Elementary Mathematics Teacher Educators. The proposed working group will be led by the authors, who collectively have taught elementary mathematics methods for over 40 years and have published numerous research articles on activities and outcomes of their elementary methods courses.

Session Organization

Session 1 - What Do Future EMTE Need to Develop Expertise?

Our first session will be focused on introductions (10 minutes) and brainstorming what type of preparation, knowledge bases, experiences, dispositions, and support future or induction EMTE require to develop expertise for teaching prospective elementary mathematics teachers (30 minutes). We will pose the following question for each board, "What preparation (knowledge bases, experiences, dispositions, support) do EMTE require to develop expertise?" Following the group brain writing we will examine each of the different boards (25 minutes) and organize and prioritize ideas (25 minutes) across the four "needs". Our goal by the end of the session is to have a list of prioritized knowledge bases, experiences, dispositions, and supports for future and induction EMTE.

Session 2 - What Are We Currently Doing to Prepare EMTE?

We will begin with a brief presentation (20 minutes) by the organizers addressing the idea of EMTE development and the various pathways faculty members may take in arriving at an elementary preparation position. We will leave space for discussion and comments throughout the presentation (15 minutes). Following the presentation, our goal is to examine our list of prioritized "needs" (20 minutes). Through discussion, we will map these to ideas or activities currently being enacted within participants' PhD programs (35 minutes). This step will provide us a rough idea of how we are currently addressing the needs for future and induction EMTE.

Session 3 - What Can We Be Doing to Prepare EMTE?

Participants will form small groups to work on knowledge bases, experiences, dispositions and supports we have prioritized as important but not currently being addressed within PhD programs. Groups will design ideas to address their chosen priority (35 minutes). Each group will then share their ideas and solicit feedback from the group (35 minutes). The final 20 minutes will be used to discuss next steps for the working group including the submission of a conference grant to support the development of EMTE and creating research and development groups to support the development of a knowledge base and create resources for EMTE developers.

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MATHEMATICS EDUCATION RESEARCHERS' INTERDISCIPLINARY COLLABORATIONS WORKING GROUP

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Interdisciplinary collaborations involving mathematics education researchers (MERs) have become common as an effective way to address large-scale problems and enhance education. While products of such collaborations are documented, how MERs engage in and experience interdisciplinary work remains understudied. This working group will examine theoretical perspectives and disciplinary practices shaping interdisciplinary collaborations involving MERs. Participants will share experiences, map them onto identified practices, analyze theoretical perspectives, and collaboratively develop actionable outcomes. Expected deliverables include an enhanced understanding of practices in interdisciplinary settings, comparative analysis of theoretical frameworks, and designs for future research.

Keywords: Interdisciplinary Research, Mathematics Education Research Practices, Communities of Practice

Mathematics education researchers (MERs) engage in interdisciplinary collaborations to address complex problems and enhance research (Bakker et al., 2021; Bruce et al., 2017; Lannin et al., 2023). Yet, research has noted challenges MERs face when collaborating across disciplines (e.g., Bruce et al., 2017; Darragh, 2022; Goos & Bennison, 2018; Roth, 2020; Suazo-Flores & Walker, in press). This new MER Interdisciplinary Collaboration Working Group will examine two guiding questions: (1) What theoretical perspectives elucidate interdisciplinary collaborations for MERs? and (2) What roles do practices play in empowering and constraining such collaborations?

Disciplines, Practices, and Interdisciplinary Work

Disciplines are communities of practice characterized by specific objects of interest, domains of knowledge, forms of discourse, and methodological approaches (e.g., Pimm, 2009; Roth, 2020; Williams et al., 2016). Disciplinary activities empower and constrain practitioners by supporting the production of disciplinary knowledge while limiting interactions with practitioners from other disciplines (Roth, 2020; Suazo-Flores & Walker, in press; Williams et al., 2016).

As members of a community of practice (Frade et al., 2009; Goos & Bennison, 2018; Wenger, 1998), MERs use disciplinary practices to conduct scholarly work (Suazo-Flores et al., 2021, 2024). Practices are methods or techniques that become tacit, spontaneous, and automatic ways of operating or knowing (Schön, 1983; Wenger, 1998; Williams et al., 2016). We define practices as "established or emergent ways of being, operating, and interacting with others within a community" (Suazo-Flores et al., 2021, p. 24). Being practices refer to how people view themselves and others, including recognition of expertise and roles. Operating practices are agreed upon procedures that acknowledge institutional policies to complete work. Interacting practices relate to negotiating meaning, setting standards for communication, and developing consensus (Suazo-Flores & Walker, 2024). Through a thematic analysis of research literature (Suazo-Flores et al., 2021) and a qualitative content analysis of interview data (Suazo-Flores et

al., 2024), we identified practices in interdisciplinary groups containing at least one MER, organized by the being, operating, and interacting categories (Table 1).

Table 1: Practices in Interdisciplinary Collaborations Identified from Data Analysis

Category	Practice	Short description
Being	View of self	Individual identity, dispositions, and roles in the group
	View of others	Perceptions of disciplinary expertise, roles, and influence
Operating	Common goals	Shared purposes and objectives for collaborative work
	Working together	Processes used to get work done
	Common research interests	Sense of community through research interests
	Understanding institutional support	Recognition of structural and institutional norms
Interacting	Developing ideas	Negotiation of meaning, finding shared meanings
	Building trust and respect	Cultivation of understanding, appreciation of others' points of view

MER Interdisciplinary Collaboration Working Group Goals and Activities

This Working Group will convene a community of practitioners who will collaboratively review theoretical frameworks, examine practices, and develop actionable insights for creating and sustaining productive interdisciplinary collaborations. On Day 1, the organizers will present a brief synthesis of our experiences and research on practices in interdisciplinary collaborations. Participants will share their own experiences in interdisciplinary collaborations, map them onto the identified practices (Table 1), and note any missing practices. On Day 2, participants will engage in discussions of selected articles addressing interdisciplinary collaborations across three categories. First, articles on theories for understanding interdisciplinary work (e.g., boundary crossing, communities of practice, care-driven approach; Akkerman & Bakker, 2011; Bakker et al., 2021; Goos & Bennison, 2018; Noddings, 2010; Williams et al., 2016); second, articles presenting examples of collaborations (e.g., Bruce et al., 2017; Darragh, 2022; Gardner et al., 2021; Goos & Bennison, 2018; Krummheuer et al., 2013; Neihaus & Kaschner, 2023; Norton et al., 2019); and third, articles offering reflections on interdisciplinary work (e.g., Cai et al., 2020; Frade et al., 2009; Steffe, 2010). In small groups, participants will compare Day 1 experiences and examples to theoretical perspectives, describe the role of practices based on the literature, and propose additional perspectives that may enhance understanding of MERs' interdisciplinary work. On Day 3, participants will synthesize insights to develop actionable outcomes by choosing one collaborative project: (1) creating a theoretical framework for understanding practices in interdisciplinary collaborations, (2) using practices to describe MERs' practical engagement in such partnerships, or (3) developing a case study design for investigating an interdisciplinary collaboration involving MERs. Groups will present their work-in-progress and receive feedback. The session will conclude with a discussion of future research directions,

proposals for presentations at future PME-NA conferences, potential publications, and strategies for continued collaboration beyond the conference.

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CRITICAL DISABILITY STUDIES IN MATHEMATICS EDUCATION: ASSET-BASED PEDAGOGY

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The Disability Studies and Mathematics Education Working Group will be organized around the theme of Critical Disability Studies in Mathematics Education as Asset-Based Pedagogy. Grounded in the understanding that disability is a source of knowledge, innovation, and transformation in mathematics education, participants will engage in self-selected small groups around topics such as: (a) Theoretical Deep Dives into Disability Justice, Disability Critical Race Theory (DisCrit), and Critical Disability Studies; (b) Asset-Based Perspectives on Mathematical Learning and Disability; (c) Accessible and Inclusive Mathematics Classrooms and Educational Design; and (d) Higher Education, including graduate student and early career scholar development. Other topics may also emerge based on participants' interests during the working group. Everyone is welcome!

Keywords: Students with Disabilities; Culture and Asset-Based Pedagogies, Special Education; Equity, Inclusion, and Diversity

"Taking up space as a disabled person is always revolutionary" (Ho, 2020, p. 115).

History

Kai Rands and James Sheldon created this working group, which met formally from 2016–2018. During these early meetings, members discussed theoretical perspectives within their work, described current projects, and raised critical issues. This core group of faculty, graduate students, disability activists, and classroom teachers also included Amber Candela, Jessica Hunt, Rachel Lambert, Katie Lewis, Paulo Tan, and Cathery Yeh. The working group reconvened in 2022 and has met each year since. Members collaborate to conduct research, write grants, present at research and practitioner conferences, and produce manuscripts.

Goals and Effort Toward These Goals

As we prepare for PME-NA 48, we center the perspectives developed by disabled people that have been often overlooked in mathematics education (Lambert & Tan, 2020), even within calls for equity. Because mathematics classrooms are often used as tools for marginalization, we also use critical theories that have an asset-based stance within their tenets such as Disability Studies in Education, CRT, and DisCrit to offer a justice-oriented vision of mathematics education.

PME-NA 48 marks the eighth meeting of this working group. Therefore, our goals encompass the continuation of past work as well as the development of new work:

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

- Create more accessible space centering disability and mathematics education;
- Anchor our work in the 10 principles of Disability Justice (Sins Invalid, 2019);
- Generate interdisciplinary and cross-institutional collaborations; and
- Leverage the group's collective wisdom to (re)imagine solutions to persistent problems.

Our strategies to meet these goals include:

- Providing a hybrid option for the working group;
- Centering the voice and work of disabled scholars and educators;
- Designing spaces that prioritize focused, small-group work; and
- Identifying small-group leaders to facilitate the cross-pollination of ideas.

Theoretical Background

We adopt disability as asset-based pedagogy in which disabled people—researchers, practitioners, and students—offer a powerful transformational, epistemic lens. This stance draws on feminist philosopher Garland-Thompson's (2011) construct of the "productive power of misfitting;" Being disabled in a world that is not made for us illuminates structural aspects of the environment that others miss. The acts of adaptation that disabled people do to navigate a world not built for them reveal alternative ways of knowing and being.

We adopt this asset-based stance to disrupt ableism through Disability Studies in Education and Critical Special Education (Connor et al., 2008; Connor, 2019; Lewis & Mason, 2025), interrogating how ideologies of "ability" and "normalcy" (Annamma et al., 2013; Siebers, 2008) are constructed within mathematics education (Dolmage, 2017; Price, 2021; Westby, 2021).To enact this pedagogy, we draw upon the 10 principles for disability justice created by Patty Berne and others (Sins Invalid, 2019), including intersectionality, recognizing wholeness, and collective liberation. This framework situates ableism within intersecting systems. We emphasize the inherent assets of disabled people, including unique mathematical ideas, pedagogies, perspectives, and solutions. We challenge deficit narratives to unsettle entrenched structures of ableism in mathematics education (Lewis & Lynn, 2018; Yeh, 2023).

Plan for Active Engagement and Working Group Organization

In response to the growing size and diversity of our working group members, and with respect to the interests that surfaced in last year's group, we have designed a structured, flexible three-session sequence. Group members have choice in how and with which topics they engage. Small groups have autonomy to define what "meaningful outcomes" mean for their members and a timeline for achieving them. These strategies promote active and purposeful engagement.

Session 1: Community, Connections, and Co-Designing

This session orients people to the working group. We start with introductions and access needs, followed by an update about the group's accomplishments, sharing the goals of the group, and frame of disability as asset-based pedagogy as educational design and research. We then introduce small-group topics and identify additional small groups that may arise. Small-group topics include the integration of asset-based pedagogy as educational design and research, but are not limited to: (1) conference accessibility (2022–2026); (2) engaging graduate students and early career scholars (2024–2026); (3) halting state "science of math" policy (e.g., addressing the "Science of Math" website; growing out of work from 2022); (4) building accessible classrooms (2025–2026); and (5) theoretical deep dives into Disability Justice, DisCrit, and Critical Disability Studies (2024–2026). This session concludes with small-groups articulating goals.

Session 2: Generating and Planning

This session will be primarily in small groups. We briefly (re)introduce people, share access needs, and our framing of disability as asset-based pedagogy in educational design and research (15 min). Groups identify actions toward meeting goals from the first session. We transition into small groups identified in Session 1. We conclude by coming back and sharing.

Session 3: Articulating Next Steps

We begin in groups, reconvene as a whole group to share, and conclude by articulating plans for support beyond the conference. This includes ongoing collaboration and advocacy through both whole-group and subgroup venues (e.g., a 2025 subgroup already collaborating via Discord, where a conference paper has been accepted and new members invited).

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WORKING GROUP REPORT: CRITICAL DISABILITY STUDIES IN MATHEMATICS EDUCATION: EDUCATING FOR CHANGE

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In continuing with past working groups on research at the intersections of Disability Studies and Mathematics Education, this working group met at PME-NA 47 to share new developments from group members, form new connections, and advance shared goals. Folks engaged in self-selected small groups around these topics: (a) Research Colloquium About Conference Accessibility, (b) Engaging Graduate Students and Early Career Scholars in this Network, (c) Building Accessible Classrooms, (d) Halting State Level "Science of Math" Policy; (e) Book Group .

Keywords: Students with Disabilities; Special Education; Disability Studies; Equity, Inclusion, and Diversity

History

Kai Rands and James Sheldon conceptualized and convened this working group, which met formally from 2016–2018, although other informal meetings occurred before and after these dates. In these early sessions, group members explored different theoretical perspectives within their work, shared ongoing projects, and raised critical issues. This collective of faculty, graduate students, disability activists, and classroom teachers included: Amber Candela, Jessica Hunt, Rachel Lambert, Katie Lewis, Paulo Tan, and Cathery Yeh. The working group convened in 2022 and has met yearly at PME-NA, collaborating on research, grants writing manuscripts and book publications, and engaging with the Mathematics Education, Special Education, and disability communities.

Work at PME-NA 47

Across the three sessions at PME-NA 47, approximately 30 people engaged in our hybrid working group. This hybrid approach expanded access to participate and engage for those who would have otherwise been excluded from an in-person-only space. The decision to adopt a hybrid model emerged from group members' concerns about COVID-19, travel, in-town transportation, and the challenges caregivers and disabled scholars and educators face when being away from home for extended periods. Our group reflected on how hybrid access fostered greater inclusivity, particularly for disabled, immunocompromised, and otherwise marginalized scholars (Doran et al., 2024; Joo, 2021). Hybrid access also enabled organizers and participants to build relationships and collaborations before, during, and after the conference. Across three days, we built community, met in small groups, and developed plans extending beyond the

Leatham, K. R., Peterson, B. E., Siebert, D. K., & Teuscher, D. (Eds.). (2026). *Proceedings of the forty-eighth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Brigham Young University. https://doi.org/10.51272/pmena.48.2026

conference itself. We consciously took steps to structure communication across modalities, demonstrating that collaborative, generative work can thrive in hybrid environments. We created a set of shared documents to capture meeting notes, conversations, and ideas.

Day 1: Community, Accomplishments, Small Groups

Our group began with introductions and an explicit invitation to share access needs, a practice emphasized by activists and scholars (e.g., Reinholz & Ridgway, 2021; Sins Invalid, 2019). Participants named their roles, pronouns, and needs. For example, "I am [name], and I am at [university]. My pronouns are she, her, hers, and it's important that I am able to get up and stretch; I need to take breaks periodically," or, "My access needs are being met through having the virtual environment." These openings foster camaraderie as members recognize shared needs and collectively consider how to meet them. These introductions also create opportunities to connect around professional roles or interests, where surfacing how taken-for-granted professional norms can unintentionally reproduce ableism. One of the common access needs involved attending in "nonstandard" ways; for example, for many people, attending virtually was an access need that was being met by this group but not at the conference more generally.

During these introductions, we documented names and contact information in a shared electronic file to support connections during and after the conference. One commonly expressed need was the availability of a virtual participation option. Throughout the session, organizers regularly checked that access needs were being met and that communication across both modalities remained smooth and inclusive. One purposeful action to improve access and inclusion was to invite all in-person attendees to join the same Zoom call as the virtual attendees. This move facilitated more equitable participation for everyone. This intentional structuring affirmed virtual participants as full contributing members of the community and prompted critical reflection on the implicit ableism that virtual participation is often viewed. Several members noted disappointment that similarly accessible structures were not available across the broader conference, limiting participation for many marginalized scholars. We updated attendees on the progress the group had made since the last meeting. Accomplishments included: (1) Meetings with PME-NA and other conference organizers to address issues of conference accessibility in which update was shared that, AMTE and RUME were offering a virtual option for their upcoming conferences, led by Alison Mirin and Paulo Tan and coordinating with Zareen Aga, Ayman Aljarrah, and Sabrina Zarza. (2) Update on book proposal on disability as driver of change, led by Cathery Yeh, Katie Lewis, and Paulo Tan. (4) A manuscript which employs Discourse Analysis to make sense of the "Science of Math" website led by Erica Mason and Katie Rupe is currently under review for publication.

Five small groups formed and set their own agendas for subsequent meetings: (a) a research colloquium on conference accessibility (b) engaging graduate students and early career scholars in this network; (c) building accessible classrooms, and (d) halting state level "Science of Math" Policy.

Day 2: Small Group Work

Our next meeting afforded the opportunity to enact the value that access is a collective responsibility, thereby moving toward a collaborative actualization of Disability Justice (Piepzna-Samarasinha, 2019). One small concrete example of this occurred as people arrived at the session. During Day 1, a group member explained how dimmed lighting improved access for them. This need was preemptively included on Day 2 as well without needing to be requested again. They did this because increasing access meant we all had a greater understanding of each other's access needs and were thus all able to honor those needs and engage in deeper work. In-

person attendees also continued the practice of joining the Zoom meeting and encouraging new attendees to also join the Zoom meeting.

With several new attendees, we began by (re)introducing ourselves and conducting another access check-in. We then worked in small groups for the majority of the session before reconvening as a whole group to share and brainstorm next steps.

Day 3: Small Group Work and Articulating Next Steps

On our final day we dedicated the majority of the time to small group work. Each small group shared aloud to the whole group. Below are the key takeaways from each group.

Research colloquium about conference accessibility. The small group highlighted that conference accessibility must move toward concrete, operationalized change that broadens participation. Their central aim is to develop actionable recommendations and a research colloquium that reframes accessibility as a collective responsibility that includes virtual options, intentional norm-setting around access needs, inclusive networking structures, and responsiveness to accommodation requests without debate. The group discussed different math education organization conferences and examining lessons learned from their virtual or hybrid conferences. The group seeks to produce tangible demands, a formal report, and sustained scholarship that aligns conference practices to expand access for disabled, caregiving, queer/trans, and financially constrained scholars.

Engaging graduate students and early career scholars in this network. This group centered on building intellectual community and infrastructure for graduate students working at the intersection of disability studies and mathematics education. The group discussed developing shared grounding in key literature to establish an academic lineage and collective scholarly identity, while fostering collaboration through shared research interests, writing partnerships, and contact networks. A major outcome is the plan to launch an open-access, graduate student–led journal (with Paulo Tan as Editor-in-Chief and faculty support) and potentially housed in an existing digital repository. The journal will welcome diverse formats (completed research, works in progress, theoretical pieces, and multimodal submissions such as audio or video) and will be supported by an editorial board, a reviewer pool, and a defined review process. The group also discussed sustaining the community through regular meetings, networking to identify contributors and collaborators, and exploring models such as workshops (e.g., TODOS) that provide structured feedback for emerging scholars.

Building accessible classrooms. This small group focused on critically examining the implementation of *Building Thinking Classrooms* (BTC) to ensure that practices are accessible, equitable, and grounded in social justice. Members emphasized understanding the principles behind BTC, rather than treating it as a checklist, and highlighted the ways uncritical adoption can harm marginalized students, including disabled, Black, and Brown students. The group discussed producing publications that make these insights accessible to others with key priorities centering intersectionality, acknowledging historical inequities in classroom practices, and framing accessibility as a collective responsibility. Actionable steps involve developing article outlines, coordinating collaborative writing, maintaining shared resources via a Google Drive and Discord, and ensuring ongoing discussion of equity, inclusion, and accessibility in both research and classroom practice. This work has resulted in an accepted presentation for TODOS 2026.

Halting state level "Science of Math" policy. The group focused on responding to state-level "Science of Math" (SOM) policies, highlighting that SOM promotes a narrow, misleading, and one-size-fits-all approach to mathematics education that ignores the complexity of learners,

classrooms, and research evidence. Key concerns raised included selective citation practices and the elevation of certain pedagogical methods while marginalizing other approaches. Action steps include developing op-eds, conducting state-by-state analyses, leveraging existing initiatives, and creating visual summaries of research to clarify what constitutes robust, equitable, and relevant math research. The overarching goal is to provide evidence-based, inclusive alternative publications and formats that are more accessible and with broader reach.

Progress Since PME-NA 47

Small groups have continued work on several collaborative projects. At this time, there are no updates to report about the progress of conference-generated initiatives; several ongoing projects have continued to progress. (1) Erica Mason and Katie Lewis have an upcoming virtual AERA session "Unpacking Critical Special Education", (2) working group members presented together for a symposium at MES Chile, and (3) AMTE offered a virtual option for their winter 2026 conference and TODOS is working towards virtual options and improved accessibility for the Summer 2026 Conference.

Actions to Include Participants

To include as many people as possible, we implemented several intentional strategies. First, by offering a hybrid option for our working group, we signaled to our colleagues and in-person conference attendees that we prioritized enacting our values of inclusivity and access. We deliberately employed best practices to ensure that those attending virtually could fully participate and were not viewed as "lesser" or secondary to in-person attendees. Further, when people joined on the second or third days, we took care to pause to introduce the newcomers and orient them to the ongoing work. Beyond the conference, we maintained the Google Group to support continued connection, community building and recruitment, positioning PME-NA as a focal site for those interested in critically engaging with mathematics education and disability.

Questions to Address in the Future

Critical questions that our group will continue to address include:

- How can we define "critical disability studies in mathematics education" in ways that center accessibility, equity, and social justice?
- How does "critical disability studies" intersect with "critical special education", and how can we leverage insights and/or disrupt traditional practices from both to inform research and practice?
- As mathematics educators, how might we collaborate meaningfully with scholars and practitioners from other disciplines, including special education, to advance inclusive and humanizing practices?
- How strategies can this group employ to promote activism and advocacy within conferences, institutions, and broader mathematics education networks?
- What kinds of conversations about disability and mathematics education are happening in local, regional, and international contexts and how can those inform the work of this group?
- What additional approaches are needed to address disability with race, gender, and other identities in mathematics education?

- How can we meaningfully engage and mentor doctoral students and early career scholars, building a community that supports ongoing critical scholarship within and beyond mathematics education?
- How can we support novice and experienced researchers in collaborating and learning with disabled students, educators, and scholars in ways that are ethical, inclusive and rigorous?
- What role can this working group play in expanding conference accessibility and inclusive practices particularly through partnerships by aligning with organizations' commitments to equity and inclusivity?
- How can we ensure our work remains relevant and impactful in the face of shifting political and institutional pressures around diversity, equity, inclusion, and access?

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